Finite Element Computational Homogenization of Nonlinear Multiscale Materials in Magnetostatics

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The increasing use of composite materials in the technological industry (automotive, aerospace, . . .) requires the development of effective models that account for the complexity of the microstructure of these materials and the nonlinear behaviour they can exhibit. In this paper we develop a multiscale computational homogenization method for modelling nonlinear multiscale materials in magnetostatics based on the finite element method. The method solves the macroscale problem by getting data from certain microscale problems around some points of interest. The missing nonlinear constitutive law at the macroscale level is derived through an upscaling from the microscale solutions. The downscaling step consists in imposing a source term and determining proper boundary conditions for microscale problems from the macroscale solution. For a two-dimensional geometry, results are validated by comparison with those obtained with a classical brute force finite element approach and a classical homogenization technique. The method provides a good overall macroscale response and more accurate local data around points of interest.

Index Terms—**Multiscale methods, computational homogenization, finite element methods, magnetostatics, nonlinear materials.**

I. INTRODUCTION

COMPOSITE materials are gaining great importance in the scientific and industrial communities. These materials may exhibit improved physical properties (young modulus, electric and thermal conductivity, electromagnetic shielding, . . .) that depend on the complexity of their microstructure. Over the last few years, several computational multiscale methods have been proposed to characterize composite materials, mainly in the frame of mechanical, fluid dynamic and thermal problems. The classical multiscale approach consists in using analytical or empirical techniques to derive an approximate model at the coarse scale of interest that accounts for the effects of the fine scale. The most popular methods are the averaging methods [1] and the homogenization methods [2]. However, purely analytical techniques are very limited for tackling real applications while the empirical approach lacks information about how microstructure changes affect the macroscale properties.

Computational multiscale methods address these issues. Among them, it is worth mentioning the Multiscale Finite Element Methods (MsFEMs) [3], and the Heterogeneous Multiscale Methods (HMMs) [4], [5]. The former construct adapted global basis functions for the macroscale problem by solving microscale problems. The latter solve the microscale problems for determining a homogenized or average quantity of interest that is directly transferred to the macroscale problem. Both approaches take advantage of the separation of scales, with possibly different governing equations for the considered scales. However, while the HMMs yield to a greatly reduced computational cost, MsFEMs are often as expensive as a brute force technique. A very popular HMM-type method is the so-called FE^2 method that applies FE to solve the micro and macro problems [6].

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In electromagnetics, the developments found in the literature concern almost exclusively homogenization techniques relying on analytical expression (e.g., mixing formulas like the Maxwell-Garnett rules) or asymptotic expansions of fields [7]. A homogenization technique based on a polynomial expansion of the variation of the induction throughout the thickness of the laminations is applied to magnetodynamic problems in [8] and [9] for the linear and nonlinear cases, respectively.

In this paper, we apply an FE computational method within the HMM framework to a nonlinear multiscale magnetostatic problem. The method couples problems at two different scales: 1) the macroscale problem that accounts for the slowly varying component of the full solution; 2) microscale problems that fully resolve the material inhomogeneities at the smallest scale. The macroscale solution is used to impose a source term for the microscale problems. Proper boundary conditions for the microscale problems are also imposed stemming from the consistency of the magnetic field at both scales. In turn, solutions of these microscale problems allow us to calculate the effective magnetic flux density and its derivative with respect to the magnetic field needed for the Newton-Raphson iterations of the macroscale problem.

II. MAGNETOSTATIC PROBLEM

The magnetostatic problem in a bounded domain $\Omega = \Omega_s \cup \Omega_s^C \in \mathbb{R}^3$ is defined by the following Maxwell equations and constitutive law [10]:

$$\mathrm{curl}\underline{h}(\boldsymbol{x}) = \underline{j}(\boldsymbol{x}), \quad \mathrm{div}\underline{b}(\boldsymbol{x}) = 0, \quad \underline{b} = \underline{b}(\underline{h}(\boldsymbol{x})), \qquad \text{(1 a-c)}$$

with $\boldsymbol{x}$ the spatial position, $\underline{h}$ the magnetic field, $\underline{b}$ the magnetic flux density, $\underline{j}$ the electric current density. Domain Ω_s contains the sources and Ω_s^C denotes its complement. Proper boundary conditions must also be imposed. Note that the constitutive law in the linear case becomes $\underline{b} = \underline{b}(\underline{h}(\boldsymbol{x})) = \mu\underline{h}$, with μ the permeability.

We use the scalar potential formulation and we assume that there is no current source and that the domain is simply connected so that, $\underline{h}(\boldsymbol{x}) = -\mathrm{grad}\phi(\boldsymbol{x})$ where ϕ is the magnetic scalar potential. Then, the weak form of (1b) reads: find $\phi(\boldsymbol{x})$ such that

$$-(\underline{b}(-\mathrm{grad}\phi(\boldsymbol{x})), \mathrm{grad}\phi'(\boldsymbol{x}))_\Omega = 0 \quad (2)$$

holds for all test functions $\phi'(\boldsymbol{x})$ in an appropriate function space [10].

III. Computational Homogenization Model

In a multiscale material, rapid spatial variations of the material properties induce rapid variations of the magnetic scalar potential $\phi^\varepsilon(\boldsymbol{x})$. The exponent ε refers to the ratio between the scale of the microstructures and the scale of the material or the characteristic length of external loadings, hence it denotes quantities with rapid spatial variations. We impose Dirichlet boundary conditions on $\phi^\varepsilon(\boldsymbol{x})$ on some parts of the boundary which serve as a source. The multiscale magnetic field becomes $\underline{h}^\varepsilon(\boldsymbol{x}) = -\mathrm{grad}\phi^\varepsilon(\boldsymbol{x})$ and the weak form (2) reads: find $\phi^\varepsilon(\boldsymbol{x})$ such that

$$-(\underline{b}^\varepsilon(-\mathrm{grad}\phi^\varepsilon(\boldsymbol{x})), \mathrm{grad}{\phi'}^\varepsilon(\boldsymbol{x}))_\Omega = 0 \quad (3)$$

is verified for all ${\phi'}^\varepsilon(\boldsymbol{x})$ in a suitable function space.

Equation (3) can be solved in the whole domain using the finite element method. However this is very expensive in terms of memory and computation time due to the need of discretizing the unknown field at the smallest scale ε. A finite element computational homogenization method allows to overcome this problem and offers a good compromise between accuracy and computational cost. The method, based on the scale separation assumption ($\varepsilon \ll 1$), is illustrated in Fig. 1. A macroscale problem is defined on a coarse mesh covering the entire domain and many microscale problems are defined on small, finely meshed areas around some points of interest of the macroscale mesh (e.g., numerical quadrature points). In the following, the subscripts M and m refer to macroscale and microscale quantities, respectively.

A. Downscaling

From (3), the weak equation at the microscopic level reads:

$$-\left(\underline{b}^\varepsilon\left(-\mathrm{grad}\phi_m^\varepsilon(\boldsymbol{x})\right), \mathrm{grad}{\phi'}_m^\varepsilon(\boldsymbol{x})\right)_{\Omega_m} = 0 \quad (4)$$

where Ω_m is the microdomain. The microscale magnetic scalar potential ϕ_m^ε can be expressed in terms of $\phi_M^{lin}(\boldsymbol{x}) = \phi_M(\boldsymbol{x}_G) + (\boldsymbol{x}-\boldsymbol{x}_G)\cdot\mathrm{grad}\phi_M(\boldsymbol{x}_G)$, the "mean" macroscale component with slow variations linearized around the Gauss point $\boldsymbol{x}_G$, and ϕ_c^ε, a correction term that accounts for the rapid variations, i.e.,

$$\phi_m^\varepsilon(\boldsymbol{x}) = \phi_M^{\mathrm{lin}}(\boldsymbol{x}) + \phi_c^\varepsilon(\boldsymbol{x}). \quad (5)$$

For the nth multiscale iteration, $\phi_M(\boldsymbol{x})$ will be obtained through a macroscale calculation (see below). The weak equation (4) can be written as

$$-\left(\underline{b}^\varepsilon\left(-\mathrm{grad}\left(\phi_M^{\mathrm{lin}}(\boldsymbol{x}) + \phi_c^\varepsilon(\boldsymbol{x})\right)\right), \mathrm{grad}{\phi'}_m^\varepsilon(\boldsymbol{x})\right)_{\Omega_m} = 0 \quad (6)$$

for the unknown $\phi_c^\varepsilon(\boldsymbol{x})$.

This equation must be completed by the boundary conditions. To derive these conditions, we use the homogenization theory.

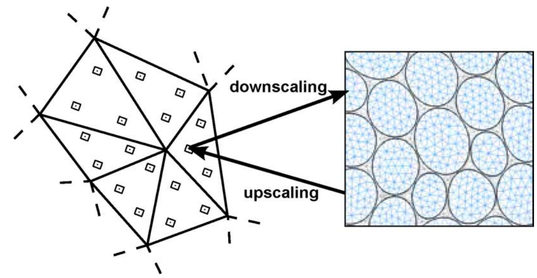

Fig. 1. Scale transitions between macroscale (left) and microscale (right) problems. Downscaling (Macro to micro): obtaining proper boundary conditions for the microscale problem from the macroscale solution. Upscaling (micro to Macro): effective quantities for the macroscale problem calculated from the microscale solution.

Applying the gradient operator to both sides of (5) and then integrating gives

$$\frac{1}{V_m}\int_{\Omega_m} \mathrm{grad}\phi_m^\varepsilon(\boldsymbol{x})\,\mathrm{d}\Omega_m = \mathrm{grad}\phi_M(\boldsymbol{x}_G) + \frac{1}{V_m}\int_{\Gamma_m} \boldsymbol{n}\phi_c^\varepsilon(\boldsymbol{x})\,\mathrm{d}\Gamma_m \quad (7)$$

where V_m and Γ_m are respectively the volume and the boundary of the microdomain Ω_m. Assuming that the average magnetic field at the microscale equals the average magnetic field of the macroscale projected onto the microdomain, we can write

$$\frac{1}{V_m}\int_{\Omega_m} \mathrm{grad}\phi_m^\varepsilon(\boldsymbol{x})\,\mathrm{d}\Omega_m = \mathrm{grad}\phi_M(\boldsymbol{x}_G) \quad (8)$$

which implies that the magnetic field is consistent between the macroscale and the microscale. The surface integral in (7) vanishes, which leads to periodic boundary conditions for the correction term ϕ_c^ε. One additional constraint must be defined to fix the level of the microscale magnetic scalar potential. We chose to cancel the mean value of the correction term

$$\frac{1}{V_m}\int_{\Omega_m} \phi_c^\varepsilon(\boldsymbol{x})\,\mathrm{d}\Omega_m = 0 \quad (9)$$

so that the macroscale solution is the average of the multiscale solution.

B. Upscaling

The upscaling consists in calculating the missing constitutive law for the macroscale model from the microscale solution. The macroscale problem is governed by the following nonlinear weak equation [3]:

$$-(\underline{b}_M(-\mathrm{grad}\phi_M(\boldsymbol{x})), \mathrm{grad}{\phi'}_M(\boldsymbol{x}))_{\Omega_M} = 0. \quad (10)$$

This equation can be solved using the Newton-Raphson method which uses the magnetic flux density $\underline{b}_M(-\mathrm{grad}\phi_M(\boldsymbol{x}))$ and its derivative with respect to the magnetic field $\underline{h}_M(\boldsymbol{x})$ as inputs. These quantities are not available at the macroscale level. They are calculated from the microscale solution as explained hereafter.

To determine the macroscale magnetic flux density, we equalize the magnetic co-energies at the microscale and the macroscale levels

$$-\frac{1}{V_m}\int_{\Omega_m} \mathrm{grad}\phi_m^\varepsilon(\boldsymbol{x}) \cdot \underline{b}_m^\varepsilon\left(-\mathrm{grad}\phi_m^\varepsilon(\boldsymbol{x})\right) \mathrm{d}\Omega_m = -\mathrm{grad}\phi_M(\boldsymbol{x}) \cdot \underline{b}_M(-\mathrm{grad}\phi_M(\boldsymbol{x})). \quad (11)$$

For a problem with a similar equation in heat conduction, it has been shown that when this condition together with the condition of consistency of the gradient of temperature (the analogue of consistency of the magnetic field (8)) are fulfilled, then the macroscale heat flux density (analogue of the macroscale flux density) could be calculated by averaging [6]. We have used this analogy to calculate the macroscale flux density as

$$\underline{b}_M(\boldsymbol{x}) = \frac{1}{V_m}\int_{\Omega_m} \underline{b}_m^\varepsilon(\boldsymbol{x})\, \mathrm{d}\Omega_m \quad (12)$$

so that the macroscale magnetic flux density is simply an average of the microscale magnetic flux density over the microdomain.

The derivative of the magnetic flux density with respect to the magnetic field $\underline{h}_M(\boldsymbol{x})$ cannot simply be averaged. We have used the numerical perturbation method [11] to calculate it. For each microscale geometry, 3 problems (4 for a three-dimensional geometry) are solved and we used Euler's method to numerically calculate the derivatives. In the case of the equation dealing with heat conduction, a semi-analytical method based on the condensation of the stiffness matrix at the microscale level was proposed [6].

IV. Application Example

As an application example, we consider a laminated core (0.2 m $\times$ 0.2 m^2) consisting of 101 laminations (thickness $d_l = 1.78$ mm) and 100 insulation layers (thickness $d_0 = 0.198$ mm, $\mu_r = 1$), so that $\varepsilon \approx 0.01$. The filling factor is $\lambda = d_l/(d_l+d_0) = 0.9$. The material of the laminations is considered as 1) linear with $\mu_r = 10$; 2) nonlinear with constitutive law

$$\underline{b}^\varepsilon(\underline{h}^\varepsilon(\boldsymbol{x})) = 1000\,\mu_0 \frac{\underline{h}^\varepsilon(\boldsymbol{x})}{(1+||\underline{h}^\varepsilon(\boldsymbol{x})||^2)^{0.485}}. \quad (13)$$

Dirichlet boundary conditions are imposed on a surface (1/3 of the width) at the center of the top and bottom laminations, respectively, $\phi = 0$ A and $\phi = 1$ A.

The reference FE solution is obtained on an extremely fine mesh of the whole stack consisting of 15 layers of 10 quadrangles for each lamination and 5 layers of 10 quadrangles for each insulation layer (i.e., 20150 elements in total). The microproblems are solved in a square domain with either two (3.96×3.96 cm^2) or three laminations and insulation layers, i.e., microdomain with dimensions 3.96×3.96 mm^2 or 5.94×5.94 mm^2. Each lamination is discretized with 13 layers of 5 quadrangles and each insulation layer with 5 layers of 5 quadrangles.

A. Linear Case

We compare our HMM-based computational homogenization approach with both a classical homogenization technique and a fine reference FE model. The coarse mesh used for both the macroscale level of the computational homogenization and the classical homogenization comprises 392 triangular elements. We consider 3 Gauss points per element, which leads to 1176 microproblems for each multiscale iteration.

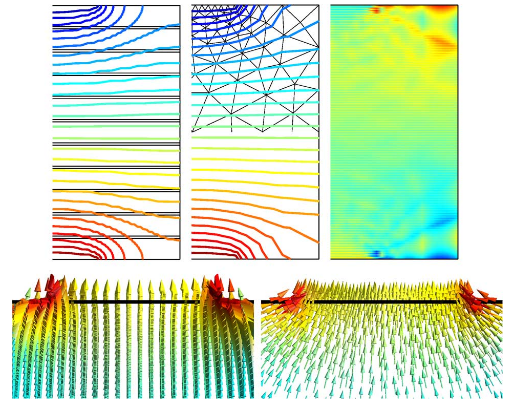

Fig. 2. Linear case—Top: Flux lines for the FE reference model (left) and the computational multiscale method (middle); error map (right). Normalised scale. Representation of the fine scale geometry (11 laminations instead of 101) and coarse mesh. Bottom: Zoom of the magnetic flux density near the top with imposed ϕ for the FE reference (left) and the computational multiscale models (right).

For the classical homogenization, we consider a homogenized domain with an anisotropic constitutive law $\underline{b} = \bar{\bar{\mu}}\, \underline{h}$ and the permeability tensor $\bar{\bar{\mu}} = (\mu_{||}, \mu_{||}, \mu_\perp, 0, 0, 0)$ with diagonal elements that account for the parallel and perpendicular fluxes, i.e., $\mu_{||}$ and $\mu_\perp$ can be written as [8]

$$\mu_{||} = \lambda\mu_l + (1-\lambda)\mu_0, \quad \frac{1}{\mu_\perp} = \frac{\lambda}{\mu_l} + \frac{1-\lambda}{\mu_0}, \quad (14)$$

where μ_l is the permeability of the laminations.

Flux lines obtained with the FE reference model and the computational multiscale approach are depicted in Fig. 2 (top-left and middle). The difference between the computational approach and the reference FE model is shown as well in Fig. 2(top-right): it is in interval $[-1.3, 1.6]\%$, with an average value equal to 0.299%. The magnetic flux density is also represented in 2(bottom). It is worth mentioning that the error in the vicinity of the surfaces with imposed ϕ is higher. A finer macroscale mesh would help enhancing this solution.

In Fig. 3, we show the magnetic scalar potential along a cut at $x = 0.875$ m. In this linear case, the classical homogenization gives an average result that follows the behaviour of the reference solution slightly better. However, the computational homogenization solution captures the variations of the solution of the microscale problem.

B. Nonlinear Case

In this case, the coarse mesh used for the macroscale level of the computational homogenization counts 160 triangular elements. We consider 3 Gauss points per element, what amounts to 480 microproblems for each multiscale nonlinear iteration.

In Fig. 4, one can see the flux lines of the reference and multiscale solution together with the associated error map (top). A detail of the geometry and the coarse mesh is depicted as well. The relative error is in interval $[-0.942, 0.945]\%$ with an average value of 0.0011%, which is better than in the linear case even though the mesh is coarser. This can be explained when realising the very small variation of the flux lines with regard to a 1-D problem, i.e., flux lines are nearly horizontal: see Fig. 2(bottom).

The magnetic scalar potential along a cut at $x = 1.666$ m is represented in Fig. 5. The computational homogenization solution fits perfectly well the average of the reference FE model.

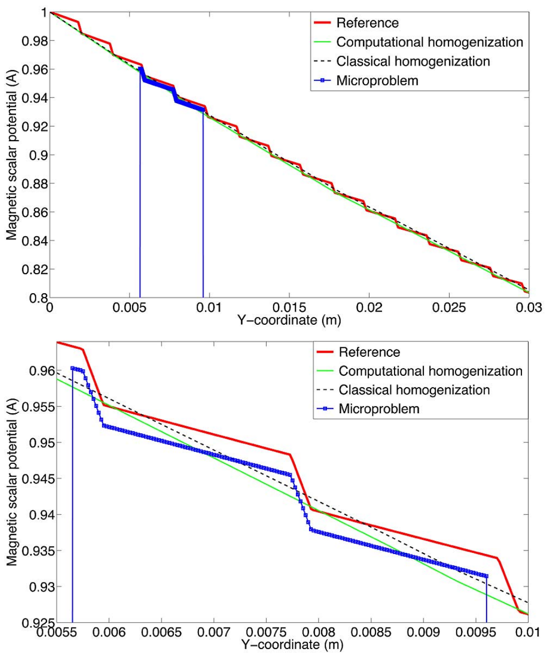

Fig. 3. Magnetic scalar potential at $x = 0.875$ m in the 3.96×3.96 cm^2 microdomain (2 laminations and 2 insulation layers) (up). Zoom between 0.05 and 0.1 (down).

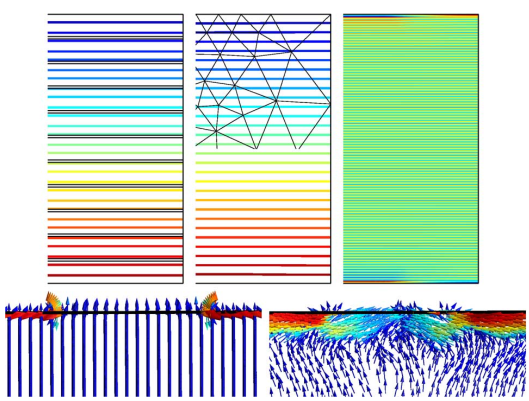

Fig. 4. Nonlinear case—Top: Flux lines for the FE reference model (left) and the computational multiscale method (middle); error map (right). Normalised scale. Representation of the fine scale geometry (11 laminations instead of 101) and coarse mesh. Bottom: Zoom of the magnetic flux density near the top with imposed ϕ for the FE reference (left) and the computational multiscale models (right).

Besides, an excellent agreement is observed between the microscale solution and the reference.

V. Conclusion

In this paper, we have developed an HMM-based computational homogenization technique for dealing with two-dimensional magnetostatic problems, which allows us to accurately determine the macroscale constitutive law. Furthermore, local data can also be recovered by means of solutions of microscale problems at points of interest. The huge computational cost (as independent microscale problems are solved at each Gauss point) can be reduced straightforwardly through parallelization of the computation of microscale problems. Further developments may concern the extension of the method to magnetodynamics.

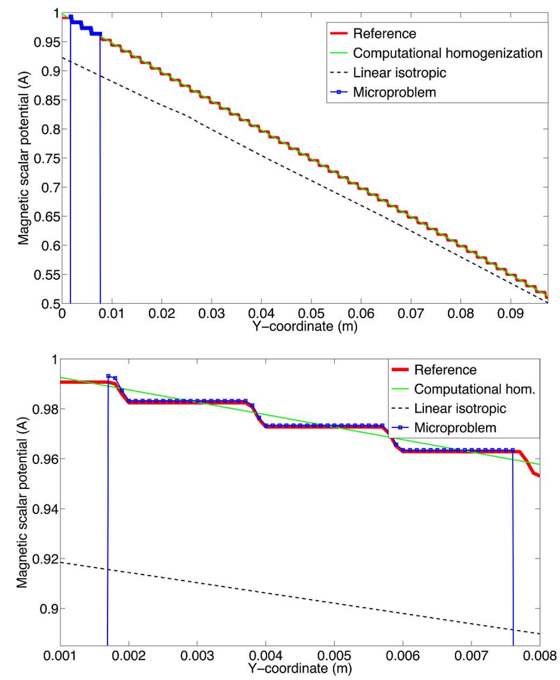

Fig. 5. Magnetic scalar potential at $x = 1.666$ m in the 5.95×5.95 cm^2 microdomain (3 laminations and 3 insulation layers) (up). Zoom between 0.0018 and 0.078 (down).

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