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STUDENTS' UNDERSTANDING OF PROPORTIONAL, INVERSE PROPORTIONAL, AND AFFINE FUNCTIONS: TWO STUDIES ON THE ROLE OF EXTERNAL REPRESENTATIONS

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ABSTRACT. We investigated students' understanding of proportional, inverse proportional, and affine functions and the way this understanding is affected by various external representations. In a first study, we focus on students' ability to model textual descriptions of situations with different kinds of representations of proportional, inverse proportional, and affine functions. Results highlight that students tend to confuse these models and that the representational mode has an impact on this confusion: Correct reasoning about a situation with 1 mathematical model can be facilitated in a particular representation, while the same representation is misleading for situations with another model. In a second study, we investigate students' ability to link representations of proportional, inverse proportional, and affine functions to other representations of the same functions. Results indicate that students make most errors for decreasing functions. The number and nature of the errors also strongly depend on the kind of representational connection to be made. Both studies provide evidence for the strong impact of representations in students' thinking about these different types of functions.

KEY WORDS: affine model, inverse proportional model, multiple representations, non-linearity, proportional model

Mathematics educators very often emphasize the (stimulating) role of multiple external representations in mathematics. As Matteson (2006) explains, learning mathematics is like learning a foreign language. External representations are key elements in the vocabulary of that language, and students need to become fluent in their use if they want to succeed in expressing and understanding mathematical ideas with correctness and precision. Vergnaud (1997) argues that external representations are inherent to the discipline of mathematics, since a characteristic of mathematical concepts is that they can *only* be communicated through their external representations. The use of multiple external representations has also been shown to facilitate mathematical problem solving (e.g. Duval, 2002; Even, 1998; Gagatsis & Shiakalli, 2004; Kaput, 1992; Yerushalmy, 2006). The NCTM Standards (1989) therefore hold a strong plea for establishing "mathematical connections" through the use of multiple external representations:

> Different representations of problems serve as different lenses through which students interpret the problems and the solutions. If students are to become mathematically powerful, they must be flexible enough to approach situations in a variety of ways and recognize the relationships among different points of view (p. 84).

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However, research has shown that students are not always sufficiently fluent in using external representations in the sense that they do not have the necessary diagrammatic knowledge to interact with the representations (de Jong, Ainsworth, Dobson, van der Hulst, Levonen, Reimann et al., 1998), to interpret representations by linking them with reality (Ainsworth, Bibby & Wood, 1998), or to translate and switch between representations within the same domain (Even, 1998). Another reason why students do not always benefit from using multiple external representations in problem solving is that they are unable to make flexible representational choices (Acevedo Nistal, Van Dooren, Clarebout, Elen & Verschaffel, 2009, Acevedo Nistal, Van Dooren, Clarebout, Elen & Verschaffel, 2010; Acevedo Nistal, Van Dooren & Verschaffel, 2012a, b).

Traditionally, making a flexible representational choice was understood and operationalized as selecting the external representation(s) which better matched the characteristics of a to-be-solved task, and consequently, research almost exclusively focused on task characteristics influencing this ability (e.g. Gilmore & Green, 1984; Sparrow, 1989; Vessey, 1991; Wickens & Andre, 1990). More recently, the role of student's own characteristics as a representational user, and of the context wherein the choice is made, has been acknowledged too (Verschaffel, Luwel, Torbeyns & Van Dooren, 2009). With respect to subject characteristics, research has for instance shown that students' conceptual and procedural knowledge about representations in general as well as their personal experience with a specific representation influence the representational choices they make and their ability to solve a mathematical task using a selected representation (Acevedo Nistal et al., 2009). A convincing example of the influence of the context on students' representational choices was provided by Acevedo Nistal et al. (2012b) who showed that students, especially from Science and Technology, felt a pressure to use formulas to solve mathematical problems in a classroom context. Tables and graphs were considered as more appropriate for informal contexts (e.g. to explain to a peer how to solve a problem), as backups (e.g. to check a solution obtained with the formula) or as mere information displays (e.g. to represent a solution obtained with the formula).

Most research on representational fluency and flexibility in mathematics learning focuses on the concept of function (e.g. DeMarois & Tall, 1999; Leinhardt, Zaslavsky & Stein, 1990) because functions are a prominent example in the domain of mathematics where different types of representations (such as graphs, formulas, and tables) can be used (Even, 1998). More specifically, most studies focus on linear functions. A main reason is that previous research has shown that students of various ages often exhibit a very limited understanding of linearity (De Bock, Van Dooren, Janssens & Verschaffel, 2007; Van Dooren, De Bock, Janssens & Verschaffel, 2008). One typical kind of error is that students tend to assume linearity in situations that are not linear at all, or assume a wrong linear relation (e.g. assume a proportional relation in an affine situation). Particularly proportional relations are frequently used outside their applicability range. For example, research has shown that a large majority of 10- to 12-year-old pupils answer 170 s to the word problem "John's best time to run 100 m is 17 s. How long will it take him to run 1 km?," whereby they assume a proportional relation in a situation that is not linear at all: The reality referred to in this problem allows no single, precise answer (Verschaffel, De Corte & Lasure, 1994; Verschaffel, Greer & De Corte, 2000). Another example was documented by Van Dooren, De Bock, Depaepe, Janssens & Verschaffel (2003): Many upper secondary students respond proportionally ($2 \times 1/6 = 2/6$) to probabilistic problems such as "The probability of getting a six in one roll with a die is 1/6. What is the probability of getting at least one six in two rolls?" Students' overreliance on linear—and specifically proportional—models has already been studied extensively in a variety of mathematical domains (e.g. elementary arithmetic, algebra, (pre)calculus, probability, and geometry) and, more recently, also in physics (De Bock, Van Dooren & Verschaffel, 2011). De Bock, Van Dooren, Janssens & Verschaffel (2002) and De Bock et al. (2007) explain this phenomenon by referring to (1) the intuitive, heuristic nature of the linear model for students, (2) students' experiences in the mathematics classroom and their beliefs toward mathematical modeling and problem solving, and (3) elements related to the mathematical particularities of the problem situation in which the linear error occurs. Efforts to remedy students' overreliance on linearity/proportionality by a concentrated and systematic instructional action (Van Dooren, De Bock, Hessels, Janssens & Verschaffel, 2004) or by creating an authentic context outside the mathematics classroom (Van Dooren, De Bock, Janssens & Verschaffel, 2007) did not yield entirely satisfactory results: Students' newly attained insights about (non-)proportionality proved to be not deep and lasting.

Some studies have pointed to the role of external representations in students' overreliance on linearity in general and specifically in students' overreliance on proportionality. In the domain of functions, for instance, the straight line graph prototype proved to be very appealing for many students. Leinhardt et al. (1990) mentioned several studies showing that students of different ages have a strong tendency to produce a linear pattern through the origin when asked to graph non-linear situations such as the growth in the height of a person from birth to the age of 30. Another example in this domain is the study by Markovits, Eylon & Bruckheimer (1986). When asked 14- to 15-year-old students to draw a graph of a function that passes through two given points, students typically drew straight lines. Similarly, Karplus (1979) found that when students interpolate between two graphed data points in a science experiment, they strongly tend to connect the points using a straight line. Although, as exemplified above, several studies referred to the influence of external representations on students' overreliance on linearity in general or on their overreliance on proportionality in particular, we are not aware of empirical studies focusing on the role of external representations on this well-known phenomenon. Consequently, the representational aspect remained a blind spot in the literature on students' overreliance on linearity.

In this article, we aim at setting a first step in unraveling the largely neglected relation between students' (lack of) mastery of external representations and their overreliance on proportionality. With that aim, we conducted two empirical studies. In the first study, which fitted into the research tradition of students' overreliance on linearity, we focused on the modeling aspect of functions. We investigated how accurate students are in connecting descriptions of realistic situations to various external representations of proportional, inverse proportional, and affine functions.[1] Since we hypothesized that external representations would play a role in students' tendency to inappropriately connect non-proportional situations to the proportional model, we also investigated in a second study how well these different representations are understood by students. In this second study, functions were no longer used as models of situations but as mathematical objects per se. More concretely, we investigated how accurate students are in connecting various representations of proportional, inverse proportional, and affine functions to other representations[2] of the same functions, without any contextualization of these functions.

Study 1: Connecting Representations of Functions to Realistic Situations

Aims and Rationale

In this study, we focused on students' modeling competencies with linear and non-linear functions. A key step in a modeling process is the translation of a problems situation in a mathematical model. Students' difficulties in selecting a mathematical model were addressed in a study with Swedish 12th graders on mathematical modeling competencies (Frejd & Ärlebäck, 2011). The framework of that study distinguished seven modeling sub-competencies. Selecting a model was one of them.

To measure this (and the other) modeling sub-competencies, a slightly adapted version of a research instrument developed by Haines, Crouch & Davis (2000) was used. In two items of their instrument, a "realistic" situation was described and students had to link this situation to an appropriate mathematical model to be chosen from five given alternatives. Models were either represented in a graphical or in a formula mode. The results indicated that selecting a model was one of the sub-competencies in which students showed the least proficiency in, both in the graphical or formula representational mode. Also the previously mentioned article by Leinhardt et al. (1990) already established a link between students' modeling competence and their preference for linear patterns.

Because the relation between students' modeling competence, their overreliance on proportionality, and their (lack of) mastery of representational modes was not yet systematically investigated, both in terms of mathematical models and in terms of accompanying representations, we set up this first study. More concretely, we investigated: (1) How accurate are students in connecting descriptions of realistic situations to proportional, inverse proportional, and affine models, and (2) do accuracy and model confusion depend on the representational mode in which a model is given? Because we expected that model confusion would be more likely to occur with models that share at least some characteristics with the proportional model and we anyway had to make a selection in the infinite domain of non-proportional models, we worked in this study with three specific types of non-proportional models that are conceptually most related to the proportional model ($y = ax$): inverse proportional models ($y = a/x$), affine models with positive slope ($y = ax + b$ with $a > 0$ and $b \neq 0$), and affine models with negative slope ($y = ax + b$ with $a < 0$ and $b \neq 0$). Arguably, these three models share some characteristics with the proportional model but not all. Affine functions for instance share with proportional functions the property that their graph has the shape of a straight line and that the same increase Δx in x always results in the same increase Δy in y. But while in proportional functions doubling x implies doubling y, this does not hold for affine functions. Inverse proportional models and proportional models share the proportionality characteristic, which means that the two variables are multiplicatively related (either the product or the ratio is constant), but, for instance, the shapes of their graphs are different.

Method

Sixty-five students from the first year of Educational Studies of the University of Leuven participated. These students had successfully finished secondary education and typically also 3 years of non-university higher education. Although they all followed the obligatory mathematics courses in secondary school, this was in most cases not the core of their curriculum. In these mathematics courses, solving realistic problems and the applicability of basic functions such as the ones central in our study receive quite some attention.

Participants were confronted with a written multiple-choice test consisting of 12 descriptions of realistic situations they had to connect with an appropriate mathematical model. For each situation, the appropriate model was either proportional, inverse proportional, affine with positive slope, or affine with negative slope. These models were given either in graphical, tabular, or formula form (each representation was provided in one third of the cases). Figure 1 exemplifies how the descriptions of situations to be connected to the appropriate model were presented to the participants for each of the three representational modes. A list of all 12 descriptions of situations is given in the "Appendix." We only provided one situation for each of the 12 categories (4 models × 3 representations) in the test instrument. The first reason for doing so was to prevent that learning effects could occur if students were repeatedly confronted with similar items. The second reason was that otherwise the test would become too long and repetitive so that participants could lose their concentration or motivation. In order to avoid disturbance from hidden variables, we opted for realistic situations one can reasonably expect that students are familiar with and we tried to keep the descriptions of these situations as simple as possible. Therefore, we also limited ourselves to situations that we could describe with a model that is situated in the first quadrant. Situations were also chosen so that there was a very strong and clear fit with only one of the provided models. We are aware that models never perfectly fit to a realistic situation, but the model we considered as the correct one for a situation provided an unquestionably better fit than the other three models. Furthermore, test items were made comparable in terms of not mentioning concrete numbers or variables in the situation formulations.

Responses of 64 participants[3] were statistically analyzed by means of the generalized estimating of equations approach within SPSS (Liang & Zeger, 1986). This procedure allows to analyze repeated (and thus possibly correlated) categorical observations within series of individuals and to appropriately correct for the correlation between measurements in order to make inferences. Given the dichotomous nature of the dependent variable (i.e. is a particular response alternative chosen or not), a logistic regression (modeling the probability that a correct response is given,

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Example 1 (situation with underlying inverse proportional model/alternative formulas)

During the war, butter was rationed. Each week, butter was delivered and fairly distributed among the people. Which formula properly denotes the relation between the number of people who wants butter and the amount of butter everybody receives?

❑ $y = 150\,x$

❑ $y = 150/x$

❑ $y = 150\,x + 30$

❑ $y = -150\,x + 30$

Example 2 (situation with underlying proportional model/alternative tables)

Jennifer buys minced meat at the butcher's shop. Which table properly denotes the relation between the amount of minced meat that Jennifer buys and the price she has to pay?

❑

x	*y*
0	8
1	-4
2	-16
3	-28
4	-40

❑

x	*y*
0	0
1	12
2	24
3	36
4	48

❑

x	*Y*
0	
1	12
2	6
3	4
4	3

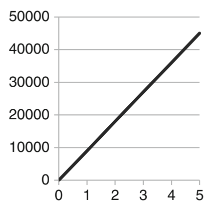
❑

x	*y*
0	8
1	20
2	32
3	44
4	56

Example 3 (situation with underlying affine model with negative slope/alternative graphs)

A chemical concern has a big cistern with hydrochloric. This morning they started to pump with a constant pace all hydrochloric out of this cistern. Which graph properly denotes the relation between elapsed time and the amount of hydrochloric that is still in the cistern?

❑



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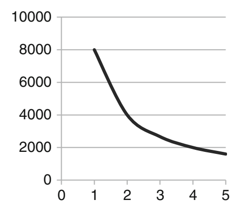

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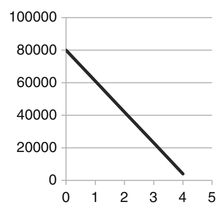

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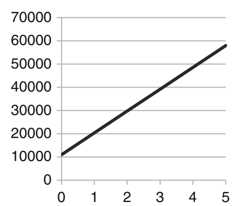

Figure 1. Example items from the multiple-choice test of study 1

TABLE 1

Percentage of correct assignments for the four models underlying the verbal descriptions and for the three representational modes

	Graph	*Table*	*Formula*	*Average*
Proportional	98	95	89	94
Inverse proportional	70	69	92	77
Affine with positive slope	81	55	66	67
Affine with negative slope	81	81	48	70
Average	83	75	74	77

depending on the type of function and the representational mode as explanatory variables) was appropriate.

Results

Table 1 shows the percentage of correct assignments for the different models underlying the verbal descriptions and for the different representational modes. The logistic regression analysis first of all revealed a main effect of model, Wald chi-square (3) = 38.472, $p < 0.00015$, indicating that students' accuracy in connecting situations to the appropriate mathematical model depended on the type of model involved: The percentage of correct matches for an underlying proportional model was significantly higher than for an inverse proportional, affine with positive slope, and affine with negative slope model. The analysis did not reveal a main effect of representation, Wald chi-square (2) = 5.297, $p = 0.071$: The percentages of correct assignments for the three representational modes did not differ significantly. Finally, but most importantly, an interaction effect between model and representation was found, Wald chi-square (6) = 45.111, $p < 0.0015$: The percentage of correct matches for a given underlying model depended on the representational mode in which that model was given, and for different underlying models, different representational modes led to higher percentages of correct matches. Although all representations were quite good to identify an underlying proportional model, the percentage of correct assignments in the formula mode was significantly lower than in the tabular ($p = 0.049$) and graphical mode ($p = 0.028$). For the inverse proportional model, the result for formula was significantly better than for table ($p < 0.0015$) and graph ($p < 0.0015$). Underlying affine models with positive slope elicited significantly more correct matches in the graph mode than in the two other representational modes (table $p < 0.0015$, formula $p = 0.014$), while underlying affine models with negative slope elicited significantly less correct matches in the formula mode than for the two other modes (table $p < 0.0015$, graph $p < 0.0015$).

These results indicate that the graph was the best representation in all cases, except for the inverse proportional relationships. For these relations, the formula proved to be more supportive. A possible explanation is that students tend to associate characteristics of linear graphs (straight lines) and linear tables (equal distances) with graphs and tables in general. Absence of these prototypical characteristics in graphs and tables of inverse proportional relationships might have refrained students from choosing these representations, leading to a decrease of correct matches for the inverse proportional model when given in graphical or tabular mode. Another observation is that formulas seem to be misleading for affine relations. This might have been a side effect of the fact that the situations to be modeled were commonly formulated in a "$y = b \pm ax$" form—first referring to an intercept and then to a slope or rate of change—while the formulas to be matched with were given in a "$y = ax + b$" form.

Based on the number of correct responses, we conclude that students can interpret all representations (since overall, correct matches for all representations varied between 83 and 74 % and mutual differences were not significant). Students are also able to identify underlying mathematical models (more than 80 % of the students detected the underlying model in at least one of the three representations). However, some representations support the identification of one model better than other models, while others put students on the wrong track. To obtain a better understanding of these findings, an error analysis was conducted.

Table 2 shows the percentage of erroneous assignments for the different models in the distinct representational modes. The error analysis revealed that in situations for which the underlying model was inverse proportional, proportional errors were made most frequently, especially in the tabular representational mode. The predomination of proportional errors is in line with the results of previous studies on students' overreliance on the proportional model (De Bock et al., 2007) when another model is appropriate.

Also in situations in which the underlying model was affine with positive slope, the most frequent type of errors was the proportional one, but here these errors mainly occurred in the formula and tabular representational mode. Apparently, for this model, which comes closest to the proportional model, students *see* the value at 0 or the *Y*-intercept in

TABLE 2

Percentage of erroneous assignments for the proportional (P), the inverse proportional (IP), the affine with positive slope (A+), and the affine with negative slope (A−) model in the different representational modes (graph, table, and formula)

Models	*Graph*			*Table*			*Formula*			*Total*		
P	IP	A+	A−	IP	A+	A−	IP	A+	A−	IP	A+	A−
	0	2	0	2	3	0	9	0	2	4	2	1
IP	P	A+	A−	P	A+	A−	P	A+	A−	P	A+	A−
	14	3	11	27	2	3	5	3	0	15	3	5
A+	P	IP	A−	P	IP	A−	P	IP	A−	P	IP	A−
	17	2	0	41	0	2	31	2	0	30	1	1
A−	P	IP	A+	P	IP	A+	P	IP	A+	P	IP	A+
	8	11	0	6	11	2	19	23	3	11	15	2

the graphical representation, while this element seems to be more often overlooked in the other representations.

Finally, in situations in which the underlying model was affine with negative slope, inverse proportional errors were most frequent, especially in the formula representational mode. A straightforward explanation refers to the decreasing nature of both models. Apparently, for many students, the independent variable in the denominator is more salient than the negative sign in the numerator. Another explanatory element refers to the attractiveness of the "doubling/halving" prototype in situations of decrease, which is most prominent in the formula representation. Students likely experience this prototype both in daily life and in school word problems for teaching inverse proportionality such as "For 4 painters, it takes 12 days to paint a bridge. How many days does it take 8 painters to do that job?" (Van Dooren, De Bock, Hessels, Janssens & Verschaffel, 2005) and tend to over-generalize this prototype to situations of decrease which are not inverse proportional.

Conclusions

Our results show that students are very proficient in relating descriptions of realistic situations to models when the situation described is a proportional one. In case of a situation with an inverse proportional or affine model with positive slope, there is, however, a strong tendency to connect the situation also to the proportional model. These results are in line with several other studies showing the "default" role of the proportional model (De Bock et al., 2007; Van Dooren et al., 2008).

Results also indicate that the representational mode has a strong impact on students' modeling accuracy and on their tendency to inappropriately connect non-proportional situations to proportional models. This last tendency was most clear for situations having an underlying model that was inverse proportional or affine with positive slope, but it was always affected by the representation in which the model was given. Apparently, a particular representation may highlight aspects of non-proportionality that are easily noticed by students (e.g. the *Y*-intercept of a graph or the distances in a table) and therefore facilitate correct reasoning, but be misleading when representing a situation with another model.

Study 2: Connecting Representations to Representations

Aims and Rationale

Study 1, as well as most research on improper proportional reasoning so far, was related to mathematical modeling, i.e. to tasks in which real-life situations had to be expressed in mathematical terms. Although students' overreliance on proportionality occurred in all representations, it did not have the same strength in all representations. Much less research exists on students' (lack of) understanding of representations of functions per se and how this might be related to their tendency toward improper proportional reasoning. Therefore, we examined in study 2 students' understanding of representations of proportional, inverse proportional, and affine functions in an abstract mathematical context, where thus no modeling of real-life situations needs to take place.

More specifically, study 2 focuses on students' fluency in linking multiple representations of functions. The importance of this skill in the mathematics curriculum is widely acknowledged (e.g. Elia, Panaoura, Gagatsis, Gravvani & Spyrou, 2006). More specifically, we hypothesize that students' limited understanding of proportional functions will lead to difficulties in distinguishing them from conceptually related functions (cf. supra). We assume that this limited understanding will interfere when students have to connect various representations of these functions to each other. Therefore, study 2 investigates (1) how accurate students are in connecting representations of proportional, inverse proportional, or affine functions to each other and (2) whether accuracy in connecting representations and the confusion between proportional and conceptually related functions depend on the nature of the external representations that have to be connected to each other.

Method

The same 65 students from study 1 also participated in study 2, but the order in which they participated in both studies was counterbalanced. They were confronted with a written multiple-choice test consisting of 24 items. In each item, a graph, formula, or table was provided that had to be linked to one of four graphs (when a formula or table was given), to one of four formulas (when a graph or table was given), or to one of four tables (when a graph or formula was given). Only one of the graphs, formulas, or tables accurately represented the same function as the given graph, formula, or table. The test offered graphs, formulas, and tables of the functions that were already used in study 1: proportional functions, inverse proportional functions, affine functions with positive slope, and affine functions with negative slope. An example item is shown in Fig. 2. The 24 items were offered in a random order to students. As in study 1, data were analyzed by means of a repeated measures logistic regression analysis followed by multiple pairwise comparisons, and an error analysis was conducted in order to investigate the most frequently chosen incorrect answering alternatives.

Results

The logistic regression first of all showed a significant main effect of the type of function, Wald chi-square (3) = 28.322, $p < 0.0015$. Pairwise comparisons indicated that accuracy was considerably higher for items dealing with proportional and positive affine functions (with average accuracy rates of 0.90 and 0.88, respectively) than for items dealing with negative affine and inverse proportional functions (with average accuracy rates of 0.73 and 0.77, respectively). Thus, students had more difficulties in appropriately linking representations of functions where a larger value of x implies a smaller value of y than representations of functions where a larger value of x implies a larger value of y.

Second, the logistic regression analysis indicated a main effect of the type of representational connection that students had to make, Wald chi-square (5) = 97.109, $p < 0.0015$. Pairwise comparisons indicated that linking a given graph to a table and linking a given table to a graph were

Choose among the four graphs below the one that describes the same mathematical relationship as the formula y = .08 x

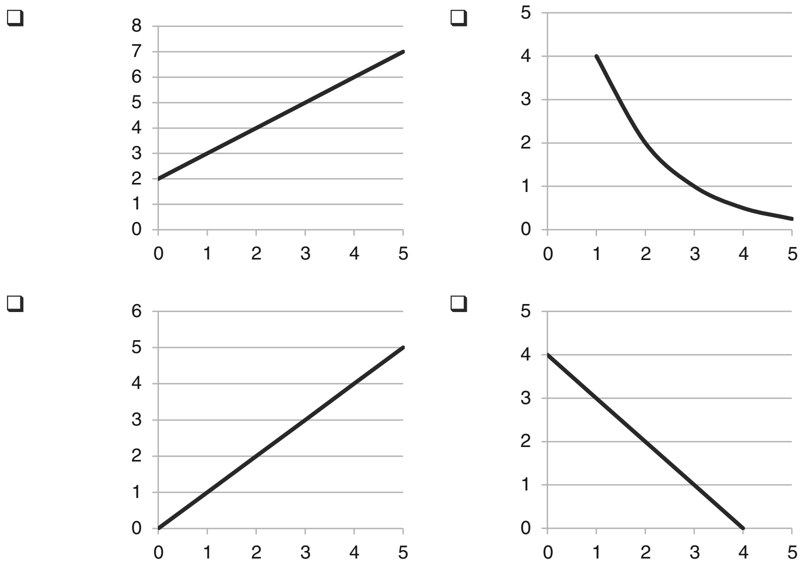

Figure 2. Example item from the multiple-choice test of study 2

done best by students (average accuracy of 0.95 in both cases), while linking a given formula to a table and linking a given table to a formula were significantly more difficult (average accuracy of 0.89 and 0.84, respectively). The lowest accuracies occurred for links where no table is involved, i.e., linking a graph to a formula and a formula to a graph (accuracies of 0.76 and 0.62, respectively). This indicates that students can deal best with representations that involve concrete function values: Given concrete function values in a table, the correct graph and/or the correct formula can be rather easily retrieved (and reversely), while the link between graphs and formulas is more difficult, probably because there are no concrete function values given which could be used as an intermediate step.

Third, and most important, a significant 4 × 6 interaction effect between the type of function and the type of representational connection was found, Wald chi-square (15) = 31.314, p = 0.006. This interaction effect globally indicates that for some functions, certain representational connections are made easier than for other functions. In order to get a good understanding of this complex interaction, the accuracy rates for the various types of items and connections between representations are summarized in Table 3. To allow a proper interpretation, the data from this table are completed with data coming from an analysis of the most frequently chosen incorrect answering alternatives. As can be seen in Table 3, the items dealing with proportional functions were generally solved rather well (average accuracy of 0.90), even though connecting a graph to a formula and vice versa led to errors in about 20 % of the cases. A further analysis showed that when students had to select a graph for a given proportional formula, they often selected an affine (positive or negative) graph, probably because they were misled by the fact that these graphs are also straight lines. When selecting a formula for a given proportional graph, affine formulas were hardly selected. Nearly all incorrectly selected formulas were inverse proportional ones. Apparently, students were aware that they had to select a formula without a constant term, but sometimes picked a function with x in the denominator of the formula.

Also the items dealing with positive affine functions were generally solved rather well (average accuracy 0.88), but choosing the right graph

TABLE 3

Overview of accuracies for different function types and representational connections

		To ...		
		Graph	*Formula*	*Table*
Proportional functions				
From …	Graph		0.82	0.97
	Formula	0.80		0.94
	Table	0.97	0.92	
Affine functions positive slope				
From …	Graph		0.88	0.97
	Formula	0.72		0.91
	Table	0.97	0.85	
Affine functions negative slope				
From …	Graph		0.60	0.91
	Formula	0.43		0.91
	Table	0.83	0.71	
Inverse proportional functions				
From …	Graph		0.66	0.95
	Formula	0.49		0.75
	Table	0.95	0.82	

for an affine formula led to a considerable number of mistakes. About half of these mistakes were due to the fact that students selected the graph of a negative affine function instead of the positive one, indicating that they were looking for a straight line graph that does not pass the origin, but not taking into account the positive value of the slope in the formula.

The affine items with a negative slope were the most difficult ones overall (average accuracy 0.73). When looking more closely at the types of representational connections, it becomes clear that not only linking graphs to formulas and vice versa led to a substantial number of errors (as for the other functions) but also linking tables to the correct graph and linking tables to the correct formula. In all these cases, about half of the errors were due to the fact that students selected the positive affine alternative. Apparently, students most often recognize the affine character of a function (be it in a graph, formula, or table), but have trouble interpreting the negative slope in the formula (a negative value of a). Also in the table representation (where the negative slope can only be noticed by seeing that larger values of x imply smaller values of y), this leads to difficulties. For the negative affine functions, a second major error was noticed: In about one third of the errors, the inverse proportional alternative was chosen. These students realized that larger values of x implied smaller values of y, but then erroneously selected a hyperbola graph instead of a straight (decreasing) line, or a formula with x in the denominator instead of a formula with x in the numerator but with a negative coefficient.

Finally, the inverse proportional items were generally solved slightly better than the negative affine ones (average accuracy 0.77). A closer look at the error patterns shows a quite diverse picture for the different representational connections. As for the other function types, connecting the right graph to a given formula and vice versa were the most difficult tasks for students. But it is remarkable that when students had to select a graph for a given inverse proportional formula, the most frequent error (about two thirds of the errors) was the choice of the proportional graph, while when students had to select a formula for a given inverse proportional graph, they chose in about two thirds of the cases for the negative affine alternative. Moreover, students had also considerable difficulties in connecting the right table to a given inverse proportional formula, while this phenomenon was not observed for the other type of functions. A closer look at the errors indicates that in almost all erroneous answers, the proportional table was chosen instead of the correct one.

Conclusions

The results of study 2 show quite a diverse picture of students' difficulties in linking representations of proportional and various kinds of conceptually related functions (inverse proportional functions, affine functions with positive slope, and affine functions with negative slope). Difficulties depend on the type of the underlying function and on the representational connection to be made. Particularly the decreasing functions (in which a larger value of x implies a smaller value of y) seem to be less well understood: Affine functions with a negative slope are often linked to representations of affine items with a positive slope (which share the straight line graph that does not pass the origin) or with representations of inverse proportional functions (which share their decreasing character). In the same way, inverse proportional functions are often linked to representations of affine functions with a negative slope, but additionally, students often link an inverse proportional function to representations of proportional functions.

The representational connection that was most difficult was the one between a formula and a graph and vice versa. Especially for the two types of decreasing functions, the "from formula to graph" direction results are particularly low (see Table 3). Students seem to be less familiar with formula-based representations of non-proportional functions, which is probably the reason why transitions starting from that representation are most difficult. All representational connections wherein a table was involved were made considerably better. Our hypothetical explanation for this last finding is the absence of concrete function values when linking graphs and formulas. The exemplary function values that are given in a table allow the student to concretely test which graph or formula fits (and similarly, a formula and graph can be concretely tested against a few alternative tables). For the items used in our test, an expert would probably be able to immediately recognize the appropriate graph for a given formula (and vice versa) without turning to concrete values, merely by comparing the formula to the global shape of the graphs.

General Discussion

The two studies reported in this article point to the important intermediate role of representations in students' understanding of proportional, inverse proportional, and affine functions. Some representations highlight aspects of non-proportionality and prevent students from over-using proportionality, while others seem to produce the opposite effect. Results also seem to suggest that it is not only the proportional model that is tempting for students. In both studies, mutual confusion between the two increasing and between the two decreasing functions was reported. This issue could be further researched in follow-up studies also including the "$y = ax$" model with $a < 0$ in order to see if confusion between this model and the inverse proportional or the negative affine model would show up too. Another parallel in both studies is that students have most difficulties with decreasing functions and their representations. However, the results of both studies also diverge in several aspects, indicating that finding a mathematical model and understanding a functional relation are two quite different things. In a mathematical modeling context (study 1), graphical representations were helpful in most cases to detect the model underlying a realistic situation, while for mutually connecting representations (study 2), tabular representations, providing concrete function values, proved to be most supportive.

Of course, these two studies also have their weaknesses affecting, to a certain extent, their internal validity and/or their wider applicability. With respect to internal validity, we remind the fact that in study 1 each category of the test instrument was only represented by one item which is an option that could be disputed. As explained above, we had good reasons to take that option, but despite these reasons, we think that including more than one item per category—probably manipulated as a between student variable, and not as a within student variable—would be useful in follow-up research. With respect to external or ecological validity, one can, for instance, point to the somewhat artificial character of the testing setting. Instead of asking students to construct a model or a representation, they only had to select the right alternative in a multiple-choice format. The list of possible alternatives was also very limited and unilateral, e.g., not including quadratic, cubic, or exponential functions that could be appropriate too on restricted domains. Moreover, participants did not have other resources, such as computers or graphing calculators, at their disposal. This is rather atypical for genuine mathematics classes in Belgium. This critique especially holds for study 1: Mathematical modeling, including the transition between a real-life situation and a mathematical model, is certainly broader and more sophisticated than just picking a representation of the right model from a given series of representations of possible models.

In mathematical practices in and outside school, even a given table or graph has different "parts" and expertise often consists in selecting the right *part* of that table or graph, i.e., the part that adequately grasps the gist of a given situation, answers a particular question that one has in mind, or clearly highlights the key features or general shape of a given relationship. For instance, as a prototype of a quadratic relationship, nobody will ever show a piece of a parabola that does not include the top of that curve. Usually computer-supported environments are employed to give representations a dynamic character and in that way support this selection and adaptation process. So, functions have *several* tables and graphs, and thus, *the* table or graph of a given function does not exist. This also holds for formulas: One formula may also appear in different forms. As indicated in study 1: Although "$y = ax + b$" and "$y = b + ax$" mathematically describe the same relationship, from a cognitive point of view they may be perceived differently by students.

Another shortcoming of our studies, which is directly related to the testing format, was the absence of process-oriented data. This type of data could reveal differences in cognitive styles and the criteria that students used to select models or representations. We hypothesize that students relied on properties of functions and their representations, but further research should confirm this. In that respect, it would be interesting to systematically investigate how students' understanding of the functions in this study and their representations is affected by such functional properties. For instance, given a representation (a table, graph, or formula) of a proportional, inverse proportional, positive affine, or negative affine function, one could ask students to indicate whether a statement as "when x doubles, y doubles" is true. One could expect that students most easily detect the incorrectness of such statement for non-linear functions in a tabular representation because this representation allows them to readily compare given function values. By involving the role of functional properties in the research on functions and representations, a more refined view on students' selection criteria and difficulties in distinguishing proportional and conceptually related functions could be achieved.

Notwithstanding the above-mentioned limitations undermining to a certain extent the internal and external validity of these studies, a preliminary implication that could be drawn for mathematics education is the need for drawing sufficient instructional attention to representations, to discuss strengths and weaknesses of various representational forms, to match representations with each other, and to link them to realistic situations, and for explicitly discussing differences between proportional and conceptually related models. Modeling tasks (study 1) as well as decontextualized matching tasks (study 2) can alternate in concrete learning environments and can both be applied to develop students' ability to recognize mathematical functions and their characteristics and to fluently and flexibly apply related procedures (or strategies). In our view, both types of tasks could and should also be used as a starting point for tasks relating more authentic real-world situations to mathematical models and with reflections on this relation (Greer, 2006; Verschaffel et al., 2000). Future interventions studies could empirically validate this type of learning environments, including the effectiveness by which different representations are used to strengthen students' insight in the domain of functions and their applications.

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Appendix

List of the 12 Descriptions of Realistic Situations as Used in Study 1

Situations with underlying proportional model

- Jennifer buys minced meat at the butcher's shop. Which representation[4] properly denotes the relation between the amount of minced meat that Jennifer buys and the price she has to pay?
- John fills up the tank of his truck with fuel. Which representation properly denotes the relation between the amount of fuel filled and the price he has to pay?
- A cruise ship crosses the ocean at a constant speed. Which representation properly denotes the relation between elapsed time and the distance travelled by the cruise ship at that time?

Situations with underlying inverse proportional model

- During the war, butter was rationed. Each week, butter was delivered and fairly distributed among the people. Which representation properly denotes the relation between the number of people who wants butter and the amount of butter everybody receives?
- For a benefit an action committee wants to peel a full container of potatoes. This job will take them several hours. Which representation

properly denotes the relation between the number of committee members who collaborates and the time needed to finish this job?
- A group of friends participates in a gambling game. When they win some money it will be shared equally among the friends. Which representation properly denotes the relation between the number of friends and the amount of money each person will receive?

Situations with underlying affine model with positive slope

- A new phone company charges for service by applying a fixed monthly subscription cost and an amount per number of minutes talked. Which representation properly denotes the relation between the number minutes talked and the monthly invoice amount?
- At a refinery fuel oil is pumped into a tank truck at a constant flow rate. This is done on a balance to prevent overloading of the truck. Which representation properly denotes the relation between the time fuel oil is pumped and the weight of the tank truck?
- A taxi company charges for a night ride a fixed fee upon entree and an amount for each kilometer driven. Which representation properly denotes the relation between the number of kilometers driven and the total price for the night ride?

Situations with underlying affine model with negative slope

- A chemical concern has a big cistern with hydrochloric. This morning they started to pump with a constant pace all hydrochloric out of this cistern. Which representation properly denotes the relation between elapsed time and the amount of hydrochloric that is still in the cistern?
- In case of an electrical breakdown a hospital needs to switch to emergency generators. These generators are connected to large fuel tanks. Which representation properly denotes the relation between the time the generators are running and the amount of fuel in the tanks?
- Thom has a mobile phone subscription, but uses prepaid reloadable cards. Per minute talked the uploaded sum decreases by a fixed amount. Which representation properly denotes the relation between the number of minutes talked and the remaining sum on the card?

NOTES

[1] In the rest of this paper, we use the term proportional to determine functions that can be described by a formula of the form "$y = ax$." Functions that can be described by a formula of the form "$y = ax + b$" with $b \neq 0$ are labeled as affine functions. Both proportional and affine functions are, at least in the Anglo-Saxon literature, denoted as "linear."

[2] In the rest of this paper, we will use the term "representation" to denote "external representation."

[3] Responses of one participant were dropped from the analysis because they were incomplete.

[4] In the test, the word "representation" was replaced either by "graph," "formula," or "table," each in one third of the cases (see also the example items in Fig. 1)

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