Special Cases in Using the Matrix Lambert W function for the Stability Analysis of High-Order Linear Systems with Time Delay ⋆

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Abstract: A recently developed methodology based on the matrix Lambert W function for the stability analysis of linear time invariant, time delay systems is based on an assumed one to one correspondence between the branches of this multi-valued function and the characteristic roots of the system. By studying a particular, yet common, second order system, we show that in general there is no such one to one correspondence. Furthermore, it is shown that under mild conditions only two branches suffice to find the complete spectrum of the system, and that the principal branch can be used not only the dominant root, as stated in previous works, but also some non dominant roots too. The results are presented analytically, and then verified by numerical experiments.

Keywords: Time-Delayed Systems; Matrix Lambert W Function.

1. INTRODUCTION

Consider a Linear Time Invariant-Time Delay System (LTI-TDS), represented by a Delay-Differential Equation (DDE) of the form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\tau) \tag{1}$$

The stability analysis and control synthesis of this class of systems is a wide open area of research. The difficulty of this problem arises from the the fact that the delay makes this class of systems infinite dimensional. A review of the recent results and challenges int his area can be found in the article by Sipahi et al. (2011). Several avenues have been followed by different researchers to find solutions to the stability and control questions. Some works study the absolute stability regions with respect to the time delay (Olgac and Sipahi, 2002; Sipahi and Olgac, 2006), and lead to control strategies that use the time delay as a stabilizing tool (Olgac et al., 2005). Other researchers focus on the numerical computation of the characteristic roots of the system. These works include approaches based on the discretization of the solution operator(Engelborghs et al., 2002; Breda, 2006; Butcher and Bobrenkov, 2011) or its infinitesimal generator (Wu and Michiels, 2012; Breda et al., 2009), and methods based on root finding of the characteristic equation (Vhylidal and Zitek, 2009). From here methods have been proposed to optimize the location of the dominant roots to guarantee a certain performance (Michiels et al., 2002; Mondié and Michiels, 2003). Finally, a Krylov method for computing characteristic roots of large-scale problems has been proposed in (Jarlebring et al., 2010).

In the past decade a framework for analyzing DDES based on the Lambert W function has been developed and presented by Asl and Ulsoy (2003), Yi (2009) and Yi et al. (2010c). It expands the earlier work of Wright (1959). The main idea of the methodology is to express the solution of a DDE as the sum of a series of infinitely many exponential functions. The characteristic roots of the system are found analytically in terms of the Lambert W function. While the problem remains infinite dimensional, a one-to-one correspondence between the characteristic roots of the system and the branches of this multi-valued function is assumed. The stability question is then solved by earmarking the dominant characteristic roots of the system with the branches of the Lambert W function corresponding to $k = 0, \pm 1, \ldots, \pm m$, where m is the nullity of the matrix $\mathbf{B}$ in (1). Therefore, only a few branches have to be considered to determine whether a solution is stable or not. Furthermore, the existence of an explicit solution expressed in terms of a power series, allows analyzing structural properties like observability and controllability for systems of DDES, as reported by Yi et al. (2008), the development of pole placement techniques for control synthesis, presented by Yi et al. (2010a,b,d, 2012b); and other applications like the estimation of decay rates shown by Duan et al. (2012) and spectrum design as presented by Wei et al. (2014).

⋆ This work was supported in part by the Coimbra Group under its program of scholarships for young professors and researchers of Latin America and by the Programme of Interuniversity Attraction Poles of the Belgian Federal Science Policy Office (IAP P6-DYSCO), by OPTEC, the Optimization in Engineering Center of KU Leuven, and by the project G.0712.11N of the Research Foundation-Flanders (FWO).

The basic foundation of his methodology, i.e., the assumption that the principal branch of the Lambert W function defines the stability of the system, is well established for first order systems, as proven by Shinozaki and Mori (2006) and Asl and Ulsoy (2003). For higher order systems, however, this result has not been extended with the same rigor, and is rather based upon observations, as stated in the works of Yi and Ulsoy (2006) and Yi et al. (2006).

In this paper, we show that the main assumption does not hold in general. We study a particular structure of second order delay systems and show that the full spectrum of a LTI-TDS of this class can be found using only two branches of the Lambert W function, i.e., there is no one-to-one correspondence between the characteristic roots and the branches of the Lambert W function. This is due to the fact that an important nonlinear equation in the approach does not have a unique solution. Furthermore, we show that the principal branch can be used to find not only the dominant root of the system, but some other roots too.

The organization of the paper is as follows. In section 2 we present the Lambert W function and its matrix version, required to work with higher order systems. Section 3 reviews the methodology to solve DDES using the Lambert W function, both in the scalar and vector cases. Section 4 presents the analysis of a second order system and contains the main results. This analysis is illustrated by numerical examples in Section 5. Finally, some conclusions of the study are given in section 6.

In the remainder of the paper, scalar quantities are denoted by italic symbols (a, b, λ) whereas vectors and matrices are represented by bold face lowercase ($\mathbf{x}$) and uppercase ($\mathbf{A}$, $\mathbf{B}$) letters, respectively. The notation e^b is used to represent the exponential function of a scalar and $\exp(\mathbf{A})$ represents a matrix exponential function. In a similar way, $W_k(z)$ represents the k-th branch of the Lambert W function of a scalar number, and $\mathbf{W}_k(\mathbf{A})$ is the matrix Lambert W function.

2. THE LAMBERT W FUNCTION

The Lambert W function is a function $W(z)$, $\mathbb{C} \mapsto \mathbb{C}$, defined as the solution to the equation

$$W(z)\, e^{W(z)} = z \tag{2}$$

This is a multi-valued function, that is, for any $z \in \mathbb{C}$ there are infinitely many solutions to (2). To identify these values a branch number is assigned, and we refer to $W_k(z)$ as the k-th branch of the Lambert W function of z. The branch cuts are defined in such a way that each branch has a precisely defined range, which can be seen in the work of Corless et al. (1996). For $z \in \mathbb{R}$, only two of the branches are real valued. The principal branch, $W_0(z)$ is real for $z \geq -1/e$ and its range is the interval $[-1, \infty)$. The branch $W_{-1}(z)$ is real for $-1/e \leq z < 0$, and its range is $(-\infty, -1]$. These branches are shown in Figure 1.

A comprehensive study of the history, definition and properties of the Lambert W function is found in the paper written by Corless et al. (1996).

The matrix Lambert W function is defined now. Consider a matrix $\mathbf{H} \in \mathbb{C}^{n\times n}$, which has the Jordan canonical

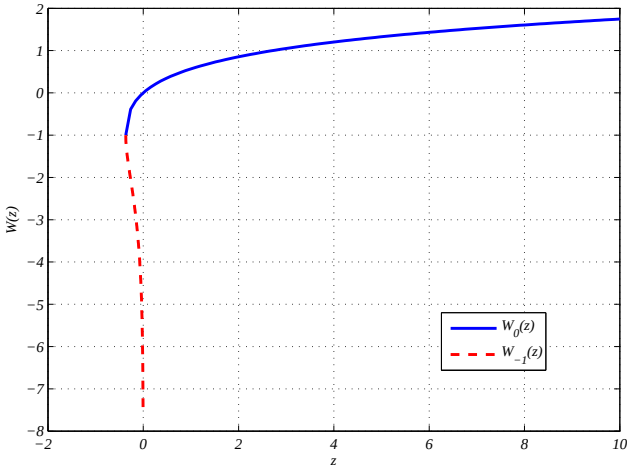

Fig. 1. The two real branches of the Lambert W function.

decomposition $\mathbf{H} = \mathbf{Z}\mathbf{J}\mathbf{Z}^{-1}$, with a block diagonal matrix $\mathbf{J} = \operatorname{diag}(\mathbf{J}_1(\lambda_1), \mathbf{J}_2(\lambda_2), \ldots, \mathbf{J}_p(\lambda_p))$. Following one of the standard definitions for a function of a matrix, as presented by Higham (2008), the Matrix Lambert W function for a Jordan block of size m is defined as:

$$\mathbf{W}_k(\mathbf{J}_i) = \begin{bmatrix} W_k(\lambda_i) & W_k'(\lambda_i) & \cdots & \frac{1}{(m-1)!}W_k^{m-1}(\lambda_i) \\ 0 & W_k(\lambda_i) & \cdots & \frac{1}{(m-2)!}W_k^{m-2}(\lambda_i) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_k(\lambda_i) \end{bmatrix} \tag{3}$$

and the matrix Lambert W function of $\mathbf{H}$ is defined as:

$$\mathbf{W}_k(\mathbf{H}) = \mathbf{Z} \operatorname{diag}(\mathbf{W}_k(\mathbf{J}_1(\lambda_1)), \mathbf{W}_k(\mathbf{J}_2(\lambda_2)), \ldots, \mathbf{W}_k(\mathbf{J}_p(\lambda_p)))\, \mathbf{Z}^{-1}. \tag{4}$$

Every matrix defined by (4), for $k = 0, \pm 1, \pm 2 \ldots$, is one particular solution to the matrix equation

$$\mathbf{W}_k(\mathbf{H}) \exp(\mathbf{W}_k(\mathbf{H})) = \mathbf{H}. \tag{5}$$

The above, standard definition implies that the same branch of the Lambert W function is used in each Jordan block. This is not necessary to have a solution of (5). Since $W_0(0) = 0$ and $W_k(0) = \infty$ for $k \neq 0$, we deviate from the standard definition to avoid the infinite value. This special case, called the *hybrid branch* case, is defined in the works of Yi (2009) and Yi et al. (2010c). More precisely, there $\mathbf{W}_k(\mathbf{H})$ uses the actual value of k for those Jordan blocks with $\lambda \neq 0$ and $k = 0$ for those blocks in which $\lambda = 0$. This definition is used in the remainder of the paper.

In a similar way, it is assumed that e^{-1} is not eigenvalue of $\mathbf{H}$ corresponding to a Jordan block of dimension higher than 1 when the principal branch is being computed. This is required to overcome the difficulty represented by the fact that $W_0'(e^{-1})$ is not defined. This limitation reduces the elegance of the definition of the matrix Lambert W function, but does not affect its usage as stated by Jarlebring and Damm (2007).

3. SOLUTION OF HIGH-ORDER, DELAY-DIFFERENTIAL EQUATIONS USING THE LAMBERT W FUNCTION

Consider a higher order DDE described by:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\tau) \tag{6}$$

with $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{A}$, $\mathbf{B} \in \mathbb{R}^{n\times n}$ and $\tau > 0$.

The following steps, introduced by Asl and Ulsoy (2003) and extended by Yi and Ulsoy (2006), Yi et al. (2006, 2010c), and Yi (2009), aim at computing characteristic roots using the matrix Lambert W function. The proposed method is based on finding a solution of the equation

$$\mathbf{S} - \mathbf{A} - \mathbf{B}\exp(-\mathbf{S}\tau) = \mathbf{0}, \tag{7}$$

where $\mathbf{S} \in \mathbb{C}^{n\times n}$. A matrix $\mathbf{Q} \in \mathbb{C}^{n\times n}$ is introduced, such that

$$\tau(\mathbf{S} - \mathbf{A})\exp((\mathbf{S} - \mathbf{A})\tau) = \tau\mathbf{B}\mathbf{Q} \tag{8}$$

is satisfied. Let us define $\mathbf{M} := \tau\mathbf{B}\mathbf{Q}$. Then, from (8) and (5),

$$\mathbf{S}_k = \frac{1}{\tau}\mathbf{W}_k(\mathbf{M}) + \mathbf{A}, \tag{9}$$

with $k \in \mathbb{Z}$, is a solution of (8). By substituting (9) into (7) the following expression is obtained,

$$\mathbf{W}_k(\mathbf{M})\exp(\mathbf{W}_k(\mathbf{M}) + \mathbf{A}\tau) - \mathbf{B} = \mathbf{0}. \tag{10}$$

Therefore, the steps to compute characteristic roots are given by the following algorithm, introduced by Yi (2009) and Yi et al. (2010c).

Algorithm 1. Repeat for $k = 0, \pm1, \pm2, \ldots$:

(1) Solve the nonlinear equation

$$\mathbf{W}_k(\mathbf{M}_k)\exp(\mathbf{W}_k(\mathbf{M}_k) + \mathbf{A}\tau) - \mathbf{B} = \mathbf{0}, \tag{11}$$

for $\mathbf{M}_k$ ($= \tau\mathbf{B}\mathbf{Q}_k$).

(2) Compute $\mathbf{S}_k$ corresponding to $\mathbf{M}_k$ as

$$\mathbf{S}_k = \frac{1}{\tau}\mathbf{W}_k(\mathbf{M}_k) + \mathbf{A}. \tag{12}$$

(3) Compute the eigenvalues of $\mathbf{S}_k$.

For system (6) to be stable, all characteristic roots must have negative real parts. Calculating the solution for all the branches is not possible. To work around this difficulty, the proposers of this methodology assume that for any branch k of the matrix Lambert W function, there is a unique solution $\mathbf{M}_k$ to (11) and a corresponding $\mathbf{S}_k$ matrix. This assumption, based on observations from many examples, leads to a stronger conjecture: when the rank of $\mathbf{B}$ in (1) is at least $n-1$, i.e., $\mathbf{B}$ does not have a repeated zero eigenvalue, the characteristic roots with largest real part correspond to the $\mathbf{S}_0$ matrix, found using the principal branch of the matrix Lambert W function in Algorithm 1. This conjecture is formally stated in the dissertation by Yi (2009) and it is the basis for several derivative works, like those of Yi et al. (2010a,b,d, 2012a,b), Duan et al. (2012) and Wei et al. (2014).

In the following section, we show that this conjecture does not hold in general. By studying a particular, yet very common, control system, we can show that (11) may have multiple solutions and there is no one to one correspondence between the branches of the matrix Lambert W function and the characteristic roots of the system. Furthermore, we show that in particular cases it is possible to find *all* characteristic roots using only *two* branches, those corresponding to $k = 0$ and $k = -1$. This contradicts the conjecture regarding stability.

4. A COMMON SPECIAL CASE IN SECOND ORDER SYSTEMS

We consider a single input, second order system under time delayed state feedback. This type of control system is ubiquitous in applications. Without loss of generality, we assume that the system matrix is in companion form. This system is represented by (6) with the following $\mathbf{A}$ and $\mathbf{B}$ matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ a_{21} & a_{22} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ b_{21} & b_{22} \end{bmatrix}. \tag{13}$$

Since $\mathbf{B}$ in (13) has nullity 1, the current theory predicts that the solutions to (11) corresponding to $k = 0$ and $k = \pm1$ generate only its dominant roots.

From the structure of $\mathbf{B}$, we can see that $\mathbf{M}_k = \tau\mathbf{B}\mathbf{Q}_k$, for any given $\mathbf{Q}_k$, has the form:

$$\mathbf{M}_k = \begin{bmatrix} 0 & 0 \\ m_{21} & m_{22} \end{bmatrix}. \tag{14}$$

Applying the definition of the matrix Lambert W function, using the hybrid branch case because $\mathbf{M}_k$ has one eigenvalue equal to zero, we get for $m_{22} \neq 0$:

$$\mathbf{W}_k(\mathbf{M}_k) = \begin{bmatrix} 0 & 0 \\ \dfrac{m_{21}}{m_{22}}W_k(m_{22}) & W_k(m_{22}) \end{bmatrix}, \tag{15}$$

and from here we obtain

$$\begin{aligned} \mathbf{S}_k &= \frac{1}{\tau}\mathbf{W}_k(\mathbf{M}_k) + \mathbf{A} \\ &= \begin{bmatrix} 0 & 1 \\ \dfrac{m_{21}}{\tau m_{22}}W_k(m_{22}) + a_{21} & \dfrac{1}{\tau}W_k(m_{22}) + a_{22} \end{bmatrix}. \end{aligned} \tag{16}$$

In case $m_{22} = 0,\ m_{21} \neq 0$, a simple computation yields

$$\mathbf{S}_k = \begin{bmatrix} 0 & 1 \\ \dfrac{m_{21}}{\tau} + a_{21} & a_{22} \end{bmatrix}, \tag{17}$$

where we used $W_0'(0) = 1$. We are now ready to state the main results of the paper.

Proposition 1. Let A and B be given by (13). Let $\{\lambda, \bar{\lambda}\}$ be any pair of complex conjugate characteristic roots of the system defined by (13). Assume their multiplicity is one. Then for either $k = 0$ or $k = -1$ there exists a real solution of (11), such that, if this solution and corresponding value of k are selected in the first step of Algorithm 1, the characteristic roots λ and $\bar{\lambda}$ are found in the last step of the algorithm.

Proof. The key idea is to perform the steps of Algorithm 1 in *reverse* order, in the course of which k is selected.

From the pair $(\lambda, \bar{\lambda})$ we first construct a real matrix $\mathbf{S}_k$, of which they are the eigenvalues, namely

$$\mathbf{S}_k = \begin{bmatrix} 0 & 1 \\ -|\lambda|^2 & 2\Re(\lambda) \end{bmatrix}. \tag{18}$$

Subsequently, we construct $\mathbf{M}_k$ from (18), where we make distinction between two cases.

Case 1: $2\Re(\lambda) \neq a_{22}$. Comparing (16) and (18), we can take $\mathbf{M}_k$ of the form (14), where $m_{21} \in \mathbb{R}$ and $m_{22} \in \mathbb{R}$ are chosen such that the following equations are satisfied:

$$W_k\left(m_{22}\right) = \tau\left(2\Re\left(\lambda\right) - a_{22}\right), \tag{19}$$

$$m_{21} = -\frac{m_{22}\left(\left|\lambda\right|^2 + a_{21}\right)}{2\Re\left(\lambda\right) - a_{22}}. \tag{20}$$

Such a choice is always possible for either $k = 0$ or $k = -1$, which then fix k. Equation (19) namely implies that $W_k\left(m_{22}\right)$ must be a real number. As mentioned earlier, the definition of the branch cuts of the Lambert W function makes this function to have two real branches: $k = 0$, the principal branch, and $k = -1$, and the union of the ranges of W_0 and W_{-1} for real valued arguments includes $\mathbb{R}$, see Figure 1. Furthermore, in the case considered, m_{22} computed from (19) is different from zero, justifying the use of (16).

Case 2: $2\Re(\lambda) = a_{22}$. We can freely choose $k \in \{0, -1\}$ and take

$$\mathbf{M}_k = \begin{bmatrix} 0 & 0 \\ -\tau(a_{21} + |\lambda|^2) & 0 \end{bmatrix}. \tag{21}$$

Finally, it remains to show that the constructed pair $(k, \mathbf{M}_k)$ is a solution of (11). Let $\mathbf{v}$, respectively $\bar{\mathbf{v}}$, be the eigenvector corresponding to λ, respectively $\bar{\lambda}$. If the characteristic root λ is simple, then the pair $(\mathbf{V}, \mathbf{\Lambda})$ is an invariant pair of (6), where

$$\mathbf{V} = [\mathbf{v}\ \bar{\mathbf{v}}], \quad \mathbf{\Lambda} = \mathrm{diag}(\lambda, \bar{\lambda}). \tag{22}$$

As a consequence (see the work of Beyn et al. (2011)) it satisfies

$$\mathbf{V\Lambda} - \mathbf{A} - \mathbf{BV}\exp(-\mathbf{\Lambda}) = 0, \tag{23}$$

The eigenvalue decomposition of $\mathbf{S}_k$ in (18) takes the form

$$\mathbf{S}_k = \mathbf{V\Lambda V}^{-1}, \tag{24}$$

from which we have

$$\exp(-\mathbf{S}_k) = \mathbf{V}\exp(-\mathbf{\Lambda})\mathbf{V}^{-1}. \tag{25}$$

It follows that

$$\mathbf{V\Lambda} = \mathbf{S}_k\mathbf{V}, \quad \mathbf{V}\exp(-\mathbf{\Lambda}) = \exp(-\mathbf{S}_k)\mathbf{V}. \tag{26}$$

Substituting the latter in (23) yields, as V is invertible,

$$\mathbf{S}_k - \mathbf{A} - \mathbf{B}\exp(-\mathbf{S}_k) = 0. \tag{27}$$

Finally, replacing $\mathbf{S}_k$ by

$$\frac{1}{\tau}\mathbf{W}_k\left(\mathbf{M}_k\right) + \mathbf{A} \tag{28}$$

results in (11). □

Proposition 2. Let A and B be given by (13). Let $\lambda_1 \neq \lambda_2$ be two real, simple characteristic roots of the system defined by (13). Then for either $k = 0$ or $k = -1$ there exists a real solution of (11), such that, if this solution and corresponding value of k are selected in the first step of Algorithm 1, the characteristic roots λ_1 and λ_2 are found in the last step of the algorithm.

Proof. The proof is completely analogous to the proof of Proposition 1. The differences are that we start by defining

$$\mathbf{S}_k = \begin{bmatrix} 0 & 1 \\ -\lambda_1\lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix}, \tag{29}$$

and that the two cases to be considered are $\lambda_1 + \lambda_2 = a_{22}$ and $\lambda_1 + \lambda_2 \neq a_{22}$. □

From Propositions 1-2 the following Corollary can be derived.

Corollary 1. Let A and B be given by (13). If all characteristic roots of (13) are simple and if the number of real characteristic roots is different from one, then *all* characteristic roots can be found using only two branches of the matrix Lambert W function in Algorithm 1, namely $k = 0$ and $k = -1$. Moreover, one can restrict to the real solutions of (11).

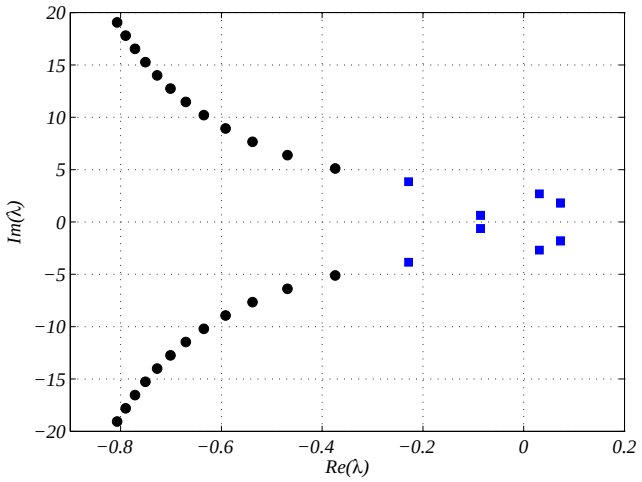

Fig. 2. Characteristic roots of the system under study. The roots represented as blue squares can be found using the principal branch of the matrix Lambert W function, whereas those in black circles are found using the branch corresponding to $k = -1$.

Remark The previous analysis shows that, with proper initial conditions, all the characteristic roots of system (13) can be found using the branches corresponding to $k = 0$ and $k = -1$. However, higher branch numbers can be used to find different pairs of roots following the same reverse engineering approach, without any particular structure.

5. NUMERICAL EXAMPLES

We consider a system defined by matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -5 & -0.5 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ -3 & -0.3 \end{bmatrix}, \tag{30}$$

with a value $\tau = 5$ for the time delay.

When the *LambertDDE* toolbox introduced by Yi et al. (2012a) was used to calculate the characteristic roots of the system, the software was not able to find a solution of (11) for any value of k with the default settings. For the numerical solution, the toolbox uses the matrix $\exp\left(-\mathbf{A}\tau\right)$ as an initial estimation of $\mathbf{Q}_k$. In this case, that value turns out not to be in the region of attraction of a solution of (11).

In order to obtain an *a posteriori* guess for $\mathbf{Q}_k$, we reverse-engineered the solution, as in the proof of Proposition 1. First, we use the QPmR algorithm of Vhylidal and Zitek (2009) to find the characteristic roots of the system in a region close to the origin of the complex plane. The roots found are shown in Figure 2.

The dominant roots of the system are $\lambda = 0.0729 \pm j1.8047$. Corresponding to these roots, we create the following $\mathbf{S}$ matrix:

$$\mathbf{S} = \begin{bmatrix} 0 & 1 \\ -3.2623 & 0.1459 \end{bmatrix} \tag{31}$$

From (16), we have that

$$\mathbf{W}_k(\mathbf{M}) = \tau(\mathbf{S}-\mathbf{A}) = \begin{bmatrix} 0 & 0 \\ 8.6886 & 3.2295 \end{bmatrix}. \quad (32)$$

This shows that $W(m_{22}) \in [-1, \infty)$, which is the range of the principal branch of the Lambert W function. There is, therefore, a matrix $\mathbf{M}$ for which (32) is satisfied for $k = 0$, and that matrix is

$$\mathbf{M}_0 = \begin{bmatrix} 0 & 0 \\ 219.5277 & 81.5962 \end{bmatrix}. \quad (33)$$

Since $\mathbf{B}$ and $\mathbf{M}$ are singular, there are an infinite number of $\mathbf{Q}_0$ matrices that satisfy $\mathbf{M}_0 = \tau\mathbf{B}\mathbf{Q}_0$, for (33). One of such matrices is

$$\mathbf{Q}_0 = \begin{bmatrix} 1 & 1 \\ -156.3518 & -64.3975 \end{bmatrix} \quad (34)$$

When this $\mathbf{Q}_0$ is used as starting value in the *Lambert-DDE* toolbox for $k = 0$, the numerical solution of (11), corresponding to the dominant roots, is found at the first iteration, as expected. Furthermore, if the matrix is slightly perturbed, the method still converges to the same solution after a few iterations.

Now, let us consider a non-dominant pair of roots: $\lambda = -0.0855 \pm j0.6319$. Following a similar reasoning, we obtain the following matrices:

$$\begin{aligned} \mathbf{S} &= \begin{bmatrix} 0 & 1 \\ -0.4067 & -0.1710 \end{bmatrix}, \\ \mathbf{W}_k(\mathbf{M}) &= \begin{bmatrix} 0 & 0 \\ 22.9667 & 1.6451 \end{bmatrix}. \end{aligned} \quad (35)$$

For this case, we again have $W(m_{22}) \in [-1, \infty)$. This implies that this pair of roots can also be found using the principal branch of the matrix Lambert W function. Indeed, if a matrix close to this one,

$$\mathbf{Q}_0 = \begin{bmatrix} 1 & 1 \\ -89.3356 & -15.6828 \end{bmatrix}, \quad (36)$$

which was created as in the previous case, is used as initial condition, the numerical routine within the *Lambert-DDE* toolbox converges to the solution *using the principal branch*.

In fact, we have observed that using suitable initial conditions, the 4 pairs of roots presented as blue squares in Figure 2 can be found using the principal branch of the matrix Lambert W function.

If we consider now the pair of roots $\lambda = -0.3740 \pm j5.1122i$, the corresponding $\mathbf{S}$ and $\mathbf{W}_k(\mathbf{M}_k)$ are

$$\begin{aligned} \mathbf{S} &= \begin{bmatrix} 0 & 1 \\ -26.2748 & -0.7481 \end{bmatrix}, \\ \mathbf{W}_k(\mathbf{M}) &= \begin{bmatrix} 0 & 0 \\ -106.3740 & -1.2404 \end{bmatrix}. \end{aligned} \quad (37)$$

In this case, $W(m_{22}) \in (-\infty, -1]$, which is the range of the branch indexed by $k = -1$. Using this branch, we obtain a $\mathbf{Q}_{-1}$ matrix,

$$\mathbf{Q}_{-1} = \begin{bmatrix} 1 & 1 \\ 10.5130 & -9.7608 \end{bmatrix}, \quad (38)$$

which is a solution to (11) for $k = -1$. Selecting an initial condition close to this matrix guarantees convergence to this solution.

This procedure can be repeated for all the roots marked as black circles in Figure 2, as well as for roots further to the left of the complex plane, always using the branch corresponding to $k = -1$. This example shows how the whole spectrum of a system with the structure given in (13) can be calculated using only two branches of the matrix Lambert W function and properly selected initial conditions for the solution of the nonlinear equation (11).

6. CONCLUDING REMARKS

The methodology for the stability analysis and control synthesis of LTI-TDS presented by Yi (2009) and Yi et al. (2010c), based on the matrix Lambert W function, assumes that there is a one to one correspondence between the characteristic roots of the system and the branches of the matrix Lambert W function. It also assumes that this correspondence is such that using the principal branch always leads to the dominant roots of the system. This paper illustrates that such a correspondence does not exist in all the cases, and that for a particular, albeit very common structure for the system, the branches corresponding to $k = 0$ and $k = -1$ can be used to find all the characteristic roots of the system. It is also illustrated that the correspondence cannot always be restored when selecting the particular initial condition $\exp(-\mathbf{A}\tau)$ in solving (11).

In our examples matrix $\mathbf{B}$ had reduced rank. An interesting path of research is to find out whether the correspondence between characteristic roots and branches of the Lambert W function, holding for scalar systems, can be extended to higher-order systems, provided *additional* structural conditions on the system are assumed.

The MATLAB code used to create the examples is available at `https://db.tt/anbyQiNf` or can be requested via email to the authors.

ACKNOWLEDGEMENTS

This paper is the result of a research visit that R. Cepeda-Gomez made to the Computer Science department at KU Leuven, funded by a Coimbra Group Scholarship for Young Professors and Researchers of Latin America. This scholarship was awarded while he was a lecturer of the Mechatronic Engineering Department at Universidad Santo Tomás in Bucaramanga, Colombia.

The work of W. Michiels work is supported by the Programme of Interuniversity Attraction Poles of the Belgian Federal Science Policy Office (IAP P6-DYSCO), by OPTEC, the Optimization in Engineering Center of KU Leuven, and by the project G.0712.11N of the Research Foundation-Flanders (FWO).

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