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Abstract

It is shown that every translation lightlike and every homothetical lightlike hypersurface in a semi-Euclidean space is a hyperplane. As a consequence, both translation and homothetical lightlike hypersurfaces are minimal.

1 Introduction

The geometry of submanifolds is extensively being studied in differential geometry with applications in several domains of which general relativity is probably one of the most important. Submanifolds of a semi-Riemannian manifold can either be spacelike or timelike, that is, non-degenerate, or can be lightlike, also called null or degenerate.

Classically, research and textbooks deal with non-degenerate submanifolds (see for instance [8]). Since the normal bundle has a non-zero intersection with the tangent bundle for a lightlike submanifold, a different approach is needed for these submanifolds. In [5], Bejancu and Duggal develop a general theory for lightlike submanifolds.

In subsequent articles, several characterizations and properties known for non-degenerate submanifolds are examined for lightlike submanifolds. In [9], for instance, totally umbilic lightlike submanifolds of Lorentzian manifolds are considered whereby a link with null strings in physics is made. The geometry of lightlike hypersurfaces in a semi-Euclidean space is studied in [3] via its geometry in a Euclidean space. Also, the Bernstein theorem for lightlike hypersurfaces of a Lorentzian space is proved there. In [12], an existence condition for a graph type lightlike hypersurface of a semi-Euclidean space is given in terms of the dimension and index of the ambient space. Properties concerning the Ricci tensor of a lightlike hypersurface in a semi-Riemannian space form, and local symmetric and totally geodesic lightlike hypersurfaces of a semi-Riemannian manifold can be found in [6]. In [11], the curvature conditions semi-symmetry and Ricci semi-symmetry for lightlike hypersurfaces of a semi-Euclidean space, and parallel and semi-parallel lightlike hypersurfaces of a semi-Euclidean space are treated.

In this article, translation lightlike and homothetical lightlike hypersurfaces of a semi-Euclidean space are considered. Concerning these, we recall that a minimal translation hypersurface in a Euclidean space is either a hyperplane or an open part of a cylinder on Scherk's surface, as proved in [4]. Also, the minimal homothetical hypersurfaces of a Euclidean space are classified in [7]. These turn out to be hyperplanes, quadratic cones, cylinders on a quadratic cone or cylinders on a helicoid. In [13], a classification of non-degenerate minimal homothetical hypersurfaces in semi-Euclidean space is given. Recently, Sağlam and Sabuncuoğlu ([10]) proved that every homothetical lightlike hypersurface in a semi-Euclidean space is minimal. In this article, we show that every translation and every homothetical lightlike hypersurface of a semi-Euclidean space is a hyperplane from which the result of Sağlam and Sabuncuoğlu in [10] follows trivially.

2 Lightlike hypersurfaces of a semi-Euclidean space

Denote with $\mathbb{E}^{n+1}_\nu$ the $(n+1)$-dimensional semi-Euclidean space of index ν with the natural metric

$$g(x,x) = -\sum_{i=1}^{\nu} x_i^2 + \sum_{i=\nu+1}^{n+1} x_i^2 \quad \text{for } x = (x_1, \ldots, x_{n+1}) \in \mathbb{E}^{n+1}_\nu.$$

Let M be a lightlike hypersurface of $\mathbb{E}^{n+1}_\nu$. Then, the normal bundle $TM^\perp$ of M is a subspace of the tangent bundle TM of M.
An orthogonal complementary non-degenerate vector bundle $S(TM)$ of $TM^\perp$ in TM is called a screen distribution on M (see [5]).
We denote by $\Gamma(E)$ the $\mathcal{F}(M)$ module of smooth sections of a vector bundle E over M, with $\mathcal{F}(M)$ the commutative ring of smooth real valued functions on M.
It is shown in [2] and [5] that for $S(TM)$, there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that for any non-zero section ξ of $TM^\perp$, there exists a unique section N of $tr(TM)$ satisfying

$$\begin{aligned} g(N,\xi) &= 1 \\ g(N,N) &= 0 \\ g(N,W) &= 0 \quad \text{for every } W \in \Gamma(S(TM)). \end{aligned} \tag{1}$$

Consequently, one has the decompositions

$$\begin{aligned} TM &= S(TM) \oplus_\perp TM^\perp \\ T\mathbb{E}^{n+1}_\nu &= S(TM) \oplus_\perp (S(TM))^\perp \\ &= S(TM) \oplus_\perp [TM^\perp \oplus tr(TM)] \\ &= TM \oplus tr(TM). \end{aligned}$$

In [3], the vector bundle $tr(TM)$ is called the lightlike transversal vector bundle of M which can be seen as the normal bundle in the case of non-degenerate hypersurfaces. In [1], a canonical screen distribution is constructed that induces a canonical lightlike transversal vector bundle for a lightlike hypersurface of a semi-Euclidean space.
From here on, let M be a Monge lightlike hypersurface determined by

$$\bar{x} : \mathbb{E}^n \to \mathbb{E}^{n+1}_\nu : (x_1, \ldots, x_n) \mapsto \bar{x}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, F(x_1, \ldots, x_n)), \tag{2}$$

with F a real valued function of n variables (see also [5]). The normal bundle of M is spanned by

$$\xi = \sum_{i=1}^{\nu} F_i' \frac{\partial}{\partial x_i} - \sum_{i=\nu+1}^{n} F_i' \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_{n+1}}, \tag{3}$$

for which F_i' denotes the partial derivative $F_i' = \frac{\partial F}{\partial x_i}$.
This leads to the following lemma (see also [3] or [5]).

Lemma 2.1. *A Monge lightlike hypersurface M given by (2) satisfies*

$$1 + \sum_{i=\nu+1}^{n} (F_i')^2 = \sum_{i=1}^{\nu} (F_i')^2.$$

Proof. A hypersurface is lightlike if and only if its normal bundle is lightlike. Therefore, the Monge hypersurface M, given by (2), is lightlike if and only if the normal vectorfield ξ, given by (3), is a null vector field. Hence the statement follows immediately. □

As in [10], but with opposite sign because a different parametrization is used, the vectorfield

$$N=\frac{\partial}{\partial x_{n+1}}-\frac{1}{2}\xi=\frac{1}{2}\left[-\sum_{i=1}^{\nu}F_i'\frac{\partial}{\partial x_i}+\sum_{i=\nu+1}^{n}F_i'\frac{\partial}{\partial x_i}+\frac{\partial}{\partial x_{n+1}}\right]$$

satisfies the conditions (1). As shown in [10], the following lemma holds.

Lemma 2.2. *The lightlike mean curvature H_ξ associated with ξ of a Monge lightlike hypersurface M given by (2) is determined by*

$$H_\xi=-\sum_{j=1}^{\nu}F_{jj}''+\sum_{j=\nu+1}^{n}F_{jj}''.$$

A lightlike hypersurface M is called minimal if and only if the lightlike mean curvature vanishes.

3 Translation and homothetical lightlike hypersurfaces of $\mathbb{E}_\nu^{n+1}$

Assume that $f_1,\ldots,f_n$ are real valued functions of one variable. Lightlike hypersurfaces parametrized by

$$\bar{x}:\mathbb{E}^n\to\mathbb{E}_\nu^{n+1}:(x_1,\ldots,x_n)\mapsto\bar{x}(x_1,\ldots,x_n)=\left(x_1,\ldots,x_n,\sum_{i=1}^{n}f_i(x_i)\right)$$

and

$$\bar{x}:\mathbb{E}^n\to\mathbb{E}_\nu^{n+1}:(x_1,\ldots,x_n)\mapsto\bar{x}(x_1,\ldots,x_n)=\left(x_1,\ldots,x_n,\prod_{i=1}^{n}f_i(x_i)\right)$$

are called translation lightlike and homothetical lightlike hypersurfaces of a semi-Euclidean space respectively.
The main results of this paper are presented in the following theorems.

Theorem 3.1. *A translation lightlike hypersurface of $\mathbb{E}_\nu^{n+1}$ is a hyperplane.*

Proof. The Monge lightlike hypersurface M determined by (2) is a translation lightlike hypersurface if $F(x_1,\ldots,x_n)=\sum_{i=1}^{n}f_i(x_i)$. From lemma 2.1, it follows that

$$\sum_{i=1}^{\nu}f_i'^2(x_i)-\sum_{i=\nu+1}^{n}f_i'^2(x_i)=1.$$

Differentiation with respect to x_i yields $f_i'(x_i)f_i''(x_i)=0$. Hence, the hypersurface must be a hyperplane. □

Corollary 3.1. *Every translation lightlike hypersurface of $\mathbb{E}_\nu^{n+1}$ is minimal.*

Theorem 3.2. *A homothetical lightlike hypersurface of $\mathbb{E}_\nu^{n+1}$ is a hyperplane.*

Proof. The Monge lightlike hypersurface M given by (2) is homothetical if $F(x_1,\ldots,x_{n+1})=\prod_{i=1}^{n}f_i(x_i)$. Consequently, from lemma 2.1, one obtains

$$\sum_{i=1}^{\nu}\frac{f_i'^2(x_i)}{f_i^2(x_i)}-\sum_{i=\nu+1}^{n}\frac{f_i'^2(x_i)}{f_i^2(x_i)}=\frac{1}{\prod_{i=1}^{n}f_i^2(x_i)}. \tag{4}$$

Taking the derivatives with respect to x_i and x_j for $i \neq j$, leads to

$$\frac{1}{\prod_{i=1}^{n} f_i^2(x_i)} \cdot \frac{f_i'(x_i) f_j'(x_j)}{f_i(x_i) f_j(x_j)} = 0,$$

from which it follows that at most one f_i' can be distinct from zero. Furthermore, not all f_i' can be zero since this contradicts equation (4). Consequently, exactly one f_i' must be non-zero. From equation (4), it then follows that $i \in \{1, \ldots, \nu\}$. One can assume that $i = 1$ for notational purposes. Therefore, equation (4) can be rewritten as

$$\left(\frac{f_1'(x_1)}{f_1(x_1)}\right)^2 = \frac{c^2}{f_1^2(x_1)} \tag{5}$$

with the constant c defined as

$$c = \frac{1}{\prod_{i=2}^{n} f_i(x_i)}.$$

From equation (5), it follows that $f_1(x_1) = \pm c x_1 + \tilde{d}$ for some $\tilde{d} \in \mathbb{R}$. Consequently,

$$F(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_i(x_i) = \frac{1}{c}(\pm c x_1 + \tilde{d}) = \pm x_1 + d$$

with $d \in \mathbb{R}$. So, every homothetical lightlike hypersurface of $\mathbb{E}_\nu^{n+1}$ is a hyperplane. $\square$

Corollary 3.2. *Every homothetical lightlike hypersurface of* $\mathbb{E}_\nu^{n+1}$ *is minimal.*

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