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Abstract

In this paper we show that under appropriate moment conditions, the supermodular ordered random vectors $\underline{X} = (X_1, X_2, \ldots, X_n)$ and $\underline{Y} = (Y_1, Y_2, \ldots, Y_n)$ with equal expected utilities (or distorted expectations) of the sums $X_1 + X_2 + \ldots + X_n$ and $Y_1 + Y_2 + \ldots + Y_n$ for an appropriate utility (or distortion) function, must necessarily be equal in distribution, that is $\underline{X} \overset{\mathrm{d}}{=} \underline{Y}$. The results in this paper can be considered as generalizations of the results of Cheung (2010), who presents necessary conditions related to the distribution of $X_1 + X_2 + \ldots + X_n$ for the random vector $\underline{X} = (X_1, X_2, \ldots, X_n)$ to be comonotonic.

Keywords: supermodular order, concordance order, expected utility, distorted expectation, comonotonicity.

1 Introduction

Both expected utility theory and distorted expectation theory offer a framework to describe how economic agents make choices under risk. In these theories, a random variable (r.v.) X describing a random future wealth is transformed into an expected utility $\mathbb{E}[u(X)]$ or a distorted expectation $\rho_g[X]$, respectively. Preferences between random wealths X and Y are then based on comparing the real numbers $\mathbb{E}[u(X)]$ and $\mathbb{E}[u(Y)]$ (or $\rho_g[X]$ and $\rho_g[Y]$) corresponding to the alternatives.

In this paper, we investigate how the expected utility $\mathbb{E}[u(X)]$ can be decomposed in terms of the upper - and lower tails $\mathbb{E}\left[(X-K)_+\right]$ and $\mathbb{E}\left[(K-X)_+\right]$ of X, as well as how the distorted expecation $\rho_g[X]$ can be expressed in terms of the Tail Values-at-Risk of X. Furthermore, we find ordering conditions under which equality of the expected utililties

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$\mathbb{E}[u(X)]$ and $\mathbb{E}[u(Y)]$ (or of the distorted expectations $\rho_g[X]$ and $\rho_g[Y]$) of two r.v.'s X and Y implies that both are equally distributed.

Cheung (2010) shows that under mild conditions, a random vector $\underline{X} = (X_1, X_2, \ldots, X_n)$ is comonotonic provided the sum $S = X_1 + X_2 + \ldots + X_n$ has the same distribution as the sum $S^c = X_1^c + X_2^c + \ldots + X_n^c$ of the comonontonic counterpart $\underline{X}^c = (X_1^c, X_2^c, \ldots, X_n^c)$ of $\underline{X}$. Furthermore, he shows that under the appropriate conditions, equality of the expected utilities $\mathbb{E}[u(S)]$ and $\mathbb{E}[u(S^c)]$ (or of the distorted expectations $\rho_g[S]$ and $\rho_g[S^c]$) implies that the random vector $\underline{X}$ is comonotonic.

In this paper, we generalize Cheung's results by establishing ordering conditions under which two random vectors $\underline{X}$ and $\underline{Y}$ having equally distributed sums $S_X = X_1 + X_2 + \ldots + X_n$ and $S_Y = Y_1 + Y_2 + \ldots + Y_n$ are equal in distribution. We also show that under the appropriate conditions, equality of $\mathbb{E}[u(S_X)]$ and $\mathbb{E}[u(S_Y)]$ (or of $\rho_g[S_X]$ and $\rho_g[S_Y]$) implies that $\underline{X}$ and $\underline{Y}$ are equally distributed.

The remainder of this paper is organised as follows. In Section 2 it is shown that expected utilities can be expressed as linear combinations of upper - and lower tails, while in Section 3, it is shown that distorted expectations can be expressed in terms of Tail Values-at-Risk. Some useful expressions for the expected value of a function of the sum of the components of a random vector, in terms of its multivariate distribution, are derived in Section 4. In Section 5 condition are considered under which convex ordered r.v.'s are necessary equal in distribution. Multivariate extensions of this result are derived in Section 6, where conditions are considered under which supermodular ordered random vectors are equal in distribution. Finally, Section 7 considers the special case where a random vector and its comonotonic modification are compared.

2 Expected utilities and stop-loss premiums

Throughout this paper, we will use the notation I to denote an interval of the real line. Furthermore, $\inf I = \inf\{x \mid x \in I\}$ and $\sup I = \sup\{x \mid x \in I\}$. The interval I may be bounded or not, implying that $\inf I$ and $\sup I$ may be finite or infinite. Hereafter, we will often consider functions $f : I \longrightarrow \mathbb{R}$ with absolutely continuous derivative. This means that f' is continuous on I, that f' has a derivative f'' a.e. on I, and that for any elements x and a of I, we have that $f'(x) = f'(a) + \int_a^x f''(K)\mathrm{d}K$. Continuity in an eventual real-valued lower or upper endpoint of I has to be understood as right or left continuity, whereas differentiability in such an endpoint means right - or left differentiability. Differentiability in a point means that the derivative (resp. right - or left derivative) is well-defined and finite. Finally, notice that all integrals in this paper have to be interpreted as Lebesgue integrals.

In the following lemma, it is shown that any sufficiently smooth function $u(x)$ can be expressed as a mixture of right and left tail functions of the form $(x-K)_+$ and $(K-x)_+$, $K \in \mathbb{R}$. Furthermore, $\mathbb{E}[u(X)]$ can be expressed in terms of the corresponding right - and left tails $\mathbb{E}\left[(X-K)_+\right]$ and $\mathbb{E}\left[(K-X)_+\right]$.

Lemma 1 *Let X be a r.v. with finite expectation and I an interval such that* $\Pr[X \in I] = 1$*. Furthermore, consider the function $u : I \longrightarrow \mathbb{R}$ with absolutely continuous derivative u'.*
(1) For any $a \in I$, the r.v. $u(X)$ can be expressed as

$$u(X) = u(a) + u'(a)(X-a) + \int_{\inf I}^{a} u''(K)(K-X)_+ \, \mathrm{d}K + \int_{a}^{\sup I} u''(K)(X-K)_+ \, \mathrm{d}K \qquad a.s. \tag{1}$$

(2) In case $\int_{\inf I}^{\sup I} u''(K)\mathrm{d}K$ is well-defined, we have that $\mathbb{E}[u(X)]$ is well-defined as well, and for any $a \in I$ it can be expressed as

$$\mathbb{E}[u(X)] = u(a) + u'(a)(\mathbb{E}[X]-a) + \int_{\inf I}^{a} u''(K)\,\mathbb{E}\left[(K-X)_+\right] \mathrm{d}K + \int_{a}^{\sup I} u''(K)\,\mathbb{E}\left[(X-K)_+\right] \mathrm{d}K. \tag{2}$$

Proof. (1) The absolute continuity of u' implies that also u is absolutely continuous and can be expressed as

$$u(x) = u(a) - \int_{x}^{a} u'(K)\,\mathrm{d}K, \qquad x \in I.$$

Partial integration leads to

$$u(x) = u(a) + xu'(x) - au'(a) + \int_{x}^{a} Ku''(K)\,\mathrm{d}K, \qquad x \in I. \tag{3}$$

Considering the cases $x < a$ and $x \geq a$ separately, one easily finds that (3) can be rewritten as

$$u(x) = u(a) + u'(a)(x-a) + \int_{\inf I}^{a} u''(K)(K-x)_+ \, \mathrm{d}K + \int_{a}^{\sup I} u''(K)(x-K)_+ \, \mathrm{d}K, \qquad x \in I.$$

Replacing x by X in this expression leads to the a.s. equality (1).

(2) Decomposing u'' into its positive and negative parts, i.e.

$$u''(K) = (u''(K))_+ - (u''(K))_-, \tag{4}$$

where $(y)_+ = \max(y,0)$ and $(y)_- = -\min(y,0)$, we find that

$$\int_{\inf I}^{\sup I} u''(K)\,\mathrm{d}K = \int_{\inf I}^{\sup I} (u''(K))_+ \, \mathrm{d}K - \int_{\inf I}^{\sup I} (u''(K))_- \, \mathrm{d}K \overset{\text{not.}}{=} I_1 - I_2.$$

The assumption that the integral $\int_{\inf I}^{\sup I} u''(K)\mathrm{d}K$ is well-defined implies that either

$$0 \leq I_1 < +\infty \text{ and } 0 \leq I_2 < +\infty$$

or

$$I_1 = +\infty \text{ and } 0 \leq I_2 < +\infty$$

or

$$0 \leq I_1 < +\infty \text{ and } I_2 = +\infty$$

must hold. Using (4), we can rewrite (1) as

$$\begin{aligned} u(X) = {} & u(a) + u'(a)(X-a) \\ & + \int_{\inf I}^{a} \left(u''(K)\right)_+ (K-X)_+ \,\mathrm{d}K - \int_{\inf I}^{a} \left(u''(K)\right)_- (K-X)_+ \,\mathrm{d}K \\ & + \int_{a}^{\sup I} \left(u''(K)\right)_+ (X-K)_+ \,\mathrm{d}K - \int_{a}^{\sup I} \left(u''(K)\right)_- (X-K)_+ \,\mathrm{d}K. \end{aligned}$$

Taking expectations and applying Fubini's theorem to any of the four integrals on the right hand side of this expression leads to

$$\begin{aligned} \mathbb{E}[u(X)] = {} & u(a) + u'(a)(\mathbb{E}[X]-a) \\ & + \int_{\inf I}^{a} \left(u''(K)\right)_+ \mathbb{E}\left[(K-X)_+\right] \mathrm{d}K - \int_{\inf I}^{a} \left(u''(K)\right)_- \mathbb{E}\left[(K-X)_+\right] \mathrm{d}K \\ & + \int_{a}^{\sup I} \left(u''(K)\right)_+ \mathbb{E}\left[(X-K)_+\right] \mathrm{d}K - \int_{a}^{\sup I} \left(u''(K)\right)_- \mathbb{E}\left[(X-K)_+\right] \mathrm{d}K. \end{aligned} \tag{5}$$

Notice that any of the integrals in the right-hand side of this expression takes a value in $[0,+\infty]$. Furthermore, the linear combination of these integrals, and hence, $\mathbb{E}[u(X)]$ is well-defined. Consider e.g. the situation where $I_1 = +\infty$ and $0 \leq I_2 < +\infty$. Then we find that

$$0 \leq \int_{\inf I}^{a} \left(u''(K)\right)_- \mathbb{E}\left[(K-X)_+\right] \mathrm{d}K \leq \mathbb{E}\left[(a-X)_+\right] \int_{\inf I}^{\sup I} \left(u''(K)\right)_- \mathrm{d}K < +\infty$$

and

$$0 \leq \int_{a}^{\sup I} \left(u''(K)\right)_- \mathbb{E}\left[(X-K)_+\right] \mathrm{d}K \leq \mathbb{E}\left[(X-a)_+\right] \int_{\inf I}^{\sup I} \left(u''(K)\right)_- \mathrm{d}K < +\infty,$$

so that the right-hand side of (5) is well-defined in this case. The other cases lead to the same conclusion. ∎

When u' is absolutely continuous, convexity of u is equivalent with $u''(x) \geq 0$ a.e. on I. Hence, $\int_{\inf I}^{\sup I} u''(K)\mathrm{d}K$ is well-defined if u is convex, implying that the expression (2) for $\mathbb{E}[u(X)]$ holds in particular for convex functions u with an absolutely continuous derivative u'. The function $u(x) = \mathrm{e}^{tx}$, $t \in \mathbb{R}$, is an example of such a function. Similarly, expression (2) holds for concave functions u with absolutely continuous derivative.

In case $\int_{\inf I}^{\sup I} u''(K)\, \mathrm{d}K$ is finite, we have that $\mathbb{E}[u(X)]$ in (2) is finite as well. This condition is fulfilled in particular when I is a closed and bounded interval.

A function with a continuous derivative is absolutely continuous. This implies that a function u with continuous second derivative u'' has an absolutely continuous derivative u'. Hence, Lemma 1 holds in particular for the class of functions u with continuous second derivative u''. The convex function $u(x) = (x - K)^2$, with K a given real number, is an example of such a function. On the other hand, Lemma 1 also holds for the convex function $u(x) = (x - K)_+^2$, where the notation y_+^2 is used for $[\max(y, 0)]^2$, although its second second derivative does not exist in K.

Formula (2) is well-known in the actuarial literature for the special case when $u(x) = (x - \mathbb{E}[X])^2$ and $a = \mathbb{E}[X]$, leading to an expression for the variance of a r.v. in terms of its stop-loss premiums; see e.g. Kaas et al. (2008).

Föllmer and Schied (2004) derive an expression similar to (1) for increasing and convex functions u with right-hand derivative; see also Cheung (2010).

Formula (1) has a 'natural' interpretation in terms of contingent claims and hedging; see e.g. Carr and Madan (2001). Indeed, suppose that X is the price of a traded asset at a future time T. The right hand side of expression (1) is the pay-off at time T of a static investment position, taken at time 0. Indeed, the first term is the pay-off of a long position in $u(a)$ zero coupon bonds, each with pay-off an amount of 1 at time T. The second term corresponds to the pay-off of a long position in $u'(a)$ calls with strike a and a short position in $u'(a)$ puts with strike a. The third term is the pay-off of a long position in $u''(K)\mathrm{d}K$ puts for all strikes less than a, while the fourth term is the pay-off of a long position in $u''(K)\mathrm{d}K$ calls for all strikes greater than a, We can conclude that the right hand side of formula (1) corresponds with the pay-off at time T of a model-free static replicating strategy for the contingent claim with pay-off $u(X)$ at T.

3 Distorted expectations and Tail Values-at-Risk

Lemma 1 can be interpreted in the framework of expected utility theory. In this section, we consider a related lemma which allows an interpretation in terms of the dual theory of choice under risk, where distorted expectations are used instead of expected utilities.

A distortion function is defined as a non-decreasing function $g : [0, 1] \to [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$. For any r.v. X, the distorted expectation associated with distortion function g, notation $\rho_g[X]$, is defined by

$$\rho_g[X] = -\int_{-\infty}^{0} \left[1 - g\left(\overline{F}_X(x)\right)\right] \mathrm{d}x + \int_{0}^{+\infty} g\left(\overline{F}_X(x)\right) \mathrm{d}x, \tag{6}$$

provided at least one of the two integrals in (6) is finite.

If g is absolutely continuous, it can be proven that $\rho_g[X]$ can be expressed as

$$\rho_g[X] = \mathbb{E}\left[F_X^{-1}(U) g'(1 - U)\right], \tag{7}$$

where U is a r.v. which is uniformly distributed on the unit interval and

$$F_X^{-1}(p) = \inf \{x \mid F_X(x) \geq p\}, \qquad p \in [0,1],$$

with $\inf \varnothing = +\infty$ by convention; see e.g. Dhaene, Kukush, Linders and Tang (2012).

The distortion function g defined by

$$g(q) = \min\left(\frac{q}{1-p}, 1\right), \qquad 0 \leq q \leq 1, \tag{8}$$

for some $p \in [0,1)$, is absolutely continuous. The related distorted expectation $\rho_g[X]$ is known as the Tail Value-at-Risk at level p, notation $\mathrm{TVAR}_p[X]$. Taking into account (7), this distorted expectation can be expressed as

$$\mathrm{TVAR}_p[X] = \frac{1}{1-p}\int_p^1 F_X^{-1}(q)\mathrm{d}q. \tag{9}$$

In the following lemma, we prove that distorted expectations related to a distortion function with absolutely continuous derivative g' can be expressed as a mixture of Tail Values-at-Risk.

Lemma 2 *Let X be a r.v. with finite expectation and g a distortion function with absolutely continuous derivative g'. In this case, the distorted expectation $\rho_g[X]$ is finite and can be expressed as*

$$\rho_g[X] = g'(1)\mathbb{E}[X] - \int_0^1 (1-p)\, g''(1-p)\mathrm{TVAR}_p[X]\,\mathrm{d}p. \tag{10}$$

Proof. Let U be a uniformly distributed r.v. on the unit interval. Taking into account that g' is absolutely continuous, $g'(1-U)$ can be written as

$$\begin{aligned} g'(1-U) &= g'(1) - \int_0^U g''(1-p)\,\mathrm{d}p \\ &= g'(1) - \int_0^1 g''(1-p)\,\mathbb{I}(U>p)\,\mathrm{d}p. \end{aligned} \tag{11}$$

As g' is absolutely continuous, also g is absolutely continuous. From (7) and (11), it follows that we can express $\rho_g[X]$ as

$$\rho_g[X] = g'(1)\mathbb{E}[X] - \mathbb{E}\left[\int_0^1 g''(1-p)\, F_X^{-1}(U)\mathbb{I}(U>p)\,\mathrm{d}p\right]. \tag{12}$$

For any distortion function g with absolutely continuous derivative g', we have that $\int_0^1 g''(p)\mathrm{d}p$ is finite. As $\mathbb{E}[X]$ is assumed to be finite as well, we find that

$$\mathbb{E}\left[\int_0^1 \left|g''(1-p)\, F_X^{-1}(U)\mathbb{I}(U>p)\right|\mathrm{d}p\right] \leq \mathbb{E}[|X|]\int_0^1 |g''(p)|\, dp < +\infty.$$

Hence, the second term in the right hand side of (12) is finite, which implies that also $\rho_g[X]$ is finite. Applying Fubini's theorem and taking into account (9) leads to (10). $\blacksquare$

Lemma 2 holds in particular for convex and concave distortion functions with an absolutely continuous derivative. On the other hand, convex and concave distortion functions with continuous derivative do not necessarily satisfy the conditions of Lemma 2. Consider e.g. the distortion function g defined by

$$g(q) = \frac{\int_0^q c(p)\mathrm{d}p}{\int_0^1 c(p)\mathrm{d}p}, \qquad 0 \leq q \leq 1,$$

where c is a singular continuous distortion function (e.g. the Cantor function). As $g'(q) = \frac{c(q)}{\int_0^1 c(p)dp}$, $0 \leq q \leq 1$, we have that g' is non-decreasing, and hence, g is convex. Further, g' is singular continuous, which means that it does not satisfy the conditions of the lemma.

Lemma 2 holds in particular for the class of distortion functions g with continuous second derivative (which implies that g' is absolutely continuous), see e.g. Property 2.6.6 in Denuit et al. (2005). For any $t \neq 0$, the function $g(q) = \frac{e^{tq}-1}{e^t-1}$, $0 \leq q \leq 1$, is an example of a such a convex (if $t > 0$) or concave (if $t < 0$) distortion function.

4 Random vectors and the sum of their components

Hereafter, we use the notation $\underline{X}$ to denote the n-vector $(X_1, X_2, \ldots, X_n)$. The sum of its components is denoted by S_X, i.e.

$$S_X = X_1 + \ldots + X_n. \tag{13}$$

The cumulative distribution function (cdf) and the decumulative distribution function (ddf) of $\underline{X}$ are denoted by $F_{\underline{X}}$ and $\overline{F}_{\underline{X}}$, respectively.

In the following lemma we derive expressions for $\mathbb{E}\left[(S_X - K)_+^{n-1}\right]$ and $\mathbb{E}\left[(K - S_X)_+^{n-1}\right]$ in terms of the ddf $\overline{F}_{\underline{X}}$ and the cdf $F_{\underline{X}}$ of $\underline{X}$, respectively. The notation y_+^s is used for $[\max(y, 0)]^s$. The first expression in the lemma was proven in Boutsikas and Vaggelatou (2002). We repeat its proof here in order to make the paper self-contained.

Lemma 3 *For any random vector $\underline{X}$ and any real K, we have that $\mathbb{E}\left[(S_X - K)_+^{n-1}\right]$ can be expressed as*

$$\mathbb{E}\left[(S_X - K)_+^{n-1}\right] = (n-1)! \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \overline{F}_{\underline{X}}\left(x_1, \ldots, x_{n-1}, K - \sum_{i=1}^{n-1} x_i\right) \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1}, \tag{14}$$

while $\mathbb{E}\left[(K - S_X)_+^{n-1}\right]$ can be expressed as

$$\mathbb{E}\left[(K - S_X)_+^{n-1}\right] = (n-1)! \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F_{\underline{X}}\left(x_1, \ldots, x_{n-1}, K - \sum_{i=1}^{n-1} x_i\right) \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1}. \tag{15}$$

Proof. (a) We first prove by induction that for any $k \geq 2$, the relation

$$\frac{\left(\sum_{i=1}^{k} x_i - K\right)_+^{k-1}}{(k-1)!} = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \mathbb{I}\left(x_1 > u_1\right) \dots \mathbb{I}\left(x_{k-1} > u_{k-1}\right) \mathbb{I}\left(x_k > K - \sum_{i=1}^{k-1} u_i\right) \mathrm{d}u_1 \dots \mathrm{d}u_{k-1} \tag{16}$$

holds for any real numbers $x_1, x_2, \dots x_k$ and K. Here, $\mathbb{I}(A)$ has value 1 if A holds, while it equals 0 in the other case.

Considering the cases $x_1 + x_2 \leq K$ and $x_1 + x_2 > K$ separately, one can easily verify that

$$\frac{(x_1 + x_2 - K)_+^j}{j} = \int_{-\infty}^{+\infty} \mathbb{I}\left(x_1 > u_1\right) \left(x_2 - (K - u_1)\right)_+^{j-1} \mathrm{d}u_1. \tag{17}$$

holds for any integer $j \geq 1$. In particular, for $j = 1$ this expression reduces to

$$\left(x_1 + x_2 - K\right)_+ = \int_{-\infty}^{+\infty} \mathbb{I}\left(x_1 > u_1\right) \mathbb{I}\left(x_2 > K - u_1\right) \mathrm{d}u_1. \tag{18}$$

We can conclude that expression (16) holds for $k = 2$ and any real numbers x_1, x_2 and K. Suppose now that for a particular value of $k \geq 2$, expression (16) holds for any real numbers $x_1, x_2, \dots, x_k$ and K. Then for any real numbers $x_1, x_2, \dots, x_{k+1}$ and K, we have that

$$\begin{aligned} &\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \mathbb{I}\left(x_1 > u_1\right) \dots \mathbb{I}\left(x_k > u_k\right) \mathbb{I}\left(x_{k+1} > K - \sum_{i=1}^{k} u_i\right) \mathrm{d}u_1 \dots \mathrm{d}u_k \\ &= \int_{-\infty}^{+\infty} \mathbb{I}\left(x_1 > u_1\right) \left\{ \int_{-\infty}^{+\infty} \mathbb{I}\left(x_2 > u_2\right) \dots \mathbb{I}\left(x_k > u_k\right) \mathbb{I}\left(x_{k+1} > K - u_1 - \sum_{i=2}^{k} u_i\right) \mathrm{d}u_2 \dots \mathrm{d}u_k \right\} \mathrm{d}u_1. \end{aligned}$$

From relation (16), we find that the right hand side of this expression can be rewritten as

$$\frac{1}{(k-1)!} \int_{-\infty}^{+\infty} \mathbb{I}\left(x_1 > u_1\right) \left(\sum_{i=2}^{k+1} x_i - (K - u_1)\right)_+^{k-1} \mathrm{d}u_1.$$

Taking into account (17), we then find that

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \mathbb{I}\left(x_1 > u_1\right) \dots \mathbb{I}\left(x_k > u_k\right) \mathbb{I}\left(x_{k+1} > K - \sum_{i=1}^{k} u_i\right) \mathrm{d}u_1 \dots \mathrm{d}u_k = \frac{\left(\sum_{i=1}^{k+1} x_i - K\right)_+^k}{k!},$$

which means that expression (16) also holds for $k + 1$.
We can conclude that expression (16) holds for any $k \geq 2$ and any real numbers $x_1, x_2, \dots, x_k$

and K. Replacing x_i by X_i and k by n in this expression, and taking expectations leads to (14).

(b) Applying expression (14) to the vector $-\underline{X}$ and the retention $-K$, we find that

$$\mathbb{E}\left[(K-S_X)_+^{n-1}\right] = (n-1)! \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \overline{F}_{-\underline{X}}\left(x_1,\ldots,x_{n-1},-K-\sum_{i=1}^{n-1} x_i\right) \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1}.$$

Taking into account that $\overline{F}_{-\underline{X}}$ has at most countably many jumps, we can replace the integrand $\overline{F}_{-\underline{X}}\left(x_1,\ldots,x_{n-1},-K-\sum_{i=1}^{n-1} x_i\right)$ by $F_{\underline{X}}\left(-x_1,\ldots,-x_{n-1},K+\sum_{i=1}^{n-1} x_i\right)$ in this expression. Substituting the x_i by $y_i = -x_i$ leads to the expression (15) for $\mathbb{E}\left[(K-S_X)_+^{n-1}\right]$. ∎

The quantities $\mathbb{E}\left[(S_X-K)_+^{n-1}\right]$ and $\mathbb{E}\left[(K-S_X)_+^{n-1}\right]$ in (14) and (15) are well-defined, but may eventually be equal to $+\infty$. Hereafter, we summarize several (necessary and) sufficient conditions for these expectations to be finite.

First, we have that

$$\mathbb{E}\left[(X_i)_+^{n-1}\right] < +\infty \text{ for all } i \Rightarrow \mathbb{E}\left[(S_X)_+^{n-1}\right] < +\infty \Leftrightarrow \mathbb{E}\left[(S_X-K)_+^{n-1}\right] < +\infty \text{ for all } K, \tag{19}$$

while

$$\mathbb{E}\left[(X_i)_-^{n-1}\right] < +\infty \text{ for all } i \Rightarrow \mathbb{E}\left[(S_X)_-^{n-1}\right] < +\infty \Leftrightarrow \mathbb{E}\left[(K-S_X)_+^{n-1}\right] < +\infty \text{ for all } K. \tag{20}$$

Here, the notation y_-^s is used for $\left[-\min(y,0)\right]^s$.

Further, the conditions

$$\mathbb{E}\left[|X_i|^{n-1}\right] < +\infty, i = 1,2,\ldots,n, \tag{21}$$

imply the first conditions in (19) and (20), whereas the condition

$$\mathbb{E}\left[|S_X|^{n-1}\right] < +\infty \tag{22}$$

implies the second conditions in (19) and (20).

Finally, notice that any of the conditions that we considered in (19), (20), (21) and (22) implies that this condition also holds if we replace the exponent $n-1$ by the exponent k, $k = 1,2,\ldots,n-1$.

In case $n = 2$, expression (15) reduces to :

$$\mathbb{E}\left[(K-S_X)_+\right] = \int_{-\infty}^{+\infty} F_{\underline{X}}(x, K-x)\,\mathrm{d}x. \tag{23}$$

Taking into account that $\mathbb{E}\left[(K-S_X)_+\right] = \mathbb{E}\left[(S_X-K)_+\right] - \mathbb{E}\left[S_X\right] + K$, this expression can be transformed in

$$\mathbb{E}\left[(S_X-K)_+\right] = \int_{-\infty}^{+\infty} F_{\underline{X}}(x, K-x)\,\mathrm{d}x + \mathbb{E}\left[S_X\right] - K,$$

which can be found e.g. in Dhaene and Goovaerts (1996).

In the following lemma we derive expressions for $\mathbb{E}\left[\mathrm{e}^{tS_X}\right]$ in terms of the cdf or the ddf of $\underline{X}$, depending on the value of t.

Lemma 4 *Consider the n-vector $\underline{X}$ and the sum of its components S_X.*
For any $t<0$, one has that $\mathbb{E}\left[\mathrm{e}^{tS_X}\right]$ can be expressed as

$$\mathbb{E}\left[\mathrm{e}^{tS_X}\right]=(-t)^n\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty}\mathrm{e}^{t(x_1+\cdots+x_n)}F_{\underline{X}}\left(\underline{x}\right)\mathrm{d}x_1\ldots\mathrm{d}x_n. \tag{24}$$

For any $t>0$, one has that $\mathbb{E}\left[\mathrm{e}^{tS_X}\right]$ can be expressed as

$$\mathbb{E}\left[\mathrm{e}^{tS_X}\right]=t^n\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty}\mathrm{e}^{t(x_1+\cdots+x_n)}\overline{F}_{\underline{X}}\left(\underline{x}\right)\mathrm{d}x_1\ldots\mathrm{d}x_n. \tag{25}$$

Proof. Let us first consider the case where $t>0$. For any i we have that

$$\begin{aligned}\mathrm{e}^{tX_i}&=t\int_{-\infty}^{X_i}\mathrm{e}^{tx_i}\mathrm{d}x_i\\&=t\int_{-\infty}^{+\infty}\mathrm{e}^{tx_i}\mathbb{I}\left(X_i>x_i\right)\mathrm{d}x_i.\end{aligned} \tag{26}$$

Taking into account these expressions for the e^{tX_i}, we can express $\mathbb{E}\left[\mathrm{e}^{tS_X}\right]$ as

$$\mathbb{E}\left[\mathrm{e}^{tS_X}\right]=t^n\mathbb{E}\left[\left(\int_{-\infty}^{+\infty}\mathrm{e}^{tx_1}\mathbb{I}\left(X_1>x_1\right)\mathrm{d}x_1\right)\cdots\left(\int_{-\infty}^{+\infty}\mathrm{e}^{tx_n}\mathbb{I}\left(X_n>x_n\right)\mathrm{d}x_n\right)\right]. \tag{27}$$

Using Fubini's theorem to interchange the order of the expectation and the integrals leads to (25).

Next, we consider the case that $t<0$. In order to prove (24) for $t<0$, consider the random vector $-\underline{X}\equiv(-X_1,-X_2,\ldots,-X_n)$. Applying expression (25) to $\mathbb{E}\left[\mathrm{e}^{-t(-S_X)}\right]$ leads to

$$\mathbb{E}\left[\mathrm{e}^{tS_X}\right]=(-t)^n\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathrm{e}^{-t(x_1+\cdots+x_n)}\overline{F}_{-\underline{X}}\left(\underline{x}\right)\mathrm{d}x_1\ldots\mathrm{d}x_n.$$

Taking into account that $\overline{F}_{-\underline{X}}$ has at most countably many jumps, we can replace $\overline{F}_{-\underline{X}}\left(\underline{x}\right)$ by $F_{\underline{X}}\left(-\underline{x}\right)$ in this expression. Subsituting the x_i by $y_i=-x_i$ leads to expression (24). ■

For any real t, the expectation $\mathbb{E}\left[\mathrm{e}^{tS_X}\right]$ in Lemma 4 is well-defined and non-negative, but eventually equal to $+\infty$. However, in case $\mathbb{E}\left[\mathrm{e}^{ntX_i}\right]<+\infty$, for all $i=1,2,\ldots,n$, the generalized Hölder inequality leads to

$$\mathbb{E}\left[\mathrm{e}^{tS_X}\right]\leq\prod_{i=1}^{n}\left(\mathbb{E}\left[\mathrm{e}^{ntX_i}\right]\right)^{1/n}<+\infty.$$

The relation between the conditions "$\mathbb{E}\left[(S_X)_+^{n-1}\right]<+\infty$" or "$\mathbb{E}\left[(S_X)_-^{n-1}\right]<+\infty$" at the one hand, and "$\mathbb{E}\left[\mathrm{e}^{tS_X}\right]<+\infty$" at the other hand is explored in the following lemma.

Lemma 5 *For any r.v. X, the following implications hold:*

$$\mathbb{E}\left[\mathrm{e}^{tX}\right] < +\infty \text{ for some } t < 0 \Rightarrow \mathbb{E}\left[(X)_-^k\right] < +\infty, \ k = 1, 2, \ldots \tag{28}$$

and

$$\mathbb{E}\left[e^{tX}\right] < +\infty \text{ for some } t > 0 \Rightarrow \mathbb{E}\left[(X)_+^k\right] < +\infty, \ k = 1, 2, \ldots \tag{29}$$

Proof. Let us first prove implication (29). For any positive real number t, we have that

$$\mathrm{e}^{t(x)_+} \leq \mathrm{e}^{tx} + 1.$$

From this inequality we find that

$$\mathbb{E}\left[\mathrm{e}^{tX}\right] < +\infty \ \Rightarrow \mathbb{E}\left[\mathrm{e}^{t(X)_+}\right] < +\infty.$$

As

$$\mathbb{E}\left[\mathrm{e}^{t(X)_+}\right] = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbb{E}\left[(X)_+^k\right] < +\infty,$$

we can conclude that the implication (29) holds.
The proof of (28) follows immediately from (29) by rewriting $\mathbb{E}\left[\mathrm{e}^{tX}\right]$ as $\mathbb{E}\left[\mathrm{e}^{(-t)(-X)}\right]$. ■

The results that we proved in this section will be used to prove results on conditions under which convex ordered r.v.'s, or supermodular order random vectors, are equal in distribution.

5 Convex order and equally distributed random variables

In this section, we consider conditions under which convex ordered r.v.'s are equal in distribution. We use the notation $\overset{\mathrm{d}}{=}$ to indicate 'equality in distribution'. We first introduce the definition of convex order.

Definition 6 *The r.v. X is smaller than the r.v. Y in convex order, notation $X \preceq_{cx} Y$, if*

$$\mathbb{E}\left[X\right] = \mathbb{E}\left[Y\right] \ \text{ and } \mathbb{E}\left[(X-K)_+\right] = \mathbb{E}\left[(Y-K)_+\right], \text{ for all } K \in \mathbb{R}. \tag{30}$$

A summary of other characterizations and properties of convex order can be found e.g. in Denuit et al. (2005).

5.1 Convex order, expected utilities and equally distributed random variables

In expected utility theory, a decision maker is risk averse if he has a (non-decreasing and) concave utility function. Consider now the r.v.'s X and Y with equal expectations. The ordering relation $X \preceq_{cx} Y$ means that all risk averse decision makers prefer random wealth X over random wealth Y. In the following theorem we show that if $X \preceq_{cx} Y$ and in addition, there is a particular risk averse decision maker (with an appropriate utility function) who is indifferent between X and Y, then we can conclude that X and Y are equal in distribution.

Theorem 7 *Consider the r.v.'s X and Y with finite expectations and the interval I with $\Pr[Y \in I] = 1$. Furthermore, let $u : I \longrightarrow \mathbb{R}$ be a strictly concave (or strictly convex) function with absolutely continuous derivative u' and such that $\mathbb{E}[u(Y)]$ is finite. Then we have that*

$$X \preceq_{cx} Y \text{ and } \mathbb{E}[u(X)] = \mathbb{E}[u(Y)] \Longrightarrow X \overset{\mathrm{d}}{=} Y. \tag{31}$$

Proof. We prove the case where u is strictly concave. The other case can be proven in a similar way. In order to prove the implication (31), notice that the convex order relation $X \preceq_{cx} Y$ implies that $\mathbb{E}[X] = \mathbb{E}[Y]$. Furthermore, the convex order relation $X \preceq_{cx} Y$ together with $\Pr[Y \in I] = 1$ implies that $\Pr[X \in I] = 1$, so that the r.v.'s $u(X)$ and $u(Y)$ are well-defined. Under the stated assumptions, we find from Lemma 1 that for $a = \mathbb{E}[X] = \mathbb{E}[Y]$, the following expressions hold for $\mathbb{E}[u(X)]$ and $\mathbb{E}[u(Y)]$, respectively:

$$\mathbb{E}[u(X)] = u(\mathbb{E}[X]) + \int_{\inf I}^{\mathbb{E}[X]} u''(K)\, \mathbb{E}\left[(K - X)_+\right] \mathrm{d}K + \int_{\mathbb{E}[X]}^{\sup I} u''(K)\, \mathbb{E}\left[(X - K)_+\right] \mathrm{d}K$$

and

$$\mathbb{E}[u(Y)] = u(\mathbb{E}[X]) + \int_{\inf I}^{\mathbb{E}[X]} u''(K)\, \mathbb{E}\left[(K - Y)_+\right] \mathrm{d}K + \int_{\mathbb{E}[X]}^{\sup I} u''(K)\, \mathbb{E}\left[(Y - K)_+\right] \mathrm{d}K,$$

where $\mathbb{E}[u(X)]$ and $\mathbb{E}[u(Y)]$, as well as all integrals in these expressions for $\mathbb{E}[u(X)]$ and $\mathbb{E}[u(Y)]$ are finite. From these expressions, taking into account the assumption that $\mathbb{E}[u(X)] = \mathbb{E}[u(Y)]$, we find that

$$\begin{aligned} 0 = & \int_{\inf I}^{\mathbb{E}[X]} u''(K) \left(\mathbb{E}\left[(K - Y)_+\right] - \mathbb{E}\left[(K - X)_+\right]\right) \mathrm{d}K \\ & + \int_{\mathbb{E}[X]}^{\sup I} u''(K) \left(\mathbb{E}\left[(Y - K)_+\right] - \mathbb{E}\left[(X - K)_+\right]\right) \mathrm{d}K. \end{aligned} \tag{32}$$

The convex order relation $X \preceq_{cx} Y$ implies that

$$\begin{aligned} \mathbb{E}\left[(X - K)_+\right] &\leq \mathbb{E}\left[(Y - K)_+\right], \text{ for all } K \in \mathbb{R}. \\ \mathbb{E}\left[(K - X)_+\right] &\leq \mathbb{E}\left[(K - Y)_+\right], \text{ for all } K \in \mathbb{R}. \end{aligned}$$

Furthermore, as the function u' is absolutely continuous, we have that the condition that u is strictly concave on I is equivalent with $u''(x) < 0$ a.e. on I. Hence, both integrands in (32) are non-positive. Furthermore, as $\mathbb{E}\left[(K-Y)_+\right] - \mathbb{E}\left[(K-X)_+\right]$ and $\mathbb{E}\left[(Y-K)_+\right] - \mathbb{E}\left[(X-K)_+\right]$ are continuous functions of K, we can conclude that

$$\begin{aligned} \mathbb{E}\left[(K-Y)_+\right] &= \mathbb{E}\left[(K-X)_+\right], \text{ for all } K < \mathbb{E}[X], \\ \mathbb{E}\left[(Y-K)_+\right] &= \mathbb{E}\left[(X-K)_+\right], \text{ for all } K \geq \mathbb{E}[X], \end{aligned}$$

which is equivalent with

$$\mathbb{E}\left[(Y-K)_+\right] = \mathbb{E}\left[(X-K)_+\right], \text{ for all } K \in \mathbb{R}, \tag{33}$$

which in turn is equivalent with $X \stackrel{\mathrm{d}}{=} Y$; see Property 1.7.3 in Denuit et al. (2005). ■

A proof of Theorem 7 for the narrower class of twice continuously differentiable functions u can be found in Cheung (2010).

Consider a r.v. X with finite expectation $\mathbb{E}[X]$. The function $(x - \mathbb{E}[X])^2$ is an example of a strictly convex function u with absolutely continuous derivative. From Theorem 7, we can conclude that for two r.v.'s X and Y with finite expectations and such that $\mathrm{Var}[Y]$ is finite, we have that

$$X \preceq_{cx} Y \text{ and } \mathrm{Var}[X] = \mathrm{Var}[Y] \Rightarrow X \stackrel{\mathrm{d}}{=} Y. \tag{34}$$

Similarly, for any r.v.'s X and Y with finite expectations and such that $\mathbb{E}\left[\mathrm{e}^{tY}\right]$ is finite for some $t \neq 0$, we have that

$$X \preceq_{cx} Y \text{ and } \mathbb{E}\left[\mathrm{e}^{tX}\right] = \mathbb{E}\left[\mathrm{e}^{tY}\right] \Rightarrow X \stackrel{\mathrm{d}}{=} Y. \tag{35}$$

Remark that the condition that u is strictly concave (resp. strictly convex) on I in Theorem 7 cannot be weakened to the condition that u is concave (resp. convex), i.e. $u''(K) \geq 0$ a.e. (resp. $u''(K) \leq 0$ a.e.), as in this case we cannot conclude from (32) that $\mathbb{E}\left[(Y-K)_+\right] = \mathbb{E}\left[(X-K)_+\right]$ holds for all K.

5.2 Convex order, distorted expectations and equally distributed random variables

In the dual theory of choice under risk, a decision maker is risk averse if he has a (non-decreasing and) convex distortion function. Consider now the r.v.s X and Y with equal expectations. Also in this dual theory, the ordering relation $X \preceq_{cx} Y$ means that all risk averse decision makers prefer gain X over gain Y. In the following theorem we show that if $X \preceq_{cx} Y$ and in addition, there is a particular risk averse decision maker (with an appropriate distortion function) who is indifferent between X and Y, then we can conclude that X and Y are equal in distribution.

Theorem 8 *Consider the r.v.'s X and Y with finite expectations. Furthermore, let g be a strictly convex (or strictly concave) distortion function with absolutely continuous derivative g'. Then we have that*

$$X \preceq_{cx} Y \text{ and } \rho_g[X] = \rho_g[Y] \Longrightarrow X \overset{\mathrm{d}}{=} Y. \tag{36}$$

Proof. From Lemma 2, we know that $\rho_g[X]$ and $\rho_g[Y]$ are finite.

The convex order relation $X \preceq_{cx} Y$ means that $\mathbb{E}[X] = \mathbb{E}[Y]$ and

$$\mathrm{TVAR}_p[X] \leq \mathrm{TVAR}_p[Y], \text{ for all } p \in (0,1),$$

see e.g. Dhaene et al. (2006). Taking into account Lemma 2, we have that

$$\begin{aligned} 0 &= \rho_g[Y] - \rho_g[X] \\ &= \int_0^1 (1-p)\, g''(1-p) \left(\mathrm{TVAR}_p[Y] - \mathrm{TVAR}_p[X]\right) \mathrm{d}p. \end{aligned} \tag{37}$$

As the derivative g' is absolutely continuous, we have that the condition that g is a strictly convex distortion function is equivalent with $g''(p) > 0$ almost everywhere. Hence, the integrand is non-negative. Furthermore, as the function $\mathrm{TVAR}_p[Y] - \mathrm{TVAR}_p[X]$ is a continuous function of p, we can conclude that

$$\mathrm{TVAR}_p[X] = \mathrm{TVAR}_p[Y], \text{ for all } p \in (0,1). \tag{38}$$

It is well-known that

$$\mathbb{E}\left[(X-K)_+\right] \leq \mathbb{E}\left[(Y-K)_+\right], \text{ for all } K \in \mathbb{R}$$

is equivalent with

$$\mathrm{TVAR}_p[X] \leq \mathrm{TVAR}_p[Y], \text{ for all } p \in (0,1),$$

see e.g. Proposition 3.4.8 in Denuit et al. (2005). Hence, (38) is equivalent with

$$\mathbb{E}\left[(X-K)_+\right] = \mathbb{E}\left[(Y-K)_+\right], \text{ for all } K \in \mathbb{R},$$

which in turn is equivalent with $X \overset{\mathrm{d}}{=} Y$. ∎

A proof of Theorem 8 can also be found in Cheung (2010), under the following weaker conditions on the distortion function: g is continuously differentiable and strictly convex (or strictly concave). Although somewhat less general, the proof presented here is more transparent and elegant than the original proof.

The function $g(q) = \frac{\mathrm{e}^{tq}-1}{\mathrm{e}^{t}-1}$, $0 \leq q \leq 1$ is an example of a distortion function satisfying the conditions of Theorem 8. For any $t > 0$, we have that g is strictly convex, while for any $t < 0$, it is strictly concave.

Remark that the condition that g is strictly convex (resp. strictly concave) in Theorem 8 cannot be weakened to the condition that g is convex (resp. concave), i.e. $g''(q) \geq 0$ a.e. (resp. $g''(q) \leq 0$ a.e.), as in this case, we cannot conclude from (37) that $\mathrm{TVAR}_p[X]$ and $\mathrm{TVAR}_p[Y]$ are equal for all values of p.

6 Supermodular order and equally distributed random vectors

Hereafter, we use the notations $\underline{X}$ and $\underline{Y}$ to denote the n-vectors $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_n)$, respectively. The sums of their components are denoted by S_X and S_Y, respectively:

$$S_X = X_1 + \ldots + X_n \text{ and } S_Y = Y_1 + \ldots + Y_n. \tag{39}$$

We start this section by repeating the definitions of some well-known orders between (distributions of) random vectors.

Definition 9 *Consider the random vectors $\underline{X}$ and $\underline{Y}$.*
(a) $\underline{X}$ is smaller than $\underline{Y}$ in the lower orthant order, notation $\underline{X} \preceq_{LO} \underline{Y}$, if

$$F_{\underline{X}}\left(\underline{x}\right) \leq F_{\underline{Y}}\left(\underline{x}\right), \qquad \text{for all } \underline{x} \in \mathbb{R}^n. \tag{40}$$

(b) $\underline{X}$ is smaller than $\underline{Y}$ in the upper orthant order, notation $\underline{X} \preceq_{UO} \underline{Y}$, if

$$\overline{F}_{\underline{X}}\left(\underline{x}\right) \leq \overline{F}_{\underline{Y}}\left(\underline{x}\right), \qquad \text{for all } \underline{x} \in \mathbb{R}^n. \tag{41}$$

(c) $\underline{X}$ is smaller than $\underline{Y}$ in the concordance order, notation $\underline{X} \preceq_{C} \underline{Y}$, if both $\underline{X} \preceq_{LO} \underline{Y}$ and $\underline{X} \preceq_{UO} \underline{Y}$ hold.

The concordance order was introduced by Joe (1997). The order relation $\underline{X} \preceq_C \underline{Y}$ implies that $X_i \overset{\mathrm{d}}{=} Y_i$, $i = 1, 2, \ldots, n$. Intuitively, $\underline{X} \preceq_{LO} \underline{Y}$ means that joint small outcomes are more likely to occur for $\underline{Y}$ than for $\underline{X}$, while $\underline{X} \preceq_{UO} \underline{Y}$ means that joint large outcomes are more likely to occur for $\underline{Y}$ than for $\underline{X}$. The ordering relation $\underline{X} \preceq_C \underline{Y}$ means that joint small outcomes as well as joint large outcomes are more likely to occur for $\underline{Y}$ than for $\underline{X}$. In this sense, $\underline{X} \preceq_C \underline{Y}$ can be interpreted as $\underline{Y}$ is 'more positive dependent' than $\underline{X}$.

Before introducing the supermodular order, we have to define supemodular functions. For any arbitrary function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, real-valued n - vector $\underline{x} = (x_1, x_2 \ldots, x_n)$, integer $i \in \{1, 2, \ldots, n\}$ and positive real number ε, the notation $\Delta_i^\varepsilon f\left(\underline{x}\right)$ is defined by

$$\Delta_i^\varepsilon f\left(\underline{x}\right) = f\left(x_1, x_2 \ldots, x_i + \varepsilon, x_{i+1}, \ldots, x_n\right) - f\left(x_1, x_2, \ldots, x_n\right).$$

Definition 10 (Supermodular function) *A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be supermodular if*

$$\Delta_j^\delta \Delta_i^\varepsilon f\left(\underline{x}\right) \geq 0$$

holds for every $\underline{x} \in \mathbb{R}^n$, $1 \leq i < j \leq n$ and all $\delta, \varepsilon > 0$.

We are now ready to define the supermodular order.

Definition 11 (Supermodular order) *Consider two random vectors $\underline{X}$ and $\underline{Y}$. Then $\underline{X}$ is said to be smaller in the supermodular order than $\underline{Y}$, notation $\underline{X} \preceq_{SM} \underline{Y}$, if*

$$\mathbb{E}\left[f\left(\underline{X}\right)\right] \leq \mathbb{E}\left[f\left(\underline{Y}\right)\right]$$

holds for all supermodular functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ for which the expectations exist.

It is well-known that supermodular order implies concordance order:

$$\underline{X} \preceq_{SM} \underline{Y} \Rightarrow \underline{X} \preceq_{C} \underline{Y}, \tag{42}$$

see e.g. Corollary 6.3.10 in Denuit et al. (2005). Since Tchen (1980), it is known that supermodular order and concordance order are equivalent in the bivariate case. In this case case, both orders also coincide with the correlation order; see e.g. Dhaene and Goovaerts (1996, 1997). Joe (1997) has shown that supermodular order is not equivalent with concordance order for dimension $n \geq 4$. Müller and Scarsini (2000) have shown that the non-equivalence statement also holds for dimension $n = 3$.

Supermodular order implies convex order of the sums of the respective components:

$$\underline{X} \preceq_{SM} \underline{Y} \Rightarrow S_X \preceq_{cx} S_Y,$$

see e.g. Proposition 6.3.9 in Denuit et al. (2005). On the other hand, Müller (1997) has shown that this implication can in general not be strenghtened to the concordance order:

$$\underline{X} \preceq_{C} \underline{Y} \nRightarrow S_X \preceq_{cx} S_Y.$$

In the following theorem we prove that under the appropriate conditions, lower orthant orthant (resp. upper orthant) ordered random vectors with equally distributed sums are equal in distribution.

Theorem 12 *Consider the n-vectors $\underline{X}$ and $\underline{Y}$, as well as the respective sums of their components S_X and S_Y which have finite expectations.*
In case $\mathbb{E}\left[(S_Y)_-^{n-1}\right] < +\infty$, we have that

$$\underline{X} \preceq_{LO} \underline{Y} \text{ and } S_X \overset{\mathrm{d}}{=} S_Y \Longrightarrow \underline{X} \overset{\mathrm{d}}{=} \underline{Y}. \tag{43}$$

In case $\mathbb{E}\left[(S_Y)_+^{n-1}\right] < +\infty$, we have that

$$\underline{X} \preceq_{UO} \underline{Y} \text{ and } S_X \overset{\mathrm{d}}{=} S_Y \Longrightarrow \underline{X} \overset{\mathrm{d}}{=} \underline{Y}. \tag{44}$$

Proof. Let us prove (43), assuming that $\mathbb{E}\left[(S_Y)_-^{n-1}\right] < +\infty$. From (20) and $S_X \overset{\mathrm{d}}{=} S_Y$ we find that $\mathbb{E}\left[(K - S_Y)_+^{n-1}\right]$ and $\mathbb{E}\left[(K - S_X)_+^{n-1}\right]$ are equal and finite, for any value of K. Expression (15) in Lemma 3 leads to the following relation, which holds for any particular value of K:

$$\begin{aligned} 0 &= \left[(K - S_Y)_+^{n-1}\right] - \mathbb{E}\left[(K - S_X)_+^{n-1}\right] \\ &= (n-1)! \int_{-\infty}^{+\infty} \mathrm{d}x_1 \dots \int_{-\infty}^{+\infty} \mathrm{d}x_{n-1} \left(F_{\underline{Y}}\left(x_1, \dots, x_{n-1}, x_n^*\right) - F_{\underline{X}}\left(x_1, \dots, x_{n-1}, x_n^*\right)\right), \end{aligned} \tag{45}$$

with

$$x_n^* = K - \sum_{i=1}^{n-1} x_i.$$

The ordering relation $\underline{X} \preceq_{LO} \underline{Y}$ implies that the integrand in (45) is always non-negative. Hence, we have that

$$F_{\underline{X}}\left(x_1, \dots, x_{n-1}, x_n^*\right) = F_{\underline{Y}}\left(x_1, \dots, x_{n-1}, x_n^*\right), \text{ for any } \left(x_1, \dots, x_{n-1}\right) \in \mathbb{R}^{n-1} \backslash A_K,$$

where λ_{n-1} is the Lebesgue measure on $\mathbb{R}^{n-1}$ and A_K is a subset of $\mathbb{R}^{n-1}$ such that $\lambda_{n-1}(A_K) = 0$. Taking into account the continuity from above of the integrand in (45), we find that

$$F_{\underline{X}}\left(x_1, \dots, x_{n-1}, x_n^*\right) = F_{\underline{Y}}\left(x_1, \dots, x_{n-1}, x_n^*\right), \text{ for any } \left(x_1, \dots, x_{n-1}\right) \in \mathbb{R}^{n-1} \text{ and any } K.$$

This observation implies that $F_{\underline{X}}(\underline{x}) = F_{\underline{Y}}(\underline{x})$, for any $\underline{x} \in \mathbb{R}^n$, which means that $\underline{X} \overset{\mathrm{d}}{=} \underline{Y}$.

The proof of (44), under the assumption that $\mathbb{E}\left[(S_Y)_+^{n-1}\right] < +\infty$ holds, follows the same lines as the proof above, taking into account expression (14) in Lemma 3.

∎

Suppose that one of the moment conditions in Theorem 12 holds true. Then it follows from the same theorem that in case $\underline{X} \preceq_{SM} \underline{Y}$, it suffices to verify whether S_X and S_Y are equally distributed in order to be able to conclude that the random vectors $\underline{X}$ and $\underline{Y}$ are equal in distribution.

Both moment conditions in Theorem 12 can be replaced by the stronger, but somewhat more obvious finite moment conditions (21) or (22); see the discussion after Lemma 3.

Combining Theorem 7 and Theorem 12, while taking into account the properties of supermodular and concordance order that were mentioned above, leads to the following theorem.

Theorem 13 *Consider the n-vectors $\underline{X}$ and $\underline{Y}$, with respective sums S_X and S_Y which are assumed to have finite expectations. Furthermore, consider the interval I with $\Pr[S_Y \in I] = 1$, and the strictly concave (or strictly convex) function $u : I \longrightarrow \mathbb{R}$ with absolutely continuous derivative u' such that $\mathbb{E}[u(S_Y)]$ is finite. Finally, suppose that either*

$$\mathbb{E}\left[(S_Y)_-^{n-1}\right] < +\infty \textit{ or } \mathbb{E}\left[(S_Y)_+^{n-1}\right] < +\infty. \tag{46}$$

Then we have that

$$\underline{X} \preceq_{SM} \underline{Y} \textit{ and } \mathbb{E}[u(S_X)] = \mathbb{E}[u(S_Y)] \Longrightarrow \underline{X} \overset{\mathrm{d}}{=} \underline{Y}. \tag{47}$$

The theorem above can be interpreted in terms of expected utility theory. It states that under the appropriate conditions and when $\underline{X} \preceq_{SM} \underline{Y}$, it suffices to verify whether there is a particular risk averse decision maker who is indifferent between S_X and S_X, in order to be able to conclude whether $\underline{X}$ and $\underline{Y}$ are equal in distribution or not.

Combining Theorem 8 and Theorem 12 leads to the following theorem which has a similar interpretation in terms of the dual theory of choice under risk.

Theorem 14 *Consider the n-vectors $\underline{X}$ and $\underline{Y}$ with sums S_X and S_Y with finite expectations. Furthermore, let g be a strictly convex (or strictly concave) distortion function with absolutely continuous derivative g'. Finally, suppose that*

$$\mathbb{E}\left[(S_Y)_-^{n-1}\right] < +\infty \text{ or } \mathbb{E}\left[(S_Y)_+^{n-1}\right] < +\infty. \tag{48}$$

Then we have that

$$\underline{X} \preceq_{SM} \underline{Y} \text{ and } \rho_g[S_X] = \rho_g[S_Y] \Longrightarrow \underline{X} \overset{\mathrm{d}}{=} \underline{Y}. \tag{49}$$

Notice that a slightly stronger version of both Theorems 13 and 14 may be derived in the sense that when $\mathbb{E}\left[(S_Y)_-^{n-1}\right] < +\infty$, we have that $\underline{X} \preceq_{SM} \underline{Y}$ may be replaced by the weaker conditions that $\underline{X} \preceq_{LO} \underline{Y}$ and $S_X \preceq_{cx} S_Y$. Similarly, when $\mathbb{E}\left[(S_Y)_+^{n-1}\right] < +\infty$, we have that $\underline{X} \preceq_{SM} \underline{Y}$ may be replaced by $X \preceq_{UO} \underline{Y}$ and $S_X \preceq_{cx} S_Y$.

From Lemma 5 it follows that the condition $\mathbb{E}\left[(S_Y)_-^{n-1}\right] < +\infty$ (resp. $\mathbb{E}\left[(S_Y)_+^{n-1}\right] < +\infty$) in Theorem 12, Corollary 13 and Corollary 14 can be replaced by the stronger condition $\mathbb{E}\left[\mathrm{e}^{tS_Y}\right] < +\infty$ for some $t < 0$ (resp. $\mathbb{E}\left[\mathrm{e}^{tS_Y}\right] < +\infty$ for some $t > 0$). Notice that a direct proof of Theorem 12 under these stronger conditions follows from Lemma 4.

Suppose that the r.v. S_Y is bounded from below by a real number a. This situation will occur in particular when all marginals of $\underline{Y}$ are bounded from below. In this case, we have that $0 \leq \mathbb{E}\left[\mathrm{e}^{-S_Y}\right] \leq \mathrm{e}^{-a} < +\infty$ and hence, from Lemma 5 we find that $\mathbb{E}\left[(S_Y)_-^{n-1}\right] < +\infty$. We can conclude that the moment condition (46) in Theorem 13 and Theorem 14 is always satisfied in case S_Y is bounded from below. In a similar way, we can prove that $\mathbb{E}\left[(S_Y)_+^{n-1}\right] < +\infty$ in case S_Y is bounded from above.

Finally, let us have a look at the bivariate case. Consider the random couples (X_1, X_2) and (Y_1, Y_2), which are supermodular ordered such that $(X_1, X_2) \preceq_{SM} (Y_1, Y_2)$. If S_X and S_Y have finite expectations, we also have that $\mathbb{E}\left[(S_Y)_+\right] < +\infty$ and $\mathbb{E}\left[(S_Y)_-\right] < +\infty$. Hence, in the bivariate case, the only moment conditions that have to be assumed for the Theorems 12, 13 and 14 to hold is that S_X and S_Y have finite expectations.

7 Characterizing comonotonic random vectors

A subset A of $\mathbb{R}^n$ is said to be comonotonic if any elements $\underline{x}$ and $\underline{y}$ of A are ordered componentwise, i.e. either $x_i \leq y_i$ for $i = 1, 2, \ldots, n$, or $x_i \geq y_i$ for $i = 1, 2, \ldots, n$ must hold. A random vector $\underline{X} = (X_1, \ldots, X_n)$ is said to be comonotonic if it has a comonotonic support. For a random vector $\underline{X}$, we define its *comonotonic modification* $\underline{X}^c$ as

$$\underline{X}^c = \left(F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right), \tag{50}$$

where the r.v. U has a uniform$(0,1)$ distribution. The comonotonic modification $\underline{X}^c$ has the same marginal distributions as $\underline{X}$, while its components are 'maximally dependent'. Hereafter, we will denote the sum of the components of $\underline{X}$ and $\underline{X}^c$ by S and S^c, respectively:

$$S = X_1 + \ldots + X_n \qquad \text{and} \qquad S^c = F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \ldots + F_{X_n}^{-1}(U). \tag{51}$$

For an overview of the theory of comonotonicity, we refer to Dhaene et al. (2002a). Financial and actuarial applications are described in Dhaene et al. (2002b). An updated overview of applications of comonotonicity can be found in Deelstra et al. (2010).

It is well-known that the comonotonic modification $\underline{X}^c$ exceeds $\underline{X}$ in supermodular order sense, hence

$$\underline{X} \preceq_{SM} \underline{X}^c, \tag{52}$$

see e.g. Proposition 6.3.7 in Denuit et al. (2005). Taking into account this result, we find from Theorem 12 that, provided S has a finite expectation and the moment condition (46) holds, we have that

$$S \stackrel{\mathrm{d}}{=} S^c \Longrightarrow \underline{X} \text{ is comonotonic.} \tag{53}$$

Similarly, from Theorem 13 we find that, provided S has a finite expectation, u is strictly concave (or strictly convex) with absolutely continuous derivative, $\mathbb{E}[u(S)]$ is finite and the moment condition (46) is satisfied, the following implication holds:

$$\mathbb{E}[u(S)] = \mathbb{E}[u(S^c)] \Longrightarrow \underline{X} \text{ is comonotonic.} \tag{54}$$

Finally, from Theorem 14 we find that, provided S has a finite expectation, g is strictly convex (or strictly concave) with absolutely continuous derivative and the moment condition (46) holds, we have that

$$\rho_g[S] = \rho_g[S^c] \Longrightarrow \underline{X} \text{ is comonotonic.} \tag{55}$$

The implications (53), (54) and (55) remain to hold under weaker conditions than the moment condition (46). Indeed, Cheung (2010) proves that implication (53) holds when the marginals have finite first moments and the underlying probability space is atomless. Mao and Hu (2011) prove Cheung's result without having to assume that the underlying probability space is atomless. Cheung (2010) also proves the implications (54) and (55) under less stringent conditions, in particular without having to assume the moment condition (46). Below, we give a simplified proof of Cheung's theorem concerning the implication (53). We first recall the following result, which is proven in Mao and Hu (2011).

Lemma 15 (Mao and Hu) *Consider the n-vector $\underline{X} = (X_1, \ldots, X_n)$. In case all expectations $\mathbb{E}[X_i]$, $i = 1, 2, \ldots, n$, are finite, we have that*

$$X_1 + \ldots + X_n \overset{\mathrm{d}}{=} X_1^c + \ldots + X_n^c \Rightarrow X_1 + \ldots + X_{n-1} \overset{\mathrm{d}}{=} X_1^c + \ldots + X_{n-1}^c.$$

In the following theorem, we give a new proof of Cheung's theorem, based on the previous lemma.

Theorem 16 *Consider the n-vector $\underline{X} = (X_1, \ldots, X_n)$. In case all expectations $\mathbb{E}[X_i]$, $i = 1, 2, \ldots, n$, are finite, we have that*

$$\underline{X} \text{ is comonotonic} \Leftrightarrow S \overset{\mathrm{d}}{=} S^c.$$

Proof. The $\Rightarrow$ - implication is straightforward. It remains to prove the other implication. First, as $\underline{X} \preceq_{SM} \underline{X}^c$ we have that

$$(X_i, X_j) \preceq_{SM} \left(X_i^c, X_j^c\right), \qquad \text{for all } i \neq j \text{ in } \{1, 2, \ldots, n\}.$$

Furthermore, as $S \overset{\mathrm{d}}{=} S^c$, we find from Lemma 15 that

$$X_i + X_j \overset{\mathrm{d}}{=} X_i^c + X_j^c, \qquad \text{for all } i \neq j \text{ in } \{1, 2, \ldots, n\}.$$

From Theorem 12 it follows then that

$$(X_i, X_j) \overset{\mathrm{d}}{=} \left(X_i^c, X_j^c\right), \qquad \text{for all } i \neq j \text{ in } \{1, 2, \ldots, n\}.$$

As comonotonicity of the n-vector $\underline{X}$ is equivalent with comonotonicity of all couples (X_i, X_j), see Theorem 3 in Dhaene et al. (2002a), we can conclude that $\underline{X}$ is comonotonic.

■

In the following theorem, a proof for implication (53) is given in case the second-order moments of all components X_i are finite. Although the conditions are stronger than the conditions in Cheung (2010) and Mao and Hu (2011), we present this proof here because of its amazing simplicity.

Theorem 17 *Consider the n-vector $\underline{X} = (X_1, \ldots, X_n)$. In case all second-order moments $\mathbb{E}[X_i^2]$, $i = 1, 2, \ldots, n$, are finite, we have that*

$$\underline{X} \text{ is comonotonic} \Leftrightarrow S \overset{\mathrm{d}}{=} S^c \Leftrightarrow \mathrm{Var}[S] = \mathrm{Var}[S^c].$$

Proof. The second-order moment conditions imply that all covariances $\mathrm{cov}[X_i, X_j]$ and $\mathrm{cov}\left[X_i^c, X_j^c\right]$, $i, j = 1, 2, \ldots, n$, as well as $\mathrm{Var}[S]$ and $\mathrm{Var}[S^c]$ are finite. The proof of the $\Rightarrow$ - implications is trivial. It remains to prove that $\mathrm{Var}[S] = \mathrm{Var}[S^c]$ implies that $\underline{X}$ is comonotonic.

We first recall that $\mathrm{cov}[X_i, X_j] \leq \mathrm{cov}\left[X_i^c, X_j^c\right]$ always holds. Furthermore, we have that comonotonicity can be characterized as follows:

$$\underline{X} \text{ is comonotonic} \Leftrightarrow \ \mathrm{cov}\left[X_i, X_j\right] = \mathrm{cov}\left[X_i^c, X_j^c\right] \ \text{ for all } i \neq j \text{ in } \ \{1, 2, \ldots, n\}\,.$$

A proof of the above-mentioned properties can be found e.g. in in Dhaene et al. (2002a). Taking into account these properties, we find that

$$\begin{aligned}
&\mathrm{Var}\left[S\right] = \mathrm{Var}\left[S^c\right] \\
&\Rightarrow \sum_{i,j=1}^{n} \mathrm{cov}\left(X_i, X_j\right) = \sum_{i,j=1}^{n} \mathrm{cov}\left(X_i^c, X_j^c\right) \\
&\Rightarrow \mathrm{cov}\left(X_i, X_j\right) = \mathrm{cov}\left(X_i^c, X_j^c\right), \text{ for all } i \neq j \text{ in } \ \{1, \ldots, n\} \\
&\Rightarrow \left(X_1, \ldots, X_n\right) \ \text{ is comonotonic.}
\end{aligned}$$

■

The bivariate special case of Theorem 17 can be found in Dhaene et al. (2002b). A proof of the equivalence $S \overset{\mathrm{d}}{=} S^c \Leftrightarrow \mathrm{Var}[S] = \mathrm{Var}[S^c]$ can also be found in Cheung and Vanduffel (2012). This equivalence was also derived in Dhaene, Linders, Schoutens and Vyncke (2012) for the case where all marginals are non-negative.

We end this section by showing that the condition that $\mathbb{E}\left[S\right]$ is finite is essential for the implication (53) to hold. In particular, we will show that this implication does not hold for r.v.'s with a Cauchy distribution.

Recall that a r.v. X has a Cauchy distribution with location parameter 0 and scale parameter σ if its probability density function f is given by

$$f_X\left(x\right) = \frac{1}{\pi\sigma\left(1 + \left(\frac{x}{\sigma}\right)^2\right)}, \qquad x \in \mathbb{R},$$

or, equivalently, its characteristic function is given by

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}tX}\right] = \mathrm{e}^{-\sigma|t|}.$$

It is well-known that the expected value of the Cauchy distributed r.v. X does not exist.

Consider now the random couple $\underline{X} = (X_1, X_2)$. Suppose that X_1 and X_2 have a standard Cauchy distribution, which means that their scale and location parameters are 0 and 1, respectively. Furthermore, suppose that X_1 and X_2 are mutually independent. The characteristic function of the sum $S = X_1 + X_2$ is given by

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}tS}\right] = \left(\mathbb{E}\left[e^{\mathrm{i}tX_1}\right]\right)^2 = \mathrm{e}^{-2|t|},$$

which means that S has a generalized Cauchy distribution with location parameter $\mu = 0$ and scale parameter $\sigma = 2$.

Next, consider the comonotonic modifiation $\underline{X}^c = (X_1^c, X_2^c)$ of $\underline{X}$ and its sum $S^c = X_1^c + X_2^c$. As $S^c \overset{\mathrm{d}}{=} 2X_1$, we find that the characteristic function of S^c is given by

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}tS^c}\right] = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}t2X_1}\right] = \mathrm{e}^{-2|t|}.$$

Equality of the characteristic functions of S and S^c allows us to conclude that $S \overset{\mathrm{d}}{=} S^c$.

From Luan (2001), we know that (X_1, X_2) can only have the independent and the comonotonic copula at the same time in case the marginal distribution F_{X_1} is degenerate. Obviously, this condition is not fulfilled here. Hence, for the couple (X_1, X_2) of independent Cauchy distributed r.v.'s we have that $S \overset{\mathrm{d}}{=} S^c$, but $(X_1, X_2) \overset{\mathrm{d}}{\neq} (X_1^c, X_2^c)$. This means that the first order moment condition is crucial for the implication (53) to hold.

We can conclude that when considering a random vector $\underline{X}$ and its comonotonic modification $\underline{X}^c$, Theorems 12, 13 and 14 remain to hold under weaker moment conditions than (46), although the condition that S has a finite expectation cannot be relaxed. These observations make us presume that the implications in Theorems 12, 13 and 14 might remain to hold under weaker moment conditions than (46). Investigating this presumption is a topic of future research.

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References

Boutsikas, M. and Vaggelatou, E. (2002), 'On the distance between convex-ordered random variables', *Advances in Applied Probability* **34**, 349–374.

Carr, P. and Madan, D. (2001), Towards a theory of volatility trading, *in* 'Option Pricing, Interest Rates and Risk Management', Cambridge University Press, pp. 458–476.

Cheung, K. C. (2010), 'Characterizing a comonotonic random vector by the distribution of the sum of its components', *Insurance: Mathematics & Economics* **47**(2), 130 – 136.

Cheung, K. C. and Vanduffel, S. (2012), 'Bounds for sums of random variables when the marginal distributions and the variance of the sum are given', *Scandinavian Actuarial Journal* . To be published.

Deelstra, G., Dhaene, J. and Vanmaele, M. (2010), An overview of comonotonicity and its applications in finance and insurance, *in* B. Oksendal and G. Nunno, eds, 'Advanced Mathematical Methods for Finance', Springer, Germany (Heidelberg).

Denuit, M., Dhaene, J., Goovaerts, M. and Kaas, R. (2005), *Actuarial Theory for Dependent Risks: Measures, Orders and Models*, John Wiley & Sons, Ltd, Chichester, UK.

Dhaene, J., Denuit, M., Goovaerts, M., Kaas, R. and Vyncke, D. (2002a), 'The concept of comonotonicity in actuarial science and finance: theory', *Insurance: Mathematics & Economics* **31**(1), 3–33.

Dhaene, J., Denuit, M., Goovaerts, M., Kaas, R. and Vyncke, D. (2002b), 'The concept of comonotonicity in actuarial science and finance: applications', *Insurance: Mathematics & Economics* **31**(2), 133–161.

Dhaene, J. and Goovaerts, M. (1996), 'Dependency of risks and stop-loss order', *Astin Buleltin* **26**(2), 201–212.

Dhaene, J. and Goovaerts, M. (1997), 'On the dependency of risks in the individual life model', *Insurance: Mathematics and Economics* **19**(3), 243 – 253.

Dhaene, J., Kukush, A., Linders, D. and Tang, Q. (2012), 'Remarks on quantiles and distortion risk measures', *European Actuarial Journal* **2**, 319–328.

Dhaene, J., Linders, D., Schoutens, W. and Vyncke, D. (2012), 'The herd behavior index: A new measure for the implied degree of co-movement in stock markets', *Insurance: Mathematics and Economics* **50**(3), 357–370.

Dhaene, J., Vanduffel, S., Goovaerts, M., Kaas, R., Tang, Q. and Vyncke, D. (2006), 'Risk measures and comonotonicity: a review', *Stochastic Models* **22**, 573–606.

Föllmer, H. and Schied, A. (2004), *Stochastic finance*, De Gruyter studies in mathematics, 2., rev. and extended ed. edn, de Gruyter.

Joe, H. (1997), *Multivariate Models and Dependence Concepts*, Chapman & Hall, London.

Kaas, R., Goovaerts, M., Dhaene, J. and Denuit, M. (2008), *Modern Actuarial Risk Theory Using R*, Springer. 2nd edition.

Luan, C. (2001), 'Insurance premium calculation with anticipated utility theory', *ASTIN Bulletin* **31**(1), 23–25.

Mao, T. and Hu, T. (2011), 'A new proof of cheung's characterization of comonotonicity', *Insurance: Mathematics and Economics* **48**(2), 214 – 216.

Müller, A. (1997), 'Stop-loss order for portfolios of dependent risks', *Insurance: Mathematics and Economics* **21**(3), 219 – 223.

Müller, A. and Scarsini, M. (2000), 'Some remarks on the supermodular order', *Journal of Multivariate Analysis* **73**(1), 107 – 119.

Tchen, H. A. (1980), 'Inequalities for distributions with given marginals', *Annals of probability* **8**(4), 814–827.

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