Orthogonal bases in system identification

Improving the condition number of the iterations

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Summary

- Identification problem

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- Orthogonal Vector Polynomials

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- Orthogonal Rational Functions

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- ORF via inverse EVP

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Stable Linear Algebra techniques
to improve numerical stability

Systems

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$G = Y/U$ transfer function

Identification

- Given: measurements

$$\{G(s_k)\}_{k=1}^{N} \quad \text{or} \quad \{U(s_k), Y(s_k)\}_{k=1}^{N}, \quad s_k \in j\mathbb{R}$$

+ noise/uncertainty information.

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- Find estimate $\hat{G}$ or $\hat{Y} = \hat{G}U$ such that ...

$$\|Y - \hat{Y}\|_{w_y} \quad \text{or} \quad \|G - \hat{G}\|_{w_g} = \min$$

  for some weights w_y or w_g.

Norms

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Discrete

$$\|f\|_w^2 = \sum_{k=1}^{N} |f(s_k)|^2 w_k, \quad s_k = j\omega_k.$$

Weights

Data dependent

Measurements $\{G(s_k)\}$, with variance $\{\sigma_k\}$:

$$\|G - \hat{G}\|_w^2 = \sum_{k=1}^{N} \frac{|G(s_k) - \hat{G}(s_k)|^2}{\sigma_k^2}, \quad w_k = \sigma_k^{-2}$$

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$$\begin{aligned} \|Y-\hat{Y}\|_{w_y} &= \|(G-\hat{G})U\|_{w_y} \\ &= \|G-\hat{G}\|_{w_g}, \quad w_g = w_y|U|^2. \end{aligned}$$

Weights

Solution dependent

$$\|G - \hat{G}\|_{w_g} = \left\| \frac{Y}{U} - \frac{B}{A} \right\|_{w_g} = \left\| \frac{YA - UB}{AU} \right\|_{w_g}$$

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$$= \left\| \begin{bmatrix} \frac{Y}{AU} & \frac{-U}{AU} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \right\|_{w_g}, \quad P = \begin{bmatrix} A \\ B \end{bmatrix}$$

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$$W = [Y \quad -U] \frac{\sqrt{w_g}}{AU}$$

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$$= \|P\|_w, \qquad w = W^H W$$

vector valued polynomial P matrix valued weight w

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- Start: with $W_k = W_k(1)$

  Iterate: solve for A, B, and set $W_k = W_k(A)$
  adapt basis to weights

  Close: Solve ML problem with fixed basis,
  condition number grows moderately

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- The monic orthogonal polynomial of degree n.

Proof

$$\langle f, g \rangle_w = \sum_k f(s_k)^H w_k g(s_k) = \Sigma(- \square |)$$

$$= \sum_k f(s_k)^H W_k^H W_k g(s_k) = \Sigma(- | - |)$$

Switch to z for "complex variable"
since derivation is general.

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$$\dim \mathbb{P}_n^{2\times 1} = 2(n+1),$$

$$\left[\begin{array}{cccc|cccc} 1 & z & \cdots & z^n & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & z & \cdots & z^n \end{array}\right]$$

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How to construct the vector OP?

Recurrence

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Now write this down for $\mathbf{z} = [z_1, \ldots, z_N]^T$

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Setting

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Multiply first eq. from the left with $\mathbf{W}$, $\quad \mathbf{W}\tilde{\mathbf{Z}} = \mathbf{Z}\mathbf{W}$

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Replace $\mathbf{W}\boldsymbol{\Phi}$ by $\mathbf{Q}$

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Multiply from the left with $\mathbf{Q}^H$

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use orthogonality!

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Inverse eigenvalue problem

$$\mathbf{Q}^H \mathbf{Z} \mathbf{Q} = \mathbf{H} \Leftrightarrow \mathbf{Z} = \mathbf{Q} \mathbf{H} \mathbf{Q}^H$$

Note symmetry of $\mathbf{H}$

Inverse eigenvalue problem

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$$\mathbf{Q}^H \begin{bmatrix} W_1 \\ \vdots \\ W_N \end{bmatrix} = \mathbf{\Phi}^H \mathbf{W}^H \mathbf{W} \begin{bmatrix} I_2 \\ \vdots \\ I_2 \end{bmatrix}$$

ϕ_0 is a constant, hence $\phi_0 \phi_0^{-1} = I_2$ and thus

$$\phi_0(\mathbf{z}) \phi_0^{-1} = \begin{bmatrix} I_2 \\ \vdots \\ I_2 \end{bmatrix}$$

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$$\|\phi_0\|_w = I_2 \Rightarrow \eta_{00}^H \eta_{00} = \sum w_k = \sum W_k^H W_k$$

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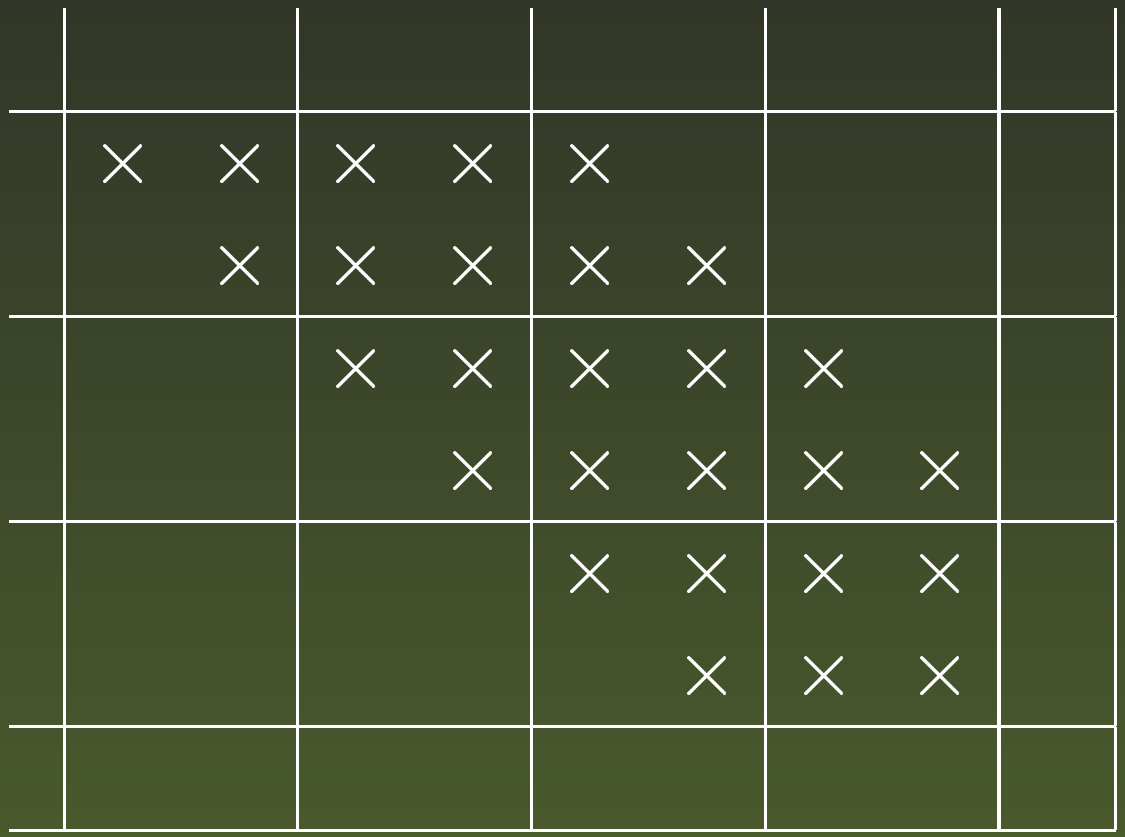
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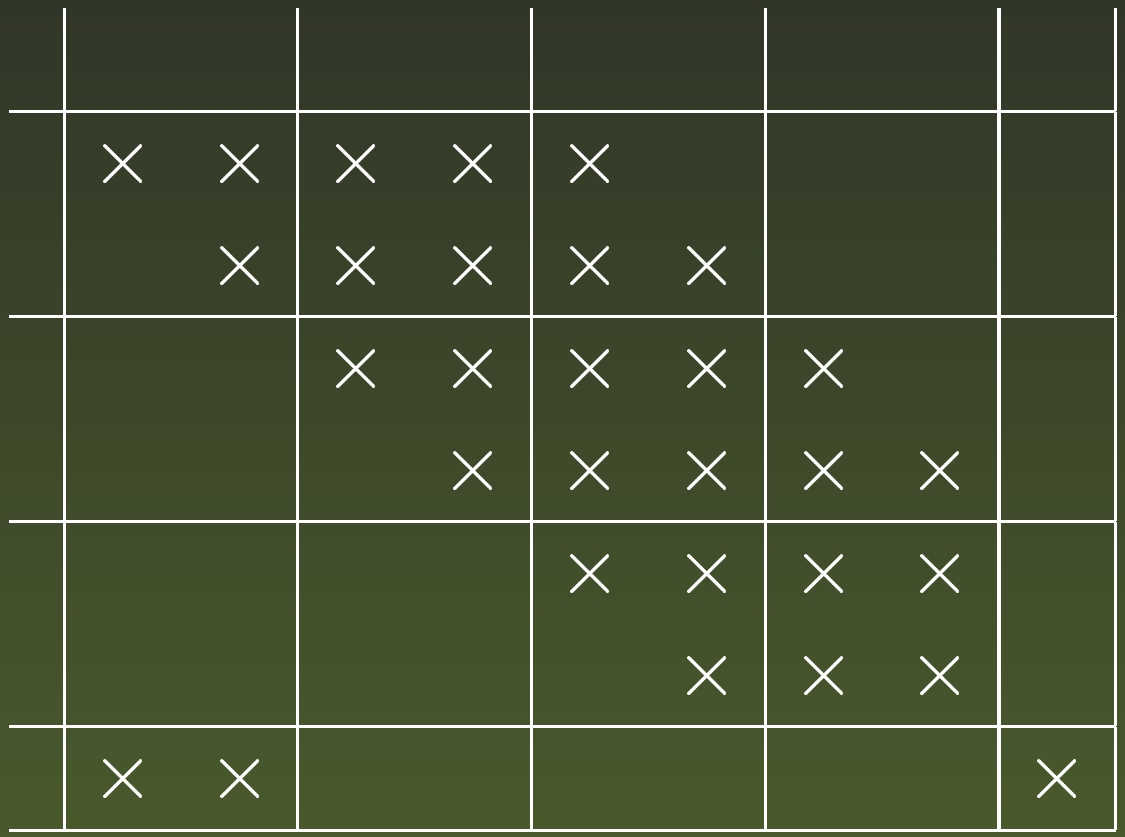
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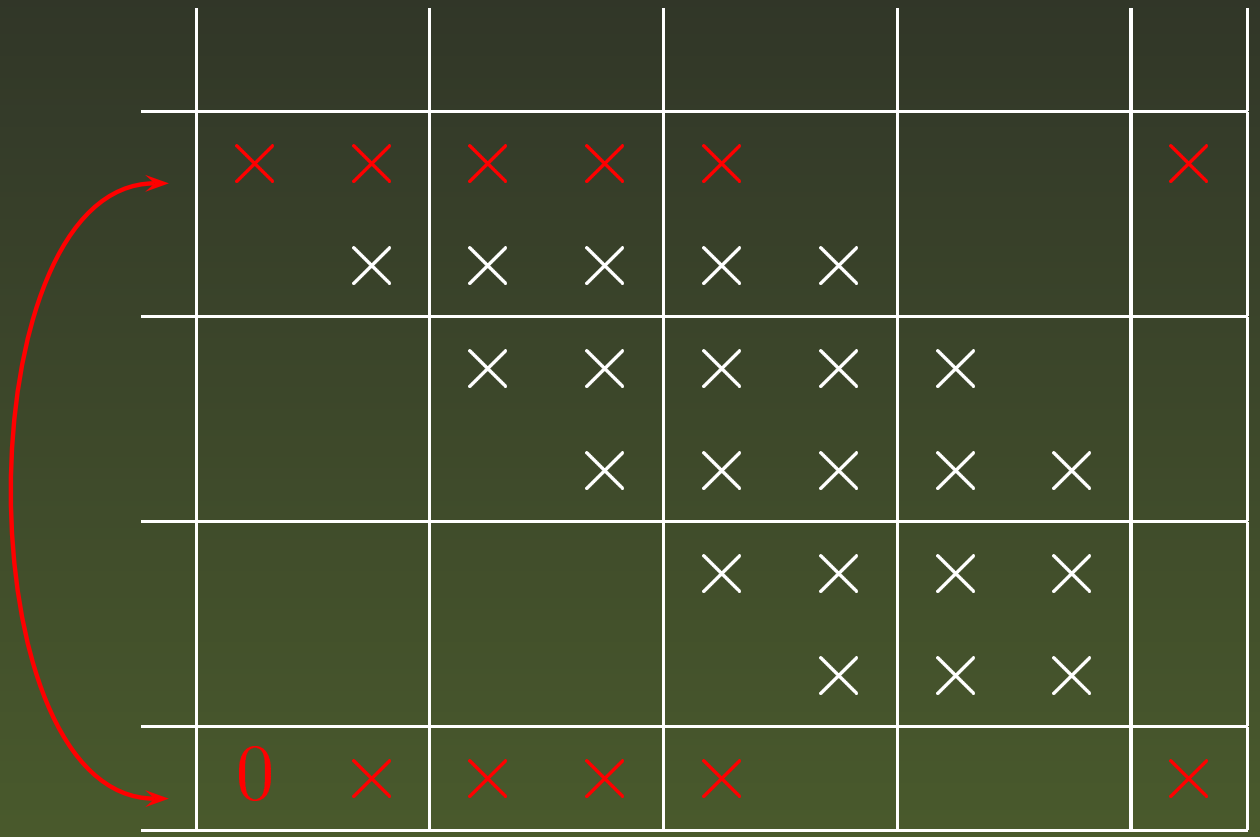
Algorithm



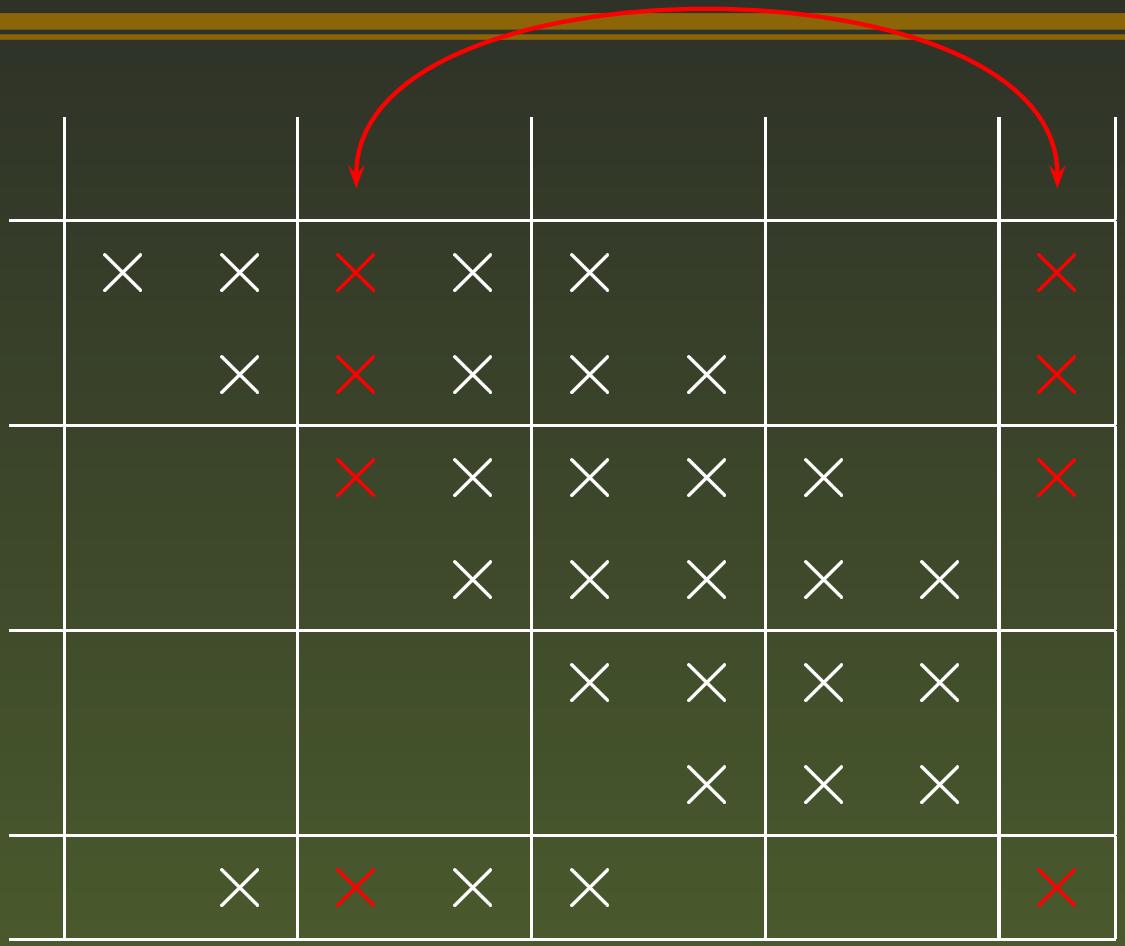
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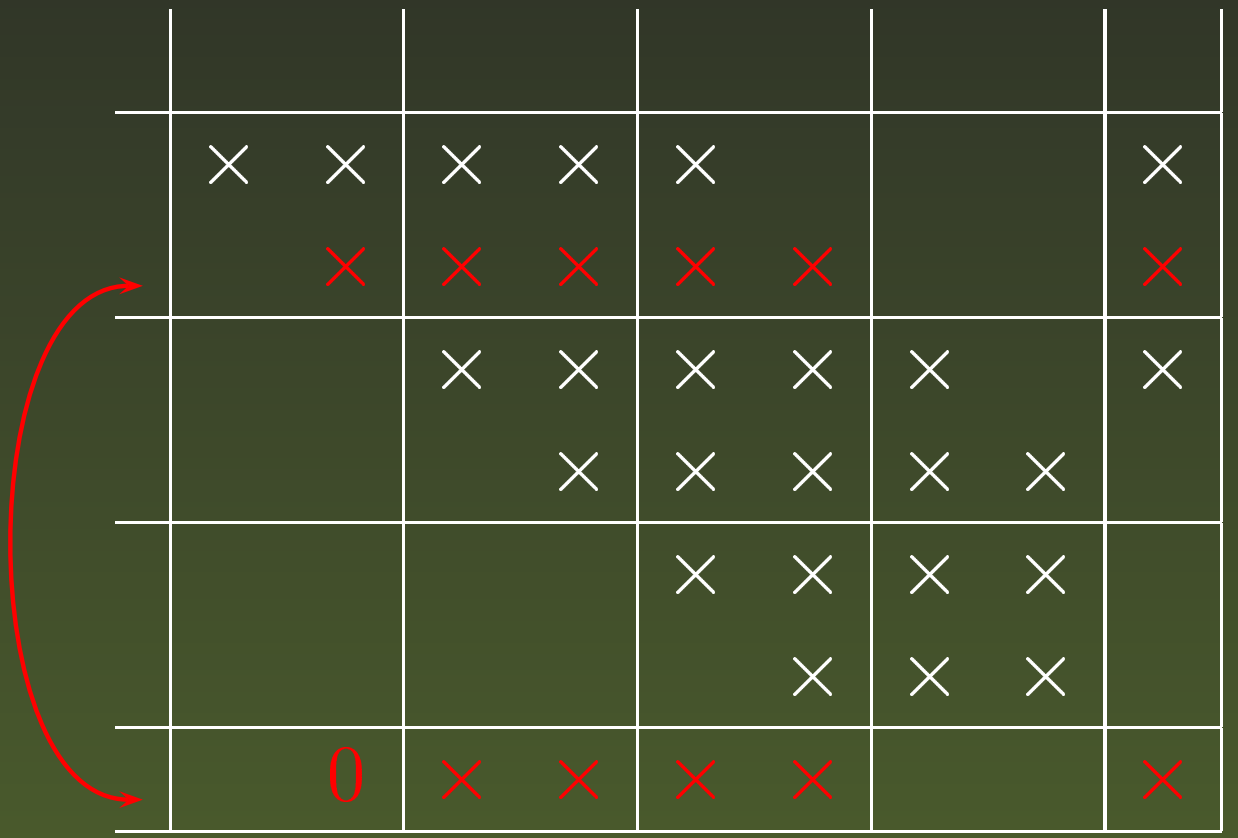
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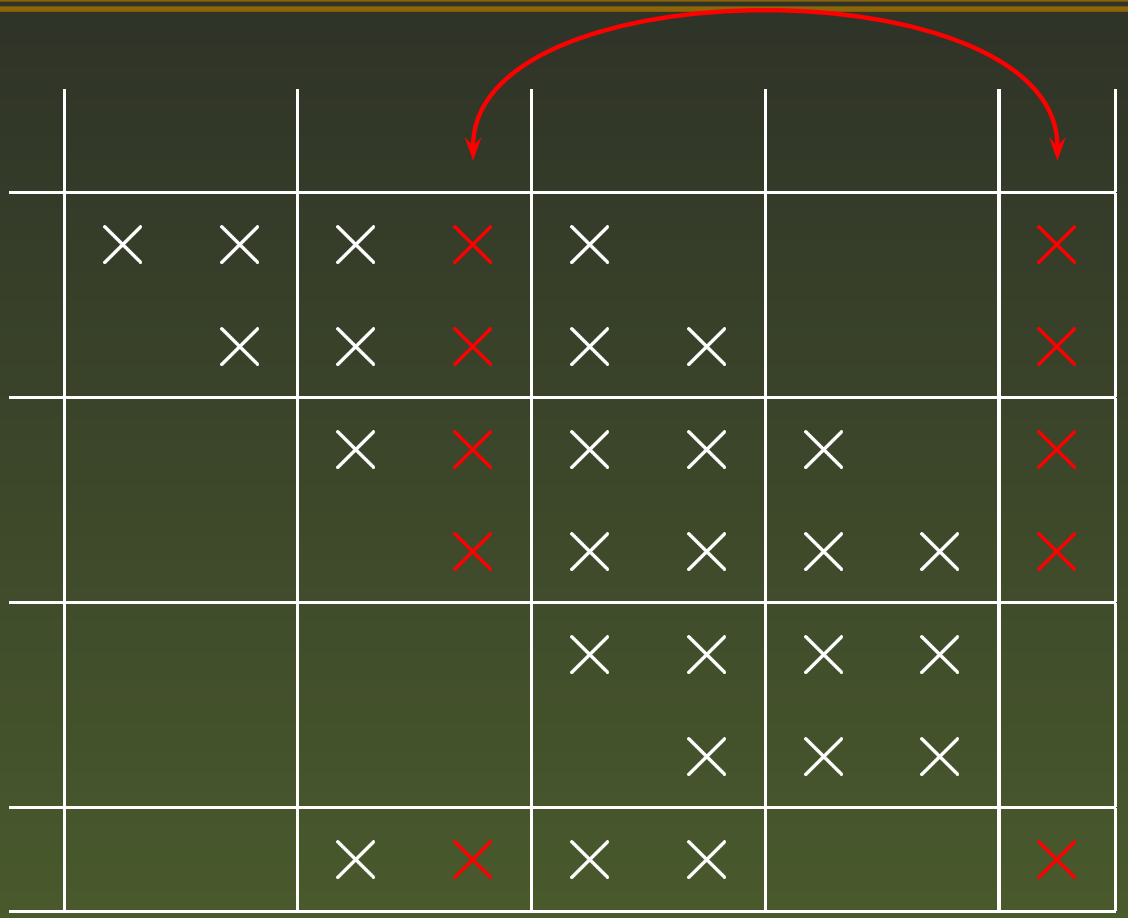
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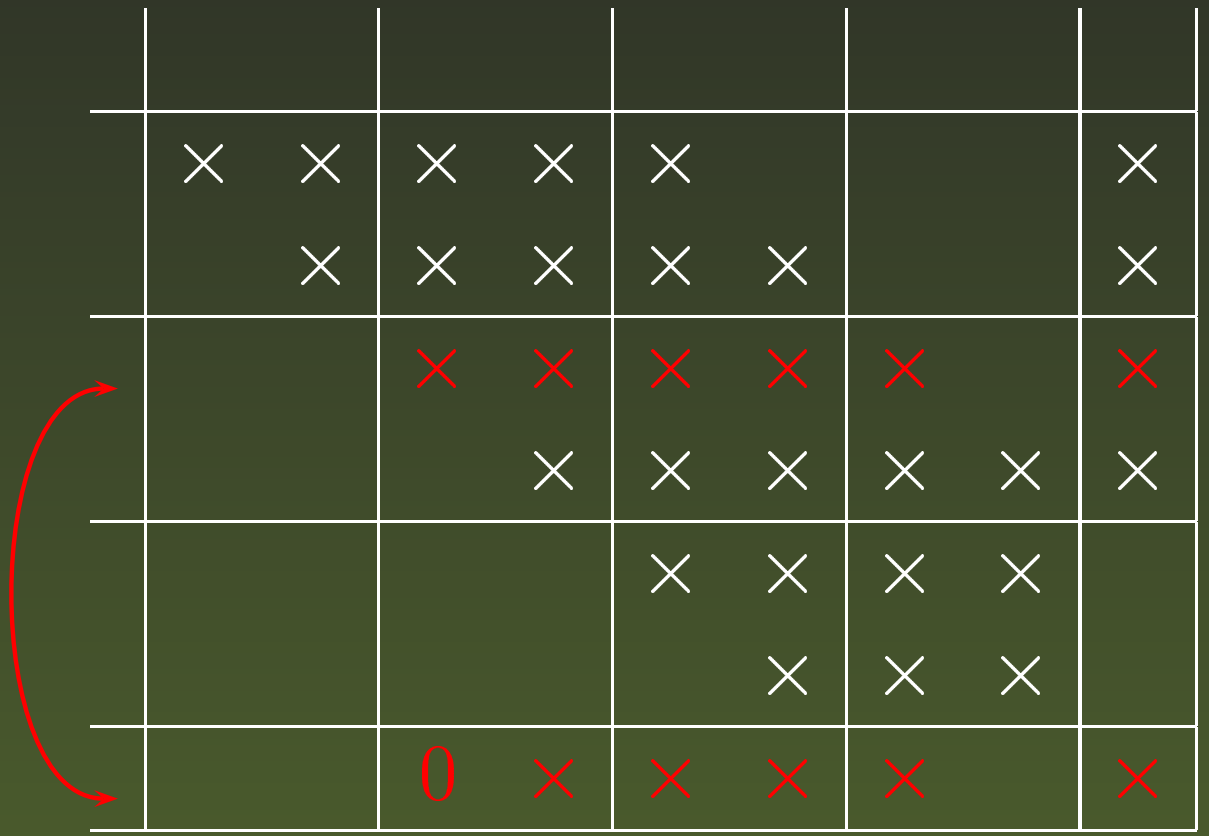
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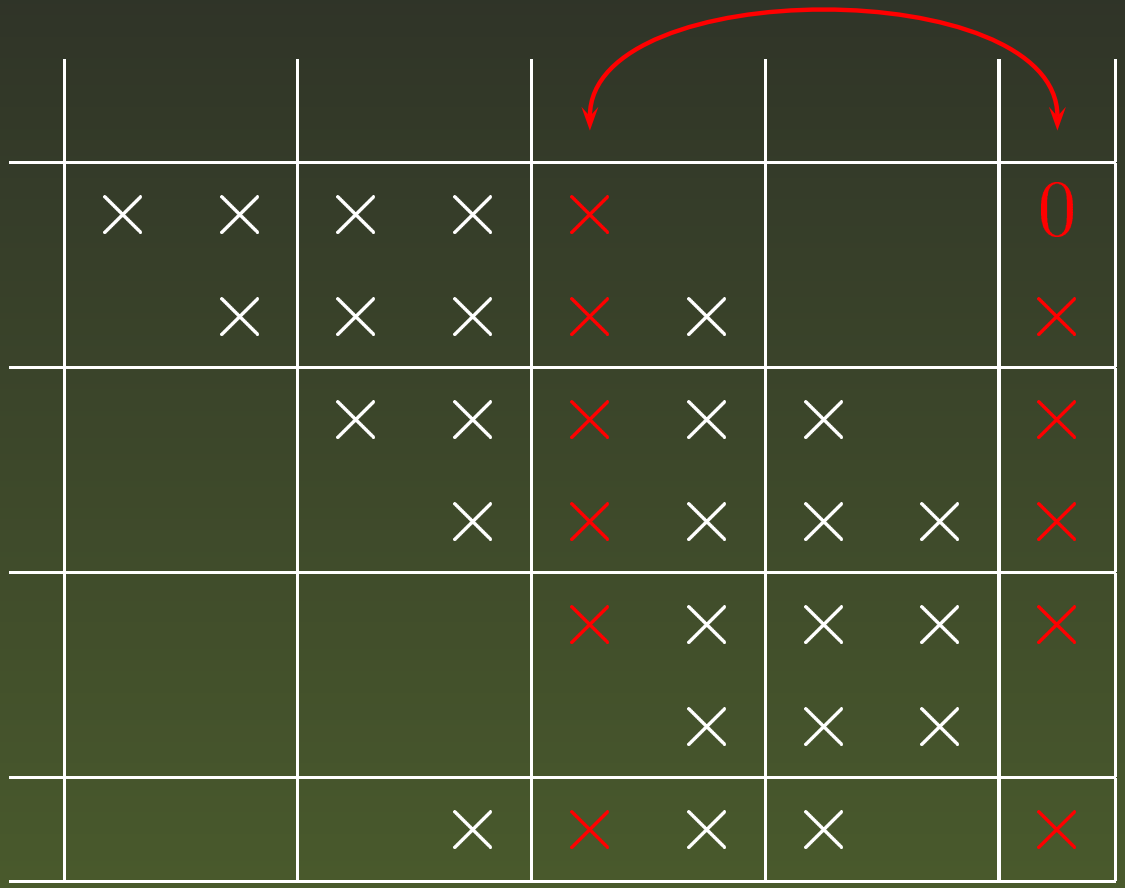
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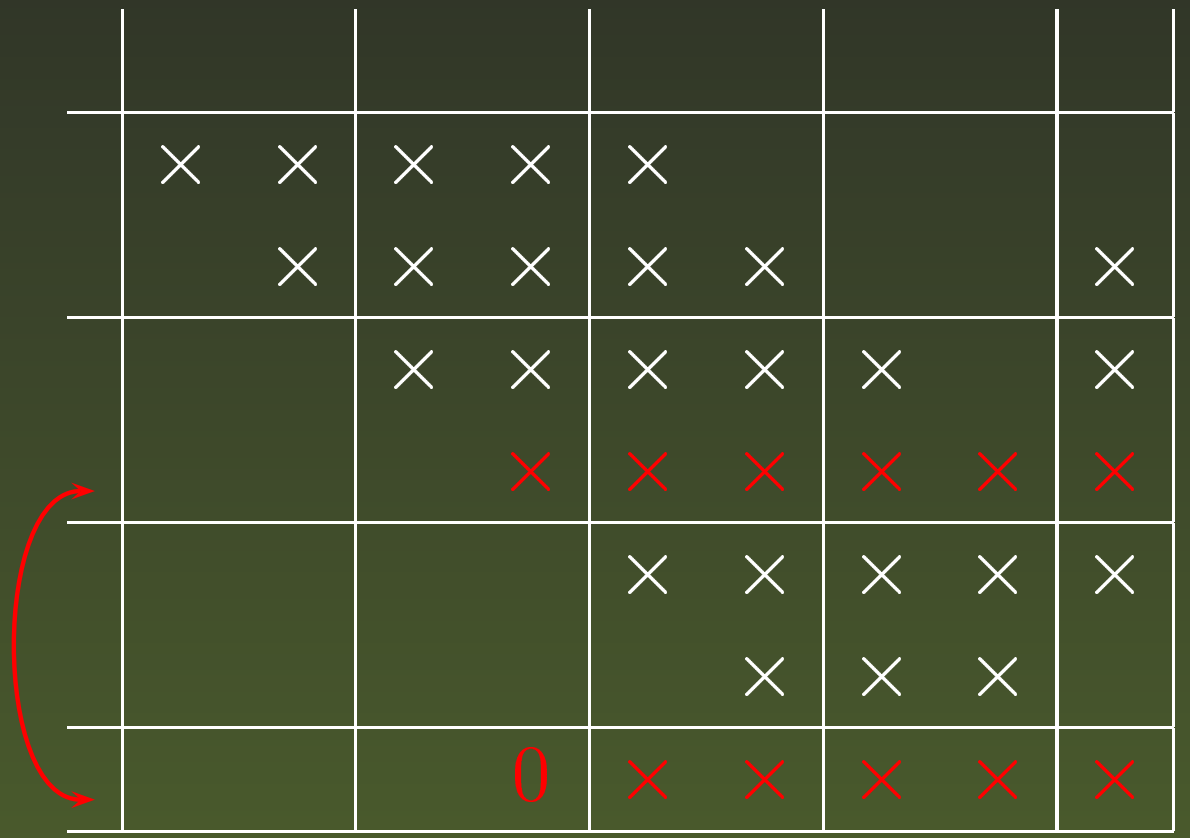


Algorithm



zero by symmetry

Algorithm

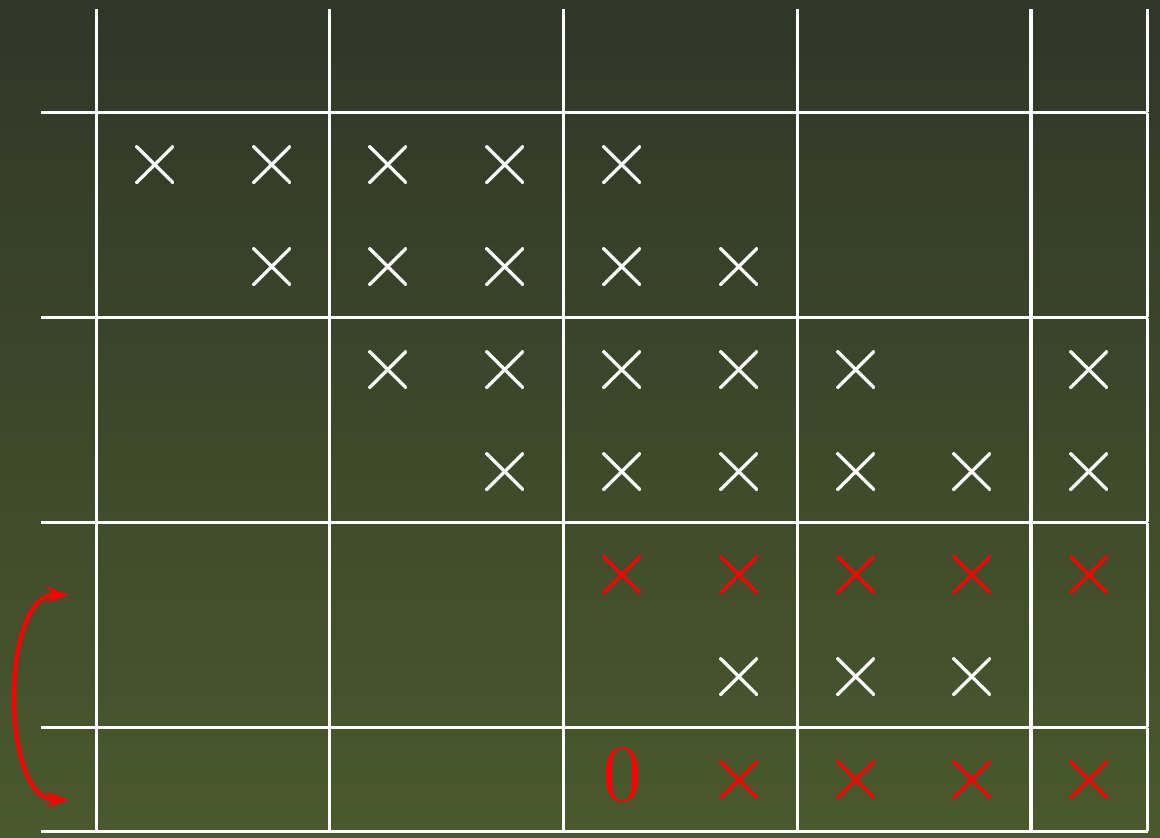


Algorithm



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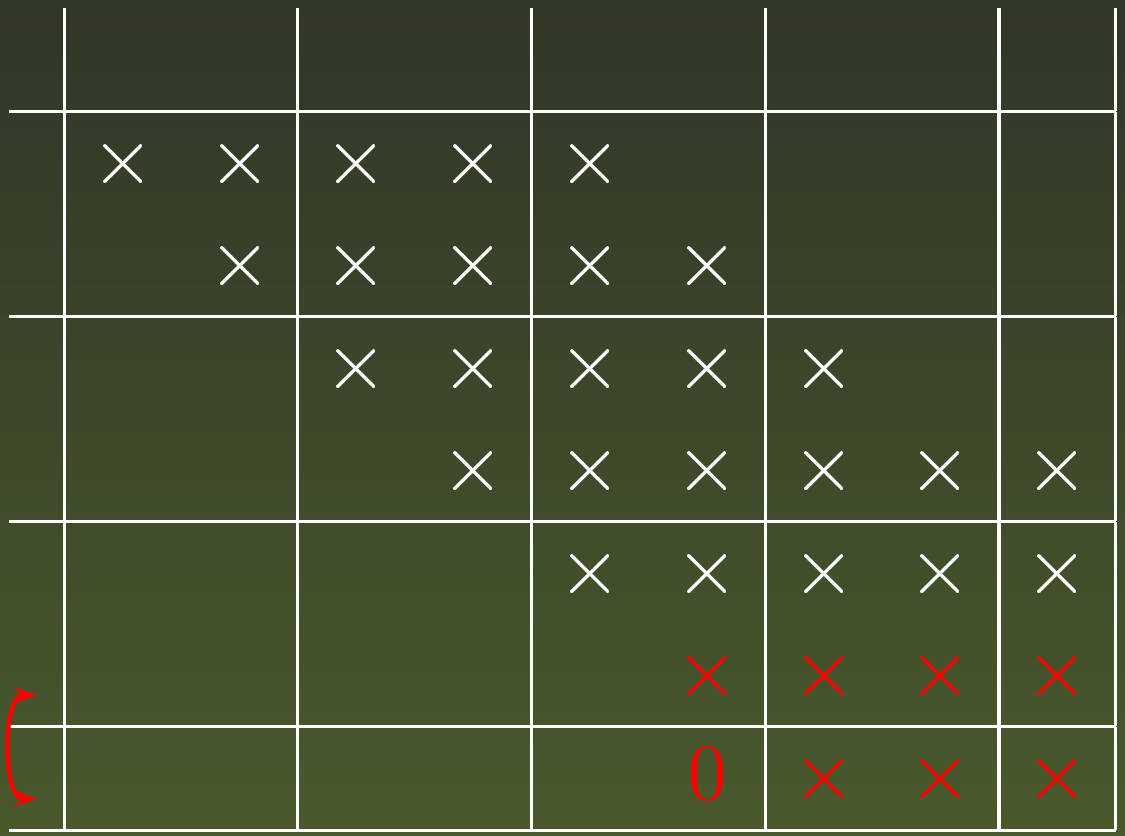


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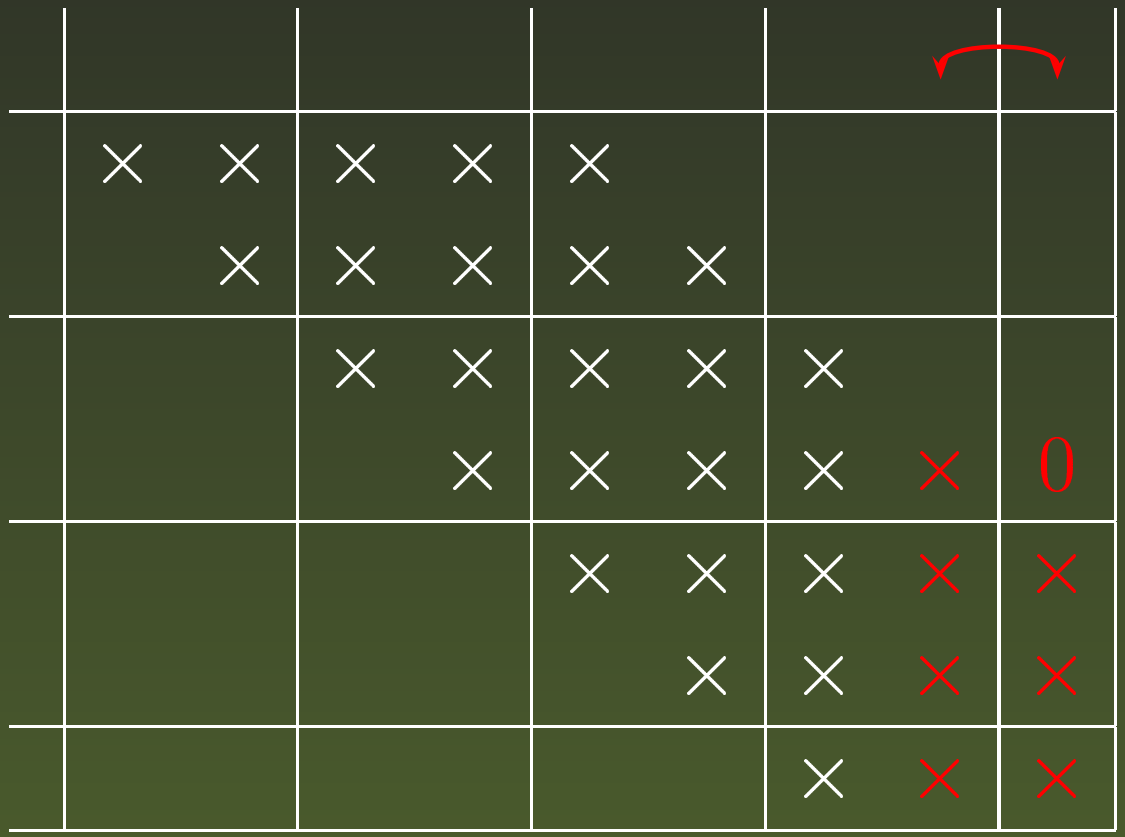


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- Similar for discrete time systems ($z_i \in \mathbb{T}$)

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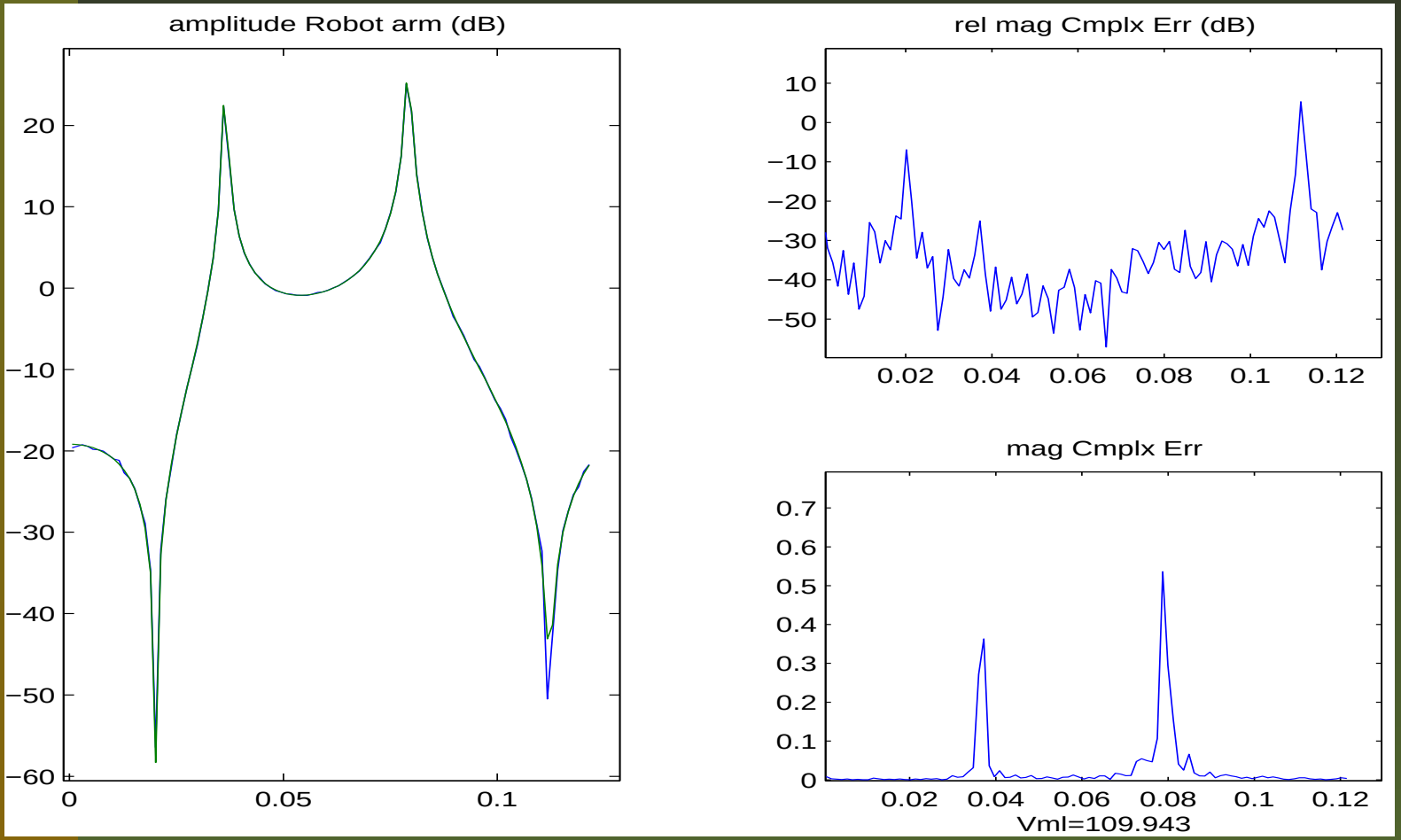
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- Several degree structures are possible (column permutations)

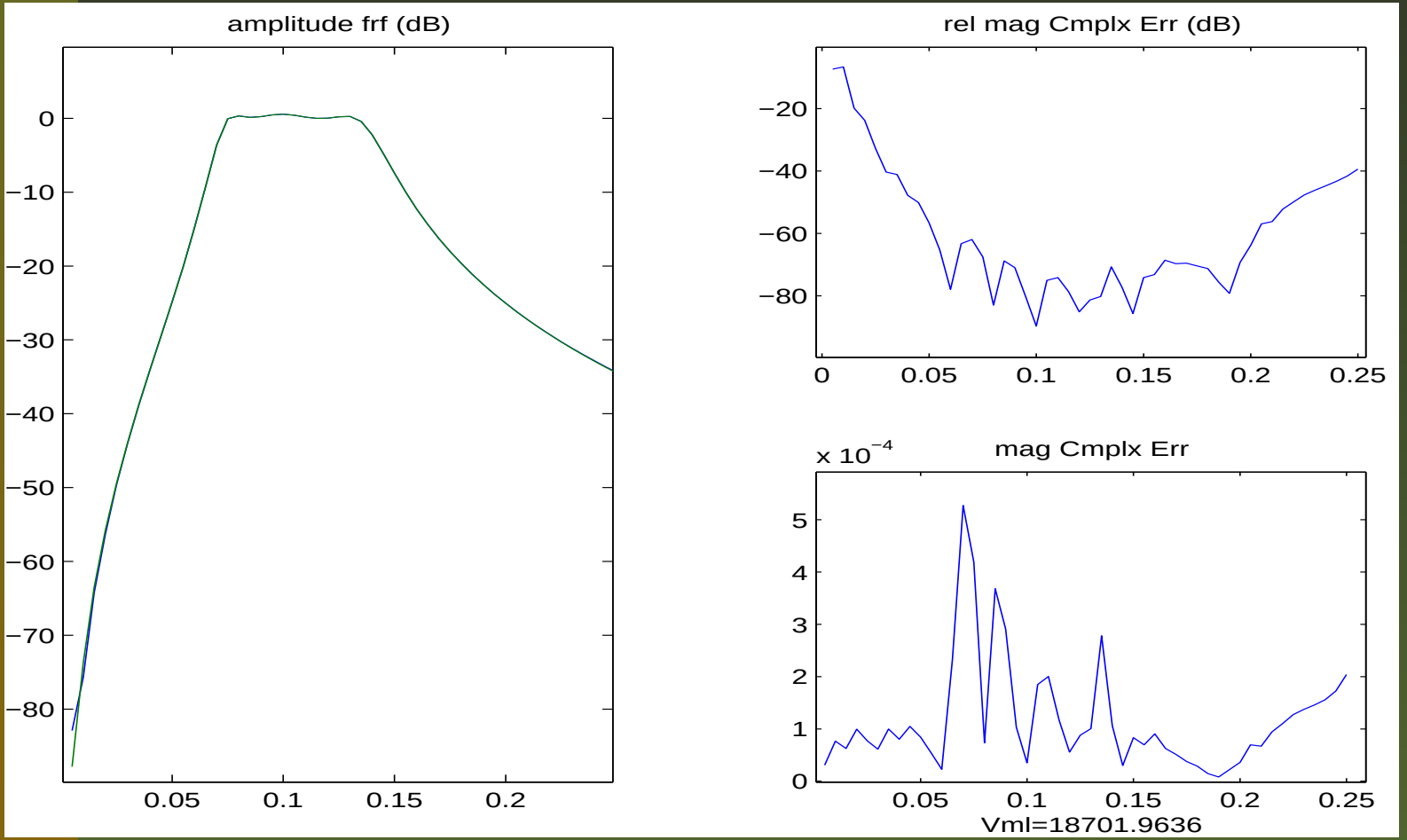
Robotarm

order [6/6], 100 data points, 200Hz, discrete time



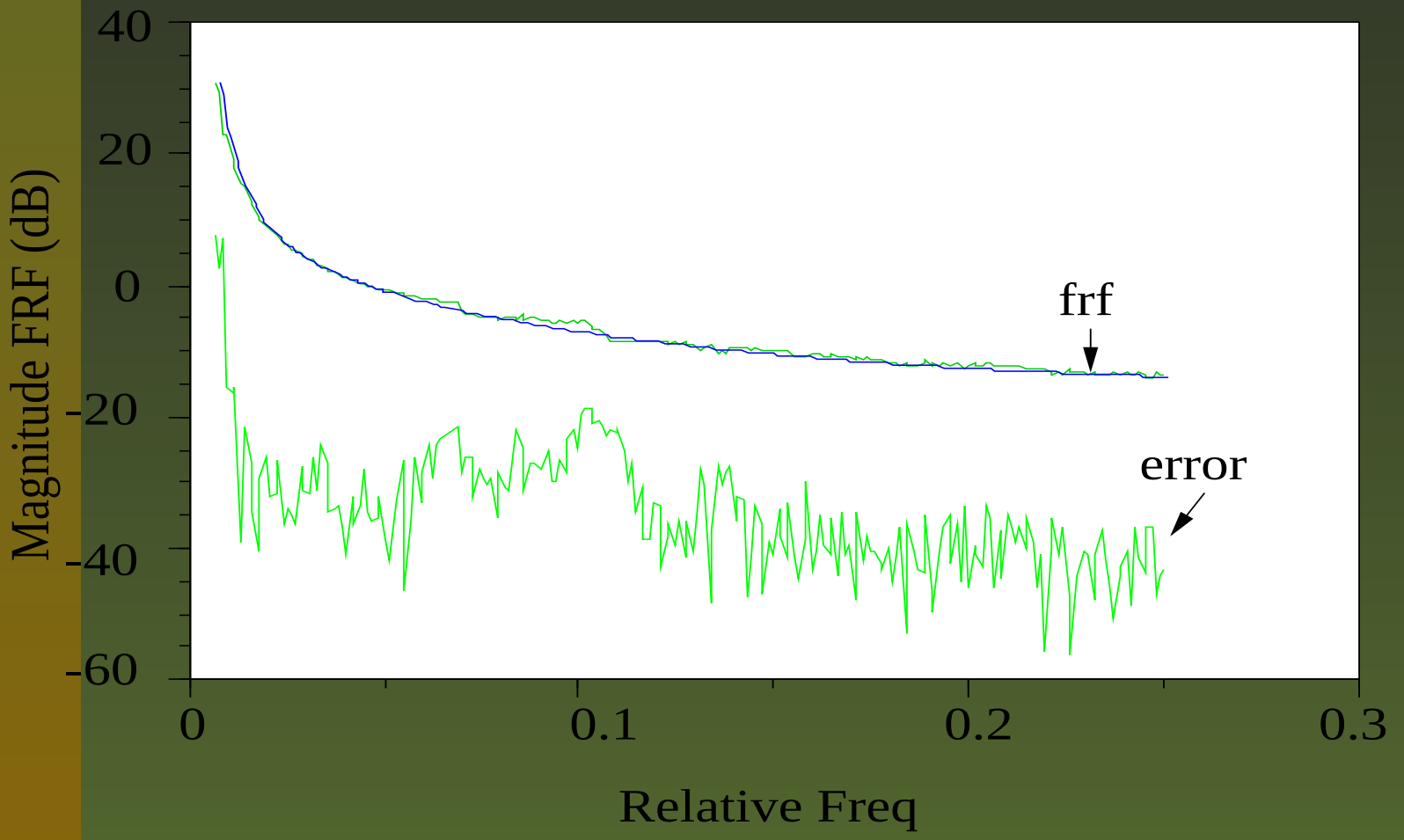
Bandpass filter

order [6/6], 50 data points, 20KHz, discrete time



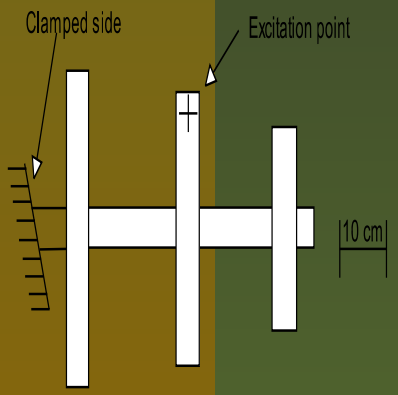
CD's radial servo system

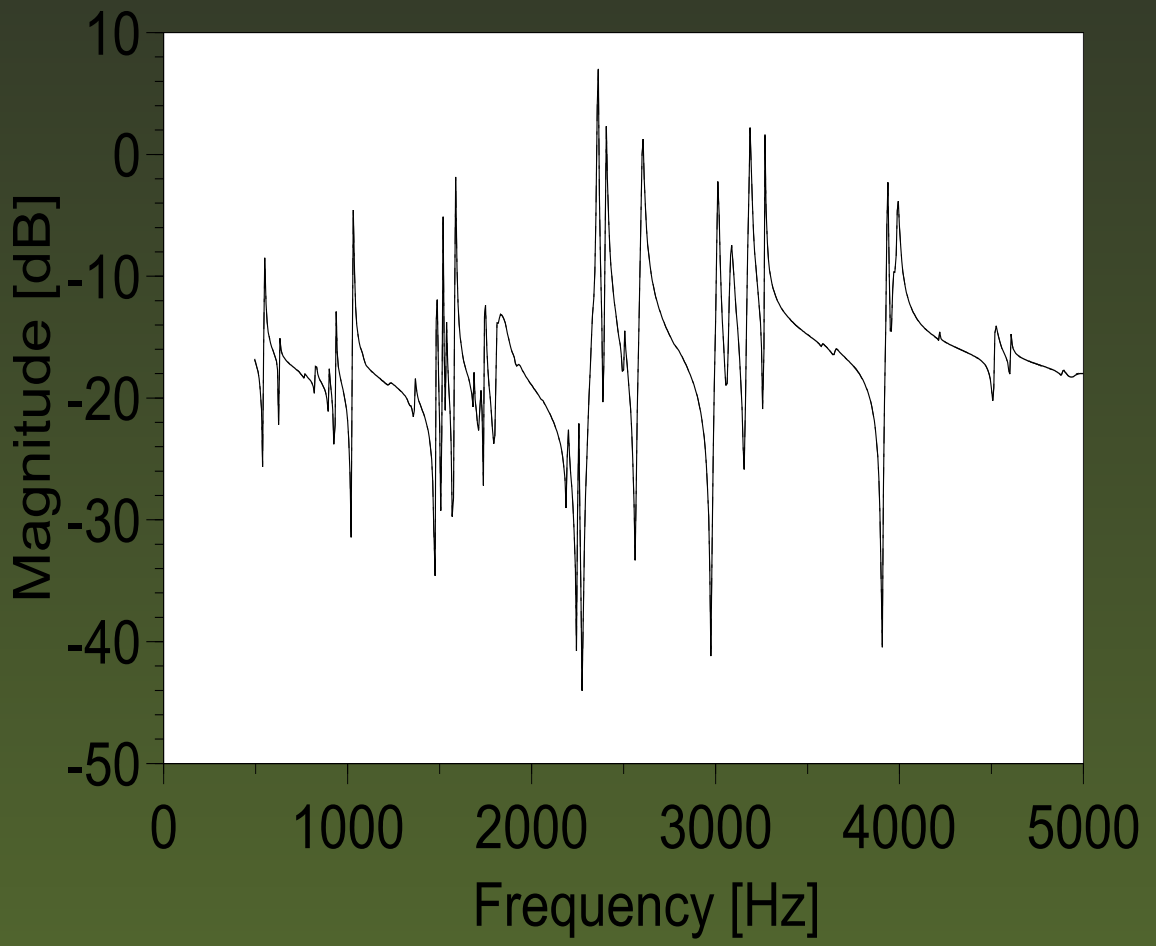
sampling frequency 9.7 kHz, order [5/5]



Mechanical structure

order [120/120]





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- Choice of poles $\boldsymbol{\alpha} = \{\alpha_k\} \Rightarrow$ linear problem

Takenaka-Malmquist basis

- For discrete systems:

$$\phi_k = \frac{\sqrt{1-|\alpha_k|^2}z}{z-\alpha_k}\prod_{l=1}^{n-1}\frac{1-\overline{\alpha}_l z}{z-\alpha_l}$$

is orthonormal w.r.t.

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$$\langle f,g\rangle_w = \sum_{k=1}^{N} f(z_k)w_k\overline{g(z_k)}$$

Nonlinear problem

- if α_k given, finding $\frac{B}{A} = \sum_{l=0}^{n} \phi_l p_l$ is linear

$$\sum \phi_l p_l = [\phi_0 \dots \phi_n]\mathbf{p} = \Phi(\cdot;\boldsymbol{\alpha})\mathbf{p}$$

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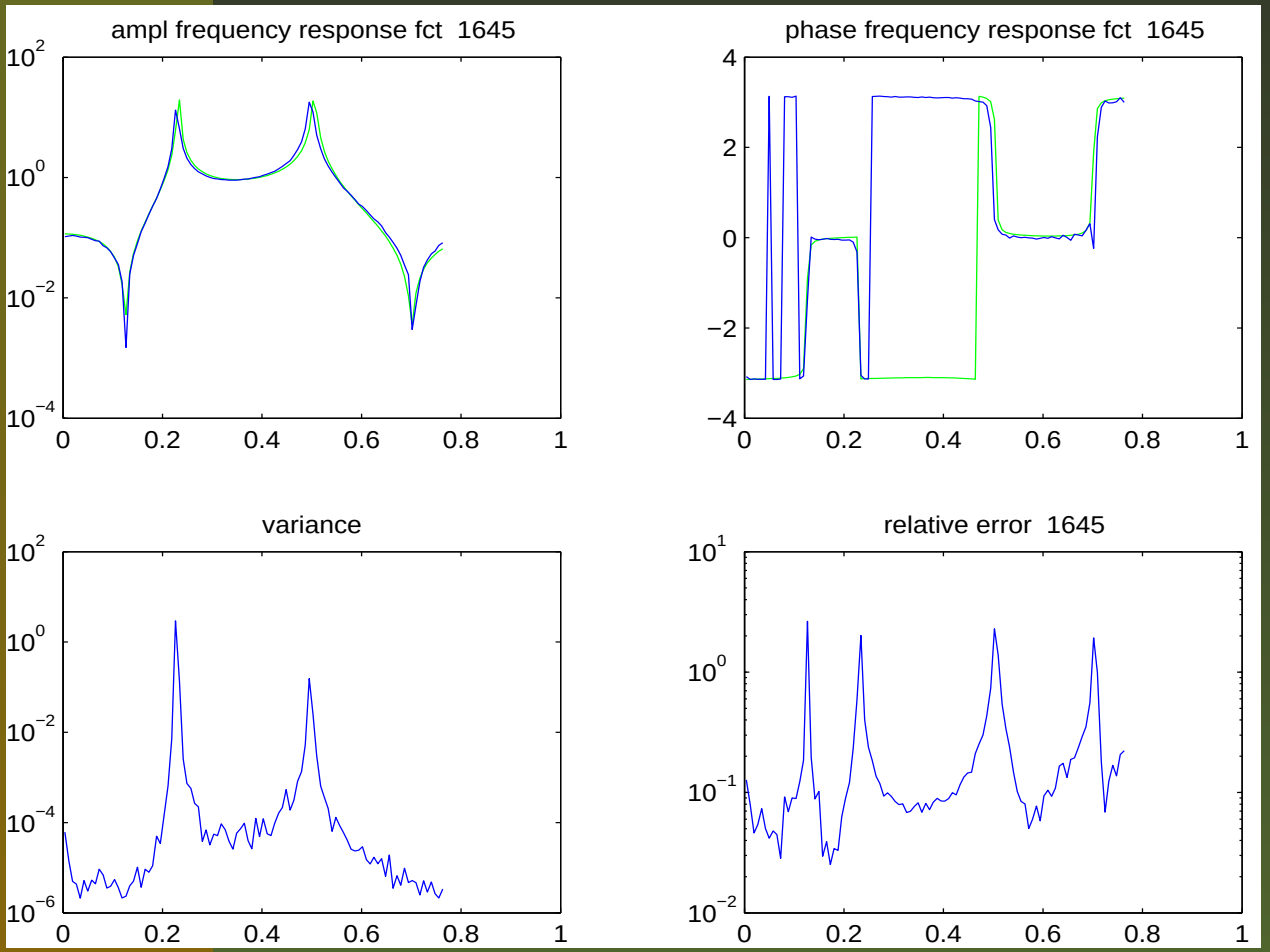
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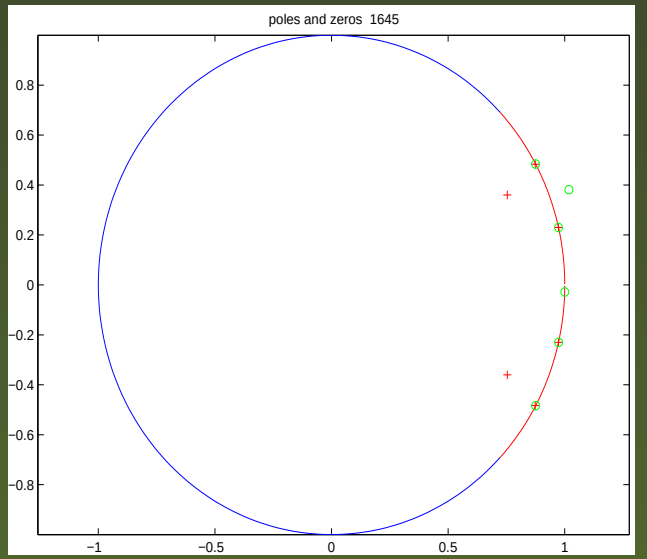
$$\Phi = [\phi_0, \ldots, \phi_n], \mathbf{A} = \text{diag}(1, \alpha_1, \ldots, \alpha_n),$$

$$\mathbf{T} = \begin{bmatrix} a_0 & b_0 & & & \\ b_0 & a_1 & b_1 & & \\ & b_1 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix}$$

Robotarm

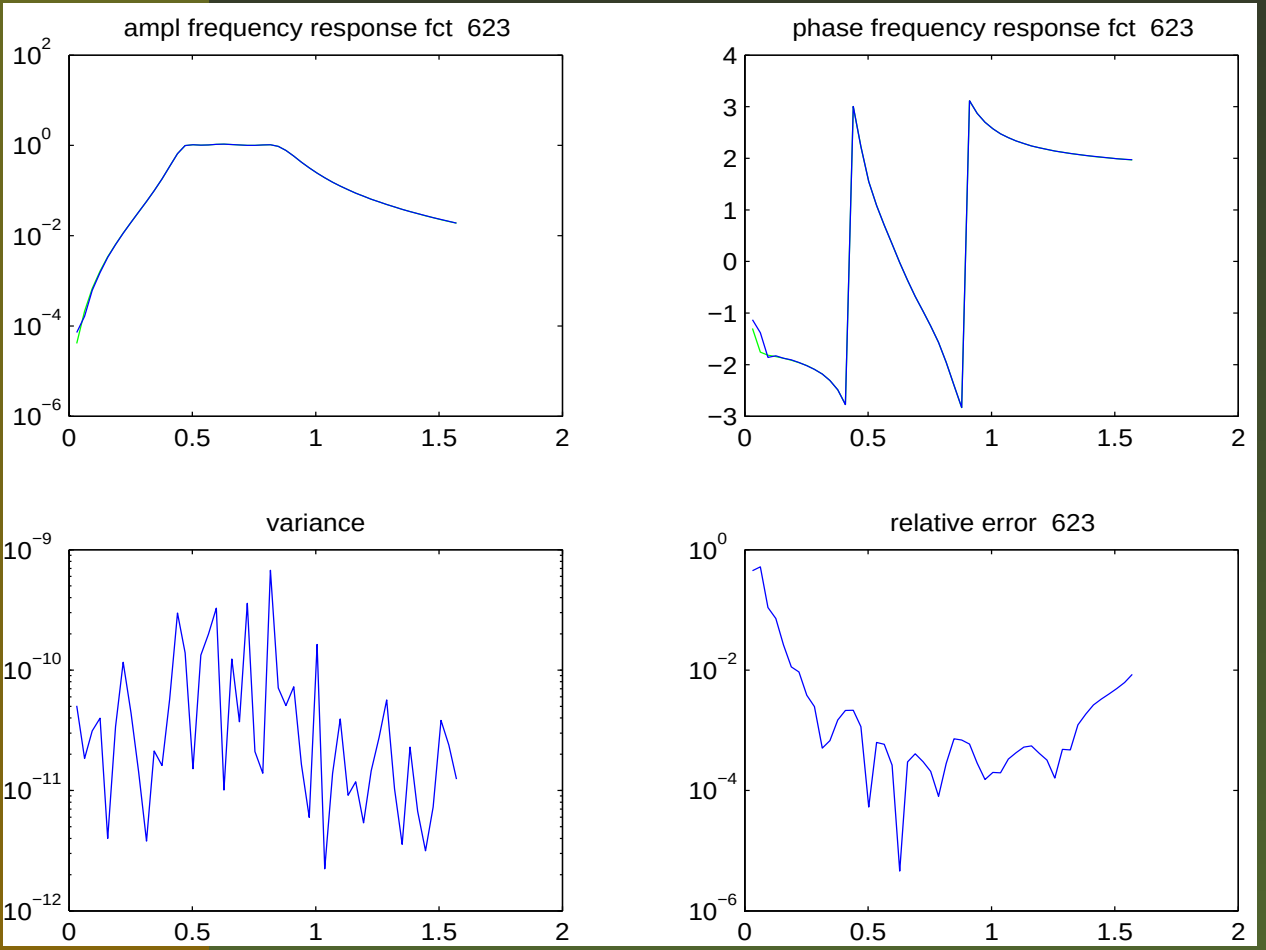
order [6/6], 100 data points, 200Hz, discrete time

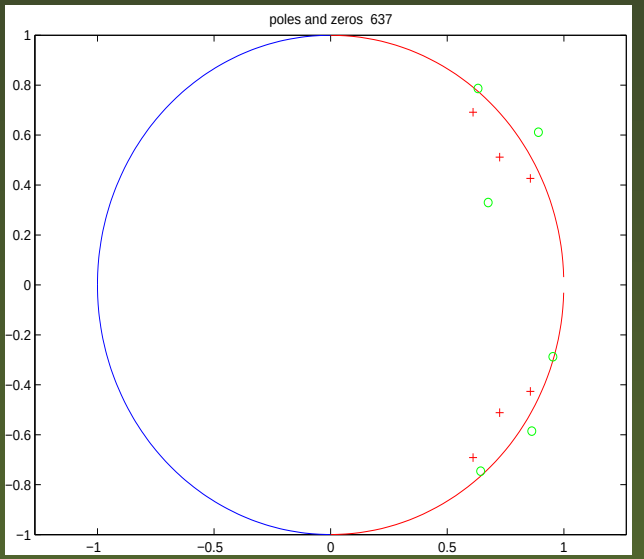




Bandpass filter

order [6/6], 50 data points, 20KHz, discrete time





Recurrence for ORF (continued)

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- $\mathbf{ZQ} = (\mathbf{Q} - \mathbf{ZQA}^{-1})\mathbf{T} + [0 \ldots 0 \ *]$
  multiply with $\mathbf{W} = \mathrm{diag}(W_1, \ldots, W_N)$
  and set $\mathbf{Q} = \mathbf{W\Phi}$.

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- $\mathbf{ZQ} = (\mathbf{Q} - \mathbf{ZQA}^{-1})\mathbf{T}+0$

$\phi_{N+1}(\mathbf{z})$ has zero norm (does not exist), give it zeros $z_1, \ldots, z_N \Rightarrow \phi_{N+1}(\mathbf{z}) = 0$

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- $\mathbf{T}$ symmetric tridiagonal, irreducible (generic case)
  Suppose it is invertible.
  Its inverse is a symmetric SemiSeparable (SS) matrix: $\mathbf{S}$

$$\mathbf{S} = \mathbf{S}\begin{pmatrix} u_1, \ldots, u_n \\ v_1, \ldots, v_n \end{pmatrix}, \mathbf{S} = \mathbf{S}^T,\ \mathbf{S}_{ij} = u_i v_j, i \geq j$$

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Inverse eigenvalue problem

- given W, $\mathbf{A}$, $\mathbf{Z}$

$$\mathbf{Q}\left[\begin{array}{c|c} W_1 & \\ \vdots & \mathbf{Z}^{-1} \\ W_N & \end{array}\right]\left[\begin{array}{c|c} 1 & \\ \hline & \mathbf{Q}^H \end{array}\right] = \left[\begin{array}{c|c} \phi_0^{-1} & \\ & \\ 0 & \mathbf{S}+\mathbf{A}^{-1} \end{array}\right]$$

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- stable efficient recursive algorithm $O(N^2)$ or $O(nN)$
- can do real operations combining conjugate pair
- similar for discrete time systems

Inverse eigenvalue problem

- given W, $\mathbf{A}$, $\mathbf{Z}$

$$\mathbf{Q}\left[\begin{array}{c|c} W_1 & \\ \vdots & \mathbf{Z}^{-1} \\ W_N & \end{array}\right]\left[\begin{array}{c|c} 1 & \\ \hline & \mathbf{Q}^H \end{array}\right] = \left[\begin{array}{c|c} \phi_0^{-1} & \\ & \\ 0 & \mathbf{S}+\mathbf{A}^{-1} \end{array}\right]$$

- stable efficient recursive algorithm $O(N^2)$ or $O(nN)$
- can do real operations combining conjugate pair
- similar for discrete time systems
- also for MIMO systems

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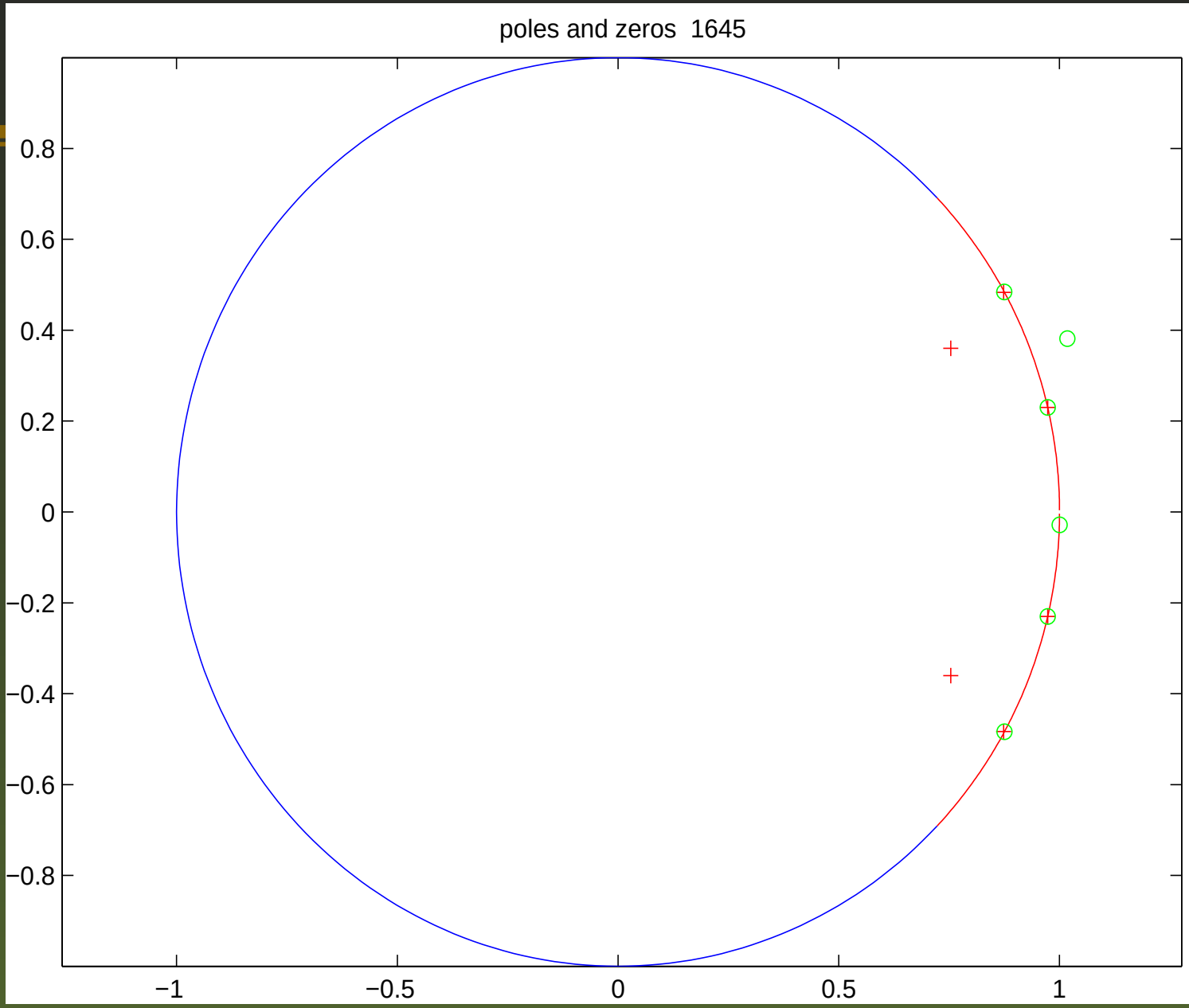
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Eventually...



Robotarm



Bandpass

