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The "look-ahead" philosophy applied to matrix rational interpolation problems

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In [7], we developed a general module theoretic framework to solve rather general matrix rational interpolation problems (MRIPs). The MRIP is linearized in an interpolation problem of the following form:

Definition 1 (vector M-Padé approximation) *We look for the set S of all polynomial vectors $\vec{p} \in \mathbb{F}[z]^{m\times 1}$ ($\mathbb{F}$ is an arbitrary finite or infinite field) such that they satisfy some vector interpolation conditions for $z \to \alpha$*

$$F(z-\alpha)\vec{p}(z) = O_+((z-\alpha)^{\vec{\sigma}^{(\alpha)}}), \quad \forall \alpha \in I \tag{1}$$

with I a finite set of interpolation points $\alpha \in \mathbb{F}$, $F(z-\alpha)$ a formal $p \times m$ Taylor series ($p < m$) around α and $\vec{\sigma}^{(\alpha)} \in \mathbb{N}^{p\times 1}$, the order indices for the interpolation point α which can be different for each row index i. The set $\mathbb{N} = \{0, 1, 2, \ldots\}$.

The set S of all solutions $\vec{p}$ of (1) forms an $\mathbb{F}[z]$-submodule of $\mathbb{F}[z]^m$. This set can be represented in a compact way by a basis matrix B.

Definition 2 (basis matrix B) *All solutions $\vec{p}$ of (1) can be written as $\vec{p} = B\vec{c}_B$ with $B \in \mathbb{F}^{m\times m}[z]$, $\vec{c}_B \in \mathbb{F}^{m\times 1}[z]$.*
B is called a basis matrix and $\vec{c}_B$ the coordinates of $\vec{p}$ with respect to this basis matrix. These coordinates are unique, once B is given.

To represent all solutions having a certain degree structure in an efficient way, we use the concept of the $\vec{\tau}$-degree.

Definition 3 ($\vec{\tau}$-degree, $\vec{\tau}$-column reduced) *Given shift parameters $\vec{\tau} \in \mathbb{Z}^{m\times 1}$ ($\mathbb{Z}$ is the set of integer numbers). Every $\vec{0} \neq \vec{p} \in \mathbb{F}^{m\times 1}[z]$ can be written as*

$$\vec{p} = z^{\vec{\tau}} O_-^{\vec{s}}(z^0) z^{\alpha}$$

with

- $z^{\vec{\tau}} = \mathrm{diag}(z^{\tau_1}, z^{\tau_2}, \ldots, z^{\tau_m})$ *and*
- $O^s_-(z^0) = \vec{p}_0 z^0 + \vec{p}_{-1} z^{-1} + \ldots$ *with* $\vec{p}_0 \neq \vec{0}$ *and* $\vec{p}_i \in \mathbb{F}^{m \times 1}$.

The integer number α *is called the* $\vec{\tau}$*-degree of* $\vec{p}$ *(* $\vec{\tau}$-deg $\vec{0} = -\infty$ *). A polynomial matrix* $P \in \mathbb{F}^{m \times q}[z]$ *is called* $\vec{\tau}$*-(column) reduced if it can be written as*

$$P = z^{\vec{\tau}} O^s_-(z^0) z^{\vec{\alpha}}$$

with $O^s_-(z^0) = P_0 z^0 + P_{-1} z^{-1} + \ldots$ *with* $P_0 \in \mathbb{F}^{m \times q}$ *having full rank.*

We can now represent all solutions of (1) having a limited $\vec{\tau}$-degree using the following theorem.

Theorem 1 *If the basis matrix* $B = [\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_m]$ *is* $\vec{\tau}$*-column reduced, all solutions* $\vec{p}$ *of (1) having* $\vec{\tau}$*-deg* $\vec{p} \leq \alpha$ *can be represented as* $\vec{p} = B\vec{c}$ *with* $\deg c_i \leq \alpha - \alpha_i$ *and* $\alpha_i = \vec{\tau}$-deg *of the* i*-th column* $\vec{b}_i$ *of* B.

In [7], we have given a recursive method to compute such a $\vec{\tau}$-reduced basis B. Starting from a trivial $\vec{\tau}_0$-reduced basis matrix B_0 the required B is computed using 3 possible elementary steps:

a) an updating step, incrementing one of the order indices $\sigma_i^{(\alpha)}$.

b) a change of the shift parameters $\vec{\tau}$:

   b1) incrementing one of the τ_i;

   b2) decrementing one of the τ_i.

Each of these elementary steps is equivalent to multiplying the previous basis matrix to the right by a simple polynomial matrix. In [7], we have shown that performing k of these elementary steps requires $O(k^2)$ FLOPS. For related results concerning the minimal partial realization problem and scalar M-Padé approximation problem, we refer to Antoulas [1] and Beckermann [2] respectively.
The correctness of the three elementary steps was proved by using the following characterization for the basis matrix B.

Theorem 2 (characterization of a basis matrix) *A polynomial matrix* $B \in \mathbb{F}^{m \times m}[z]$ *is a basis matrix for* S *iff*

- *the* m *columns* $\vec{b}_j$ *of* B *belong to* S*, i.e. satisfy (1),*
- $\deg \det B = k_S$ *with* k_S *the number of independent equations of the set of linear homogeneous equations (1) defining the submodule* S.

This characterization can be used to design methods for performing bigger jumps than the elementary ones.
Instead of incrementing one of the order indices $\sigma_i^{(\alpha)}$, we could increase several of

them possibly at different interpolation points α. Assume B_σ to be a $\vec{\tau}$-reduced basis matrix whose columns satisfy (1), i.e.

$$F(z-\alpha)B_\sigma(z) = (z-\alpha)^{\vec{\sigma}^{(\alpha)}} R(z-\alpha)$$

with R called the residual series around α.
B_σ itself has the form

$$B_\sigma(z) = z^{\vec{\tau}} O_-^{\vec{s}}(z^0) z^{\vec{\alpha}}.$$

We are adding interpolation conditions, i.e. we take $\vec{\sigma}^{(\alpha)} + \vec{\nu}^{(\alpha)}$ instead of $\vec{\sigma}^{(\alpha)}$ leading to the submodule $S_{\sigma+\nu}$ instead of S_σ.

Theorem 3 (adding interpolation conditions) *Given a $\vec{\tau}$-reduced basis matrix for S_σ. The polynomial matrix $B_{\sigma+\nu} = B_\sigma B_\nu$ is a $\vec{\tau}$-reduced basis matrix for $S_{\sigma+\nu}$ if B_ν is a $(-\vec{\alpha})$-reduced basis matrix for the submodule S_ν with*

$$S_\nu = \{\vec{p} \mid R(z-\alpha)\vec{p}(z) = O_-((z-\alpha)^{\vec{\nu}^{(\alpha)}}), \qquad \forall \alpha \in I\}.$$

Instead of incrementing or decrementing one of the shift parameters τ_i, we can change them in an arbitrary way.

Theorem 4 (changing the shift parameters) *The polynomial matrix $B_{\tau'}$ is a $\vec{\tau}'$-reduced basis matrix for S iff*

- *$B_{\tau'}$ is $\vec{\tau}'$-reduced and*
- *can be written as $B_{\tau'} = B_\tau V$ with V unimodular and B_τ a $\vec{\tau}$-reduced basis matrix for the same submodule S.*

The last two theorems 3 and 4 can be used to "look-ahead", i.e. to go directly to $\vec{\sigma}^{(\alpha)}$ and/or $\vec{\tau}$ values in which one is interested.
This allows to jump from one perfect point in the "solution table" to the next perfect one or from one well-conditioned point to the next one finding back or generalizing existing methods like e.g. in Cabay, Labahn and Beckermann [5] for perfect points in the Padé-Hermite table and in Gutknecht [6] for well-conditioned points in the Padé approximation table.
Also superfast algorithms can be built using the updating formulas getting back or extending existing methods like e.g. in Gutknecht [6] and in Beckermann and Labahn [3].
For example, using the characterization of B and the updating formula of theorem 3, choosing the basis matrix B in a perfect point as in theorem 5, the algorithm described in [5] computing consecutive "vector Padé-Hermite systems" lying on a piecewise linear path in the vector Padé-Hermite table can be generalized in another way compared to Cabay and Labahn [4].

Definition 4 (vector Padé-Hermite approximation) *The vector Padé-Hermite problem is a special case of the vector M-Padé approximation problem where the set of interpolation points contains only one point which is taken equal to zero without loss of generality, i.e. $I = \{0\}$.*

We take the interpolation conditions in their natural order, i.e. if the total number of interpolation conditions is t then $\sigma_1 = \sigma_2 = \cdots = \sigma_r = q+1$ and $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_p = q$ with $t = qp + r$ and $0 \leq r < p$. The following theorem allows us to compute a basis matrix B.

Theorem 5 (basis matrix B for the vector Padé-Hermite system) *Given $\vec{\tau} \in \mathbb{N}^{m\times 1}$. Assume that the formal power series F has a constant coefficient of full rank. Suppose that $\vec{\tau}$ is a perfect point, i.e. there is no nontrivial polynomial vector with $\vec{\tau}$-degree < 0 satisfying the first $t = \sum_{i=1}^{m} \tau_i$ interpolation conditions (this is equivalent to the nonsingularity of a generalized block Sylvester matrix connected with the given formal McLaurin series F and the shift parameters $\vec{\tau}$). A $\vec{\tau}$-reduced basis matrix B for the submodule S of all polynomial vectors satisfying the first $t + p$ interpolation conditions can be constructed as*

- *$m-p$ basis vectors, being solution vectors of $\vec{\tau}$-degree ≤ 0, satisfying the first $t+p$ interpolation conditions and having linearly independent constant coefficients;*
- *p basis vectors, being solution vectors multiplied by z^2, having $\vec{\tau}$-degree < 0, satisfying the first $t-p$ interpolation conditions and having linearly independent residual coefficients (of the next p interpolation conditions).*

This result, in turn, can be used to solve rectangular-block Hankel systems of linear equations.

References

[1] A.C. Antoulas. On recursiveness and related topics in linear systems. *IEEE Trans. Automat. Control*, AC-31:1121–1135, 1986.

[2] B. Beckermann. A reliable method for computing M-Padé approximants on arbitrary staircases. *J. Comput. Appl. Math.*, 40:19–42, 1992.

[3] B. Beckermann and G. Labahn. A uniform approach for the fast computation of matrix-type Padé approximants. *SIAM J. Matrix Anal. Appl.*, 1993. To appear.

[4] S. Cabay and G. Labahn. A superfast algorithm for multi-dimensional Padé systems. *Numerical Algorithms*, 2:201–224, 1992.

[5] S. Cabay, G. Labahn, and B. Beckermann. On the theory and computation of nonperfect Padé-Hermite approximants. *J. Comput. Appl. Math.*, 39:295–313, 1992.

[6] M.H. Gutknecht. Stable row recurrences for the Padé table and generically superfast lookahead solvers for non-Hermitian Toeplitz systems. *Linear Algebra and its Appl.*, 188/189:351–422, 1993.

[7] M. Van Barel and A. Bultheel. A general module theoretic framework for vector M-Padé and matrix rational interpolation. *Numerical Algorithms*, 3:451–462, 1992.