Parametric model order reduction of damped mechanical systems via the block Arnoldi process

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Abstract

This paper proposes a block Arnoldi method for parameterized model order reduction. This method works when design parameters have only low-rank impacts on the system matrix. The method preserves all design parameters in the reduced model and is easy to implement. Numerical results show that the block Arnoldi process outperforms some existing methods up to a factor of ten.

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1. Introduction

Large-scale dynamical systems often arise from the discretization of partial differential equations (PDEs). These systems are computationally expensive to solve due to their large orders. If we want to optimize l design parameters $\gamma \in \mathbb{R}^l$, the computational cost is even higher since we must simulate the system for many γ values. The goal of model order reduction (MOR) is to construct a low order *reduced model* to approximate the large-scale *original model* with high accuracy to reduce the computational cost. For large-scale systems, Krylov-Padé type MOR methods are of special interest because their computational costs are relatively low [1, 10]. The classical MOR methods reduce systems parameterized only with the frequency or the time, which means if we have design parameters, we must fix them to a given value, say $\gamma^{(0)}$, and use MOR to reduce on the frequency or the time to obtain a reduced model valid for $\gamma = \gamma^{(0)}$. Recently, Parameterized MOR (PMOR) [13, 3, 9] has been introduced to also reduce on design parameters for further speedups. In general, however, for a system with many design parameters, a reduction on all these parameters is not practical because 1) reducing on more parameters generally results in a drastic increase in the order of a Krylov-Padé type reduced model because of the combinatorial explosion in the number of (cross)moments, so both building and evaluating such a reduced model becomes more expensive; 2) implementing PMOR that reduces several variables can be very challenging.

This paper will show that for a special case, where design parameters have only low-rank impacts on the system matrix, we can use the block Arnoldi method [8], which is commonly used for MOR of multiple-input-multiple-output (MIMO) systems [2], to build a parametric reduced model preserving all parameters. This method is efficient as our numerical results will show in §3, and at the same time easy to implement.

2. Block Krylov methods for low-rank parametric structures

Consider the following dynamical system, which is described in the frequency domain and depends on design parameters $\gamma \in \mathbb{R}^l$:

$$\begin{cases} (K - \omega^2 M + BC(\omega, \gamma)B^*)x = f, \\ y = \psi(x), \end{cases} \tag{1}$$

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where $K \in \mathbb{C}^{n\times n}$, $M \in \mathbb{C}^{n\times n}$, $f \in \mathbb{C}^n$, $B \in \mathbb{C}^{n\times k_C}$ and $C(\omega,\gamma)$ is a $k_C \times k_C$ matrix parameterized with ω and γ. The function ψ describes how the output value y is extracted from the state vector x. We assume that design parameters have only low-rank impacts on the system matrix, i.e., $k_C \ll n$, typically of order 10 at most.

The goal of this chapter is to derive a reduced model of (1) for both ω and γ. If we write the first equation in (1) as

$$\left(K - \omega^2 M\right) x = f - BC(\omega,\gamma)B^* x,$$

we derive that $x = \left(I - \omega^2 K^{-1}M\right)^{-1}\left(K^{-1}f - K^{-1}BC(\omega,\gamma)B^* x\right)$. When ω^2 is sufficiently small, we have

$$x = \sum_{i=0}^{\infty} \omega^{2i}\left(K^{-1}M\right)^i \left(K^{-1}f - K^{-1}BC(\omega,\gamma)B^* x\right)$$

Therefore, it is reasonable to approximate x in the following *block Krylov subspace* [5, 7, 2]:

$$\mathcal{K}_k(K^{-1}M, K^{-1}F) \triangleq \text{colspan}\{K^{-1}F, (K^{-1}M)K^{-1}F, \ldots, (K^{-1}M)^{k-1}K^{-1}F\}, \qquad \text{with} \quad F = [f, B]. \tag{2}$$

2.1. A brief introduction of the block Arnoldi process

If all Krylov vectors are linearly independent, $\mathcal{K}_k(K^{-1}M, K^{-1}F)$ is of dimension $k(k_C + 1)$, which is then also the dimension of the reduced model. As is well known, directly computing (block) Krylov vectors is numerically unstable. Therefore, we use the *block Arnoldi process*, which conducts orthonormalization at each step.

Algorithm 1 (The Block Arnoldi Process)**.** *To generate the block Krylov subspace $\mathcal{K}_k(A, b)$ with $b = [b_1, b_2, \ldots, b_{k_C+1}]$:*

1. *Compute an orthonormal basis for b, namely $v_1^{(0)}, v_2^{(0)}, \ldots, v_{k_C+1}^{(0)}$. Set $V = v^{(0)} = \left[v_1^{(0)}, v_2^{(0)}, \ldots, v_{k_C+1}^{(0)}\right]$.*
2. *For $j = 1, \ldots, k-1$ do:*
   - 2.1. *Compute $\widetilde{w}^{(j)} = Av^{(j-1)}$.*
   - 2.2. *For $i = 1, 2, \ldots, k_C + 1$, do:*
     - 2.3.1. *Orthogonalize $\widetilde{w}_i^{(j)}$ with each column vector of V to get $w_i^{(j)}$.*
     - 2.3.2. *Normalization: $v_i^{(j)} = {}^{w_i^{(j)}}/_{\|w_i^{(j)}\|_2}$.*
     - 2.3.3. *Update V: $V = \left[V, v_i^{(j)}\right]$.*

   End for.

In Algorithm 1, $v^{(j)}, \widetilde{w}^{(j)}, w^{(j)} \in \mathbb{C}^{n\times(k_C+1)}$ are block vectors and the subscript i picks up the i-th column. Algorithm 1 only shows the basic principles of the block Arnoldi process. In practice, we need to deal with such details as deflation, i.e., when $\|w_i^{(j)}\|_2 = 0$ in Step 2.3.2 [12]. In implementing the block Arnoldi process, it is also possible to use more block operations. See [8] for the technical details. The block Arnoldi process leads to the relationship

$$AV^{(j)} = V^{(j+1)}\underline{H}^{(j)}, \tag{3}$$

where $V^{(j)} = [v^{(0)}, v^{(1)}, \ldots, v^{(j-1)}] \in \mathbb{C}^{n\times j(k_C+1)}$ is the basis at the end of the j-th iteration, and $\underline{H}^{(j)} \in \mathbb{C}^{(j+1)(k_C+1)\times j(k_C+1)}$ is a band Hessenberg matrix whose nonzero elements are computed via the orthogonalization in Step 2.3.1 and the normalization in Step 2.3.2. When K, M and $C(\omega,\gamma)$ are all Hermitian, we can use the block Lanczos method instead [8].

2.2. Two approaches in obtaining the reduced model

After we obtained V via the block Arnoldi process, we have two approaches in obtaining the reduced model.

1. **The implicit projection approach** defines $\widehat{M} = V^*K^{-1}MV, \widehat{B} = V^*K^{-1}B, \widetilde{B} = V^*B, \widehat{f} = V^*K^{-1}f$, and constructs the reduced model as

$$\begin{cases} (I - \omega^2\widehat{M} + \widehat{B}C(\omega,\gamma)\widetilde{B}^*)\widehat{x} &= \widehat{f}, \\ y &= \psi(V\widehat{x}). \end{cases} \tag{4}$$

   Note that 1) $\widehat{M}$ is the block Hessenberg matrix of the block-Arnoldi method; 2) $\widehat{B}$ and $\widehat{f}$ can be computed from the orthonormalization of the first $r + 1$ columns of V and are therefore readily available. Hence, only $\widetilde{B}$ needs to be computed explicitly, which is not expensive.

2. **The explicit projection approach** defines $\widehat{K} = V^*KV, \widehat{M} = V^*MV, \widehat{B} = V^*B, \widehat{f} = V^*f$, and constructs the reduced model as

$$\begin{cases} (\widehat{K} - \omega^2\widehat{M} + \widehat{B}C(\omega,\gamma)\widehat{B}^*)\widehat{x} = \widehat{f}, \\ y = \psi(V\widehat{x}), \end{cases} \tag{5}$$

Although this approach is more expensive than the implicit approach, it has the following two advantages over the implicit approach: 1) when the original system is Hermitian, i.e., K, M, $C(\omega,\gamma)$ are Hermitian, the reduced model is also Hermitian; 2) it can also be easily extended to deal with an additional Rayleigh damping term, as we will discuss in detail in §3.

2.3. The moment matching properties

Using the above block Krylov subspace for projection, the resulting reduced model is expected to be a good approximation on the entire parameter space if the order of the Krylov subspace is sufficiently large. To prove this, we assume that the state vector $x(\omega,\gamma)$ is analytic and can be expanded as

$$x(\omega,\gamma) = \sum_{i=0}^{\infty} \omega^i x_i(\gamma),$$

i.e., a series in terms of ω with its Taylor coefficients x_i as functions of γ. We call $x_i(\gamma)$ the i-th moment w.r.t. ω throughout this paper. The following theorem shows the moment matching property of $x(\omega,\gamma)$ w.r.t. ω, independent of γ. Here, we consider the implicit projection approach. The moment matching property for the explicit projection approach can be proven similarly.

Theorem 1. *Consider system* (1)*. Assume that K is nonsingular,* $\mathrm{colspan}\{V\} \subseteq \mathcal{K}_k(K^{-1}M, K^{-1}F)$*, $x(\omega,\gamma)$, $\widehat{x}(\omega,\gamma)$ and $C(\omega,\gamma)$ are analytic, and we expand them as*

$$x(\omega,\gamma) = \sum_{i=0}^{\infty} \omega^i x_i(\gamma), \quad \widehat{x}(\omega,\gamma) = \sum_{i=0}^{\infty} \omega^i \widehat{x}_i(\gamma), \quad \textit{and} \quad C(\omega,\gamma) = \sum_{i=0}^{\infty} \omega^i C_i(\gamma).$$

Then we have

1. *$x_i(\gamma) \in \mathcal{K}_k(K^{-1}M, K^{-1}F)$ for $i = 0, 1, 2, \ldots, 2k-1$;*
2. *for the block Arnoldi reduced model* (4) *where $\mathcal{K}_k(K^{-1}M, K^{-1}F) \subseteq \mathrm{colspan}\{V\}$, if $\left(I + \widehat{B}C_0(\gamma)\widehat{B}^*\right)$ is nonsingular, we also have $x_i(\gamma) = V\widehat{x}_i(\gamma)$ for $i = 0, 1, 2, \ldots, 2k-1$.*

Proof.

1. For simplicity, we assume that $C_0(\gamma)$ is nonsingular for the time being. By grouping the terms with equal power in $(K - \omega^2 M + BC(\omega,\gamma)B^*)x = f$, we obtain the following recurrence:

$$(K + BC_0(\gamma)B^*)\, x_0(\gamma) = f, \tag{6}$$

$$(K + BC_0(\gamma)B^*)\, x_1(\gamma) = -BC_1(\gamma)B^* x_0(\gamma), \tag{7}$$

$$(K + BC_0(\gamma)B^*)\, x_i(\gamma) = Mx_{i-2}(\gamma) - \sum_{j=0}^{i-1} BC_{i-j}(\gamma)B^* x_j(\gamma). \tag{8}$$

Now we prove that $x_i(\gamma) \in \mathcal{K}_k(K^{-1}M, K^{-1}F)$ holds for $2 \le i \le 2k-1$ by mathematical induction on k.

(a) First we prove that $x_0(\gamma), x_1(\gamma) \in \mathcal{K}_1(K^{-1}M, K^{-1}F)$. According to the Sherman-Morrison-Woodbury formula [6], we have

$$\begin{aligned} x_0(\gamma) &= \left(K^{-1} - K^{-1}B(C_0^{-1}(\gamma) + B^*K^{-1}B)^{-1}B^*K^{-1}\right) f \in \mathrm{colspan}\{K^{-1}[f, B]\} = \mathcal{K}_1(K^{-1}M, K^{-1}F), \\ x_1(\gamma) &= (-K^{-1} + K^{-1}B(C_0^{-1}(\gamma) + B^*K^{-1}B)^{-1}B^*K^{-1})BC_1(\gamma)B^*x_0(\gamma) \\ &\in \mathrm{colspan}\{K^{-1}B\} \subseteq \mathrm{colspan}\{K^{-1}[f, B]\} = \mathcal{K}_1(K^{-1}M, K^{-1}F). \end{aligned}$$

(b) We assume that Statement 1 holds for k, i.e., $x_i(\gamma) \in \mathcal{K}_{k+1}(K^{-1}M, K^{-1}F)$ holds for $0 \le i \le 2k-1$. We now prove that it also holds for $k+1$. For $i = 2k$ and $i = 2k+1$, we derive

$$x_i(\gamma) = \left(K^{-1} - K^{-1}B(C_0^{-1}(\gamma) + B^*K^{-1}B)^{-1}B^*K^{-1}\right) \cdot \left(Mx_{i-2}(\gamma) - \sum_{j=0}^{i-1} BC_{i-j}(\gamma)B^*x_j(\gamma)\right)$$
$$\in \text{colspan}\{K^{-1}Mx_{i-2}(\gamma), K^{-1}B\} \subseteq \mathcal{K}_{i+1}(K^{-1}M, K^{-1}F).$$

If $C_0(\gamma)$ is singular, there exists $\bar{B}\bar{C}_0(\gamma)\bar{B}^* = BC_0(\gamma)B^*$, where $\bar{C}_0(\gamma)$ is nonsingular. Therefore, Statement 1 also holds when $C_0(\gamma)$ is singular, since the proof above follows if we replace $BC_0(\gamma)B^*$ with $\bar{B}\bar{C}_0(\gamma)\bar{B}^*$.

2. For simplicity of the proof, we define $x_{-1}(\gamma) = 0$ and $\widehat{x}_{-1}(\gamma) = 0$ here. To prove Statement 2, we first derive the following recurrence by equaling the term with the same order w.r.t. ω in the first equation of (5):

$$\left(I + \widehat{B}C_0(\gamma)\widetilde{B}^*\right)\widehat{x}_0(\gamma) = \widehat{f}, \tag{9}$$

$$\left(I + \widehat{B}C_0(\gamma)\widetilde{B}^*\right)\widehat{x}_i(\gamma) = \widehat{M}\widehat{x}_{i-2}(\gamma) - \sum_{j=0}^{i-1} \widehat{B}C_{i-j}(\gamma)\widetilde{B}^*\widehat{x}_j(\gamma). \tag{10}$$

Now we prove Statement 2 by mathematical induction.

(a) First we show that $x_i(\gamma) = V\widehat{x}_i(\gamma)$ holds for $i = -1, 0$. For $i = -1$, the proof is trivial since $x_{-1}(\gamma) = V\widehat{x}_{-1}(\gamma) = 0$. For $i = 0$, since $x_0(\gamma) \in \mathcal{K}_k(K^{-1}M, K^{-1}F)$, it follows that $x_0(\gamma) = VV^*x_0(\gamma)$. Using this equation and left multiplying both sides of (6) with V^*K^{-1}, we obtain $V^*\left(I + K^{-1}BC_0(\gamma)B^*\right)VV^*x_0(\gamma) = V^*K^{-1}f$, or

$$\left(I + \widehat{B}C_0(\gamma)\widetilde{B}^*\right)V^*x_0 = \widehat{f}. \tag{11}$$

Comparing (9) with (11), we see that $V^*x_0(\gamma)$ and $\widehat{x}_0(\gamma)$ are both the solution of the same equation. Since $\left(I + \widehat{B}C_0(\gamma)\widetilde{B}^*\right)$ is nonsingular, this equation has a unique solution, which implies that $V^*x_0(\gamma) = \widehat{x}_0(\gamma)$.

(b) Assuming that $x_i(\gamma) = V\widehat{x}_i(\gamma)$ holds for $j = -1, 0, \ldots, i$ $(i < 2k-1)$, now we show that it also holds for $j = i+1$. According to Statement 1, we have $x_j(\gamma) = VV^*\widehat{x}_j(\gamma) = 0$ for $j = -1, 0, \ldots, i$. Using this relationship and left multiplying both sides of (8) by V^*K^{-1}, we obtain

$$V^*\left(I + K^{-1}BC_0(\gamma)B^*\right)VV^*x_i(\gamma) = V^*K^{-1}MVV^*x_{i-2}(\gamma) - \sum_{k=0}^{i-1} V^*K^{-1}BC_{i-k}(\gamma)B^*VV^*x_k(\gamma),$$

which can be rewritten as

$$\left(I + \widehat{B}C_0(\gamma)\widetilde{B}^*\right)V^*x_i = \widehat{M}\widehat{x}_{i-2}(\gamma) - \sum_{k=0}^{i-1} \widehat{B}C_{i-k}(\gamma)\widetilde{B}^*\widehat{x}_k(\gamma). \tag{12}$$

Comparing (12) with (10), we see that $V^*x_i(\gamma)$ and $\widehat{x}_i(\gamma)$ are both solutions of the same equation. Since $\left(I + \widehat{B}C_0(\gamma)\widetilde{B}^*\right)$ is nonsingular, this equation has a unique solution, which implies that $V^*x_i(\gamma) = \widehat{x}_i(\gamma)$.

This proves Statement 2. □

3. Numerical results for two applications arising from mechanical engineering

In this section, we apply the block Arnoldi method to two design optimization problems arising from vibration and structures. Our objective is to reduce the total energy of the system output over a frequency range, i.e.,

$$\min_{\gamma} \quad g(\gamma) \triangleq \int_{\omega_L}^{\omega_H} |y(\omega, \gamma)|^2 \, \mathrm{d}\omega. \tag{13}$$

Here, we give a brief introduction of the problems. For detailed descriptions, see [15, 14].

3.1. A brief description of the test problems

The objective of the *floor damper optimization problem* is to use n_d tuned mass dampers to reduce the vibration of a floor. The mathematical formulation is

$$\begin{cases} \left(K + \sum_{j=1}^{n_d} (k_j + i\omega c_j) v_j v_j^* - \omega^2 M\right) x = f, \\ y = -\omega^2 \ell^* x, \end{cases} \tag{14}$$

where k_j and c_j are the stiffness and the damping coefficient of the j-th damper, respectively. Assume that both the stiffness matrix and the mass matrix obtained from finite element discretization are of size n. Then system (14) is of order $n + n_d$. The vector $v_j \in \mathbb{R}^{n+n_d}$ is a vector that describes the interaction between the j-th damper and the floor:

$$v_j(i) = \begin{cases} 1, & \text{for } i \text{ corresponding to the attached element,} \\ -1, & \text{for } i = n + j, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

We can rewrite system (14) in the form of (1) by assigning

$$B = [v_1, v_2, \ldots, v_{n_d}] \quad \text{and} \quad C(\omega, \gamma) = \operatorname{diag}\left[k_i + i\omega c_i\right]_{n_d \times n_d}. \tag{16}$$

In our numerical test, we used an order 29800 finite element model to describe the floor, used one damper placed at the central point of the floor, and used the implicit projection approach to construct the reduced model.

The objective of the *footbridge damper optimization problem* is to use four tuned mass dampers to reduce the vibration of a footbridge with its mathematical formulation

$$\begin{cases} \left(K + i\omega(\lambda_1 K + \lambda_2 M) - \omega^2 M + BC(\omega, \gamma)B^*\right) x = f, \\ y = \ell^* x, \end{cases} \tag{17}$$

where B and $C(\omega, \gamma)$ are also formulated as (16) and λ_1 and λ_2 are two real constants. Although (17) has an additional Rayleigh damping term $i\omega(\lambda_1 K + \lambda_2 M)$ compared with (1), we can actually still use the block Arnoldi method to reduce it since the first equation can be rewritten to

$$\left(K - \frac{\omega^2 - i\omega\lambda_2}{1 + i\omega\lambda_1} M\right) = \frac{1}{1 + i\omega\lambda_1} \left(f - BC(\omega, \gamma)B^*)x\right) \in \operatorname{colspan}\{f, B\},$$

which means that we can build the Krylov subspace $\mathcal{K}_k(K^{-1}M, K^{-1}[f, B])$ to reduce (17). Using the basis V built by the block-Arnoldi process, we can easily use the explicit projection approach to obtain a reduced model.

3.2. Numerical results and conclusions

We compare the block Arnoldi method with the MOR Framework proposed in [15], the Error-based Trust Region method (ETR) and the Error-based Penalty method (EP) proposed in [14], all of which employ (P)MOR to speed up design optimization. We also use the direct method, which uses a sparse solver for all computations, for a reference.

Table 1 shows numerical results for both test problems. We observe that the block Arnoldi method outperforms the MOR Framework, ETR and EP. The main reason is that the block Arnoldi method enables us to use only one reduced model for the design optimization, while the other methods require building a reduced model for each iteration. For the footbridge damper optimization problem, we have experienced that if we use the direct method, a single evaluation of $g(\gamma)$ costs about 540 s. Numerical results also indicate that the block Arnoldi reduced models have good approximation accuracy.

In computing a basis for the block Krylov subspace (2), another two alternatives to the block Arnoldi process are the Lanczos method with multiple right-hand sides [11] and the tangential interpolation method [4].

Table 1: Numerical results

	Floor Damper Optimization			Footbridge damper Optimization		
	Order	Optimum	CPU Time	Order	Optimum	CPU Time
Direct Method	29800	1.31621e+10	7202 s	-	-	-
The MOR Framework	7	1.31621e+10	89 s	12	24.7775	879 s
ETR	10	1.31621e+10	35 s	20	24.7859	205 s
EP	10	1.31621e+10	41 s	20	24.7763	295 s
Block Arnoldi	8	1.31629e+10	3.08 s	20	24.7810	6.83 s
Block Arnoldi	16	1.31621e+10	3.97 s	30	24.7775	20.4 s

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