CONVERGENCE OF SCHUR PARAMETERS AND TRANSMISSION ZEROS OF A MEROMORPHIC SPECTRUM

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It will be shown that under certain conditions, the ratio of two successive Schur parameters of a meromorphic spectrum will converge to a transmission zero. It will also be shown how other transmission zeros can be found from sequences of Schur parameters or from Rutishauser polynomials.

1. MOTIVATION

Let x(t) be a real, stationary, zero-mean, scalar stochastic process with covariance function

$$f_k = E[x(t)\ x(t-k)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}\ F(e^{i\theta})d\theta$$

with $F(e^{i\theta})$ the spectral density. Let H be a spectral factor of F, defined in the following theorem [3]

Theorem 1.1

Let $F(e^{i\theta})=F(e^{-i\theta}) \geq 0$, $\theta \in [-\pi,\pi]$ and suppose F and log F $\in L_1$ (Lebesgue integrable on the unit circle). Then there exists an H , outer in H_2[10], such that $F(e^{i\theta}) = |H(e^{i\theta})|^2$. This factor H is given by

$$H(e^{i\phi}) = \lim_{r\nearrow 1} \exp\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\phi}+re^{i\theta}}{e^{i\phi}+re^{i\theta}} \log F(e^{i\theta})d\theta\}.$$

The Fourier coefficients h_k of $H : H(e^{i\theta}) = \sum_0^\infty h_k e^{ik\theta}$ are real and $h_0>0$.

Under roomyness conditions [4], F and H may be defined in the whole complex plane and $F(z) = H(z)\ H(1/\bar{z})$.

Proof: See [3] □

The Schur-Levinson-Szegö recursions and the Pick-Nevanlinna algorithm give step by step solutions to the problem: given

$\{f_k\}_{-\infty}^{\infty}$,find H. These find successive rational approximants H_n of degree n, outer in H_2, such thas they solve the inverse least squares problem which finds the the minimum of

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| 1 - \frac{H_n(0)\ H(e^{i\theta})}{H_n(e^{i\theta})\ H(0)} \right|^2 d\theta$$

over all rational H_n of the form $H_n(z) = c_n \prod_1^n (1-z/\zeta_k) \ / \ Q_n(z)$

where the zeros ζ_k, k=1,....,n are given and where Q_n is a free polynomial of degree n. The zeros of Q_n and all ζ_k are outside the closed unit disc. The Pick-Nevanlinna algorithm [8,9] finds the successive H_n by recursive rational interpolation at the points ζ_k, k=1,2,.... If all ζ_k are at infinity, this algorithm

simplifies to the Schur-Levinson-Szegő [12,7,14] recursions. See e.g. [2] for a brief survey of these results.
If H were rational of degree n with zeros z_k, k=1,...,n (transmission zeros), the Pick-Nevanlinna algorithm would never find H exactly after a finite number of steps, unless you take $\zeta_k=z_k$, k=1,...,n. Only in that case would it find $H_n=H$ in the nth step. Thus the Levinson algorithm would never find the even simplest $H(z)=1-z/\zeta_1$ in a finite number of steps and if $|\zeta_1|$ were close to 1, it would take a long time before a reasonable approximant were found. For this and other reasons it is important to know the transmission zeros. This brings us the problem of finding the zeros of H (or F), given the Fourier coefficients of F.
It will turn out that the Schur parameters generated by Levinson-Schur type algorithms performed on the sequence $\{f_k\}$ and its shifted versions can give you such information.
These algorithms will be briefly reviewed in section 2.
A key element in the analysis will be the asymptotic behavior of Toeplitz determinants based on $\{f_k\}$. These asymptotics can be given in terms of the zeros of F when F is meromorphic. These results are summarized in section 3. If F has equimodular zeros, a more general analysis, involving Rutishauser polynomials will be in order. The classical theory of the polynomials for a power series is surveyed in section 4 and this is generalized in section 5 for Laurent series from which the results for equimodular transmission zeros are found, given in section 6.
Before we start the analysis, it should be noted that, when applying this theory in numerical computations, we can expect severe problems since finding the transmission zeros from $\{f_k\}$ may be very ill-conditioned [2].

2. A SHIFTED LEVINSON-SCHUR-SZEGO ALGORITHM

Suppose for the time being that $\{f_k\}_{-\infty}^{\infty}$ is a formal sequence of complex numbers. Associate with it the Toeplitz matrices $T_n^{(m)} = (f_{m+i-j})_{i,j=0}^{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$.
For our first development suppose also that the sequence $\{f_k\}$ is normal, by which we mean that all determinants $\det T_n^{(m)}$ are nonzero for all m and n. Normality of the sequence is a severe restriction and the covariance sequence will in general not be normal. We shall adapt the results for nonnormal sequences lateron.
We give a recursive relation for formal generalizations of the Szegő polynomials. The recursion forms the basis of $O(n^2)$ algorithms to solve Toeplitz systems à la Trench-Zohar [15-17]. Suppose you want to solve the systems

$$T_n^{(m)} Q_n^{(m)} = [\, v_n^{(m)} 0 \ \dots\ 0]^T \quad \text{and} \quad T_n^{(m)} \hat{Q}_n^{(m)} = [\, 0 \ \dots\ 0\ \hat{v}_n^{(m)}]^T \tag{2.1}$$

for fixed m and for n = 0,1,2,.... $v_n^{(m)}$ and $\hat{v}_n^{(m)}$ are nonzero constants defined by the conditions $[\,10\dots0]\, Q_n^{(m)} = 1$ and $[\,0\dots01]\, \hat{Q}_n^{(m)} = 1$.
If is a well known fact that the polynomials
$Q_n^{(m)}(z) = [\,1z\dots z^n]\, Q_n^{(m)}$ and $\hat{Q}_n^{(m)}(z) = [\,1z\dots z^n]\, \hat{Q}_n^{(m)}$ satisfy a recurrence relation

$$[\, Q_{n+1}^{(m)}(z) \quad \hat{Q}_{n+1}^{(m)}(z)] = [\, Q_n^{(m)}(z) \ \hat{Q}_n^{(m)}(z)] \begin{bmatrix} 1 & -\tilde{\alpha}_{n+1}^{(m)} \\ -z\alpha_{n+1}^{(m)} & z \end{bmatrix} \tag{2.2}$$

with

$$\alpha_n^{(m)} = (-1)^{n-1} \det T_{n-1}^{(m+1)} \,/\, \det T_{n-1}^{(m)} \tag{2.3a}$$

and

$$\hat{\alpha}_n^{(m)} = (-1)^{n-1} \det T_{n-1}^{(m-1)} \,/\, \det T_{n-1}^{(m)} \tag{2.3b}$$

For the special case that m=0 and $f_{-k}=\overline{f_k}$, we have that $\hat{\alpha}_n^{(0)} = \overline{\alpha_n^{(0)}}$ which are the reflection coefficients or Schur parameters associated with $\{f_k\}$, while the polynomials $\hat{Q}_n^{(0)}(z) = z^n \overline{Q_n^{(0)}(1/\bar{z})}$ are the classical Szegő polynomials and (2.2) is

the recursion of the classical Levinson-Szegö relation [7,14].
The parameters $\alpha_n^{(m)}$ and $\hat{\alpha}_n^{(m)}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$ will be called (generalized) Schur parameters for F or $\{f_k\}$. There are several methods to compute them. We shall not go into this. Note that in general $\alpha_n^{(m)}$ and $\hat{\alpha}_n^{(m)}$ need not be bounded by 1 in modulus as are the classical parameters (for m=0). Note also that for a nonnormal sequence some of them may not exist. As you see from (2.3), this depends on det $T_{n-1}^{(m)}$ being zero or not.
There is a classical result [6] saying that if F is positive on the unit circle and log $F \in L_1$, then $\sum|\alpha_n^{(0)}|^2$ converges, and hence $\alpha_n^{(0)} \to 0$. This analysis is not applicable in the more general situation of $\alpha_n^{(m)}$ with $m \neq 0$. As (2.3) suggests, we shall approach this problem by using asymptotics for the Toeplitz determinants det $T_n^{(m)}$ when n tends to infinity. If we want to bring zeros of F into these expressions we shall have to restrict F to a class of meromorphic functions. This class, the asymptotics for Toeplitz determinants and some simple corollaries for the Schur parameters are given in the next section.

3. MEROMORPHIC FUNCTIONS AND ASYMPTOTICS FOR TOEPLITZ DETERMINANTS

Suppose $F(z)=H(z)H_\star(z)$ with $H_\star(z)=\overline{H(1/\bar{z})}$, $H(0) > 0$, and suppose $H(z)$ is outer in H_2[10] and meromorphic in $\{z:0 \leq |z| < R \leq \infty\}$. Let the zeros of $H(z)$ be ζ_k, $k=1,2,..,Z$ and its poles be π_k, $k=1,2,..,P$. They are all in $|z| \geq r > 1$ and they are supposed to be ordered with respect to a nondecreasing modulus. If multiple poles or zeros occur, they are repeated as many times as indicated by their multiplicity. Z or P could be infinite. In that case there is an accumulation point at infinity, but all poles and zeros with a finite index are finite. If Z is finite, we introduce an artificial number ζ_{Z+1} between $|\zeta_Z|$ and R. The other end of the sequence is bordered by a number ζ_0 which is arbitrary between 1 and r. Whence we always have $|\zeta_0| < |\zeta_1|$ and for finite Z : $|\zeta_Z| < |\zeta_{Z+1}|$.
Of course poles and zeros of $H_\star$ are obtained by the transformation $z \to 1/\bar{z}$.
Since the poles and zeros of F are the poles and zeros of H and $H_\star$, it should be clear that the series $F(z)=\sum_{-\infty}^{\infty} f_k z^k$ is a Laurent series, which will converge in $\{z:r^{-1} < |z| < r\}$ where it has no poles and no zeros and that F(z) is meromorphic in $\{z:R^{-1} < |z| < R\}$. From now on, if F is a function of the class we have just described, this will be indicated by $F \in \mathcal{R}(\zeta_1,..,\zeta_Z;\pi_1,..,\pi_P;r,R)$.
For this type of functions we can specify the results on the asymptotics of Toeplitz determinants that were derived in [1]. Because of the symmetry in our problem ($f_{-k}=\bar{f}_k$), we need only consider the determinants det $T_n^{(m)}$ for $m \geq 0$.
From [1] we get the following theorem : (note that ind $F = \frac{1}{2\pi}[\arg F(e^{i\theta})]_0^{2\pi} = 0$)

Theorem 3.1

Let F be in $\mathcal{R}(\zeta_1,..,\zeta_Z;\pi_1,..,\pi_P;r,R)$ and define

$$G = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \log F(e^{i\theta})\, d\theta\right] = H(0)^2.$$

Then, for n sufficiently large and c, a constant not depending on n:

$$\det T_n^{(0)} = cG^{n+1} \{1 + o(1)\} \tag{3.1a}$$

and

$$\det T_n^{(m)} = cG^{n+1} [(-\zeta_1)(-\zeta_2)\ldots(-\zeta_m)]^{-n} \{1 + O(|\zeta_m/\sigma|^n)\} \tag{3.1b}$$

for $0 < m \leq Z$ and $|\zeta_m| < \sigma < |\zeta_{m+1}|$.

Proof See [1]. □

As we have said before, if F is as above and $F(z) = \sum_{-\infty}^{\infty} f_k z^k$, then the sequence $\{f_k\}$ need not be normal. But if n is large enough, the Schur parameters will always be defined for certain m as can be seen in the following consequence of theorem 3.1

Corollary 3.2

Let F be as in theorem 3.1. Then for $0 \leq m \leq Z$ and $|\zeta_m| < |\zeta_{m+1}|$ and n sufficiently large we have: $\det T_n^{(m)} \neq 0$ and hence $\alpha_n^{(m)}$ and $\hat{\alpha}_n^{(m)}$, the Schur parameters of F, are well defined.

Proof

This is trivial since the correction terms will almost vanish in the asymptotic expressions for the Toeplitz determinants. □

On condition that the zeros of F separate properly in modulus, we can give some simple results on the limit behavior of Schur parameters. This is given in the following theorem

Theorem 3.3

Let F be in $\mathcal{R}(\zeta_1,\ldots,\zeta_Z;\pi_1,\ldots,\pi_P;r,R)$. Then

(1) if $|\zeta_m| < |\zeta_{m+1}|$, $0 \leq m \leq Z-1$:

$$\lim_{n\to\infty} \alpha_n^{(m)} \hat{\alpha}_n^{(m)} = 0, \tag{3.2}$$

(2) if $|\zeta_m| < |\zeta_{m+1}| < |\zeta_{m+2}|$, $0 \leq m \leq Z-1$:

$$\lim_{n\to\infty} \alpha_{n-1}^{(m)} / \alpha_n^{(m)} = \zeta_{m+1}. \tag{3.3}$$

Proof Formula (3.2) is simply verified by filling in the definitions (2.3). Then use Jacobi's identity for Toeplitz matrices:

$$(\det T_n^{(m)})^2 = \det T_n^{(m-1)} \det T_n^{(m+1)} + \det T_{n+1}^{(m)} \det T_{n-1}^{(m)}$$

to find that

$$\alpha_n^{(m)} \hat{\alpha}_n^{(m)} = 1 - \det T_{n+1}^{(m)} \det T_{n-1}^{(m)} / (\det T_n^{(m)})^2.$$

The limit is then easily found with the expressions (3.1). Formula (3.3) is even simpler to prove with (2.3) and (3.1).

□

This is about as far as we can go for the moment. As you will have noticed, it is essential that certain zeros of F seperate in modulus for the formula (3.1b) to be

applicable. However, if F is a real spectrum, complex zeros will always appear as complex conjugate pairs, which means that these zeros do not separate in modulus. Thus we will never be able to find them by a limit like (3.3).
Therefore we shall have to introduce Rutishauser polynomials for F. This will be done in two steps. We start by reviewing the classical theory of Rutishauser polynomials in next section.

4. RUTISHAUSER POLYNOMIALS FOR POWER SERIES

Rutishauser polynomials were introduced in connection with the qd algorithm [11] which can be used to find the poles of a meromorphic function given by its McLaurin series. If this series represents the inverse of a polynomial it gives the zeros of a polynomial. A problem appears there for equimodular poles which is of the same type as the one mentioned at the end of previous section .
Rutishauser never proved the convergence of these polynomials but gives the result without proof. A proof was recently given by Seewald [13]. Gragg[5] gives similar results for a variant of the qd algorithm, called the $\pi\zeta$ algorithm. It is also shown there that, by using symmetry relations between the Padé table of a power series and its inverse, we can obtain symmetric versions of the Rutishauser polynomials, giving the zeros of the function directly (without inverting the series).
In our development we will follow the approach of Gragg.
Let $G(z) = \sum_0^\infty g_k z^k$ be a meromorphic function in $\{z: 0 \leqslant |z| < R\}$ which is nonzero and analytic at the origin. Suppose that $\{g_k\}_0^\infty$ is a normal sequence, which now means that all the Toeplitz determinants $\det T_n^{(m)}(G)$ with $T_n^{(m)}(G) := (g_{m+i-j})_{i,j=0}^{n}$ are nonzero for $n,m \geqslant 0$ (whenever g_k with $k<0$ appears in this section, it is supposed to be zero). It is clear that, in the normal case, all the numbers

$$a_{n+1}^{(m)}(G) := -\frac{\det T_{n-2}^{(m)}(G)\ \det T_n^{(m)}(G)}{\det T_{n-1}^{(m-1)}(G)\ \det T_{n-1}^{(m)}(G)} \tag{4.1}$$

do exist and we can construct the monic Rutishauser polynomials $P_n^{(m,k)}(z;G)$ of degree k for k=0,1,... by the recursive definitions

$$P_n^{(m,k)}(z;G) = z\,P_n^{(m,k-1)}(z;G) - a_{n+1}^{(m+k-1)}(G)\,P_{n+1}^{(m,k-1)}(z;G) \tag{4.2a}$$

with initial conditions

$$P_n^{(m,0)}(z;G) = 1 \qquad ,n=0,1,.... \tag{4.2b}$$

This may be visualized by the flow graph in fig.1 on next page.
At node (m+k, n-j) you will find the polynomial $P_{n-j}^{(m,k)}(z;G)$ if all nodes at row m are initialized at 1, all vertical transmittances are equal to z and the transmittance for the branch from node (m+k, n-j) to node (m+k+1, n-j-1) is $-a_{n-j}^{(m+k)}$ for j=0,1,..,n-1 and k=0,1,..,j. So for each m and n, we get a triangle of polynomials $P_{n,j}^{(m,k)}(z;G)$. Since they are all related to the same G, they are not independent from each other. We have the following property which gives the main idea used in [13].

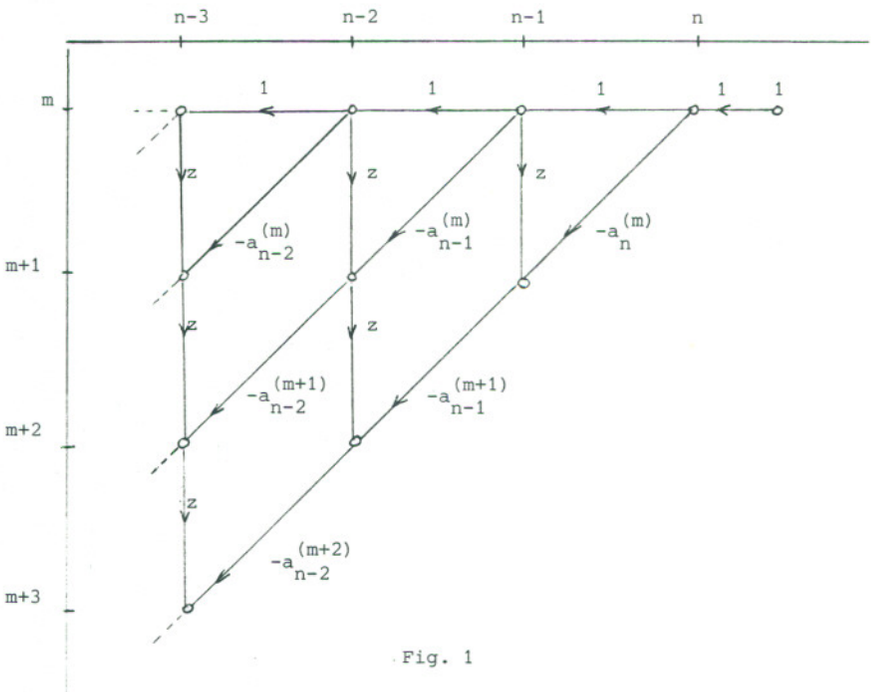

Fig. 1

Theorem 4.1

Let $\{g_k\}_0^\infty$ be a normal sequence and define the Rutishauser polynomials $P_n^{(m,k)}(z;G)$ by (4.2). Then the following relation exists:

$$P_{n-m-k}^{(0,m+k)}(z;G) = \sum_{j=0}^{k} q_j z^j \, P_{n-m-j}^{(0,m)}(z;G) \tag{4.3a}$$

with

$$P_{n-m-k}^{(m,k)}(z;G) = \sum_{j=0}^{k} q_j z^j . \tag{4.3b}$$

Proof

This directly follows from the flow graph in fig.1 constructing the polynomials. Indeed, if $P_{n-m-k}^{(m,k)}(z;G) = \sum_0^k q_j z^j$ is constructed with (4.2a) with initial conditions $P_{n-m-j}^{(m,0)} = 1$ for $j=0,1,\ldots,k$, then it is a simple matter to verify that if $P_{n-m-j}^{(m,0)} = 1$ is modified to some number ρ_j say, then the only change resulting in $P_{n-m-k}^{(m,k)}(z;G)$ will be that q_j is replaced by $q_j\rho_j$ (see fig.1).

Thus, if all initial values $P_{n-m-j}^{(m,0)}$ are replaced by $P_{n-m-j}^{(0,m)}(z;G)$, then we get by definition $P_{n-m-k}^{(0,m+k)}(z;G)$ at the bottom of the triangle.(see fig.2) and this results in replacing q_j by $q_j \, P_{n-m-j}^{(0,m)}(z;G)$.

□

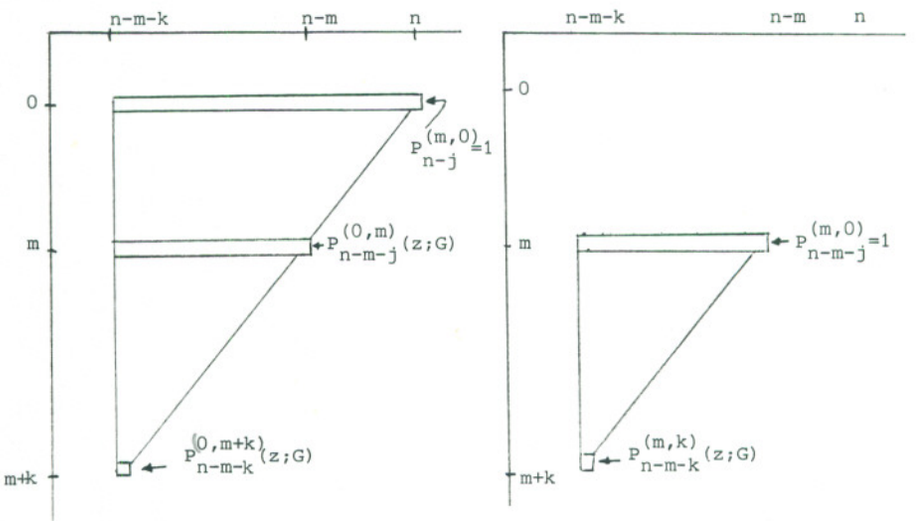

Fig. 2

It is a well known fact that the Rutishauser polynomials $P_n^{(0,m)}(z;G)$ are the (monically normalized) numerators of the (m/n) Padé approximants of G [5] and therefore, they are given by the following determinant expression.

Theorem 4.2

The Rutishauser polynomials $P_n^{(0,m)}(z;G)$ are given by

$$P_n^{(0,m)}(z;G) = \det \begin{bmatrix} G_{m+n-1}(z) & \cdots & z^n G_m(z) \\ \hline & & f_{m-n+2} \\ T_{n-2}^{(m+1)}(G) & & \vdots \\ & & f_m \end{bmatrix} / \det T_{n-1}^{(m)}(G) \qquad (4.4)$$

with $G_k(z) = \sum_0^k g_j z^j$.

Proof

All necessary elements to give the proof can be found in [5].

□

It follows from this theorem that $P_n^{(0,m)}(z;G)$ is defined as long as $\det T_{n-1}^{(m)}$ is nonzero.

As it was stated in corollary 3.2 for F we can also conclude in a similar way (see [1]) that if ζ_k, k=1,...,Z are the zeros of G ordered such that

$|\zeta_k| \leq |\zeta_{k+1}|$, then, if $|\zeta_m| < |\zeta_{m+1}|$ we also have that for n sufficiently large, $\det T_n^{(m)}(G) \neq 0$.

Thus in that case will $P_n^{(0,m)}(z;G)$ be a well defined monic polynomial of (strict) degree m. If we combine this with the result of theorem 4.1, we see that if G is not normal but its zeros satisfy $|\zeta_m| < |\zeta_{m+1}|$ and $|\zeta_{m+k}| < |\zeta_{m+k+1}|$, then in formula (4.3a), the $P_{n+m+k}^{(0,m+k)}(z;G)$ and $P_{n+m+k-j}^{(0,m)}(z;G)$, $j=0,\ldots k$ polynomials are well defined for n sufficiently large. Thus, there must exist a unique set of numbers $q_j, j=0,\ldots,k$, relating them like in (4.3a). This unique set defines the polynomial $P_{n+m+k}^{(m,k)}(z;G)$ as in (4.3b) and thus defines and proves the existence of certain Rutishauser polynomials for a nonnormal sequence.

Now we are ready to give the following classical convergence result for Rutishauser polynomials associated with a meromorphic functions given in terms of its McLaurin series.

Theorem 4.3

Let $G(z) = \sum_0^\infty g_k z^k$ be a meromorphic functions in $\{z: 0 \leq |z| < R\}$ with zeros ζ_k, $k=1,\ldots,Z$ ordered such that

$$0 < \zeta_0 < |\zeta_1| \leq \ldots \leq |\zeta_m| < |\zeta_{m+1}| \leq \ldots \leq |\zeta_{m+k}| < |\zeta_{m+k+1}| \leq \ldots \leq |\zeta_Z| < \zeta_{Z+1} < R$$

for $0 \leq m < m+k \leq Z$. (Note that we introduced artificial numbers ζ_0 and ζ_{Z+1} as we did for F.) Then the Rutishauser polynomials $P_n^{(m,k)}(z;G)$ exist for n sufficiently large and

$$|P_n^{(m,k)}(z;G) - P^{(m,h)}(z;G)| \leq c\lambda^n, \quad \forall z \in K, \text{ n large, where c is a constant}$$

independent of n, K is a compact subset of $\{z: 0 \leq |z| < R\}$,

$$1 > \lambda > \max\{|\zeta_m/\zeta_{m+1}|, |\zeta_{m+k}/\zeta_{m+k+1}|\}$$

and $P^{(m,k)}(z;G)$ is defined as

$$P^{(m,k)}(z;G) = (z-\zeta_{m+1})\ldots(z-\zeta_{m+k}).$$

Proof: See [5,13]. □

This theorem thus says that the Rutishauser polynomials $P_n^{(m,k)}(z;G)$ converge uniformly to a polynomial $P^{(m,k)}(z;G)$ whose zeros are the zeros of G that could have an equal modulus.

5. RUTISHAUSER POLYNOMIALS FOR LAURENT SERIES

The previous section would in principle solve the problem of finding the transmission zeros if we would know the McLaurin coefficients of the spectral factor H of the spectral density F. However, we don't know this factor but we only know the Laurent coefficients of F.

Rutishauser polynomials $P_n^{(m,k)}(z) := P_n^{(m,k)}(z;F)$ however may be obtained for the normal sequence $\{f_k\}_{-\infty}^{\infty}$ just as we did in the previous section by replacing G by F everywhere and by using the earlier defined Toeplitz determinants $T_n^{(m)} := T_n^{(m)}(F)$ instead of $T_n^{(m)}(G)$ (note that $f_k \neq 0$ for $k < 0$ now). Theorem 4.1 remains true but we get stuck at theorem 4.2 since we don't have a power series now.

We shall get around this problem by condidering the Rutishauser polynomials for the factor H with $F(z)=H(z)H_*(z)$ as in section 3. The subject of this section will be to show that we have the following remarkable relation.

Theorem 5.1

Let $F(z)=H(z)H_*(z)$ be in $\mathcal{R}(\zeta_1,\ldots,\zeta_Z;\pi_1,..\pi_P;r,R)$ and suppose $|\zeta_m|<|\zeta_{m+1}|$ and $|\zeta_{m+h}|<|\zeta_{m+h+1}|$ for $0\leq m<m+h\leq Z$. Then for n large enough the Rutishauser polynomials for F and H do exist and they are related by

$$P_n^{(m,h)}(z;F) = P_n^{(m,h)}(z;H)\ \{1+o(1)\} \tag{5.1}$$

□

We shall prove this theorem by using the asymptotic expressions for Toeplitz determinants again. As a first step we quote a lemma which was given in [1].

Lemma 5.2

Let $F(z) = H(z)H_*(z)$ be in $\mathcal{R}(\zeta_1,..,\zeta_Z;\pi_1,\ldots,\pi_P;r,R)$. Then there exist constants c and λ, only depending on F such that for $m \geq 0$.

$$\det T_n^{(m)}(F) = c\lambda^{n+1} \det T_n^{(m)}(H)\ \{1+o(1)\} \qquad \text{as } n\to\infty \tag{5.2}$$

Proof: See [1]. □

From this lemma, we can derive the following consequence.

Lemma 5.3

Let $F(z) = H(z)H_*(z)$ be in $\mathcal{R}(\zeta_1,..,\zeta_Z;\pi_1,\ldots,\pi_P;r,R)$.
Then the numbers $a_n^{(m)}$, defined by (4.1) where G can be either F or H, are related by

$$a_n^{(m)}(F) = a_n^{(m)}(H)\{1 + o(1)\} \quad \text{as } n\to\infty, \tag{5.3}$$

Proof:

Take (4.1) with G=F and replace the Toeplitz determinants for F by expressions like (5.2) to get

$$a_n^{(m)}(F) = -\frac{c\lambda^{n-1}\det T_{n-2}^{(m+1)}(H)\ c\lambda^{n+1}\det T_n^{(m)}(H)}{c\lambda^{n}\det T_{n-1}^{(m+1)}(H)\ c\lambda^{n}\det T_{n-1}^{(m)}(H)}\{1+o(1)\}.$$

This gives (5.3). □

Proof of theorem 5.1

Under the conditions of theorem 5.1, the polynomials $P_n^{(m,h)}(z;H)$ will exist as was explained in previous section. The coefficients of these Rutishauser polynomials are made up of sums and products of $a_n^{(m)}(H)$ numbers. As we saw, some of them maynot exist but their combinations that make up the coefficients of $P_n^{(m,h)}(z;H)$ do exist as long as n is large enough. Since the numbers $a_n^{(m)}(F)$ are only o(1) corrections to $a_n^{(m)}(H)$, also $P_n^{(m,k)}(z;F)$ must exist and we have (5.1) for any finite z. □

6. EQUIMODULAR TRANSMISSION ZEROS

We are now ready to give the analog of theorem 4.3 for Laurent series:

Theorem 6.1

Let F be in $\mathcal{R}(\zeta_1,..\zeta_Z;\pi_1,...,\pi_P;r,R)$ and suppose $|\zeta_m|<|\zeta_{m+1}|$ and $|\zeta_{m+k}|\ |\zeta_{m+k+1}|$, $0\leq m<m+k\leq Z$. Define the Rutishauser polynomials $P_n^{(m,k)}(z)$ as in section 5 and define the polynomial $P^{(m,k)}(z) := (z-\zeta_{m+1}).....(z-\zeta_{m+k})$. Then

$$|P_n^{(m,k)}(z) - P^{(m,k)}(z)| \leq c\lambda^n \ , \ \forall z\in K,\ n \text{ large} \tag{6.1}$$

where c is a constant, K a compact subset of $\{z: r<|z|<R\}$ and $1>\lambda>\max\{|\zeta_m/\zeta_{m+1}|,|\zeta_{m+k}/\zeta_{m+k+1}|\}$

Proof:

Theorem 4.3 shows that this theorem is true for $P_n^{(m,k)}(z)$ replaced by $P_n^{(m,k)}(z;H)$ while theorem 5.1 states that this replacement results only in an o(1) correction. □

From theorem 6.1 we can now rederive the results of theorem 3.3 and some more general results on how to compute equimodular transmission zeros as well. E.g.

Theorem 6.2

Let F be in $\mathcal{R}(\zeta_1,..,\zeta_Z;\pi_1,....,\pi_P;r,R)$ and denote its Schur parameters by $\alpha_n^{(m)}$, $\tilde{\alpha}_n^{(m)}$ and let the numbers $a_n^{(m)}$ be defined by

$$a_{n+1}^{(m)} = -\frac{\det T_{n-2}^{(m+1)}\ \det T_n^{(m)}}{\det T_{n-1}^{(m+1)}\ \det T_{n-1}^{(m)}} = (1-\alpha_n^{(m)}\tilde{\alpha}_n^{(m)})\ \alpha_{n-1}^{(m)}\ /\ \alpha_n^{(m)} \tag{6.2}$$

then the following statements are true:

1. If $|\zeta_m|<|\zeta_{m+1}|$ for some m : $0\leq m\leq Z$, then

$$\lim_{n\to\infty} \alpha_n^{(m)}\ \tilde{\alpha}_n^{(m)} = 0 \tag{6.3}$$

2. If $|\zeta_m|<|\zeta_{m+1}|<|\zeta_{m+2}|$ for some m : $0\leq m\leq Z-1$, then

$$\lim_{n\to\infty} a_n^{(m)} = \lim_{n\to\infty} \alpha_{n-1}^{(m)}/\alpha_n^{(m)} = \lim_{n\to\infty} \tilde{\alpha}_n^{(m+1)}/\tilde{\alpha}_{n-1}^{(m+1)} = \zeta_{m+1} \tag{6.4}$$

3. If $|\zeta_m|<|\zeta_{m+1}|\leq|\zeta_{m+2}|<|\zeta_{m+3}|$ for some m : $0\leq m\leq Z-2$, then

$$\lim_{n\to\infty} (a_n^{(m)} + a_n^{(m+1)}) = \zeta_{m+1}+\zeta_{m+2} \tag{6.5a}$$

and

$$\lim_{n\to\infty} a_{n+1}^{(m)}\ a_n^{(m+1)} = \zeta_{m+1}\ \zeta_{m+2}\ . \tag{6.5b}$$

Proof:

(6.3) was found as relation (3.2) and (6.4) partly as (3.3). The latter can be rederived as a consequence of theorem 6.1 by taking k=1. Then, by definition, $P_n^{(m,1)}(z) = z-a_{n+1}^{(m)}$ which will converge to $z-\zeta_{m+1}$ for $n\to\infty$.

Hence $\lim_{n\to\infty} a_n^{(m)} = \zeta_{m+1}$. Now $a_{n+1}^{(m)}$ can be expressed in terms of the Schur parameters as in (6.2) which is trivial when filling in the determinant expressions for the Schur parameters. This gives also $\alpha_n^{(m)} = 1/\hat{\alpha}_n^{(m+1)}$. Now, using (6.3) in the limit of (6.2) gives (6.4). (6.5) can be shown by taking k=2 in theorem 6.1. We then have $P_n^{(m,2)}(z) = z^2 - (a_n^{(m)} + a_n^{(m+1)})\, z + a_{n+1}^{(m)}\, a_n^{(m+1)}$ with limit $z^2 - (\zeta_{m+1}+\zeta_{m+2})\, z + \zeta_{m+1}\zeta_{m+2}$. Since convergence is uniform, hence coefficientwise we get the results (6.5.)

□

7. EXAMPLES

We shall give some simple examples to illustrate the theorems. We shall take H(z) to be a polynomial to cut down on the computations. Although the examples are very simple, they will be sufficient to illustrate the main points of our analysis.

Example 1

Take $H(z)=(z-2)/\sqrt{5}$. The spectrum is $F(z)=1-\frac{2}{5}(z+z^{-1})$. Even for this simple example, the Levinson algorithm needs an infinite number of steps. Indeed, there is only one real zero $\zeta_1=2$, but it is not at infinity. All the Toeplitz determinants $\det T_n^{(m)}$ are zero for $|m| \geqslant 2$ and nonzero for $|m| < 2$. Some nonzero determinants are given in table 1. From these determinants, $\alpha_n^{(0)}$ and $\hat{\alpha}_n^{(0)}$ can be computed, as well as $\alpha_n^{(0)}\, \hat{\alpha}_n^{(0)}$ and $a_n^{(0)}$. The results are summarized in table 1.

n	$\det T_n^{(m)}$	
	m=0	m=1
0	.1000000e+01	-.4000000e+00
1	.8400000e+00	.1600000e+00
2	.6800000e+00	-.6400001e-01
3	.5456001e+00	.2560000e-01
4	.4368001e+00	-.1024000e-01
5	.3495041e+00	.4096001e-02
6	.2796161e+00	-.1638401e-02
7	.2236954e+00	.6553604e-03
8	.1789569e+00	-.2621442e-03
9	.1431656e+00	.1048577e-03
10	.1145325e+00	-.4194308e-04
11	.9162607e-01	.1677723e-04
12	.7330086e-01	-.6710893e-05
13	.5864070e-01	.2684367e-05
14	.4691257e-01	-.1073751e-05

n	$\alpha_n^{(0)}$	$\alpha_n^{(0)}\,\bar{\alpha}_n^{(0)}$	$a_n^{(0)}$
1	.4000000e+00	.1600000e+00	
2	.1904761e+00	.3628118e-01	.2100000e+01
3	.9411766e-01	.8858133e-02	.2023809e+01
4	.4692082e-01	.2201563e-02	.2005882e+01
5	.2344322e-01	.5495848e-03	.2001466e+01
6	.1171946e-01	.1373459e-03	.2000366e+01
7	.5859465e-02	.3433333e-04	.2000091e+01
8	.2929698e-02	.8583134e-05	.2000023e+01
9	.1464845e-02	.2145771e-05	.2000005e+01
10	.7324221e-03	.5364421e-06	.2000001e+01
11	.3662109e-03	.1341104e-06	.2000000e+01
12	.1831054e-03	.3352761e-07	.2000000e+01
13	.9155272e-04	.8381899e-08	.2000000e+01
14	.4577652e-04	.2095490e-08	.2000000e+01

Table 1

As you would expect by theorem 6.2 $\alpha_n^{(0)}\bar{\alpha}_n^{(0)}$ converges to zero because $1<\zeta_0<\zeta_1=2$. When $a_n^{(0)}$ is computed, it converges to 2 as predicted by theorem 6.2. The only Rutishauser polynomial which is meaningful in this case is $P_n^{(0,1)}(z) = z-a_n^{(0)}$. This converges to z-2. The (1/n) Padé approximant of H is of course $(z-2)/\sqrt{5}$ and this shows that $P_n^{(0,1)}(z)$ is an o(1) correction to the numerator of this Padé approximant.

Example 2

Now we take $H(z)=(z^2+2)/\sqrt{5}$. The corresponding spectral density function is $F(z) = 1 + \frac{2}{5}(z^2+z^{-2})$. We should look for two purely imaginary zeros: $\zeta_{1,2}=\pm\sqrt{2}i$. The table of Toeplitz determinants looks as in table 2.

	$\det T_n^{(m)}$		
n	m=0	m=1	m=2
0	.100000e+00	.000000e+00	.400000e+00
1	.100000e+01	-.400000e+00	.160000e+00
2	.840000e+00	.000000e+00	.640000e-01
3	.705600e+00	.134400e+00	.256000e-01
4	.571200e+00	.000000e+00	.102400e-01
5	.462400e+00	-.435200e-01	.409600e-02
6	.371008e+00	.000000e+00	.163840e-02
7	.297679e+00	.139673e-01	.655360e-03
8	.238318e+00	.000000e+00	.262144e-03
9	.190794e+00	-.447283e-02	.104857e-03
10	.152663e+00	.000000e+00	.419430e-04

Table 2

Of course $\det T_n^{(-m)} = \det T_n^{(m)}$ and $\det T_n^{(m)} = 0$ for $m \geq 3$. Note that $\det T_{2n}^{(1)} = 0$ for n=0,1,2,... Therefore some of the $\alpha_n^{(1)}$ are infinite. It can be verified that $\alpha_{2n}^{(1)} = \alpha_{2n}^{(0)}$ for n=1,2,.... while $\alpha_{2n+1}^{(0)} = 0$ and $\alpha_{2n+1}^{(1)} = \infty$ for n=0,1,2,...(See table 3)

n	$\alpha_n^{(0)}$	$\alpha_n^{(0)} \bar{\alpha}_n^{(0)}$	$a_{n+1}^{(1)} . a_{n+2}^{(0)}$
1	.000000e+00	.000000e+00	.210000e+01
2	-.400000e+00	.160000e+00	.202380e+01
3	.000000e+00	.000000e+00	.202380e+01
4	.190476e+00	.362811e-01	.200588e+01
5	.000000e+00	.000000e+00	.200588e+01
6	-.941176e-01	.885813e-02	.200146e+01
7	.000000e+00	.000000e+00	.200146e+01
8	.469208e-01	.220156e-02	.200036e+01
9	.000000e+00	.000000e+00	.200036e+01
10	-.234432e-01	.549584e-03	.200009e+01

Table 3

Because $1 < \zeta_0 < |\zeta_1| = \sqrt{2}$, the $\alpha_n^{(0)} \bar{\alpha}_n^{(0)}$ should converge to zero and they do, as can be seen in the second column of table 3. However $\alpha_n^{(1)} \bar{\alpha}_n^{(1)}$ need not converge to zero and it can be verified that $\alpha_{2n}^{(1)} \bar{\alpha}_{2n}^{(1)} = 1$ while $\alpha_{2n+1}^{(1)} \bar{\alpha}_{2n+1}^{(1)}$ is infinite for integer n. Because of the special structure of table 2, we find that $a_n^{(0)}$ and $a_n^{(1)}$ are alternatingly zero and infinite, even if n tends to infinity, meaning that there is no convergence at all. Now we are confronted with the definition of the Rutishauser polynomial $P_n^{(0,2)}(z)$. Our recursive definition doesn't work because of all the infinite numbers $a_n^{(0)}$ and $a_n^{(1)}$. However, we would find

$$P_n^{(0,2)}(z) = z^2 - z(a_{n+1}^{(0)} + a_{n+1}^{(1)}) + a_{n+1}^{(1)} a_{n+2}^{(0)}$$

Now, using the definitions of the a-numbers in terms of Toeplitz determinants we get:

$$a_{n+1}^{(1)} a_{n+2}^{(0)} = \frac{\det T_{n-2}^{(2)} \det T_n^{(1)} \det T_{n-1}^{(1)} \det T_{n+1}^{(0)}}{\det T_{n-1}^{(2)} \det T_{n-1}^{(1)} \det T_n^{(1)} \det T_n^{(0)}}$$

$$= \frac{\det T_{n-2}^{(2)} \det T_{n+1}^{(0)}}{\det T_{n-1}^{(2)} \det T_n^{(0)}} .$$

And this is well defined and converges to 2 as you can see in the third column of table 3.

The other coefficient of $P_n^{(0,2)}$ is

$$a_{n+1}^{(0)} + a_{n+1}^{(1)} = - \frac{\det T_{n-2}^{(1)} \det T_n^{(0)} \det T_{n-1}^{(2)} + \det T_{n-2}^{(2)} \det T_n^{(1)} \det T_{n-1}^{(0)}}{\det T_{n-1}^{(0)} \det T_{n-1}^{(1)} \det T_{n-1}^{(2)}}$$

For n even, there is no problem, since the numerator is zero and the denominator is not zero. For n odd, however will det $T_{n-1}^{(1)}$ be zero and the numerator will vanish too.

Using extensively Jacobi's identity for Toeplitz determinants, it can be shown that $a_{n+1}^{(0)} + a_{n+1}^{(1)} = - C/ [(\det T_n^{(0)})^2 \det T_{n-1}^{(2)}]$

with $C = \det T_n^{(0)} \det T_{n-2}^{(1)} \det T_{n-1}^{(1)} + \det T_n^{(1)} \det T_{n-2}^{(2)} \det T_{n-1}^{(-1)}$

$- \det T_{n-3}^{(1)} \det T_n^{(0)} \det T_n^{(1)}$.

The denominator of this expression is nonzero, while the numerator is zero since $\det T_{n-1}^{(1)} = \det T_{n-1}^{(-1)} = \det T_{n-3}^{(1)} = 0$. So we may conclude that $a_{n+1}^{(0)}+a_{n+1}^{(1)} = 0$ for all n. Therefore $P_n^{(0,2)}$ converges to z^2+2 as it should.

Example 3

A last, less trivial example is $H(z) = (z^3+18z^2-38z+40)/\sqrt{3369}$.

$$F(z)= 1 - \frac{2186}{3369} (z+z^{-1}) + \frac{682}{3368} (z^2+z^{-2}) + \frac{40}{3368} (z^3+z^{-3}).$$

(The normalization was always chosen to make $f_0=1$).

The zeros are now $\zeta_{1,2} = 1 \pm i$ and $\zeta_3 = - 20$. Now $\det T_n^{(m)}$ will be zero for $|m| > 3$. Computing the other determinants as before and computing from these the Schur parameters $\alpha_n^{(m)}$ gives us table 4.

	$\alpha_n^{(m)}$		
n	m=0	m=1	m=2
1	.648857e+00	.311985e+00	-.586510e-10
2	.377526e+00	.222723e+00	.289559e-02
3	.105040e+00	.382723e+00	-.144837e-03
4	-.781225e-01	-.248870e+00	.724207e-05
5	-.127770e+00	-.740448e-01	-.362101e-06
6	-.893280e-01	-.523916e-01	.181050e-07
7	-.260309e-01	-.896527e-01	-.905254e-09
8	.185525e-01	.627695e-01	.452627e-10
9	.315247e-01	.184394e-01	-.226313e-11
10	.222569e-01	.130500e-01	.113156e-12

Table 4

Table 5 gives the products $\alpha_n^{(m)} \hat{\alpha}_n^{(m)}$.

As you can see, $\alpha_n^{(0)}\hat{\alpha}_n^{(0)}$ and $\alpha_n^{(2)}\hat{\alpha}_n^{(2)}$ will converge to zero, but $\alpha_n^{(1)}\hat{\alpha}_n^{(1)}$ does not.

	$\alpha_n^{(m)}$ $\delta_n^{(m)}$		
n	m=0	m=1	m=2
1	.421015e+00	.480822e+00	-.187992e+00
2	.142526e+00	.589955e+00	.130008e-01
3	.110334e-01	.364359e+01	-.378440e-03
4	.610312e-02	.318563e+01	-.290998e-04
5	.163253e-01	.579513e+00	.489030e-05
6	.797950e-02	.586508e+00	-.345571e-06
7	.677610e-03	.344408e+01	.100973e-07
8	.344195e-03	.338334e+01	.721093e-09
9	.993808e-03	.584918e+00	-.122733e-09
10	.495371e-03	.586335e+00	.867098e-11

Table 5

The $a_n^{(m)}$ numbers are given in table 6

	$a_n^{(m)}$		
n	m=0	m=1	m=2
2	.892313e+00	.166410e+01	-.202552e+02
3	.147374e+01	.574378e+00	-.199919e+02
4	.355446e+01	-.153842e+01	-.199995e+02
5	-.133635e+01	.336117e+01	-.200001e+02
6	.601445e+00	.141328e+01	-.200000e+02
7	.141894e+01	.584384e+00	-.200000e+02
8	.342928e+01	-.142828e+01	-.200000e+02

Table 6

As you can see $a_n^{(2)}$ converges quite rapidly to -20, which is ζ_3. To find $\zeta_{1,2}$, we have to compute $a_{n+1}^{(1)}$ $a_{n+2}^{(0)}$ and $a_{n+1}^{(0)} + a_{n+1}^{(1)}$, which goes without problems in this example. They are given in table 7.

n	$a_{n+1}^{(0)}+a_{n+1}^{(1)}$	$a_{n+1}^{(1)}$ $a_{n+2}^{(0)}$
2	.255642e+01	.245247e+01
3	.204812e+01	.204160e+01
4	.201603e+01	.205588e+01
5	.202481e+01	.202156e+01
6	.201473e+01	.200537e+01
7	.200332e+01	.200402e+01
8	.200100e+01	.200333e+01
9	.200148e+01	.200134e+01
10	.200089e+01	.200034e+01
11	.200019e+01	.200026e+01
12	.200005e+01	.200022e+01
13	.200008e+01	.200009e+01
14	.200004e+01	.200003e+01
15	.200000e+01	.200003e+01

Table 7

The first column shows the convergence of $a_{n+1}^{(0)} + a_{n+1}^{(1)}$ to 2 which is $\zeta_1 + \zeta_2 = 2\mathrm{Re}\ \zeta_{1,2}$, while the second column shows the convergence of $a_{n+1}^{(1)} a_{n+2}^{(0)}$ to 2 which is $\zeta_1\zeta_2 = |\zeta_{1,2}|^2$. Thus $P_n^{(1,2)}(z) = z^2-2z+2$ having the zeros ζ_1 and ζ_2, while $P_n^{(2,3)}(z) = z-a_{n+1}^{(2)}$ converges to z+20, giving the zero ζ_3.

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