Positive quadrature formula on an arc of the unit circle

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Abstract

We analyze positive quadrature formulas associated with Geronimus measures supported on an arc of the unit circle that exactly integrate certain subspaces of Laurent polynomials with the highest possible dimension. The quasi-paraorthogonal polynomials associated with these measures will play a fundamental role in the case of quadrature rules with one node (Szegő-Radau) or two nodes (Szegő-Lobatto) preselected at the end points of the arc.

Keywords: arc of the unit circle, Geronimus measure, positive quadrature formula

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1 Introduction

When considering n-point quadrature formulas $J_n(f) = \sum_{k=1}^{n} \lambda_k f(x_k)$ to approximate the integral $J(f) = \int_a^b f(x)\mathrm{d}\nu(x)$, ν being a finite positive Borel measure on an interval $[a, b]$ of the real line, Gauss, Gauss-Radau and Gauss-Lobatto formulas are well known and extensively studied. Gauss-Radau and Gauss-Lobatto mean that one or both end points of $[a, b]$, respectively, have

been fixed as nodes in an n-point quadrature formula $J_n(f)$ while the remaining nodes and all the weights are determined by imposing that $J(p) = J_n(p)$ for any polynomial p of degree up to $2n-1-r$ with $r \in \{0,1,2\}$. The case $r = 0$ produces the n-point Gauss or Gauss-Christoffel formula without a node fixed in advance.

The authors in [1, 2] introduced a nested sequence of subspaces of Laurent polynomials that allows us to obtain properties analogous to those on the real line for the quadrature formulas on the unit circle. In addition they introduce the quasi-paraorthogonal polynomials as nodal polynomials to provide positive quadrature rules with maximal domain of exactness. In the present paper, we focus on the analysis of the positive quadrature formulas on an arc of the unit circle $\mathbb{T}$, with maximal domain of exactness and possibly some preassigned nodes.

Concretely, let μ be a finite positive Borel measure supported on an arc $[a,b] = \{e^{i\theta} : \theta_a \le \theta \le \theta_b,\ \theta_b - \theta_a < 2\pi\} \subset \mathbb{T}$, the main concern of this paper is the estimation of integrals of the form

$$I(f) = \int_{\theta_a}^{\theta_b} f(e^{i\theta}) \mathrm{d}\mu(\theta),$$

where $g(\theta) = f(e^{i\theta})$ is a Riemann-Stieltjes integrable function with respect to μ. For this purpose, we will consider an n-point quadrature formula like,

$$I_n(f) := \sum_{k=1}^{n} \lambda_k f(x_k), \quad \{x_k\}_{k=1}^n \subset [a,b], \quad x_j \neq x_k \ \text{ if } \ j \neq k, \quad \{\lambda_k\}_{k=1}^n \subset \mathbb{R}. \tag{1}$$

For convergence and stability reasons, it is preferable to have positive rules, i.e., with positive weights λ_k. The usual strategy for determining weights $\{\lambda_k\}_{k=1}^n$ and nodes $\{x_k\}_{k=1}^n$ of the quadrature formula is to impose $I(g) = I_n(g)$ for all functions g belonging to some subspace with the highest possible dimension that is in some sense close to the class of integrands f that one has in mind. It is well known that polynomials are dense in the space of continuous functions on an arc of $\mathbb{T}$ with respect to the uniform norm. However, to take full advantage of the theory of paraorthogonal polynomials, it is much more effective to use Laurent polynomials as approximating functions, which obviously are also dense with respect to the uniform norm. So, we define a nested sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of subspaces of Laurent polynomials, such that $\mathcal{L}_0 = \mathrm{span}\{1\}$, $\mathcal{L}_{n-1} \subset \mathcal{L}_n$, $\dim(\mathcal{L}_n) = n+1$ and $\bigcup_{n\in\mathbb{N}} \mathcal{L}_n = \mathcal{L}$, the space of all Laurent polynomials. This can be done by starting from a sequence $\{\omega_k\}_{k\ge 1} \subset \mathbb{T}$ and defining for $k \ge 0$

$$\mathcal{L}_{2k} := \mathrm{span}\{z^{-k}, \dots, z^k\}, \quad \mathcal{L}_{2k-1} = \mathcal{L}_{2k-1}(\omega_k) := \mathcal{L}_{2(k-1)} \oplus \mathrm{span}\left\{z^k + \frac{\omega_k}{z^k}\right\}. \tag{2}$$

Note that $\mathcal{L}_n$ is invariant under the involution $L \mapsto L_*$, that is $L(z) \in \mathcal{L}_n \Leftrightarrow \overline{L(1/\overline{z})} \in \mathcal{L}_n$. As we have pointed out, this allows many of the classical properties in the real case to be kept on the unit circle, see [2]. Namely:

1. The weights are always real, and at least half of them are positive.
2. There always exist n-point positive rules whose maximal domain of validity has dimension $2n$.
3. An analogue of the well known Jacobi theorem can be stated in order to characterize the nodal polynomial by its orthogonality conditions.

We will show that the true similarity with the interval $[a, b] \subset \mathbb{R}$ is the arc of the unit circle. Thus, an important aim of this paper is to set a first step in the analysis and construction of quadrature rules on an arc of the unit circle. This paper is the third in this context of paraorthogonality, see [1, 2]. The paper is organized as follows. In Section 2 and Section 3 we give a background by recalling the most relevant results for our objectives and we guarantee the existence of a paraorthogonal polynomial with all its zeros in the support of a measure when it belongs to a class considered by Geronimus, respectively. We consider positive quadrature rules with a maximal domain of exactness, called Szegő quadrature in Section 4. The construction of positive quadrature rules with maximal domain of exactness when one (Szegő-Radau) or both (Szegő-Lobatto) end points of $[a, b]$ are fixed as nodes is presented in Sections 5 and 6, respectively. Several illustrative numerical examples are presented in Section 7. Finally, in the Addendum we collect all the technical details for greater transparency in the principal text.

We make use of the following notation. $\mathbb{P}_k$ will denote the space of polynomials of degree k at most, $k = 0, 1, \ldots$ and $\mathbb{P}$ the space of all polynomials, $\mathcal{L}$ is the space of all Laurent polynomials and the subspaces $\mathcal{L}_n$ are defined in (2). We define the reversed polynomial as $P^*(z) = z^n \overline{P(1/\overline{z})}$ when P has degree n and it is said to be invariant if $P = \tau P^*$ with $\tau \in \mathbb{T}$. $\lfloor r \rfloor$ denotes the integer part of a real number. $\mathbb{D}$ will denote the unit disk in the complex plane and $\mathbb{T}$ its boundary. If $c, d \in \mathbb{T}$, then $[c, d]$ will be the arc on $\mathbb{T}$ oriented counterclockwise from c to d and we write $c \prec d$ or $d \succ c$. When c and d are not included in the arc, we write (c, d). If $\Gamma = [a, b] \subset \mathbb{T}$ is an (oriented) arc and the function $F : \mathbb{T} \to \mathbb{T}$ is one-to-one, continuous and orientation preserving then $F(\Gamma)$ denotes the (oriented) arc $[F(a), F(b)]$. If $S \subset \mathbb{C}$ then $\mathring{S}$ denotes its interior and $\hat{S}$ its closure. We denote by $\{\rho_n\}_{n\in\mathbb{N}}$ the sequence of monic orthogonal polynomials of degree n associated to μ, the corresponding Schur parameters will be denoted by $\{\delta_k\}_{k=0}^{\infty}$ where $\delta_0 = 1$ and $\delta_k = \rho_k(0) \in \mathbb{D}$, i.e. the ρ_k satisfy the recurrence relation $\rho_k(z) = z\rho_{k-1}(z) + \delta_k \rho_{k-1}^*(z)$ for $k \geq 1$. The parameters $-\overline{\delta_k}$ are also known as Verblunsky coefficients. In the appendix we shall need recurrence relation in its backward form: $z\rho_{k-1}(z) = \frac{1}{1-|\delta_k|^2}(\rho_k(z) - \delta_k \rho_k^*(z))$, $k \geq 1$. We also use the automorphism of $\mathbb{D}$, denoted by $\varphi_u(z) = \frac{z+u}{1+\overline{u}z}$, $u \in \mathbb{D}$.

2 Background

Let μ a finite positive Borel measure on an arc $[a, b]$ and fix the sequence of nested subspaces of Laurent polynomials $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ as in (2). We say that a

quadrature formula $I_n(f)$ given by (1) is exact in $\mathcal{L}_k$ if and only if

$$I(L) = I_n(L), \quad \forall L \in \mathcal{L}_k.$$

The following result gives us a connection between the number of positive weights and the dimension of the domain of exactness of an n-point rule.

Theorem 2.1 *([2, Theorem 2.3]) Let $I_n(f)$ be the quadrature formula given by (1), exact in $\mathcal{L}_\ell$, with $n-1 \leq \ell \leq 2n-1$. Then, the weights $\{\lambda_k\}_{k=1}^n$ are all real and at least $\lfloor \frac{\ell}{2} \rfloor + 1$ of them are positive.*

Corollary 2.1 *([2, Corollary 2.4]) Under the conditions of Theorem 2.1 if $\ell \in \{2n-2, 2n-1\}$ then the quadrature formula given by (1) is positive.*

Remark 2.1 Like for Gauss quadrature formulas there exist formulas for the weights in terms of (reproducing) kernel functions which clearly show that they are positive. However, it is numerically preferred to use a Golub-Welsch algorithm to compute nodes and weights by solving an eigenvalue problem related to the Jacobi matrix, where the weights appear as squares (hence positive) of the first component of the eigenvectors. Our numerical computations in Section 7 will use an analog procedure for the circle which is an eigenvalue problem related to the CMV matrix, giving in an analogous way the positive weights.

Nevertheless, for theoretical interest, we briefly recall here how the kernel relates to our results below. A (reproducing) kernel is defined as

$$\begin{aligned} K_{n-1}(w,z) &= \sum_{k=0}^{n-1} \frac{\overline{\rho_k(w)}\rho_k(z)}{\|\rho_k\|^2} = \frac{\overline{\rho_n^*(w)}\rho_n^*(z) - \overline{\rho_n(w)}\rho_n(z)}{(1-\overline{w}z)\|\rho_n\|^2} \\ &= \frac{\overline{\rho_{n-1}^*(w)}\rho_{n-1}^*(z) - \overline{w}z\overline{\rho_{n-1}(w)}\rho_{n-1}(z)}{(1-\overline{w}z)\|\rho_{n-1}\|^2} \end{aligned}$$

and it reproduces in the sense that $\int P(t)\overline{K_{n-1}(z,t)}d\mu(\theta) = P(z)$, $t = e^{i\theta}$ for all $P \in \mathbb{P}_{n-1}$. If $w \in \mathbb{T}$, then obviously $\tau = -F_n(w) := -w\frac{\overline{\rho_{n-1}^*(w)}}{\rho_{n-1}(w)} \in \mathbb{T}$ and the numerator in the previous expressions for $K_{n-1}(w,z)$ is proportional to a paraorthogonal polynomial B_n with $B_n(0) = \tau$ (more on paraorthogonality in Section 3 below). The zeros of B_n are w and the $n-1$ zeros of $K_{n-1}(z,w)$. These zeros are all simple and on $\mathbb{T}$. At most one of them can be outside the support, which happens if $-\tau \notin (F_n(a), F_n(b))$ (see [1, Theorem 3.4]). In Section 4 we shall discuss n-point Szegő quadrature formulas and in Section 5, Szegő-Radau quadrature formulas, where the nodes are zeros of $(1-\overline{w}z)K_{n-1}(w,z)$ for an appropriate $w \in \mathbb{T}$ such that all the nodes $\{x_{k,n}\}_{k=1}^n$ are in (a,b) in the Szegő case, and where a or b is one of the nodes in the Szegő-Radau case. Since w is one of the nodes, just set $w = a$ or $w = b$ to get Szegő-Radau. Both quadratures are exact in $\mathcal{L}_{2(n-1)}$. The properties of the kernel imply that $K_{n-1}(x_{j,n}, x_{k,n}) = 0$ if $j \neq k$. Applying this quadrature (recall K_{n-1} is a polynomial in z of degree n in z) for the following integral where $t = e^{i\theta}$ and $j = 1, \ldots, n$ gives the first equality by the reproducing property and the second

by exactness of the quadrature

$$1 = \int_a^b \overline{K_{n-1}(x_{j,n}, t)} \mathrm{d}\mu(\theta) = \sum_{k=1}^{n} \lambda_{k,n} K_{n-1}(x_{j,n}, x_{k,n}) = \lambda_{j,n} K_{n-1}(x_{j,n}, x_{j,n}).$$

This shows that the Christoffel weights $\{\lambda_{j,n}\}_{j=1}^n$ are given by

$$\lambda_{j,n}^{-1} = \sum_{k=0}^{n-1} \frac{|\rho_k(x_{j,n})|^2}{\|\rho_k\|^2} = K_{n-1}(x_{j,n}, x_{j,n}), \quad j = 1, \ldots, n.$$

See also Golinskii [3].

In the sequel, we will be mainly concerned with the construction of n-point quadrature rules $I_n(f)$ with distinct nodes in $[a, b]$ and with a domain of exactness whose dimension is $2n - 2$ or more. There are at least two fundamental reasons for wanting positive weights. On the one hand, as we have pointed out, to guarantee stability and on the other hand, as we will shown later on, $I_n(f)$ can be considered as a Riemann-Stieltjes sum for $I(f)$, allowing us to enlarge as much as possible the class of integrands where the convergence of $\{I_n(f)\}_{n\geq 1}$ to $I(f)$ can be assured. In this respect, we have the following characterization theorem.

Theorem 2.2 (Jacobi) *([2, Theorem 2.6]) Let $W_n(z) = \prod_{k=1}^n (z - x_k)$ be a nodal polynomial with n distinct zeros $\{x_k\}_{k=1}^n \subset \mathbb{C}\backslash\{0\}$, and such that $W_n(0) \in \mathbb{T}$. Then the quadrature formula given by (1) is exact in $\mathcal{L}_{n-1+s}$, with $1 \leq s \leq n$, if and only if,*

1. *$I_n(f)$ is exact in $\mathcal{L}_{n-1}$.*
2. *The nodal polynomial W_n satisfies*

$$\langle W_n, P\rangle = 0, \qquad \forall\, P \in \begin{cases} \mathrm{span}\{z^{\ell+1}, \ldots, z^{n-(\ell+1)}\}, & \text{if } n - s = 2\ell + 1, \\ \mathrm{span}\{z^{\ell+1}, \ldots, z^{n-(\ell+1)}, z^{n-\ell} + \frac{\omega_\ell}{W_n(0)} z^\ell\}, & \text{if } n - s = 2\ell, \end{cases} \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $L_\mu^2 := L_\mu^2([a, b])$, the Hilbert space of μ-square integrable function on $[a, b]$, $a = e^{i\theta_a}$ and $b = e^{i\theta_b}$, i.e.,

$$\langle f, g\rangle = \int_{\theta_a}^{\theta_b} f(e^{i\theta}) \overline{g(e^{i\theta})} \mathrm{d}\mu(\theta), \quad f, g \in L_\mu^2.$$

This theorem says that the nodal polynomial W_n is a quasi-paraorthogonal polynomial of order $2\ell+1$ (see [1]). The paraorthogonal polynomials correspond to $\ell = 0$. Furthermore, we have

Theorem 2.3 *[1, Theorem 3.2]. Note that a quasi-paraorthogonal polynomial of degree $n \geq 2$ and order $2\ell + 1$, $0 \leq \ell \leq \lfloor \frac{n}{2} \rfloor - 1$, with respect to a finite positive Borel measure supported on $[a, b] \subset \mathbb{T}$, has at least $n - 2\ell$ zeros of odd multiplicity on $\mathbb{T}$ and $n - 2\ell - 1$ are in (a, b).*

In the sequel it will be important to guarantee that all these nodes are simple and in $[a, b]$.

3 Paraorthogonal polynomial of minimum norm

Given a finite positive Borel measure μ on the arc $[a, b]$, a paraorthogonal polynomial B_n of degree n is orthogonal to $\text{span}\{z, z^2, \ldots, z^{n-1}\}$. Up to normalization, it can be characterized by its minimal-norm property. More precisely, B_n is the unique monic polynomial of degree n with $B_n(0) \in \mathbb{T}$ preassigned if only if

$$\|B_n\| \leq \|z^n + zq(z) + B_n(0)\|, \quad \forall q \in \mathbb{P}_{n-2}, \tag{4}$$

where $\|\cdot\|$ is the L^2_μ norm. We are interested in the minimum norm, that is, $\inf_{B_n(0)\in\mathbb{T}} \|B_n\|$ when the corresponding Schur parameters $\{\delta_k\}_{k\in\mathbb{N}}$ are non-vanishing, $\delta_k \neq 0$, $k \in \mathbb{N}$. We define

$$F(B_n(0)) = \int_{\theta_a}^{\theta_b} |B_n(e^{i\theta})|^2 \mathrm{d}\mu(\theta),$$

where B_n is monic. We know, see [4], that

$$B_n(z) = z\rho_{n-1}(z) + B_n(0)\rho^*_{n-1}(z).$$

Therefore

$$F(B_n(0)) = 2\|\rho_{n-1}\|^2(1 - \Re(\delta_n\overline{B_n(0)})), \tag{5}$$

where $\delta_n = \rho_n(0)$ and thus

$$\min_{B_n(0)\in\mathbb{T}} F(B_n(0)) = 2\|\rho_{n-1}\|^2 \min_{B_n(0)\in\mathbb{T}} (1 - \Re(\delta_n\overline{B_n(0)})).$$

If $B_n(0) = e^{i\tau}$ and $\delta_n = |\delta_n|e^{i\theta_n}$, then $\Re(\delta_n\overline{B_n(0)}) = |\delta_n|\cos(\theta_n - \tau)$, $\tau \in (-\pi, \pi]$. Because $\|\rho_n\| = \|\rho^*_n\|$ and ρ^*_n is orthogonal to $\text{span}\{z, z^2, \ldots, z^n\}$, it is easy to see that the minimum is $F(\frac{\delta_n}{|\delta_n|}) = 2\|\rho_{n-1}\|^2(1 - |\delta_n|)$ and the paraorthogonal polynomial with minimal norm is

$$\tilde{B}_n(z) = z\rho_{n-1}(z) + \frac{\delta_n}{|\delta_n|}\rho^*_{n-1}(z).$$

This minimal norm property will be important in this paper dealing with quadrature on the unit circle like it is for quadrature on the real line, but note that on the unit circle, unlike the real line situation, there is also a paraorthogonal polynomial with maximal norm property:

$$\hat{B}_n(z) = z\rho_{n-1}(z) - \frac{\delta_n}{|\delta_n|}\rho^*_{n-1}(z)$$

because $F(-\frac{\delta_n}{|\delta_n|}) = 2\|\rho_{n-1}\|^2(1+|\delta_n|)$, is the maximal value of $F(z)$, $z \in \mathbb{T}$. Furthermore, there is another close link between them.

Theorem 3.1 *The paraorthogonal polynomials $\tilde{B}_n$ and $\hat{B}_n$ with minimal and maximal norm are orthogonal*

Proof This follows by direct computation:

$$\begin{aligned}\langle \tilde{B}_n, \hat{B}_n\rangle &= \langle z\rho_{n-1}(z) + \frac{\delta_n}{|\delta_n|}\rho^*_{n-1}(z), z\rho_{n-1}(z) - \frac{\delta_n}{|\delta_n|}\rho^*_{n-1}(z)\rangle \\ &= \|\rho_{n-1}\|^2 - \|\rho_{n-1}\|^2 - \frac{\overline{\delta}_n}{|\delta_n|}\langle z\rho_{n-1}, \rho^*_{n-1}\rangle + \frac{\delta_n}{|\delta_n|}\overline{\langle z\rho_{n-1}, \rho^*_{n-1}\rangle} \\ &= -\frac{\overline{\delta}_n}{|\delta_n|}(-\delta_n\|\rho_{n-1}\|^2) + \frac{\delta_n}{|\delta_n|}(-\overline{\delta}_n\|\rho_{n-1}\|^2) = 0,\end{aligned}$$

which concludes the proof. □

Analogously to the real case, it would be most interesting to obtain the relevant property that $\tilde{B}_n$ has all its simple zeros on (a,b). In general, we can only guarantee that $\tilde{B}_n$ has $n-1$ of its zeros in (a,b), see Theorem 2.3. However, by [1, Theorem 3.4], if $-\frac{\delta_n}{|\delta_n|} \in (F_n(a), F_n(b))$, $\tilde{B}_n$ has all its simple zeros in (a,b), where $F_n(z) = \frac{z\rho_{n-1}(z)}{\rho^*_{n-1}(z)}$. In this section we will verify that $\tilde{B}_n$ has all its simple zeros in (a,b) when μ is absolutely continuous with respect to the Lebesgue measure, such that the associated weight function belonging to the class of Geronimus weight functions. Specifically,

$$\mu'(\theta) = \frac{\sqrt{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta+\alpha}{2}}}{\sin\frac{\theta-\beta}{2}}, \quad \begin{cases} \theta \in (\alpha, 2\pi-\alpha), \\ \beta = 2\arg(1-\delta) \in (-\alpha, \alpha), \\ \sin(\frac{\alpha}{2}) = |\delta|, \end{cases} \tag{6}$$

where $\delta = \rho_n(0) \in \mathbb{G} = \{z \in \mathbb{C} : |z - \frac{1}{2}| < \frac{1}{2}\}$, for all $n \in \mathbb{N}$. Its support is $\Gamma = [a, \overline{a}]$ with $a = e^{i\alpha}$. Although for the Geronimus measure, the support is always of the form $[a, \overline{a}]$, we will continue using $[a,b]$, since it is more general, it is silently understood that for a Geronimus measure $b = \overline{a}$. The discussion of the properties of these measures is technical and we have moved it to Addendum A.3. The main conclusions for our goal are the following results.

Theorem 3.2 *Let μ be a Geronimus measure with support $[a,b]$, then for all $n \in \mathbb{N}$, the zeros of the paraorthogonal polynomials $\tilde{B}_n(z) = z\rho_{n-1}(z) + d\rho^*_{n-1}(z)$ with $d = \frac{\delta}{|\delta|}$ are simple and they are in (a,b).*

Proof The zeros of $\tilde{B}_n$ satisfy $F_n(z) = -d$, and since $-d \in (F_n(a), F_n(b))$ for all $n \in \mathbb{N}$, see Remark A.1, they are given by the preimages $F_n^{-1}(-d) \subset (a,b)$. □

Theorem 3.3 *Let μ be a Geronimus measure with support $[a,b]$ with $\delta \in \mathbb{G}$ such that $|\delta| \geq \frac{1}{2}$, then the paraorthogonal polynomials $\hat{B}_n(z) = z\rho_{n-1}(z) - d\rho^*_{n-1}(z)$ with $d = \frac{\delta}{|\delta|}$ have $n-1$ zeros in the support (a,b) and one in $[b,a]$. For any other $\delta \in \mathbb{G}$, there exists an integer $n_0 = n_0(\delta)$ such that for all $n \geq n_0$, the zeros of $\hat{B}_n$ have the same property.*

Proof The zeros of $\hat{B}_n$ are given as the preimages $F_n^{-1}(d)$, and since $d \in [F_n(b), F_n(a)]$ for all $\delta \in \mathbb{G}$ with $|\delta| \geq \frac{1}{2}$ then from Remark A.1, $\hat{B}_n$ has $n-1$ zeros in the support (a,b) and one in $[b,a]$. For any other $\delta \in \mathbb{G}$, there exists an $n_0 = n_0(\delta)$ such that the zeros of $\hat{B}_n$ have the same property if $n > n_0$. □

4 Szegő quadrature formulas

Our purpose in this section is to analyze under what circumstances there is an exact quadrature formula in a space of dimension $2n$, which is reasonable since we have $2n$ parameters available: the n nodes and their corresponding weights. As a consequence of Theorem 2.2 with $s = n$, we obtain the following.

Theorem 4.1 *Let $I_n(f)$ be the quadrature formula given by (1). It is exact in $\mathcal{L}_{2n-1}(\omega)$ if and only if*

1. *$I_n(f)$ is exact in $\mathcal{L}_{n-1}$.*
2. *The nodal polynomial $W_n(z) = \prod_{k=1}^n (z - x_k)$ satisfies*

$$\left\langle W_n, z^n + zq(z) + \frac{\omega}{W_n(0)} \right\rangle = 0, \quad \forall q \in \mathbb{P}_{n-2}. \tag{7}$$

From a constructive point of view, we have to make sure that the nodal polynomial W_n has exactly n distinct nodes on the arc $[a,b]$. The following result gives a first positive answer.

Theorem 4.2 *Let μ be a Geronimus measure with support $[a,b]$, then there exists a unique quadrature formula $I_n(f)$ of the form (1) that is exact in $\mathcal{L}_{2n-1}(-d^2)$. Its nodal polynomial is $\tilde{B}_n$.*

Proof It is a direct consequence of the previous Theorem 4.1, Theorem 3.1, and Theorem 3.2, since $\tilde{B}_n$ is orthogonal to $\hat{B}_n$ and $-d = \omega\overline{d}$. □

The corresponding quadrature formula is called Szegő quadrature and we will denote it by $I_n^S(f)$. The weights of $I_n^S(f)$ are positive by Corollary 2.1 and its nodes satisfy the following interlacing property:

Theorem 4.3 *For each n let $\{x_{j,n}^S\}_{j=1}^n$ denote the nodes of the n-point Szegő quadrature $I_n^S(f)$. Then,*

$$a \prec x_{i,n+1}^S \prec x_{i,n}^S \prec x_{i+1,n+1}^S \prec b, \quad i = 1, \dots, n.$$

Proof Let's compare the zeros of $G_n(z) + d$ and $zG_n(z) + d$ where $G_n(z) = \frac{\rho_n(z)}{\rho_n^*(z)}$. Then

$$\frac{1 + zG_n(z)\overline{d}}{(1-z)(1+G_n(z)\overline{d})} = \frac{1}{2}\left(\frac{1 - G_n(z)\overline{d}}{1+G_n(z)\overline{d}} + \frac{1+z}{1-z}\right).$$

Both terms in the right-hand side are of the form $\frac{1-S(z)}{1+S(z)}$ with $S(z)$ a finite Blaschke product with all its zeros on $\mathbb{D}$. Therefore they are rational Carathéodory functions meaning that they are analytic in $\mathbb{D}$ with positive real part in $\mathbb{D}$ and they vanish on $\mathbb{T}\backslash\{\text{poles}\}$. As the sum of two such functions is again a Carathéodory function, the rational function on the left-hand side is a Carathéodory function too. Then its zeros and poles interlace (see for example [3, Theorem 1]). As $1 \in (b, a)$, we have our interlacing property because the zeros of $zG_n(z) + d$ coincide with zeros of $G_{n+1}(z) + d$, since $\varphi_\delta(zG_n(z)) = G_{n+1}(z)$ and $\varphi_\delta(-d) = -d$. □

Theorem 4.4 (Chebyshev inequality) *Let $I_n^S(f) = \sum_{k=1}^n f(x_{k,n}^S)\lambda_{k,n}^S$ n-point Szegő quadrature with nodes $x_{k,n} = e^{i\theta_{k,n}}$ and define*

$$V(\theta) = \int_{\theta_a}^{\theta} \mathrm{d}\mu(\vartheta).$$

Then, there exist $\phi_{k,n} \in (\theta_{k,n}, \theta_{k+1,n})$ such that $V(\phi_{k,n}) = \sum_{j=1}^k \lambda_{j,n}^S$ and

$$V(\theta_{k,n}) < V(\phi_{k,n}) = \sum_{j=1}^{k} \lambda_{j,n}^S < V(\theta_{k+1,n}), \quad k = 1, \dots, n-1.$$

Proof The proof is an adaptation of the one provided by Szegő [5, Section 3.412] to the case of Laurent polynomials. We know

$$\int_{\theta_a}^{\theta_b} L(e^{i\theta})\mathrm{d}\mu(\theta) = \sum_{k=1}^{n} L(x_{k,n}^S)\lambda_{k,n}^S, \quad \forall L \in \mathcal{L}_{2(n-1)}. \tag{8}$$

We introduce the distribution functions

$$V_n(\theta) = \begin{cases} 0, & \theta_a \le \theta < \theta_{1,n}, \\ \lambda_{1,n}^S, & \theta_{1,n} \le \theta < \theta_{2,n}, \\ \lambda_{1,n}^S + \lambda_{2,n}^S, & \theta_{2,n} \le \theta < \theta_{3,n}, \\ \vdots & \vdots \\ \sum_{k=1}^{n-1} \lambda_{k,n}^S, & \theta_{n-1,n} \le \theta < \theta_{n,n}, \\ \sum_{k=1}^{n} \lambda_{k,n}^S, & \theta_{n,n} \le \theta \le \theta_b, \end{cases}$$

and define $S = V - V_n$, then from (8),

$$\int_{\theta_a}^{\theta_b} L(e^{i\theta}) \mathrm{d}S(\theta) = 0, \quad \forall L \in \mathcal{L}_{2(n-1)}.$$

Note that $S(\theta)$ is a non-decreasing and continuous function on each interval $(\theta_a, \theta_{1,n}), (\theta_{n,n}, \theta_b)$, $(\theta_{j,n}, \theta_{j+1,n})$, $j = 1, \ldots, n-1$ because V_n is constant in these intervals. In addition, $S(\theta)$ is positive on $(\theta_a, \theta_{1,n})$ and negative on $(\theta_{n,n}, \theta_b)$, then there must be at least one point $\zeta \in [\theta_{1,n}, \theta_{n,n}]$ where $S(\zeta + \epsilon) < 0 < S(\zeta - \epsilon)$. Such a point must coincide with one of the $\theta_{k,n}$ since S is monotonically increasing outside those points. Thus there can be at most $\ell \leq n$ such points $\{\theta_{k_j,n}\}_{j=1}^{\ell}$ where S changes sign from positive to negative. We note first that $\mathrm{d}S(\theta)$ can be written as

$$\mathrm{d}S(\theta) = \mathrm{d}h(\theta) + \sum_{k=1}^{n} m_k \delta_{\theta_{k,n}}(\theta), \quad m_k \in \mathbb{R},$$

where $h(\theta)$ is an increasing function and by our assumption, $m_j < 0$ for $j \in N = \{k_1, k_2, \ldots, k_\ell\}$, $\ell \leq n$. We are going to prove that $\ell = n$, i.e., that $m_k < 0$, for all $k = 1, \ldots, n$. Define the polynomial of degree $n - \ell \leq n - 1$

$$P(z) = \prod_{s=1, s \notin N}^{n} (z - e^{i\theta_{s,n}}),$$

then $|P(z)|^2 \in \mathcal{L}_{2(n-1)}$ and thus

$$0 = \int_{\theta_a}^{\theta_b} |P(z)|^2 \mathrm{d}S(\theta) = \int_{\theta_a}^{\theta_b} |P(z)|^2 \mathrm{d}h(\theta) + \sum_{s=1, s \notin N}^{n} m_s |P(e^{i\theta_{s,n}})|^2 > 0.$$

This is a contradiction and thus $\ell = n$. Hence we have that

$$S(\theta_{k,n} + \epsilon) < 0 < S(\theta_{k,n} - \epsilon), \quad k = 1, \ldots, n,$$

or equivalently (assuming $\sum_{k=1}^{0} \cdot = 0$),

$$V(\theta_{j,n}) - \sum_{k=1}^{j} \lambda_{k,n}^S < 0 < V(\theta_{j,n}) - \sum_{k=1}^{j-1} \lambda_{k,n}^S, \quad j = 1, \ldots, n,$$

and in particular,

$$\sum_{k=1}^{j-1} \lambda_{k,n}^S < V(\theta_{j,n}) < \sum_{k=1}^{j} \lambda_{k,n}^S, \quad j = 1, \ldots, n,$$

and

$$V(\theta_{j,n}) < \sum_{k=1}^{j} \lambda_{k,n}^S < V(\theta_{j+1,n}), \quad j = 1, \ldots, n-1.$$

Finally, as $S(\theta)$ is non-decreasing and continuous function in each interval $(\theta_{j,n}, \theta_{j+1,n})$, as $S(\theta_{j,n} + \epsilon) < 0$ and $S(\theta_{j+1,n} - \epsilon) > 0$, there exists a point $\phi_{j,n} \in (\theta_{j,n}, \theta_{j+1,n})$ where $S(\phi_{j,n}) = 0$. Thus

$$V(\phi_{j,n}) = \sum_{k=1}^{j} \lambda_{k,n}^S, \quad j = 1, \ldots, n-1. \tag{9}$$

This concludes the proof. □

Theorem 4.5 *Let $I_n^S(f) = \sum_{j=1}^{n-1} \lambda_{j,n}^S f(x_{j,n})$, $n = 1, 2, \ldots$ be the sequence of Szegő formulas. Then $\lim_{n\to\infty} I_n(f) = I(f)$ for any function f that is Riemann-Stieltjes integrable with respect to μ.*

Proof We know $\lim_{n\to\infty} \lambda_{j,n}^S = 0$ uniformly in j, since $\sum_{j=1}^{n} \lambda_{j,n}^S = \mu_0$. We take any f that is Riemann-Stieltjes integrable with respect to μ. As $V(\theta) = \int_{\theta_a}^{\theta} \mathrm{d}\mu(\theta)$ is strictly increasing in (θ_a, θ_b) and $V(\theta_b) = \int_{\theta_a}^{\theta_b} \mathrm{d}\mu(\theta) = \mu_0$. We define $\sigma_{j,n} = V(\theta_{j,n})$, $j = 1, \ldots, n$, $n \in \mathbb{N}$. Then, by Theorem 4.4, it is seen that

$$0 < \sigma_{1,n} < \lambda_{1,n}^S < \sigma_{2,n} < \lambda_{1,n}^S + \lambda_{2,n}^S < \sigma_{3,n} < \lambda_{1,n}^S + \lambda_{2,n}^S + \lambda_{3,n}^S < \cdots$$
$$\cdots < \sum_{j=1}^{n-1} \lambda_{j,n}^S < \sigma_{n,n} < \sum_{j=1}^{n} \lambda_{j,n}^S = \mu_0.$$

By the previous Theorem there are $\phi_{j,n} \in (\theta_{j,n}, \theta_{j+1,n})$ such that

$$\tilde{\sigma}_{j,n} := \lambda_{1,n}^S + \lambda_{2,n}^S + \cdots + \lambda_{j,n}^S = V(\phi_{j,n}), \quad j = 1, \ldots, n-1.$$

Set $\tilde{\sigma}_{0,n} = 0$ and $\tilde{\sigma}_{n,n} = \sum_{k=1}^{n} \lambda_{k,n}^S = \mu_0$ then for all $j = 1, 2, \ldots, n$, $\tilde{\sigma}_{j,n} - \tilde{\sigma}_{j-1,n} = \lambda_{j,n}^S \to 0$ uniformly as $n \to \infty$. Now the function $\tilde{f}(u) = f(e^{iV^{-1}(u)})$ is Riemann integrable on $[0, \mu_0]$, and since $\sum_{j=1}^{n} \tilde{f}(\sigma_{j,n})(\tilde{\sigma}_{j,n} - \tilde{\sigma}_{j-1,n})$ is a Riemann sum on $[0, \mu_0]$, we can write

$$I(f) = \int_{\theta_a}^{\theta_b} f(e^{i\theta}) \mathrm{d}\mu(\theta) = \int_0^{\mu_0} \tilde{f}(y) \mathrm{d}y$$
$$= \lim_{n\to\infty} \sum_{j=1}^{n} \tilde{f}(\sigma_{j,n})(\tilde{\sigma}_{j,n} - \tilde{\sigma}_{j-1,n}) = \lim_{n\to\infty} \sum_{j=1}^{n} \lambda_{j,n}^S f(x_{j,n}) = \lim_{n\to\infty} I_n(f).$$

This concludes the proof. □

Remark 4.1 Observe that in the previous result improper Riemann-Stietjes functions are also being considered, that is, with integrable singularities.

Moreover, we can give information about the quality of convergence in terms of the regularity of the integrand. We consider $\mathfrak{L}(k, \gamma)$, the space of functions $g(\theta) = f(e^{i\theta})$, $\theta \in (\theta_a, \theta_b)$, whose k-th derivatives with respect to θ are Lipschitz continuous of order $0 < \gamma < 1$, see [6, Theorem 1].

Theorem 4.6 (Jackson-type) *Let $f \in \mathfrak{L}(k, \gamma)$. Then, exist a Laurent polynomial of the form $L_{2n} \in \mathcal{L}_{2n}$ such that*

$$|f(z) - L_{2n}(z)| \leq \frac{B}{n^{k+\gamma}}, \quad \forall z \in [a, b], \quad B > 0.$$

Let $\mathcal{C}([a, b])$ be the space of all real-valued continuous functions on $[a, b]$, then the best (uniform) approximation of a given function $f \in \mathcal{C}([a, b])$, by elements from $\mathcal{L}_n$, will play a key role. Define

$$E_n(f) := \inf_{L \in \mathcal{L}_n} \|f - L\|_\infty.$$

If we denote by $R_n(f)$ the remainder term, that is $I(f) = I_n^S(f) + R_n(f)$, we see that for $L \in \mathcal{L}_{2(n-1)}$

$$\begin{aligned} R_n(f) &= I(f) - I_n^S(f) = \big(I(f) - I(L)\big) - \big(I_n^S(f) - I_n(L)\big) \\ &= I\,(f - L) - I_n^S(f - L) \\ &= I(f - L) - \textstyle\sum_{k=1}^{n} \lambda_{k,n}^S \Big(f(x_{k,n}^S) - L(x_{k,n}^S)\Big). \end{aligned}$$

So, taking modulus and uniform norm, and using $\sum_{k=1}^{n} \lambda_{k,n}^S = \mu_0$, we get

$$|R_n(f)| \leq 2\mu_0 E_{2(n-1)}(f). \tag{10}$$

By Theorem 4.6 we conclude the following result

Corollary 4.1 *Let $f \in \mathfrak{L}(k, \gamma)$. Then, $|R_n(f)| \leq \frac{C}{(n-1)^{k+\gamma}}$ for some $C > 0$.*

5 Szegő-Radau quadrature formulas

In this section we will be mainly concerned with the existence of n-point rules where a node has been previously fixed. More precisely, given $n > 1$, we will try to find a quadrature formula

$$I_n^R(f) = \sum_{j=1}^{n} \lambda_{j,n} f(x_{j,n}) \tag{11}$$

with n distinct nodes $x_{1,n}, x_{2,n}, \ldots, x_{n,n}$ in $[a, b]$, and n weights $\lambda_{1,n}, \lambda_{2,n}, \ldots, \lambda_{n,n}$, where either $x_{1,n} = a$ or $x_{n,n} = b$ is fixed while all the others nodes are inside (a, b). So there are $2n - 1$ free parameters left which allow us to make the quadrature formula exact for any function f in the subspace $\mathcal{L}_{2n-2}$ of dimension $2n - 1$. Its characterization is a consequence of Theorem 2.2 with $s = n - 1$, we obtain the following.

Theorem 5.1 *The quadrature formula $I_n(f)$ of the form (11) is exact in $\mathcal{L}_{2n-2}$ if and only if,*

1. *$I_n(f)$ is exact in $\mathcal{L}_{n-1}$.*
2. *The nodal polynomial $W_n(z) = \prod_{k=1}^{n}(z - x_{k,n})$ satisfies*

$$\langle W_n, zq \rangle = 0, \quad \forall q \in \mathbb{P}_{n-2}. \tag{12}$$

We must assure that the nodal polynomial W_n has exactly n distinct nodes on $[a, b]$ one of them being at a or at b. Note that W_n is a paraorthogonal polynomial. The existence is guaranteed by [1, Theorem 3.4].

Theorem 5.2 *For $n \geq 2$, the quadrature formula $I_n(f)$ given by (11) whose nodal polynomial is*

$$B_n(z) = z\rho_{n-1}(z) - \frac{x_R\rho_{n-1}(x_R)}{\rho^*_{n-1}(x_R)}\rho^*_{n-1}(z),$$

is exact in $\mathcal{L}_{2n-2}$ and $x_R \in \{a, b\}$ is one of the nodes while the others are distinct and on (a, b).

This quadrature formula is called Left Szegő-Radau if $x_{1,n} = a$ and Right Szegő-Radau quadrature formula if $x_{n,n} = b$.

Clearly, the weights of $I^R_n(f)$ are positive by Corollary 2.1 and its nodes satisfy the following interlacing property, see [3]:

Theorem 5.3 *For $n \in \mathbb{N}$, let $\{x^{LR}_{j,n}\}_{j=2}^{n}$ with $x^{LR}_{1,n} = a$ denote the nodes of Left Szegő-Radau quadrature formula and let $\{x^{RR}_{j,n}\}_{j=1}^{n-1}$ with $x^{RR}_{n,n} = b$ be the nodes of the Right Szegő-Radau quadrature formula. Then,*

$$x^{LR}_{i,n+1} \prec x^{LR}_{i,n} \prec x^{LR}_{i+1,n+1}, \quad i = 2, \ldots, n,$$

and

$$x^{RR}_{i,n+1} \prec x^{RR}_{i,n} \prec x^{RR}_{i+1,n+1}, \quad i = 1, \ldots, n-1,$$

respectively.

Theorem 5.4 (Chebyshev inequality) *Let $n \geq 2$ and let $I^{LR}_n(f) = \sum_{k=1}^{n} f(x^{LR}_{k,n})\lambda^{LR}_{k,n}$ be the n-point Left Szegő-Radau quadrature with nodes $x^{LR}_{k,n} = e^{i\theta_{k,n}}$, $k = 1, \ldots, n$, and $x^{LR}_{1,n} = a$. Then using the definition*

$$V(\theta) = \int_{\theta_a}^{\theta} \mathrm{d}\mu(\vartheta),$$

there exist $\phi_{k,n} \in (\theta_{k,n}, \theta_{k+1,n})$ such that $V(\phi_{k,n}) = \sum_{j=1}^{k} \lambda^{LR}_{j,n}$ and

$$V(\theta_{k,n}) < V(\phi_{k,n}) < V(\theta_{k+1,n}), \quad k = 1, 2, \ldots, n-1.$$

Similarly, let $I^{RR}_n(f) = \sum_{k=1}^{n} f(x^{RR}_{k,n})\lambda^{RR}_{k,n}$ be the n-point Right Szegő-Radau quadrature with nodes $x^{RR}_{k,n} = e^{i\theta_{k,n}}$, $k = 1, \ldots, n$, and $x^{RR}_{n,n} = b$. Then there exist $\phi_{k,n} \in (\theta_{k,n}, \theta_{k+1,n})$ such that $V(\phi_{k,n}) = \sum_{j=1}^{k} \lambda^{RR}_{j,n}$ and

$$V(\theta_{k,n}) < V(\phi_{k,n}) < V(\theta_{k+1,n}), \quad k = 1, 2, \ldots, n-1.$$

Proof For the n-point Szegő-Radau quadrature we know that

$$\int_{\theta_a}^{\theta_b} L(e^{i\theta})\mathrm{d}\mu(\theta) = \sum_{k=1}^{n} L(x^R_{k,n})\lambda^R_{k,n}, \quad \forall L \in \mathcal{L}_{2(n-1)}. \tag{13}$$

For the Left Radau case we define

$$V_n(\theta) = \begin{cases} 0, & \theta_a \le \theta < \theta_{1,n}^a, \\ \lambda_{1,n}^{LR}, & \theta_{1,n}^a \le \theta < \theta_{2,n}, \\ \lambda_{1,n}^{LR} + \lambda_{2,n}^{LR}, & \theta_{2,n} \le \theta < \theta_{3,n}, \\ \vdots & \vdots \\ \sum_{k=1}^{n} \lambda_{k,n}^{LR}, & \theta_{n,n} \le \theta \le \theta_b, \end{cases}$$

where $\theta_{1,n}^a$ is such that $V(\theta_{1,n}^a) < \lambda_{1,n}^L$. Because $S(\theta) := V(\theta) - V_n(\theta)$ is positive in $(\theta_a, \theta_{1,n}^a)$ and negative in $(\theta_{n,n}, \theta_b)$, S drops from positive to negative in at least one and at most n points in the interval $[\theta_{1,n}^a, \theta_{n,n}]$. The rest of the proof is like in Theorem 4.4.

For the Right Szegő-Radau quadrature we proceed similarly. By introducing the auxiliary point $\theta_{n,n}^b$ near θ_b and we define $V_n(\theta) = \sum_{k=1}^{n} \lambda_{k,n}^{RR}$, $\theta \in (\theta_{n,n}^b, \theta_b)$, we can conclude again that there are again at least one and at most n points in the interval $[\theta_{1,n}, \theta_{n,n}]$ where S changes sign from positive to negative and we can again apply the proof of Theorem 4.4. □

Theorem 5.5 *Let $\{I_n^R(f)\}_{n\ge1}$ denote the sequence of Szegő-Radau formulas. Then, it holds that,*

$$\lim_{n\to\infty} I_n^R(f) = I(f)$$

for any function f which is Riemann-Stieljes integrable with respect to μ.

Proof The proof proceeds essentially the same as in the proof of Theorem 4.5, with the appropriate adaptations. □

Corollary 5.1 *Let $f \in \mathfrak{L}(k,\gamma)$ and $R_n^R(f)$ is the error in the approximation of Szegő- Radau quadrature formula. Then, $\left|R_n^R(f)\right| \le \frac{A}{(n-1)^{k+\gamma}}$ for some $A > 0$.*

Remark 5.1 Note that this section is valid for any finite positive Borel measure μ supported on the arc $[a, b]$.

6 Szegő-Lobatto quadrature formulas

The main aim of this Section is the construction of quadrature rules

$$I_n(f) = \sum_{j=1}^{n} \lambda_{j,n} f(x_{j,n}) \tag{14}$$

with two preassigned nodes at end points $x_{1,n} = a$ and $x_{n,n} = b$ of the integration arc $[a, b]$. Then, we will try to find the $n-2$ remaining mutually

distinct nodes in (a, b) and the n weights, so that the n-point rule satisfies $I(L) = I_n(L)$ for any L in $\mathcal{L}_{2n-3}(\omega)$ since in (14) we have $2n-2$ free parameters at our disposal. If such quadrature formula exists, it will be called an n-point Szegő-Lobatto quadrature. Concerning the positivity of the weights, it should be observed that, as a consequence of Theorem 2.1, we can assure that only at least $n-1$ weights are positive. As for the construction of the above formula, we must make use of the following characterization, see Theorem 2.2 with $s = n-2$.

Theorem 6.1 *The quadrature formula $I_n(f)$ given by (14) is exact in $\mathcal{L}_{2n-3}(\omega)$ if and only if*

1. *$I_n(f)$ is exact in $\mathcal{L}_{n-1}$.*
2. *The nodal polynomial $W_n(z) = \prod_{k=1}^{n}(z - x_{k,n})$ satisfies*

$$\left\langle W_n, z^{n-1} + z^2 q(z) + z\frac{\omega}{W_n(0)} \right\rangle = 0, \quad \forall q \in \mathbb{P}_{n-4}. \tag{15}$$

That is, the nodal polynomial W_n must be a quasi-paraorthogonal of order 3 with simple zeros in (a, b), fixing a and b in advance. However, we know that all quasi-paraorthogonal polynomials of degree n and order 3 have at least $n-3$ zeros of odd multiplicity on (a, b), see Theorem 2.3. The following result gives us a positive answer.

Theorem 6.2 *Let μ be a Geronimus measure with support $[a, b]$. If $\delta \in \mathbb{G}$ such that $|\delta| \geq \frac{1}{2}$, then there exists an $\eta \in \mathbb{D}$ such that the polynomial $B_n(z) = z(z-\eta)\rho_{n-1}(z) + d(1-\overline{\eta}z)\rho^*_{n-1}(z)$ is quasi-paraorthogonal of order 3 with zeros in a and b while the $n-2$ remaining zeros are simple and in (a, b). If $\delta \in \mathbb{G}$ and $|\delta| < \frac{1}{2}$, then there exists an $n_0 = n_0(\delta)$ such that if $n \geq n_0$, then B_n has the same properties.*

Proof We know that every polynomial $(n, 3)$-quasi-paraorthogonal can be expressed in the following way, see [1, equations (5)-(10)]

$$B_n(z) = z(z-\eta)\rho_{n-2}(z) + t(1-\overline{\eta}z)\rho^*_{n-2}(z), \quad t \in \mathbb{T}.$$

If we want to have a and b as zeros, then $\eta = \eta(t)$ must belong to the circle

$$\mathbb{T}_{ab} = \frac{aG(a) - bG(b)}{G(a) - G(b)} - t\frac{a-b}{G(a) - G(b)}, \quad t \in \mathbb{T}, \quad G(z) = \frac{\rho_{n-2}(z)}{\rho^*_{n-2}(z)}.$$

Clearly $\mathbb{T} \cap \mathbb{T}_{ab} = \{a, b\}$, since there exist $t_a = G(b)$ and $t_b = G(a)$ such that

$$a = \frac{aG(a) - bG(b)}{G(a) - G(b)} - t_a\frac{a-b}{G(a) - G(b)} \quad \text{and} \quad b = \frac{aG(a) - bG(b)}{G(a) - G(b)} - t_b\frac{a-b}{G(a) - G(b)}.$$

B_n has its simple zeros on $\mathbb{T}$ being a and b two of them and $n-3$ on (a, b) for all $t \in (G(b), G(a))$. Thus if $\eta = \eta(d)$, then $\eta \in \mathbb{D}$ if and only if $t_a \prec d \prec t_b$ or $G(b) \prec d \prec G(a)$ or $d \in (G(b), G(a))$.

From Lemma A.2 we know that $d \in (G(b), G(a))$ for all $\delta \in \mathbb{G}$ with $|\delta| \geq \frac{1}{2}$ and more generally for any $\delta \in \mathbb{G}$ and for all $n \geq n_0$ for some finite $n_0 = n_0(\delta)$ (see Remark A.1).

We shall now prove that all the zeros of B_n different from a and b are simple and are in (a, b). To do this we compare B_n with $B^S_{n-1}(z) = z\rho_{n-2}(z) + d\rho^*_{n-2}(z)$ that we know has its $n-1$ simple zeros in (a, b) by Theorem 3.2.

$$\begin{aligned} B_n(z) = 0 \quad &\Leftrightarrow \quad \varphi_{-\eta}(z) z G(z) \overline{d} + 1 = 0, \\ B^S_{n-1}(z) = 0 \quad &\Leftrightarrow \quad z G(z) \overline{d} + 1 = 0. \end{aligned}$$

We define the rational function

$$\frac{1 + \varphi_{-\eta}(z)\overline{d} z G(z)}{(1 - \varphi_{-\eta}(z))(1 + \overline{d} z G(z))} = \frac{1}{2}\left(\frac{1 - \overline{d} z G(z)}{1 + \overline{d} z G(z)} + \frac{1 + \varphi_{-\eta}(z)}{1 - \varphi_{-\eta}(z)}\right).$$

We can rewrite the last term as

$$\frac{1 + \varphi_{-\eta}(z)}{1 - \varphi_{-\eta}(z)} = \frac{(1 + \overline{\eta} z) + (z - \eta)}{(1 - \overline{\eta} z) - (z - \eta)} = -\frac{(1 - \eta) + (1 - \overline{\eta}) z}{(1 + \eta) - (1 + \overline{\eta}) z}.$$

The same argument as in the Szegő case can be used, the only difference is that the factor $1 - z$ is here replaced by $1 - \varphi_{-\eta}(z)$ which has a zero $\tilde{\eta} = \frac{1+\eta}{1+\overline{\eta}}$. The zeros and poles of the rational function on the left-hand side interlace (see for example [3, Theorem 1]). Thus by the interlacing property all the zeros of B_n are simple and in $[a, b]$ if $\tilde{\eta} \in (b, a)$. The latter is true, since $\tilde{\eta}$ belongs to the straight line connecting -1 and η, because

$$-1 = t_0 \eta + (1 - t_0)\frac{1 + \eta}{1 + \overline{\eta}} \quad \text{for} \quad t_0 = \frac{2(1 + \Re\, \eta)}{1 - |\eta|^2} \in \mathbb{R}.$$

The condition $\tilde{\eta} \in (b, a)$ is immediate because the chord of the unit circle that determines the arc $\overset{\frown}{a\eta b}$ coincides with the chord that determines (b, a).

Figure 1 illustrates our situation. □

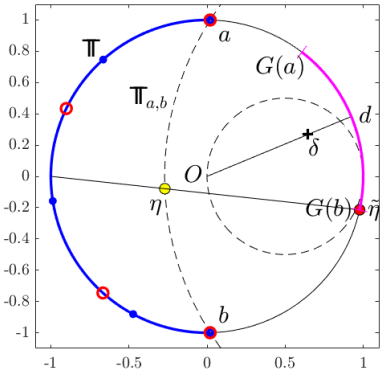


Fig. 1 This is the situation for $\delta = 0.7e^{i\pi/8}$ and $n = 4$. $d = \frac{\delta}{|\delta|} \in (G(b), G(a))$, $\tilde{\eta} = \frac{1+\eta}{1+\overline{\eta}} \in (b, a)$, the blue arc is the support $[a, b]$, the magenta arc is $[G(b), G(a)]$, the blue dots are the zeros of B^S_3 (they are all in (a, b)), the red circles are the zeros of B_4 (they are all in $[a, b]$ and a and b are two of them).

Corollary 6.1 *Let μ be a Geronimus measure with support $[a,b]$. If $\delta \in \mathbb{G}$ and $|\delta| \geq \frac{1}{2}$, then there exists an n-point Szegő-Lobatto quadrature formula with positive weights that is exact in $\mathcal{L}_{2n-3}(\omega)$ where $\omega = -d^2 \frac{|\delta|+\overline{\eta}_n}{|\delta|+\eta_n}$ with $d = \frac{\delta}{|\delta|}$. If $\delta \in \mathbb{G}$ and $|\delta| < \frac{1}{2}$, then this holds for all $n \geq n_0$ for a certain $n_0 = n_0(\delta)$.*

Proof By theorems 6.1 and 6.2, it is sufficient to show that B_n is orthogonal to $z^{n-1} + z^2 q(z) - \tau_n z$ for all $q \in \mathbb{P}_{n-4}$ with $\tau_n = d\frac{|\delta|+\overline{\eta}_n}{|\delta|+\eta_n}$. That can be derived from [1] where the relation between the invariance parameter $B_n(0)$ and the orthogonality parameter τ_n of a general quasi-paraorthogonal polynomial B_n is given. We will give a simplified derivation in Lemma A.3.
It remains to prove that the weights are positive. Note that

$$B_n(z) = z\hat{\rho}_{n-1}(z) + d\hat{\rho}^*_{n-1}(z), \quad \text{with} \quad \hat{\rho}_{n-1}(z) = z\rho_{n-2}(z) - d\overline{\eta}\rho^*_{n-2}(z), \quad d\overline{\eta} \in \mathbb{D}.$$

The Schur parameters $\{\delta_1, \delta_2, \ldots, \delta_{n-2}\}$ haven't changed, by Favard's theorem on the unit circle, we conclude that $\hat{\rho}_{n-1}$ is a monic Szegő polynomial and B_n is a monic paraorthogonal associated with a positive modified measure $\hat{\mu}$, then our $I_n(f)$ is also a Szegő-Radau q.f. associated to $\hat{\mu}$ that coincides with our q.f. in $\mathcal{L}_{2n-3}(\omega)$, and hence it is a positive q.f. □

Remark 6.1 If $\eta_n \in \mathbb{R}$, then Szegő-Lobatto quadrature $I_n^L(f)$ is exact in $\mathcal{L}_{2n-3}(-d^2)$.

Furthermore, they also satisfy the following interlacing property.

Theorem 6.3 *Let μ be a Geronimus measure with support $[a,b]$. If $\delta \in \mathbb{G}$ such that $|\delta| \geq \frac{1}{2}$ and $d = \frac{\delta}{|\delta|}$. Let $\{x_{j,n}^L\}_{j=1}^n$ with $x_{1,n}^L = a$ and $x_{n,n}^L = b$ be the nodes of the n-point Szegő-Lobatto quadrature $I_n^L(f)$. Then*

$$x_{i,n+1}^L \prec x_{i,n}^L \prec x_{i+1,n+1}^L, \quad i = 2, \ldots, n-2.$$

If $\delta \in \mathbb{G}$ and $|\delta| < \frac{1}{2}$, then there exists a finite $n_0 := n_0(\delta)$ such that for all $n \geq n_0$ we obtain the same property.

Proof We shall reduce the proof to the case of Szegő quadrature by showing that the nodes of I_n^L and I_{n-1}^L that differ from a and b are zeros of polynomials P_{n-2} and P_{n-3} respectively that are paraorthogonal with respect to an auxiliary measure.
We know that P_{n-2} defined by $B_n(z) = P_{n-2}(z)(z-a)(z-b)$ with $P_{n-2}(0) = d$ has all its simple zeros on (a,b) and

$$\int_{\theta_a}^{\theta_b} P_{n-2}(z)(z-a)(z-b)\left(\overline{z^{n-1} + z^2 q(z) - \tau_n z}\right) d\mu(\theta) = 0, \quad \forall q \in \mathbb{P}_{n-4}, \quad z = e^{i\theta},$$

with $\tau_n = d\frac{|\delta|+\overline{\eta}_n}{|\delta|+\eta_n}$ (see Lemma A.3). Equivalently,

$$\int_{\theta_a}^{\theta_b} P_{n-2}(z)\left(\overline{z^{n-2} + zq - \tau_n}\right) d\hat{\mu}(\theta) = 0, \quad \forall q \in \mathbb{P}_{n-4}, \quad z = e^{i\theta},$$

where $d\hat{\mu}(\theta) = (z-a)(b-z)\overline{z}d\mu(\theta) = 4\sin\left(\frac{\theta-\theta_a}{2}\right)\sin\left(\frac{\theta_b-\theta}{2}\right)d\mu(\theta)$ for $z = e^{i\theta}$, $\theta_a = \alpha$, $\theta_b = 2\pi - \alpha$. Therefore, $P_{n-2}(z)$ is a paraorthogonal polynomial of degree $n-2$ with respect to the measure $d\hat{\mu}$, then $P_{n-2}(z) = z\hat{\rho}_{n-3} + d\hat{\rho}^*_{n-3}(z) = c_{n-2}\left[\hat{\rho}_{n-2}(z) + \tau_n\hat{\rho}^*_{n-2}(z)\right]$, $\varphi_{-\hat{\rho}_{n-2}(0)}(d) = \tau_n$, where $\{\hat{\rho}_n\}_{n\in\mathbb{N}}$ is the sequence of orthogonal polynomials associated to $d\hat{\mu}(\theta)$. We denote the zeros of P_{n-2} by $\{z_{i,n-2}\}_{i=1}^{n-2}$, where

$$a \prec z_{1,n-2} \prec \cdots \prec z_{2,n-2} \prec z_{n-2,n-2} \prec b.$$

In a similar way, defining P_{n-3} by $B_{n-1}(z) = P_{n-3}(z)(z-a)(z-b)$ with $P_{n-3}(0) = d$, we obtain that P_{n-3} is a paraorthogonal polynomial of degree $n-3$ for $d\hat{\mu}$. Hence we know that all the zeros of P_{n-2} and P_{n-3} are simple zeros and they are in (a,b). Next we have to prove that these zeros interlace. Since

$$P_{n-3}(z) = z\hat{\rho}_{n-4}(z) + d\hat{\rho}^*_{n-4}(z) = c_{n-3}[\hat{\rho}_{n-3}(z) + \tau_{n-1}\hat{\rho}^*_{n-3}(z)],$$

then

$$\begin{aligned} P_{n-2}(z) = 0 &\equiv \overline{d}z\frac{\hat{\rho}_{n-3}(z)}{\hat{\rho}^*_{n-3}(z)} + 1 = 0, \\ P_{n-3}(z) = 0 &\equiv \overline{\tau}_{n-1}\frac{\hat{\rho}_{n-3}(z)}{\hat{\rho}^*_{n-3}(z)} + 1 = 0. \end{aligned}$$

We set $\hat{G}_{n-3} = \frac{\hat{\rho}_{n-3}}{\hat{\rho}^*_{n-3}}$ and define

$$R_{n-2}(z) = \frac{1 + \overline{d}z\hat{G}_{n-3}(z)}{(1-\overline{\lambda}_{n-1}z)(1+\overline{\tau}_{n-1}\hat{G}_{n-3}(z))},$$

with $\lambda_{n-1} = \frac{d}{\tau_{n-1}} = d\overline{\tau}_{n-1}$, then we can deduce that

$$R_{n-2}(z) = \frac{1}{2}\left(\frac{1-\overline{\tau}_{n-1}\hat{G}_{n-3}(z)}{1+\overline{\tau}_{n-1}\hat{G}_{n-3}(z)} + \frac{1+\overline{\lambda}_{n-1}z}{1-\overline{\lambda}_{n-1}z}\right).$$

Now, we proceed analogously to Theorem 4.3, but in this case with $\hat{G}_{n-3}$ instead of G_n, we conclude that the zeros of P_{n-2} and $(z-\lambda_{n-1})P_{n-3}$ are interlacing on $\mathbb{T}$. Then, we conclude that P_{n-2} and P_{n-3} have interlacing zeros on (a,b) if $\lambda_{n-1} \in (z_{n-2,n-2}, z_{1,n-2})$.

Let us prove that $z_{1,n-2} \prec \lambda_{n-1} \prec z_{n-2,n-2}$ leads to a contradiction. As

$$d = -z_{j,n-2}\hat{G}_{n-3}(z_{j,n-2}),\ j = 1,\dots,n-2,$$

and

$$\tau_{n-1} = -\hat{G}_{n-3}(z_{k,n-3}),\quad k = 1,\dots,n-3,$$

it follows that

$$\lambda_{n-1} = z_{1,n-2}\hat{G}_{n-3}(z_{1,n-2})\overline{\hat{G}_{n-3}(z_{1,n-3})},$$

and

$$\lambda_{n-1} = z_{n-2,n-2}\hat{G}_{n-3}(z_{n-2,n-2})\overline{\hat{G}_{n-3}(z_{n-3,n-3})}.$$

Therefore,

$$\begin{aligned} z_{1,n-2} \prec \lambda_{n-1} &\Leftrightarrow \arg(\lambda_{n-1}\overline{z_{1,n-2}}) > 0 \\ &\Leftrightarrow \arg\left(\hat{G}_{n-3}(z_{1,n-2})\overline{\hat{G}_{n-3}(z_{1,n-3})}\right) > 0 \\ &\Leftrightarrow \hat{G}_{n-3}(z_{1,n-3}) \prec \hat{G}_{n-3}(z_{1,n-2}) \\ &\Leftrightarrow z_{1,n-3} \prec z_{1,n-2}, \end{aligned}$$

since the Blaschke product $\hat{G}_{n-3}$ is orientation preserving.

Similarly, from

$$\begin{aligned}\lambda_{n-1} \prec z_{n-2,n-2} &\Leftrightarrow \arg(\lambda_{n-1}\overline{z_{n-2,n-2}}) < 0\\ &\Leftrightarrow \arg\Big(\hat{G}_{n-3}(z_{n-2,n-2})\overline{\hat{G}_{n-3}(z_{n-3,n-3})}\Big) < 0\\ &\Leftrightarrow \hat{G}_{n-3}(z_{n-2,n-2}) \prec \hat{G}_{n-3}(z_{n-3,n-3})\\ &\Leftrightarrow z_{n-2,n-2} \prec z_{n-3,n-3}.\end{aligned}$$

Thus if $z_{1,n-2} \prec \lambda_{n-1} \prec z_{n-2,n-2}$ then we have (see Addendum A.2)

$$z_{1,n-3} \prec z_{1,n-2} \prec z_{2,n-3} \prec \cdots \prec z_{n-4,n-3} \prec z_{n-2,n-2} \prec z_{n-3,n-3}$$

which contradicts the interlacing of the zeros of P_{n-2} and $(z-\lambda_{n-1})P_{n-3}$. This concludes the proof. □

Theorem 6.4 (**Chebyshev inequality**) *Let $n \geq 3$. Let $I_n^L(f) = \sum_{k=1}^n f(x_{k,n}^L)\lambda_{k,n}^L$ be the n-point Szegő-Lobatto quadrature formula with nodes $x_{k,n}^L = e^{i\theta_{k,n}}$, $k = 1, 2, \ldots, n$ where $x_{1,1}^L = a$ and $x_{n,n}^L = b$. Then there exist $\phi_{k,n} \in (\theta_{k,n}, \theta_{k+1,n})$ such that $V(\phi_{k,n}) = \sum_{j=1}^k \lambda_{j,n}^L$ and*

$$0 < V(\theta_{k,n}) < V(\phi_{k,n}) < V(\theta_{k+1,n}), \quad k = 2, \ldots, n-1, \tag{16}$$

where

$$V(\theta) = \int_{\theta_a}^{\theta} \mathrm{d}\mu(\vartheta),$$

and $\phi_{1,n}$ is defined by $V(\phi_{1,n}) = \lambda_{1,n}^L$.

Proof The proof is essentially the same as with Right Szegő-Radau quadrature. First recall that in the Lobatto-Szegő case we know that

$$\int_{\theta_a}^{\theta_b} L(e^{i\theta})\mathrm{d}\mu(\theta) = \sum_{k=1}^{n} L(x_{k,n}^L)\lambda_{k,n}^L, \quad \forall L \in \mathcal{L}_{2(n-2)}. \tag{17}$$

We start by defining the distribution function V_n for the Left version

$$V_n(\theta) = \begin{cases} 0, & \theta_a \leq \theta < \theta_{2,n},\\ \lambda_{1,n}^L + \lambda_{2,n}^L, & \theta_{2,n} \leq \theta < \theta_{3,n},\\ \lambda_{1,n}^L + \lambda_{2,n}^L + \lambda_{3,n}^L, & \theta_{3,n} \leq \theta < \theta_{4,n},\\ \vdots & \vdots\\ \sum_{k=1}^{n-1} \lambda_{k,n}^L, & \theta_{n-1,n} \leq \theta < \theta_{n,n}^b,\\ \sum_{k=1}^{n} \lambda_{k,n}^L, & \theta_{n,n}^b \leq \theta \leq \theta_b, \end{cases}$$

where we are introducing the auxiliary point $\theta_{n,n}^b$ near θ_b. Define $S(\theta) = V(\theta) - V_n(\theta)$, then for the same reasons as in the Right-Radau case, $S(\theta)$ is non-decreasing in each of the open subintervals between two successive points of increase of V_n and $S(\theta)$ is positive in $(\theta_a, \theta_{2,n})$ and negative in $(\theta_{n,n}^b, \theta_b)$. Then there must be at least one sign change from positive to negative at some point in $[\theta_{2,n}, \theta_{n,n}^b]$. Such a point

must coincide with one of the $n-1$ points $\theta_{2,n}, \theta_{3,n}, \ldots, \theta_{n-1,n}, \theta^b_{n,n}$ since S is monotonically increasing outside those points. We can now reproduce the proof of Theorem 5.4 to conclude that there exist $\phi_{k,n} \in (\theta_{k-1,n}, \theta_{k,n})$ such that

$$V(\theta_{k,n}) < V(\phi_{k,n}) = \sum_{j=1}^{k} \lambda^L_{j,n} < V(\theta_{k+1,n}), \quad k = 2, \ldots, n-1.$$

□

Theorem 6.5 *Let $\{I^L_n(f)\}_{n\geq 1}$ denote the sequence of Szegő-Lobatto formulas. Then, it holds that,*

$$\lim_{n\to\infty} I^L_n(f) = I(f)$$

for any function f which is Riemann-Stieljes integrable with respect to $\mathrm{d}\mu$ on (θ_a, θ_b)

Proof The proof is essentially a replica of that of Theorem 5.5, with appropriate adaptations. In particular, using the notation of Theorem 5.5 we conclude from (16) that

$$\begin{aligned} I(f) &= \int_{\theta_a}^{\theta_b} f(e^{i\theta})\mathrm{d}\mu(\theta) = \int_0^{\mu_0} \tilde{f}(y)\mathrm{d}y = \lim_{n\to\infty} \sum_{k=2}^{n} \tilde{f}(\sigma_{j,n})(\tilde{\sigma}_{j,n} - \tilde{\sigma}_{j-1,n}) \\ &= \lim_{n\to\infty} \sum_{k=2}^{n} f(x_{j,n})\lambda^L_{j,n} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_{j,n})\lambda^L_{j,n} = \lim_{n\to\infty} I_n(f), \end{aligned}$$

because $\lim_{n\to\infty} f(a)\lambda^L_{1,n} = 0$. This concludes the proof. □

Corollary 6.2 *Let $f \in \mathfrak{L}(k,\alpha)$ and $R_n(f)^L$ is the error in the approximation of Szegő- Lobatto quadrature formula. Then, $\left|R^L_n(f)\right| \leq \frac{A}{(n-2)^{k+\gamma}}$ for some $A > 0$.*

7 Numerical examples

In this section we give some numerical illustrations of the theory presented in the previous sections. More precisely, we consider the class of Geronimus measures where the parameter δ belongs to the disk $\mathbb{G}$ and support $\Gamma = [a, \overline{a}]$ with $a = e^{i\alpha}$ as defined in (6). All computations were performed in Matlab's double precision. The nodes and weights are computed using the CMV approach of [7] as explained in [1, Section 4].

We first compute the nodes and weights of the Szegő quadrature, see Table 1. They are given for $\delta = 0.5e^{-i\pi/6}$ and $n = 6$, corresponding to $\alpha = \pi/3$ and $\beta \approx 0.8306$. The continuous Geronimus weight and the quadrature nodes and weights are simultaneously represented graphically in Figure 2, where we also show how the nodes for $n = 6$ interlace with the nodes for $n = 7$.

Examples of Szegő-Radau formulas are illustrated in Figure 3 for $n = 6$. The Geronimus parameter is here $\delta = 0.4d$, $d = e^{-i\pi/8}$, corresponding to $\alpha \approx 0.823034$ and $\beta \approx 0.4763832$. Here the selection of the fixed node defines

Table 1 Nodes and weights of the Szegő quadrature formula for the Geronimus class with $\delta = 0.5d$, $d = e^{-i\pi/6}$ and $n = 6$.

nodes $= x_k = e^{i\theta_k}$	weights
θ_k/π	w_k
$3.950298783881980e-01$	$9.308824215457805e-02$
$5.790610857810271e-01$	$1.465542304532152e-01$
$8.288432334637249e-01$	$1.406310219821962e-01$
$1.097612946015048e+00$	$1.187503958552353e-01$
$1.358507950087968e+00$	$8.519389092187264e-02$
$1.574278239597368e+00$	$3.543905609664107e-02$

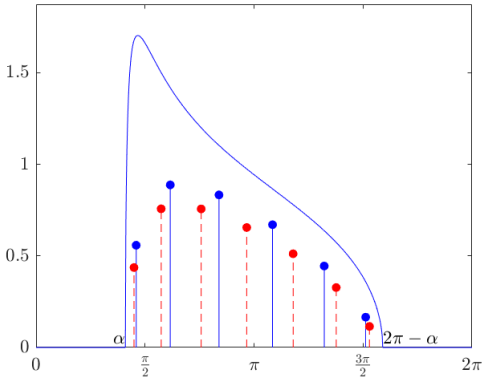


Fig. 2 This illustrates the Geronimus weight for $\delta = 0.5d$ with $d = e^{-i\pi/6}$ so that $\alpha = \frac{\pi}{3}$ and $\beta \approx 0.8306$. together with the nodes and weights of the Szegő quadrature formulas for $n = 6$ (in blue) and for $n = 7$ (in red).

the invariance parameter, so unlike for the Szegő quadrature, it will not be equal to d. The right figure shows the nodes on the unit circle. The blue nodes are the ones corresponding to fixing $b = \overline{a}$ as one of the nodes. The blue arcs are the preimages $F_6^{-1}(F_6([a, b]))$, and it is clear that all the nodes are at the end points of these arcs. The numerical values of nodes and weights of the Right Szegő-Radau formula are given in Table 2.

Finally we consider the Szegő-Lobatto quadratures for the Geronimus measure where $\delta = 0.6d$, $d = e^{i\pi/5}$. The graphical representation of nodes and weights as well as the Geronimus weight is given in Figure 4, the numerical values are in Table 3.

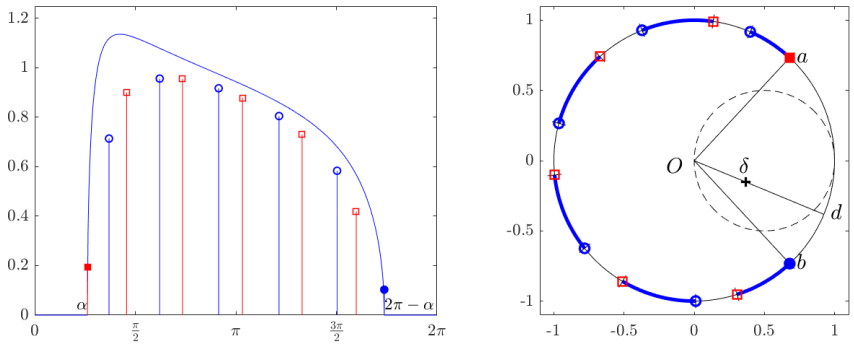


Fig. 3 This shows the 6-point Left and Right Szegő-Radau quadrature for the Geronimus class with $\delta = 0.4d$, $d = e^{-i\pi/8}$ so that $\alpha \approx 0.823034$ and $\beta \approx 0.4763832$. The left figure is analogous to Figure 2. Nodes for the Right Radau formula are marked by small blue circles, and for the Left Radau by small red squares. The right figure shows the same nodes on the unit circle. The blue arcs represent the preimages $F_6^{-1}(F_6([a,b]))$. When a (or b) is prefixed, then the other nodes are given by $F_6^{-1}(F_6(a))$ (or $F_6^{-1}(F_6(b))$).

Table 2 Nodes and weights of the 6-point Right-Szegő-Radau quadrature formula for the Geronimus class with $\delta = 0.4d$, $d = e^{-i\pi/8}$ and $n = 6$ where we fix the end point b as a node. The last node is the preselected b.

nodes $= x_k = e^{i\theta_k}$	weights
θ_k/π	w_k
$3.693195211290171e-01$	$1.135450006450852e-01$
$6.211978843029399e-01$	$1.521041325529260e-01$
$9.144238321357578e-01$	$1.458855295536433e-01$
$1.214610200677930e+00$	$1.280174474730122e-01$
$1.503362805942077e+00$	$9.285039555858803e-02$
$1.738020239131091e+00$	$1.636276300891058e-02$

Table 3 Nodes and weights of the 7-point Szegő-Lobatto quadrature formula for the Geronimus class with $\delta = 0.6e^{i\pi/5}$ and $\delta = 0.5e^{-i\pi/3}$.

$\delta = 0.6\exp(i\pi/5)$	
nodes $= z_k = \exp(i\theta_k)$	weights
θ_k/π	w_k
$4.096655293982665e-01$	$5.221708400285106e-03$
$5.675278973544569e-01$	$5.069412489693022e-02$
$7.971642608912867e-01$	$9.460732995345651e-02$
$1.049440835910682e+00$	$1.293972009937455e-01$
$1.293678216136843e+00$	$1.566907051825573e-01$
$1.492188789706731e+00$	$1.561474452060101e-01$
$1.590334470601734e+00$	$3.108409913274984e-02$

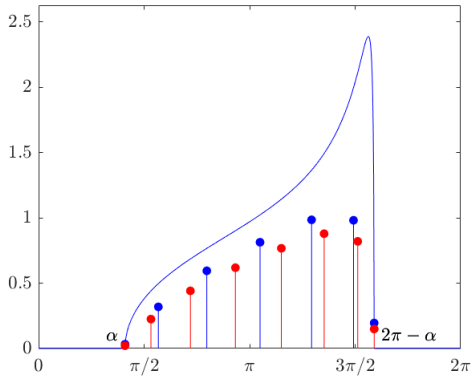


Fig. 4 This shows the 7-point (blue) and 8-point (red) Szegő-Lobatto quadrature for the Geronimus class with $\delta = 0.6d$, $d = e^{i\pi/5}$, $\alpha \approx 1.287002$ and $\beta \approx 5.081541$ where the end points a and $b = \overline{a}$ are fixed to be among the nodes.

The convergence behaviour of the Szegő and Szegő-Radau formulas is very similar. The Szegő-Radau formula seems to be less stable after convergence. It is represented in Figure 5.

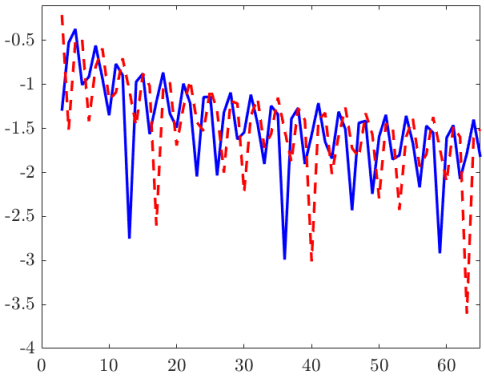


Fig. 5 This shows the 10-logarithm of the relative error of the Szegő (blue) and Left Szegő-Radau (red-dashed) quadrature for $n = 3, 4, \ldots, 65$ when it is used to approximate the integral $I = \int_{\pi/3}^{5\pi/3} \sqrt{e^{i\theta} - a}\, \mathrm{d}\mu(\theta)$. The measure is in the Geronimus class with $\delta = 0.5d$, $d = e^{-i\pi/6}$.

However, numerical rounding errors start contaminating the CMV computations used to compute the eigenvalues (nodes) and the modulus squared of the first components of the corresponding eigenvectors of the unitary Hessenberg matrix (see [1]). This is clearly illustrated by the the behaviour of the 10-logarithm of the absolute error produced by the same routine applied to the highly oscillating function $f(z) = z^{30} + z^{-30}$ which is given in Figure 6. It illustrates that both formulas are exact when $n = 30$, but while the Szegő formula remains at the machine precision, the Szegő-Radau formula deteriorates because of numerical imprecision accumulating rounding errors, which results in nodes deviating slightly from $\mathbb{T}$, and perturbation of the weights.

To test the convergence of the Szegő-Lobatto quadrature, we computed the integral

$$I = \int_{\theta_a}^{\theta_b} f(e^{i\theta}) \mathrm{d}\mu(\theta), \quad f(z) = \sqrt{(z-a)(z-b)},$$

for the Geronimus measure with parameter $\delta = 0.6d$ with $d = e^{i\pi/5}$ corresponding to $\alpha \approx 1.287002$ and $\beta \approx 5.081541$. The value of the integral is $I \approx 0.65456595 + 0.1221633\, i$. The convergence is illustrated in Figure 7. The method converges nicely but slowly. The right plot in Figure 7 is for the integrand $f(z) = z^{30} + 1/z^{30}$ which shows that this is integrated exactly (up to numerical rounding) once $n > 31$.

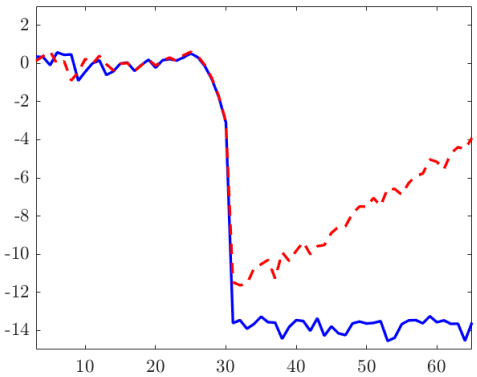


Fig. 6 This shows the 10-logarithm of the absolute error of the Szegő (blue) and Right-Szegő-Radau (dashed red) quadrature for $n = 3, 4, \ldots, 65$ when it is used to approximate the integral $I = \int_{\pi/3}^{5\pi/3} (z^{30} + z^{-30})\, \mathrm{d}\mu(\theta)$. The parameter of the Geronimus class is $\delta = 0.5d$, $d = e^{i\pi/6}$.

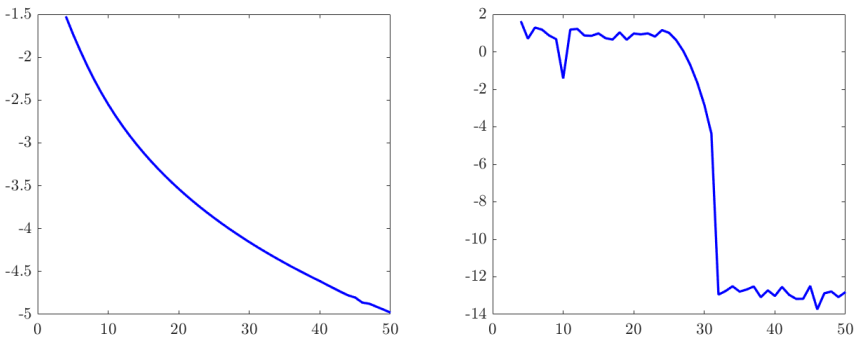


Fig. 7 This shows the 10-logarithm of the relative error of the integral $\int_{\theta_a}^{\theta_b} f(z)\mathrm{d}\mu(\theta)$ of n-point Szegő-Lobatto quadrature for the Geronimus class with $\delta = 0.6d$, $d = e^{i\pi/5}$ and $n = 4, 5, \ldots, 50$. On the left for $f(z) = \sqrt{(z-a)(z-b)}$ and on the right for $f(z) = z^{30} + 1/z^{30}$.

8 Addendum

In this section, we collect some technical issues that are necessary to obtain one of the the main conclusions in our paper.

A.1 Properties of the Möbius transform φ_δ

We give some properties of the Möbius transform $\varphi_u(z) = \frac{z+u}{1+\overline{u}z}$, $u \in \mathbb{D}$. As is well known, it is of the hyperbolic type, therefore has two fixed points, $\pm\frac{u}{|u|}$ with very different behavior, one attracting, the other repelling. This and some other relevant properties are collected in the following lemma. These properties are of general interest, and although we use the same notation as in the rest of the paper to avoid confusion, they should be understood as completely independent.

So, we use δ instead of u and briefly recall the notation of the main text $d = \frac{\delta}{|\delta|} = e^{i\kappa}$. Define $|\delta| = \cos(\tau)$, $0 < \tau < \frac{\pi}{2}$ and $t = e^{i\tau}$. The following four points, symmetric with respect to the diameter through d will be important: $\zeta_1^+ = -\overline{t}d$, $\zeta_2^+ = td$, $\zeta_1^- = -td$, and $\zeta_2^- = \overline{t}d$. It is easy to check that the chord spanning the arc $\mathcal{G}^c = (\overline{t}d, td) = (\zeta_2^-, \zeta_2^+)$ contains δ. Similarly, $-\delta$ is on the chord spanning the arc $\mathcal{F}^c = (-td, -\overline{t}d) = (\zeta_1^-, \zeta_1^+)$. Obviously, $-d \in \mathcal{F} \subset \mathcal{G}$. It is very helpful to keep their geometric meaning in mind. They are illustrated in the left plot of Figure 8.

Lemma A.1 *With the notation just introduced, the following properties hold.*

1. $\varphi_\delta(\pm d) = \pm d$.
2. $\varphi_\delta(z) = d\varphi_{|\delta|}(\overline{d}z)$.
3. $\varphi_\delta(\zeta_1^\pm) = \zeta_2^\pm$.
4. $\varphi_\delta(\mathcal{F}^c) = \mathcal{G}^c$. *In particular, as* $\mathcal{G}^c \subset \mathcal{F}^c$, $\varphi_\delta(\mathcal{G}^c) \subset \mathcal{G}^c$.
5. $|\varphi_\delta'(z)| < 1$ *for* $z \in \mathcal{F}^c$ *and* $|\varphi_\delta'(z)| > 1$ *for* $z \in \mathcal{F}$.
6. $\forall z \in \mathcal{G}^c$, $\zeta_2^- = \overline{t}d \prec \varphi_\delta(\overline{a}z) \prec z \prec \varphi_\delta(az) \prec td = \zeta_2^+$, *with* $a = \zeta_1^+\,\overline{\zeta_2^+} = -\overline{t}^2 = e^{i\alpha}$.

Proof 1. The first item is immediate.
2. As $\delta\overline{d} = |\delta|$, we have

$$\varphi_\delta(z) = \frac{z+\delta}{1+\overline{\delta}z} = d\frac{\overline{d}z + |\delta|}{1+|\delta|\overline{d}z} = d\varphi_{|\delta|}(\overline{d}z).$$

3. For the first equality, note that item 2. implies

$$\varphi_\delta(-\overline{t}d) = d\varphi_{|\delta|}(-\overline{t}) = d\frac{-\overline{t}+|\delta|}{1+|\delta|\overline{t}} = dt\frac{-\overline{t}+|\delta|}{t-|\delta|} = dt$$

where in the last step we used $t = |\delta| + i\sqrt{1-|\delta|^2}$. The other equality is obtained in a similar way.

4. This is a direct consequence of item 3: $\varphi_\delta(\zeta_1^\pm) = \zeta_2^\pm$ and $\varphi_\delta(d) = d$ and the fact that φ_δ preserves orientation.
5. If we denote $f(\theta) = \arg(\varphi_\delta(e^{i\theta}))$, it is easy to verify that

$$f'(\theta) = |\varphi_\delta'(e^{i\theta})| = \frac{1-|\delta|^2}{1+|\delta|^2+2|\delta|\cos(\theta-\kappa)}, \quad \theta \in (-\pi, \pi].$$

So, in the previous expression for $|\varphi_\delta'(e^{i\theta})|$, $\cos(\theta-\kappa)$ is monotonically decreasing from 1 to -1 and hence $|\varphi_\delta'(e^{i\theta})|$ is monotonically increasing as $e^{i\theta}$ moves from d to $-d$ on the circle. Therefore, it is clear that $|\varphi_\delta'(e^{i\theta})|$ reaches a minimum for $\theta = \kappa$, given by $|\varphi_\delta'(d)| = \frac{1-|\delta|}{1+|\delta|} < 1$ and a maximum is reached for $\theta = \kappa + \pi$ which is $|\varphi_\delta'(-d)| = \frac{1+|\delta|}{1-|\delta|} > 1$. One may check that $|\varphi_\delta'(z)| = 1$ at $z = \zeta_1^\pm$ which are the boundary points of $\mathcal{F}$. Hence the monotonicity implies that the claimed inequalities hold.

6. First note that $a\zeta_2^+ = \zeta_1^+$ and $\overline{a}\zeta_2^- = \zeta_1^-$. Then $\forall z \in (\zeta_2^-, \zeta_2^+)$, we have

$$\zeta_1^- = -td \prec \overline{a}z \prec z \prec az \prec -\overline{t}d = \zeta_1^+,$$

then

$$\varphi_\delta(\zeta_1^-) \prec \varphi_\delta(\overline{a}z) \prec \varphi_\delta(z) \prec \varphi_\delta(az) \prec \varphi_\delta(\zeta_1^+),$$

equivalently

$$\zeta_2^- = \overline{t}d \prec \varphi_\delta(\overline{a}z) \prec \varphi_\delta(z) \prec \varphi_\delta(az) \prec td = \zeta_2^+.$$

To fit z in this sequence of inequalities, we consider 3 cases: $z = d$, $z \prec d$ and $d \prec z$. If $z = d$, then $\varphi_\delta(z) = z$ and the property is proved. Next consider $d \prec z$. Let's prove that $d \prec \varphi_\delta(z) \prec z \prec td$. Define $\phi_i = \arg(\zeta_i^+)$, $i = 1, 2$ and $z = e^{i(\phi_2-\varepsilon)}$, with $0 < \varepsilon < \tau$. Since $\phi_1 - \varepsilon > \kappa = \arg(d)$, we know by the previous item 5 that $0 < f'(\theta) < 1$ for all $\theta \in (\kappa, \phi_1)$. Therefore $f(\phi_1 - \varepsilon) = f(\phi_1) - f'(\xi)\varepsilon$ with $\xi \in (\phi_1 - \varepsilon, \phi_1)$ and hence $0 < f'(\xi) < 1$. This implies $f(\phi_1) - \varepsilon < f(\phi_1 - \varepsilon) < f(\phi_1)$, since f is an increasing function. In terms of points on $\mathbb{T}$, this is equivalent with $\varphi_\delta(\zeta_1^+)e^{-i\varepsilon} = \zeta_2^+ e^{-i\varepsilon} = z \prec \varphi_\delta(az) \prec \zeta_2^+$, note that $\phi_1 = \phi_2 + \alpha$. On the other hand, as $f(\kappa) = \kappa$, $k - f(\theta_z) = f'(\upsilon)(\kappa - \theta_z)$, with $\upsilon \in (\kappa, \theta_z) \subset (\kappa, \phi_1)$, such $z = e^{i\theta_z}$, then $k - f(\theta_z) < k - \theta_z$, equivalently $d \prec \varphi_\delta(z) \prec z$. This completes the proof when $d \prec z$. The other case is obtained in a similar way. □

A.2 A note on preimages of Blaschke products

Let $\{\rho_n\}_{n\in\mathbb{N}}$ the sequence of monic orthogonal polynomials associated to a finite positive Borel measure supported on $\mathbb{T}$. We define the Blaschke products

$G_n(z) = \frac{\rho_n(z)}{\rho_n^*(z)}$ and $F_n(z) = z\frac{\rho_{n-1}(z)}{\rho_{n-1}^*(z)} = zG_{n-1}(z)$ of degree n. By the Argument Principle, both go around the unit circle exactly n times when z goes around the unit circle once (a n-to-1 map), and preserves the orientation. Since

$$\rho_n(z) = z\rho_{n-1}(z) + \delta_n\rho_{n-1}^*(z),$$
$$\rho_n^*(z) = \overline{\delta}_n z\rho_{n-1}(z) + \rho_{n-1}^*(z),$$

where $\delta_n = \rho_n(0)$, we deduce that $\varphi_{\delta_n}(F_n(z)) = G_n(z)$, so that the solutions of $F_n(z) = t$ coincide with the solutions of $G_n(z) = t'$ where $\varphi_{\delta_n}(t) = t'$. If in addition the support of the Borel measure is an arc, as it is in our case, we can say much more.

Theorem A.1 *([1, Corollary 3.3]) Let μ be a finite positive Borel measure supported on an arc $\Gamma = [a, b]$ with associated monic orthogonal polynomials $\{\rho_n\}_{n\in\mathbb{N}}$. Then for given $t \in \mathbb{T}$, $F_n(z) = t$ will have at least $n - 1$ solutions in $\mathring{\Gamma} = (a, b)$.*

We delve a little more into the behavior of the preimages of the Blaschke products considered. For every $t \in \mathbb{T}$, there are n preimages denoted by $\{z_{k,n}\}_{k=1}^n = F_n^{-1}(t)$, at least $n - 1$ of them are on (a, b). In particular, If $t = F_n(a)$ all preimages $F_n^{-1}(F_n(a))$ are in $[a, b)$ and when $t = F_n(b)$, are in $(a, b]$. Furthermore, their role is essential, define

$$\{\alpha_{k,n}\}_{k=1}^n = F_n^{-1}(F_n(a)), \quad \{\beta_{k,n}\}_{k=1}^n = F_n^{-1}(F_n(b)), \quad \alpha_{1,n} = a, \quad \beta_{n,n} = b.$$

$$\Gamma_{k,n} = [\alpha_{k,n}, \beta_{k,n}] \subset \Gamma, \quad k = 1, \dots, n$$
$$\Gamma_{k,n}^c = (\beta_{k,n}, \alpha_{k+1,n}) \subset \Gamma, \quad k = 1, \dots, n-1, \quad \Gamma_{n,n}^c = (\beta_{n,n}, \alpha_{1,n}) = \Gamma^c = (b, a).$$

We assume the cyclic order

$$a = \alpha_{1,n} \prec \beta_{1,n} \prec \alpha_{2,n} \prec \cdots \prec \alpha_{n,n} \prec \beta_{n,n} = b.$$

The $\{\Gamma_{k,n}\}_{k=1}^n$, are interlaced by the $\{\Gamma_{k,n}^c\}_{k=1}^{n-1}$ and together they fill up the complete arc Γ and $\Gamma_{n,n}^c = (b, a)$, $\forall n \in \mathbb{N}$.
Thus F_n maps (one-to-one) each $\Gamma_{n,k}$ to $[F_n(a), F_n(b)] = F_n(\Gamma)$ and each $\Gamma_{k,n}^c$ to $(F_n(b), F_n(a)) = F_n(\Gamma^c)$. Thus

$$F_n(z) \in F_n(\Gamma) \Leftrightarrow z \in F_n^{-1}(F_n(\Gamma)) = \cup_{k=1}^n \Gamma_{k,n}$$
$$F_n(z) \in F_n(\Gamma^c) \Leftrightarrow z \in F_n^{-1}(F_n(\Gamma^c)) = \cup_{k=1}^n \Gamma_{k,n}^c.$$

In particular, if $\tau^+ \in F_n(\Gamma)$ and $\tau^- \in F_n(\Gamma^c)$, and the solutions of $F_n(z) = \tau^\pm$ are given by $\{z_{k,n}^\pm\}_{k=1}^n = F_n^{-1}(\tau^\pm)$, then they will interlace

$$a = \alpha_{1,n} \prec z_{1,n}^+ \prec \beta_{1,n} \prec z_{1,n}^- \prec \cdots \prec \alpha_{k,n} \prec z_{k,n}^+ \prec \beta_{k,n} \prec z_{k,n}^- \prec \cdots$$
$$\cdots \prec \beta_{n,n} = b \prec z_{n,n}^- \prec a.$$

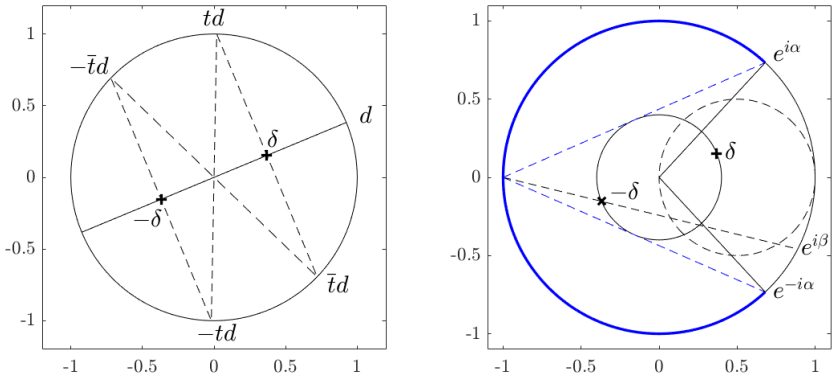


Fig. 8 On the left the representation of the points that are essential in Lemma A.1. The fix point d is an attractor and $-d$ a repellor for $\varphi_\delta(z)$. We have $|\varphi'_\delta(\zeta_1^\pm)| = 1$ and $\varphi_\delta(\zeta_1^\pm) = \zeta_2^\pm$. On the right the geometric interpretation of the points defining the Geronimus measure. The support is $[a, \overline{a}]$, where a and $\overline{a}$ are the intersections of the tangents from the point -1 to the circle $\mathbb{T}_{|\delta|}$ with radius $|\delta|$. The point $e^{i\beta}$ is the intersection of the unit circle with the line connecting -1 and $-\delta$.

If both τ^+ and τ^-, $\tau^+ \neq \tau^-$, are in $F_n(\Gamma)$ then the preimages $z_{k,n}^+$ and $z_{k,n}^-$ will both be in $\Gamma_{k,n} \subset \Gamma$ and they will interlace each other.

The previous analysis was made using $F_n(z)$ but a similar analysis can be done using $G_n(z)$.

A.3 The Geronimus class

The Geronimus class of measures was introduced in Section 3. It depends on a parameter $\delta \in \mathbb{D}$. We refer to [8–10] for more details. We shall only discuss here the case where $\delta \in \mathbb{G}$. With this restriction, the measure will have no mass point, and is absolute continuous with weight given by (6). The support is the arc $[a, b] = [a, \overline{a}]$ where

$$\begin{aligned} a &= \cos\alpha + i\sin\alpha, \quad \sin(\frac{\alpha}{2}) = |\delta|, \quad 0 < \alpha < \pi, \\ &= (1 - 2|\delta|^2) + i(2|\delta|\sqrt{1 - |\delta|^2}) \\ &= -\cos(2\tau) + i\sin(2\tau), \quad \cos(\tau) = |\delta| \\ &= -\bar{t}^2, \end{aligned}$$

We deduce that the endpoint of the support coincides with the parameter $a = \zeta_1^+\overline{\zeta_2^+}$ defined in Lemma A.1 of section A.1 Note that a and t depend on $|\delta|$, but they do not depend on $\kappa := \arg(\delta)$ and $\alpha \in (0, \pi)$ is an increasing function for $|\delta| \in (0, 1)$. Observe also that the condition $|1 - 2\delta| < 1$ is equivalent with $|\delta|^2 < \Re(\delta) = |\delta|\cos\kappa$, so that $|\kappa| < \tau$.

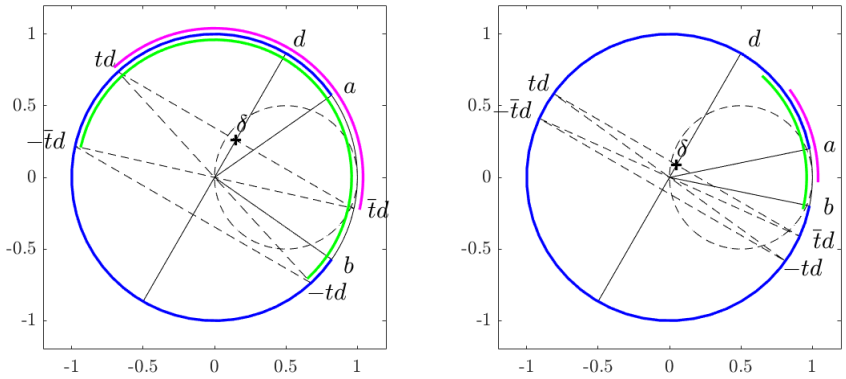


Fig. 9 In these images, the support is the arc $\Gamma = [a, b]$ indicated in blue. δ is represented by the black cross inside the circle $\mathbb{G}$, plotted in dashed lines. The diameter through δ intersects $\mathbb{T}$ at $\pm d$. On the left side, the green arc refers to $\mathcal{F}^c$ and the magenta arc to $\mathcal{G}^c$ for $\delta = 0.3e^{i\pi/3}$. They are plotted at a radius slightly different from 1 for visibility. On the right side, the green arc refers to $F_n(\Gamma^c)$ and the magenta arc to $G_n(\Gamma^c)$ for $\delta = 0.1e^{i\pi/3}$.

Some properties of φ_δ proved in the Section A.1 are essential in this subsection. In particular, we will use that φ_δ maps the arc $\mathcal{F}^c = (-t, -\overline{t})d = (\overline{at}, at)d$ onto $\mathcal{G}^c = (\overline{t}, t)d$ and that for all $z \in \mathcal{G}^c$, $\overline{t}d \prec \varphi_\delta(\overline{a}z) \prec z \prec \varphi_\delta(az) \prec td$.

We recover the definitions, $G_n(z) = \frac{\rho_n(z)}{\rho_n^*(z)}$ and $F_n(z) = \frac{z\rho_{n-1}(z)}{\rho_{n-1}^*(z)} = zG_{n-1}(z)$. Since F_n and G_n are continuous functions on $\mathbb{T}$ and since $\Gamma = [a, b]$ is closed and $\Gamma^c = (b, a)$ is open, also their images under F_n and G_n will be closed, respectively open arcs for $n > 0$. Since $G_0 = \frac{\rho_0}{\rho_0^*} = 1$ and $F_1(z) = z$, we artificially set by definition $G_0(\Gamma^c) = \{1\}$ and thus $G_0(\Gamma) = \mathbb{T}\backslash\{1\}$.

The main result of this section is the following theorem.

Theorem A.2 *With $\Gamma^c = (b, a)$ and $\Gamma = [a, b]$, we have for all $n = 1, 2, 3, \ldots$:*

$$\begin{array}{ccccccccccc} \{1\} = G_0(\Gamma^c) & \subset & \cdots & \subset & G_{n-1}(\Gamma^c) & \subset & G_n(\Gamma^c) & \subset & \cdots & \subset & \mathcal{G}^c, \\ \cap & & & & \cap & & \cap & & & & \cap \\ \Gamma^c = F_1(\Gamma^c) & \subset & \cdots & \subset & F_n(\Gamma^c) & \subset & F_{n+1}(\Gamma^c) & \subset & \cdots & \subset & \mathcal{F}^c, \end{array}$$

and for the complementary sets $F_n(\Gamma)$ and $G_n(\Gamma)$:

$$\begin{array}{ccccccccccc} \mathbb{T}\backslash\{1\} = G_0(\Gamma) & \supset & \cdots & \supset & G_{n-1}(\Gamma) & \supset & G_n(\Gamma) & \supset & \cdots & \supset & \mathcal{G}, \\ \cup & & & & \cup & & \cup & & & & \cup \\ \Gamma = F_1(\Gamma) & \supset & \cdots & \supset & F_n(\Gamma) & \supset & F_{n+1}(\Gamma) & \supset & \cdots & \supset & \mathcal{F}. \end{array}$$

where $\mathcal{G}^c$ and $\mathcal{F}^c$ are the (open) complementary arcs of the (closed) $\mathcal{G}$ and $\mathcal{F}$ respectively.

It says that the arcs $\{G_n(\Gamma^c)\}_{n\in\mathbb{N}}$ are a monotonically increasing sequence that is bounded by $\mathcal{G}^c$: $G_n(\Gamma^c) \subset \mathcal{G}^c$, $\forall n \in \mathbb{N}$. Similarly, $\{F_n(\Gamma^c)\}_{n\in\mathbb{N}}$ is

monotonically increasing sequence and it is bounded by $\mathcal{F}^c$: $F_n(\Gamma^c) \subset \mathcal{F}^c$, $\forall n \in \mathbb{N}$.

Clearly, we only have to give the proof for the maps of Γ^c because the result for the complementary arcs is immediate by inverting the inclusions. The inclusions $G_n(\Gamma^c) \subset F_{n+1}(\Gamma^c)$ are also obvious if $G_n(\Gamma^c) \subset \mathcal{G}^c$ because $F_{n+1}(\Gamma^c) = \psi_a(G_n(\Gamma^c))$ where the map ψ_a is defined by

$$\psi_a : \gamma = (p, q) \subset \mathcal{G}^c \mapsto (\overline{a}p, aq) \subset \mathcal{F}^c.$$

Thus ψ_a enlarges the arc γ by extending it with an arc of length α at both ends. Moreover $G_n(\Gamma^c) \subset G_{n+1}(\Gamma^c) \subset \mathcal{G}^c$ implies $F_{n+1}(\Gamma^c) \subset F_{n+2}(\Gamma^c) \subset \mathcal{F}^c$.

Under these conditions, there are limits $\lim_{n\to\infty} G_n(\Gamma^c) = G_\infty(\Gamma^c) \subseteq \mathcal{G}^c$ and $\lim_{n\to\infty} F_n(\Gamma^c) = F_\infty(\Gamma^c) \subseteq \mathcal{F}^c$. Moreover, it is concluded that the following limits exist,

$$G_\infty(a) = \lim_{n\to\infty} G_n(a) \quad \text{and} \quad G_\infty(b) = \lim_{n\to\infty} G_n(b),$$

and

$$F_\infty(a) = \lim_{n\to\infty} F_n(a) \quad \text{and} \quad F_\infty(b) = \lim_{n\to\infty} F_n(b).$$

Thus the following Theorem is true.

Theorem A.3 *With the previous notation used for the Geronimus measure it holds that*

$$G_\infty(a) = \zeta_2^+ \quad \textit{and} \quad G_\infty(b) = \zeta_2^-$$

while

$$F_\infty(a) = \zeta_1^+ \quad \textit{and} \quad F_\infty(b) = \zeta_1^-.$$

Equivalently

$$G_\infty(\Gamma^c) = \mathcal{G}^c \quad \textit{and} \quad F_\infty(\Gamma^c) = \mathcal{F}^c.$$

Proof As we know $\varphi_\delta(zG_n(z)) = G_{n+1}(z)$, applying limits to both sides of the equality

$$G_{n+1}(z) = \frac{zG_n(z) + \delta}{1 + \overline{\delta} z G_n(z)},$$

for $z = a$, and setting $G = G_\infty(a)$ we deduce

$$G = \frac{aG + \delta}{1 + \overline{\delta} a G} \Leftrightarrow a\overline{\delta}G^2 + (1 - a)G - \delta = 0 \Leftrightarrow G = \frac{(a-1) \pm \sqrt{(1-a)^2 + 4a|\delta|^2}}{2a\overline{\delta}}.$$

As $a - 1 = -2|\delta|\,\overline{t}$ and $(1 - a)^2 = 4|\delta|^2\,\overline{t^2} = -4|\delta|^2 a$ we obtain

$$G = \frac{a-1}{2a\overline{\delta}} = \frac{-2|\delta|\,\overline{t}}{2\overline{\delta}a} = -\overline{at}\frac{\delta}{|\delta|} = t^2\,\overline{t}d = td.$$

Analogously, for $z = b$, we deduce that $G_\infty(b)$ is given by $\overline{t}d$.

By a similar argument, using

$$F_{n+1}(z) = z\frac{F_n(z) + \delta}{1 + \delta F_n(z)},$$

we find that $F = F_\infty(a) = -\overline{t}d$ and $F_\infty(b) = -td$. □

We are now ready to prove Theorem A.2.

Proof (**of Theorem A.2**) By our previous discussion, it is clear that it only remains to prove the inclusion

$$\{1\} = G_0(\Gamma^c) \subset G_1(\Gamma^c) \subset \cdots \subset G_n(\Gamma^c) \subset G_{n+1}(\Gamma^c) \subset \cdots \subset \mathcal{G}^c.$$

The initial condition requires to show $1 \in G_1(\Gamma^c)$ since $G_0 = \frac{\rho_0}{\rho_0^*} = 1$. That follows from the Geronimus condition $\delta \in \mathbb{G}$. Indeed, $1 \in G_1(\Gamma^c)$ because

$$\frac{1-\delta}{1-\overline{\delta}} \in (b,a) = (F_1(b), F_1(a)) \Rightarrow \varphi_\delta\left(\frac{1-\delta}{1-\overline{\delta}}\right) = 1 \in (G_1(b), G_1(a)),$$

since $G_1(z) = \varphi_\delta(F_1(z))$, since $F_1(z) = z$.

To prove the induction step, we recall that $G_{n+1}(\Gamma^c) = \varphi_\delta(\psi_a(G_n(\Gamma^c)))$. That is a consequence of Lemma A.1 (6). Indeed, take $z = G_n(a)$, then this result says that $z \prec \varphi_\delta(az) \prec td$ and for $z = G_n(\overline{a})$, we get $z \succ \varphi_\delta(\overline{a}z) \succ \overline{t}d$, which is just saying that $G_n(\Gamma^c) \subset G_{n+1}(\Gamma^c) \subset \mathcal{G}^c$ and this concludes the proof. □

Remark A.1 As $d \in \mathcal{G}^c \subset \mathcal{F}^c$, in general there must exist some $n_0 = n_0(\delta)$ such that for all $n \geq n_0$, $d \in G_n(\Gamma^c) \subset F_{n+1}(\Gamma^c) \subset \mathcal{F}^c$. On the other hand, as $-d \in \mathcal{F} \subset \mathcal{G} \subset F_{n+1}(\Gamma) \subset G_n(\Gamma)$, then $-d \in F_{n+1}(\Gamma) \subset G_n(\Gamma)$ for all n.

There are some important consequences that follow from this last remark which are given in Theorems 3.2 and 3.3.

As expected, there are examples where both d and $-d$ belong to $F_n(\Gamma)$ as illustrated in the right figure of Figure 10. Observe that all the zeros of both B_3 and $\hat{B}_3$ are in $\mathring{\Gamma}$, they belong pairwise to the same subarcs $F_3^{-1}(F_3(\Gamma))$.

A.4 How to guarantee that $d \in [b, a]$?

We have seen that for the Geronimus class there is always a finite integer n_0 such that $d \in (F_n(b), F_n(a))$ for all $n > n_0$. This implies that one of the preimages $F_n^{-1}(d)$ will be in (b, a). Given that $d \in [b, a] = [F_1(b), F_1(a)] \subset [F_n(b), F_n(a)]$, $\forall\, n \in \mathbb{N}$, we can be sure that one of the preimages $F_n^{-1}(d)$ will be in $[b, a]$ for all $n > 0$. This can be obtained with an extra condition on δ as the following Lemma shows.

Lemma A.2 *Consider the Geronimus class of measures with parameter $\delta \in \mathbb{G}$, and support $[a, b]$ with $a = e^{i\alpha}$ and $b = \overline{a}$. Then $d \in (b, a)$ if $|\delta| \geq \frac{1}{2}$.*

Proof Set $d = \frac{\delta}{|\delta|} = e^{i\kappa}$. It is clear that $\delta = \frac{1}{2}e^{\pm i\pi/3} \notin \mathbb{G}$ and it is not difficult to check that with the plus sign, $d = a$ while for the minus sign $d = b$. Since α is defined by $\sin(\frac{\alpha}{2}) = |\delta|$, it only depends on $|\delta|$ and thus we will have $d \in (b, a)$ for $\delta = \frac{1}{2}e^{i\kappa}$ with $\kappa \in (-\frac{\pi}{3}, \frac{\pi}{3})$ Furthermore, if δ remains in the disk $\mathbb{G}$, and $|\delta|$ increases, then also α will increase. Thus $d \in (b, a)$ for all $\delta \in \mathbb{G}$ and $|\delta| \geq \frac{1}{2}$. □

Remark A.2 The condition $|\delta| \geq \frac{1}{2}$ in the previous lemma is sufficient, but not necessary to have $d \in (b, a)$. For example $d = 1 \in (b, a)$ for all $\delta \in (0, 1)$.

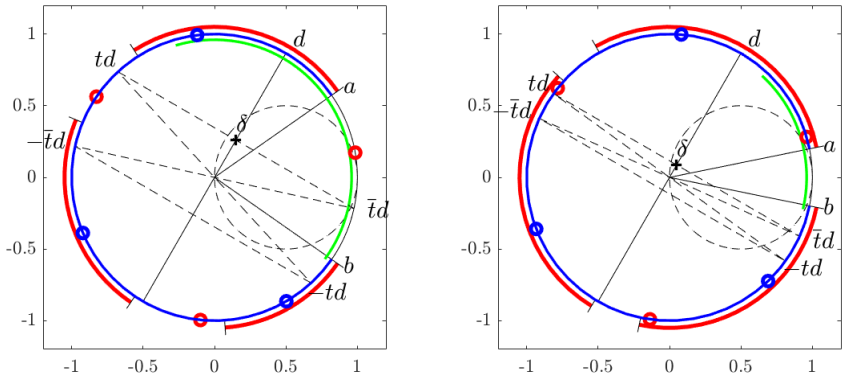


Fig. 10 These two figures correspond to the two bottom figures of figure 9. Left is for $\delta = 0.3e^{i\pi/3}$, and right for $\delta = 0.1e^{i\pi/3}$. Only $F_3(\Gamma^c)$ is plotted in green, while the magenta arc $G_3(\Gamma^c)$ is removed. Instead we have plotted the arcs $F_3^{-1}(F_3(\Gamma)) = G_3^{-1}(G_3(\Gamma))$ in red. The blue circles are the zeros of $\tilde{B}_3$ and the red circles of $\hat{B}_3$. On the left: $-d \in F_3(\check{\Gamma})$ and $d \in F_3(\Gamma^c)$. Note all the zeros of $\tilde{B}_3$ are in Γ, and one zero of $\hat{B}_3$ in Γ^c. On the right: $-d, d \in F_3(\check{\Gamma})$ and all the zeros of $\tilde{B}_3$ and of $\hat{B}_3$ are in $\check{\Gamma}$.

A.5 The orthogonality parameter for B_n

If B_n is a $(n, 2\ell+1)$-quasi-paraorthogonal polynomial that satisfies $B_n = \tau B_n^*$, then τ is called its invariance parameter; $\tau = B_n(0)$ if B_n is monic. If it is orthogonal to $z^{n-\ell} - \tilde{\tau} z^{\ell}$, then $\tilde{\tau}$ is called its orthogonality parameter, moreover if its nodes are used in a quadrature formula is exact in a the space $\mathcal{L}_{2(n-\ell)-1}(\omega)$ for $\omega = -\tau\tilde{\tau}$. See Theorem 2.2.

Lemma A.3 *Let $B_n(z) = z(z-\eta)\rho_{n-2}(z) + B_n(0)(1-\overline{\eta}z)\rho_{n-2}^*(z)$ with $\eta \in \mathbb{D}$ be a $(n,3)$-quasi-paraorthogonal polynomial where $\{\rho_n\}_{n\in\mathbb{N}}$ is the sequence of monic orthogonal polynomials associated to the Geronimus class with parameter δ, then its orthogonality parameter is $\tilde{\tau} = B_n(0)\frac{\overline{B_n(0)}\delta+\overline{\eta}}{B_n(0)\overline{\delta}+\eta}$, i.e. $\langle B_n, z^{n-1} + z^2 q(z) - \tilde{\tau} z\rangle = 0$.*

Proof We must prove that B_n admits the following expression, (see [1]):

$$B_n(z) = c_n\left[(1-\overline{\xi}z)\rho_{n-1}(z) + \tilde{\tau}(z-\xi)\rho_{n-1}^*(z)\right], \quad |\xi| > 1.$$

We consider the backward Szegő recurrence, $z\rho_{n-2}(z) = \frac{1}{1-|\delta|^2}(\rho_{n-1}(z) - \delta\rho_{n-1}^*(z))$.

$$\begin{aligned} B_n(z) &= z(z-\eta)\rho_{n-2}(z) + \tau(1-\overline{\eta}z)\rho_{n-2}^*(z), \quad \tau = B_n(0), \\ &= \frac{1}{1-|\delta|^2}\left\{(z-\eta)(\rho_{n-1}(z) - \delta\rho_{n-1}^*(z)) + \tau(1-\overline{\eta}z)(\rho_{n-1}^*(z) - \overline{\delta}\rho_{n-1}(z))\right\} \\ &= \frac{1}{1-|\delta|^2}\left\{\rho_{n-1}(z)\left[z(1+\tau\overline{\eta}\overline{\delta}) - (\eta+\tau\overline{\delta})\right] + \rho_{n-1}^*(z)\left[(\eta\delta+\tau) - (\delta+\tau\overline{\eta})z\right]\right\} \end{aligned}$$

$$= \frac{\eta+\tau\overline{\delta}}{1-|\delta|^2}\left\{\rho_{n-1}(z)\left[z\frac{1+\tau\overline{\eta}\overline{\delta}}{\eta+\tau\overline{\delta}}-1\right]+\frac{\delta+\tau\overline{\eta}}{\eta+\tau\overline{\delta}}\rho^*_{n-1}(z)\left[\frac{\eta\delta+\tau}{\delta+\tau\overline{\eta}}-z\right]\right\}$$
$$= c_n\left[(1-\overline{\xi}z)\rho_{n-1}(z)-\tilde{\tau}(z-\xi)\rho^*_{n-1}(z)\right],$$

with $\xi=\frac{\eta\delta+\tau}{\delta+\tau\overline{\eta}}$ and $\tilde{\tau}(\tau)=\tau\frac{\overline{\tau}\delta+\overline{\eta}}{\tau\overline{\delta}+\eta}$. □

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References

[1] Bultheel, A., Cruz-Barroso, R., Díaz Mendoza, C.: Zeros of quasi-paraorthogonal polynomials and positive quadrature. J. Comput. Appl. Math. **407**, 14039 (2022). https://doi.org/10.1016/j.cam.2021.114039

[2] Cruz-Barroso, R., Díaz Mendoza, C., Perdomo Pío, F.: Szegő-type quadrature formulas. J. Math. Anal. Appl. **455**(1), 592–605 (2017). https://doi.org/10.1016/j.jmaa.2017.05.055

[3] Golinskii, L.: Quadrature formula and zeros of para-orthogonal polynomials on the unit circle. Acta Math. Hungar. **96**, 169–186 (2002). https://doi.org/10.1023/A:1019765002077

[4] Jones, W.B., Njåstad, O., Thron, W.B.: Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle. Bull. London Math. Soc. **21**, 113–152 (1989). https://doi.org/10.1112/blms/21.2.113

[5] Szegő, G.: Orthogonal Polynomials, 4th edn. American Mathematical Society Colloquium Publications, vol. 23. Amer. Math. Soc., Providence, Rhode Island (1975). First edition 1939

[6] Walsh, J.L.: Degree of approximation to functions on a Jordan curve. Trans. Amer. Math. Soc. **73**(3), 447–458 (1952)

[7] Cantero, M.J., Cruz-Barroso, R., González-Vera, P.: A matrix approach to the computation of quadrature formulas on the unit circle. Appl. Numer. Math. **58**(3), 296–318 (2008). https://doi.org/10.1016/j.apnum.2006.11.009

[8] Geronimus, J.: Sur quelques propriétés des polynômes orthogonaux généralisés. Comptes Rendus de l'Académie des Sciences de l'URSS **29**(1), 5–8 (1940)

[9] Golinskii, L., Nevai, P., Van Assche, W.: Perturbation of orthogonal polynomials on an arc of the unit circle. J. Approx. Theory **83**, 392–422 (1995). https://doi.org/10.1006/jath.1995.1128

[10] Golinskii, L., Nevai, P., Pintér, F., Van Assche, W.: Perturbation of orthogonal polynomials on an arc of the unit circle, II. J. Approx. Theory **96**, 1–33 (1999). https://doi.org/10.1006/jath.1998.3208