Calculation and Visualization of Orthogonal Rational Functions

Adhemar Bultheel[1], *Department of Computer Science, University of Leuven, Celestijnenlaan 200A, B 3001 Leuven, Belgium*

Heiko Mentzel[2], *Department of Computer Science, Leipzig University, Augustusplatz 10, D 04109 Leipzig, Germany*

Karsten Müller[3], *Department of Mathematics, Technical University of Chemnitz, Reichenhainer Straße 39, D 09107 Chemnitz, Germany*

Keywords: Visualization, Orthogonal rational functions

1. What is ORF?

ORF consists of a Java applet and application which visualize orthogonal rational functions (ORFs) on the complex unit circle.

If μ is a positive measure of the complex unit circle, and $\{\alpha_k\}$ is a sequence of complex numbers inside the unit disk, then one may consider the spaces

$$\mathcal{L}_0 = \mathbf{C}, \quad \mathcal{L}_n = \{p_n(z)/\pi_n(z)\}, \quad n = 1, 2, \ldots$$

where p_n is a polynomial of degree at most n and

$$\pi_n(z) = (1 - \overline{\alpha}_1 z)(1 - \overline{\alpha}_2 z) \cdots (1 - \overline{\alpha}_n z).$$

Let these spaces be equiped with the inner product

$$\langle f, g \rangle_\mu = \int f(z)\overline{g(z)} d\mu(z).$$

The orthogonal rational functions are then a set of orthogonal basis functions for these nested sequence of rational spaces.

If you want to learn more about the mathematics of these functions, you should read Section 2.

Orthogonal rational functions are complex valued functions of a complex variable. Therefore, in the ORF application, one can choose between the three-dimensional

[1]This is the main author of these pages.

[2]This author made the Java-programmed interface for the visualization of orthogonal rational functions. Please contact him if you have problems with the Java script or Java application.

[3]The third author is responsible for these pages and for the implementation of the algorithms to compute orthogonal rational functions. He is supported by the Deutsche Akademie der Naturforscher Leopoldina with reference LPD1996 and the Deutsche Forschungsgemeinschaft with reference Mu1334/1-1.

representation of the real part, the imaginary part, the modulus or the angle (i.e. the argument) of these functions. The degree of the ORFs that can be visualized can be selected between 1 and 30.

At the moment there is the possibility to choose between the normalized Lebesgue-Borel measure and two further measures which are not absolutely continuous with respect to the normalized Lebesgue-Borel measure.

However, there is a possibility to choose among different sequences of poles.

The pictures were generated by numerical computation with Matlab5. Because the Java applet gives also the source code for the calculation of ORFs, it is practically possible to calculate ORFs with an arbitrary pre-given sequence of poles and any measure, if Matlab5 is available.

The language Matlab5 has a lot of useful techniques for visualization. Especially, neccessary features like truecolor support, phong lighting and multiple light sources for colored surfaces are used. Furthermore, the usage of Matlab5 also gives a choice between four different colouring types (Spring, Summer, Autumn, Winter). Also this choice is available to the user.

We also calculated the ORFs by using Octave which is a free clone of Matlab. Octave is able to handle most of the m-files which are written for Matlab but many features for visualization are not included.

In the Java program, the sequence of poles and the measure can be chosen via the sequence and the object panel, respectively. One can open a window which contains the unit disc with the sequence of zeros. To get an idea of the measure, a so-called Carathéodory function can be visualized. This function is obtained via the F. Riesz-Herglotz transformation and has a positive real part.

ORF is a Java-based interface which presents the three-dimensional object from the zenith of the complex plane, i.e. angle is 90 degrees, or at a tilted angle, namely 70 degrees. Hereby the user can make all the choices mentioned above by clicking the appropriate buttons. Moreover, a colorbar is added, the position of the sequence in the unit disk can be visualized and certain panels can be opened. The fields of description in the main window can be switched on and off seperately. Furthermore, the user also gets a few interactive help windows which are available in English and German at the moment.

There is also an old version of ORF which is written in Java script. Both versions can be reached through the internet, but it is also possible to use the Java application as a stand-alone program.

For improvements to be implemented later, we refer to Section 3.

2. The Mathematical Background

We give here some basic facts of the theory of orthogonal rational functions (ORFs). We assume that $\mathbf{C}$ denotes the set of complex numbers, $\mathbf{D}$ the open unit disk, $\mathbf{T}$ the unit circle, and $\mathbf{E}$ the exterior of the closed unit disk. Let μ be a probability measure on the unit circle. $L^2(\mu)$ is the Hilbert space of all square integrable functions with *inner product*

$$\langle f, g\rangle_\mu = \int_{t\in\mathbf{T}} f(t)\overline{g(t)}d\mu(t).$$

As a special case, one could choose μ to be the the normalized Lebesgue-Borel measure on the unit circle, which will be denoted by λ.

Let $\alpha \in \mathbf{D}$. Then the *Blaschke factor* $\zeta(\alpha, z)$ is defined by

$$\zeta(\alpha, z) = \frac{\overline{\alpha}}{|\alpha|} \frac{\alpha - z}{1 - \overline{\alpha} z} \tag{1}$$

where $\zeta(\alpha, z) = z$ if $\alpha = 0$. Let $\{\alpha_n\}_{n \in \mathbf{N}}$ be a sequence in $\mathbf{D}$ and let

$$\zeta_n(z) = \zeta(\alpha_n, z) \ .$$

Then the product

$$B_n(z) = \prod_{k=1}^{n} \zeta_k(z) \ , \quad n \in \mathbf{N} \tag{2}$$

is called the *n-th finite Blaschke product with respect to the sequence* $\{\alpha_n\}_{n \in \mathbf{N}}$, where $B_0(z) = 1$. (We assume everywhere that $\alpha_0 = 0$.) Define the *fundamental space* $\mathcal{L}_n$ as

$$\mathcal{L}_n = \text{span}\{B_k : k = 0, 1, ..., n\} = \{p_n/\pi_n : p_n \in \Pi_n\} \tag{3}$$

where Π_n is the space of polynomials of degree at most n and

$$\pi_n(z) = \prod_{k=1}^{n} (1 - \overline{\alpha}_k z) \ , \quad n \in \mathbf{N}$$

Thus $\mathcal{L}_n$ is the space of rational functions with poles in $\{\widehat{\alpha_k} : k = 1, ..., n\} \in \mathbf{E}$, where $\widehat{\alpha} := 1/\overline{\alpha}$. In particular, when $\alpha_n = 0$ for all n, then $\mathcal{L}_n = \Pi_n$. Note that the sequence $\{\mathcal{L}_n\}$ is nested, i.e., $\mathcal{L}_n \subset \mathcal{L}_{n+1}$.

We define the para-hermitian conjugate f_* of a function f by

$$f_*(z) = \overline{f(1/\overline{z})} \ . \tag{4}$$

The superstar f^* for a function $f \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ is defined as

$$f^*(z) = B_n(z) f_*(z) \ . \tag{5}$$

Note that this transformation depends on the degree n.

Let $\{\phi_n : n = 0, 1, 2, ...\}$ be the orthonormal sequence obtained by orthogonalization of $\{B_n : n = 0, 1, 2, ...\}$. Every such ϕ_n admits a representation

$$\phi_n(z) = \sum_{k=0}^{n} a_k^{(n)} B_k(z) \ . \tag{6}$$

The coefficient $a_n^{(n)}$ is the leading coefficient with respect to the basis $\{B_k\}$. We shall denote it by $\kappa_n = a_n^{(n)}$. The ϕ_n are uniquely defined if we require that $\kappa_n > 0$. Setting $\Phi_n = \phi_n/\kappa_n$, we obtain ORFs with leading coefficient equal to 1. We call these Φ_n the monic ORFs.

Elementary manipulations show that

$$\langle \phi_n, \phi_n \rangle = \langle \phi_n^*, \phi_n^* \rangle = 1 \tag{7}$$

and orthogonality implies that

$$\langle f, \phi_n \rangle = 0 \quad \text{and} \quad \langle g, \phi_n^* \rangle = 0 \tag{8}$$

with $f \in \mathcal{L}_{n-1}$ and $g \in \zeta_n \mathcal{L}_{n-1}$. Furthermore, the relations

$$\phi_n^*(\alpha_n) = \kappa_n \quad \text{and} \quad \Phi_n^*(\alpha_n) = 1$$

are valid.

An explicit formula for the ORFs can be obtained as follows. Let G_n be the Gram matrix

$$G_n = [\mu_{ij}]_{i,j=1}^n \quad \text{with} \quad \mu_{ij} = \langle B_i, B_j \rangle.$$

It is then not difficult to see that

$$\kappa_n^2 = \det G_{n-1} / \det G_n$$

and therefore

$$\phi_n(z) = [\det G_{n-1} \det G_n]^{-\frac{1}{2}} \cdot \det \begin{bmatrix} \mu_{00} & \cdots & \mu_{0,n-1} & B_0(z) \\ \vdots & & \vdots & \vdots \\ \mu_{n0} & \cdots & \mu_{n,n-1} & B_n(z) \end{bmatrix}. \tag{9}$$

Like for the Szegő polynomials, it holds that ϕ_n has all its zeros in the open unit disk.

There is also a Christoffel-Darboux type of relation given by

$$\sum_{k=0}^{n} \phi_k(z)\overline{\phi_k(w)} = \frac{\phi_{n+1}^*(z)\overline{\phi_{n+1}^*(w)} - \phi_{n+1}(z)\overline{\phi_{n+1}(w)}}{1 - \zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}.$$

The left hand side is the reproducing kernel for $\mathcal{L}_n$.

There are generalizations of the Szegő recurrence relations, given by

$$\phi_n(z) = \frac{\kappa_n}{\kappa_{n-1}} \left[\varepsilon_n \frac{z - \alpha_{n-1}}{1 - \overline{\alpha}_n z} \phi_{n-1}(z) + \delta_n \frac{1 - \overline{\alpha}_{n-1} z}{1 - \overline{\alpha}_n z} \phi_{n-1}^*(z) \right] \tag{10}$$

and

$$\phi_n^*(z) = -\frac{\overline{\alpha}_n}{|\alpha_n|} \frac{\kappa_n}{\kappa_{n-1}} \left[\overline{\delta}_n \frac{z - \alpha_{n-1}}{1 - \overline{\alpha}_n z} \phi_{n-1}(z) + \overline{\varepsilon_n} \frac{1 - \overline{\alpha}_{n-1} z}{1 - \overline{\alpha}_n z} \phi_{n-1}^*(z) \right]. \tag{11}$$

The recursion coefficients δ_n and ε_n are given by

$$\delta_n = \frac{(1 - \alpha_{n-1}\overline{\alpha_n})}{(1 - |\alpha_{n-1}|^2)} \frac{\phi_n(\alpha_{n-1})}{\kappa_n} \tag{12}$$

and

$$\varepsilon_n = -\frac{\overline{\alpha_n}}{|\alpha_n|} \frac{(1 - \overline{\alpha_{n-1}}\alpha_n)}{(1 - |\alpha_{n-1}|^2)} \frac{\overline{\phi_n^*(\alpha_{n-1})}}{\kappa_n}. \tag{13}$$

The practical computation of the ORFs that has been used here is based on a Nevanlinna-Pick interpolation algorithm which generates the recurrence coefficients ε_k and δ_k, which are then used in the recurrence relation.

For some history, we refer to Section 4.

For a selected bibliography, we refer to the list of references at the last page.

3. To be done

The ORF Java script and Java application are available on the World-Wide-Web via

`http://www.uni-leipzig.de/~muellerk`

or

`http://muellerk.notrix.de` .

At the moment there is only a possibility to choose between the normalized Lebesgue-Borel measure and two further measures which is not absolutely continuous with respect to the normalized Lebesgue-Borel measure. We hope to include other measures in the future.

Especially when the argument is plotted (which is highly discontinuous), some numerical inaccuracies cause some ugly effects in the picture. In the future, we want to increase the number of the grid points and to improve the whole visualization. We hope that lines which are now wrong because of the steep increase of the function in certain grid points will then be removed.

We also plan to make the pictures available for degrees up to 50 (instead of degree 30).

We have to mention that the feature *preloading for animation* is not enabled in the Java program because this could make your machine crash. This should work in the next version of the applet. Then one can get a slide-show and see the motion of the ORFs when the degree is increasing.

We also note that the Java applet can run on any platform with a Java-enabeld Web browser. Furthermore, the Java application can run on any computer where Java-Development-Kit 1.1 (not 1.0) is installed. If you have a slow internet connection or if you do not have a Java-enabled Web-Browser you can order the ORF-CD-ROM. Please send a mail to `muellerk@uni-leipzig.de`

4. Some history

It is well known that the polynomials orthogonal on the unit circle discussed in Szegő's work, are related to classical interpolation problems of Schur, Riesz, Carathéodory. Several monographs were devoted to the subject: Szegő, Grenander and Szegő, Geronimus, Freud, ...

The algorithms of Levinson and Schur are known methods to solve such classical interpolation problems. The Nevanlinna-Pick algorithm is a multipoint generalization of the Schur algorithm. Thus, there should be multipoint generalizations of the Szegő plynomials which turn out to be orthogonal rational functions (with prescribed poles that are related to the interpolation points).

These ORFs were of course implicit in the early work on the Nevanlinna-Pick algorithm, but they became only explicit in the work by Djrbashian [26, 25, 27, 28, 29, 30] (see also [31]). Also in work by Bultheel [2, 1], Bultheel-Dewilde [3], and Dewilde-Dym [23, 24] the reproducing kernels, and gradually also the ORFs played a role that became more and more important.

Work by A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad on multipoint moment problems, multipoint Padé approximation and quadrature formulas has finally brought the study of such ORFs to the foreplan in a long list of papers [4]-[21] and an upcoming monograph [22].

More recently also other people have contributed: K. Pan, Xin Li and others.

We also mention here that the ORFs discussed here are related to (but not exactly the same as) orthogonal polynomials with respect to a varying measure as they were studied by G. Lopez-Lagomassino and others.

References

[1] A. Bultheel. Orthogonal matrix functions related to the multivariable Nevanlinna-Pick problem. *Bull. Soc. Math. Belg. Sér. B*, 32(2):149–170, 1980.

[2] A. Bultheel. On a special Laurent-Hermite interpolation problem. In L. Collatz, G. Meinardus, and H. Werner, editors, *Numerische Methoden der Approximationstheorie 6*, volume 59 of *Internat. Ser. of Numer. Math.*, pages 63–79, Basel-New York-Berlin, 1981. Birkhäuser Verlag.

[3] A. Bultheel and P. Dewilde. Orthogonal functions related to the Nevanlinna-Pick problem. In P. Dewilde, editor, *Proc. 4th Int. Conf. on Math. Theory of Networks and Systems at Delft*, pages 207–212, North-Hollywood, 1979. Western Periodicals.

[4] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonal rational functions similar to Szegő polynomials. In C. Brezinski, L. Gori, and A. Ronveaux, editors, *Orthogonal Polynomials and their Applications*, volume 9 of *IMACS annals on computing and applied mathematics*, pages 195–204, Basel, 1991. J.C. Baltzer AG.

[5] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. The computation of orthogonal rational functions and their interpolating properties. *Numer. Algorithms*, 2(1):85–114, 1992.

[6] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. A Favard theorem for orthogonal rational functions on the unit circle. *Numer. Algorithms*, 3:81–89, 1992.

[7] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. A moment problem associated to rational Szegő functions. *Numer. Algorithms*, 3:91–104, 1992.

[8] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonal rational functions and quadrature on the unit circle. *Numer. Algorithms*, 3:105–116, 1992.

[9] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Moment problems and orthogonal functions. *J. Comput. Appl. Math.*, 48:49–68, 1993.

[10] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonal rational functions on the unit circle and the real line. In P. Dewilde, M.A. Kaashoek, and M. Verhaegen, editors, *Challenges of a Generalized System Theory*, volume 40 of *Essays of the Dutch Academy of Sciences, Sect. Physics, First series*, pages 241–252, Amsterdam, 1993. North Holland.

[11] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Asymptotics for orthogonal rational functions. *Trans. Amer. Math. Soc.*, 346:331–340, 1994.

[12] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Numerical quadratures on the unit circle: a survey with some new results. In M. Alfaro, A. García, C. Jagels, and F. Marcellan, editors, *Orthogonal Polynomials on the Unit Circle: Theory and Applications*, pages 1–20, 1994.

[13] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonal rational functions on the unit circle: theory, applications and generalizations. In M. Alfaro, A. García, C. Jagels, and F. Marcellan, editors, *Orthogonal Polynomials on the Unit Circle: Theory and Applications*, pages 21–42, 1994.

[14] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Quadrature formulas on the unit circle based on rational functions. *J. Comput. Appl. Math.*, 50:159–170, 1994.

[15] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Recurrence relations for orthogonal functions. In S.C. Cooper and W.J. Thron, editors, *Continued Fractions and Orthogonal Functions*, volume 154 of *Lecture Notes in Pure and Appl. Math.*, pages 24–46. Marcel Dekker, 1994.

[16] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Convergence of modified approximants associated with orthogonal rational functions. *J. Comput. Appl. Math.*, 57(1/2):77–86, 1995.

[17] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Convergence of orthogonal rational functions. In R.M. Ali, St. Ruscheweyh, and E.B. Saff, editors, *Computational Methods and Function Theory (CMFT'94)*, pages 111–123. World Scientific, 1995.

[18] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. On the convergence of multipoint Padé-type approximants and quadrature formulas associated with the unit circle. *Numer. Algorithms*, 13:321–344, 1996.

[19] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonal rational functions and modified approximants. *Numer. Algorithms*, 11:57–69, 1996.

[20] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonal rational functions and nested disks. *J. Approx. Theory*, 89:344–371, 1997.

[21] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Rates of convergence of multipoint rational approximants and quadrature formulas on the unit circle. *J. Comput. Appl. Math.*, 77:77–102, 1997.

[22] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. *Orthogonal rational functions*. Cambridge University Press, 1999. (To appear).

[23] P. Dewilde and H. Dym. Schur recursions, error formulas, and convergence of rational estimators for stationary stochastic sequences. *IEEE Trans. Inf. Th.*, IT-27:446–461, 1981.

[24] P. Dewilde and H. Dym. Lossless inverse scattering, digital filters, and estimation theory. *IEEE Trans. Inf. Th.*, IT-30:644–662, 1984.

[25] M.M. Djrbashian. Expansions in systems of rational functions on a circle with a given set of poles. *Doklady Akademii Nauk SSSR*, 143:17–20, 1962. (In Russian. Translation in Soviet Mathematics Doklady, vol. 3, 315–319, 1962).

[26] M.M. Djrbashian. Orthogonal systems of rational functions on the unit circle with given set of poles. *Doklady Akademii Nauk SSSR*, 147:1278–1281, 1962. (In Russian. Translation in Soviet Mathematics Doklady, vol. 3, 1794–1798, 1962).

[27] M.M. Djrbashian. Orthogonal systems of rational functions on the circle. *Izv. Akad. Nauk Armyan. SSR*, 1:3–24, 1966. (In Russian).

[28] M.M. Djrbashian. Orthogonal systems of rational functions on the unit circle. *Izv. Akad. Nauk Armyan. SSR*, 1:106–125, 1966. (In Russian).

[29] M.M. Djrbashian. Expansions by systems of rational functions with fixed poles. *Izv. Akad. Nauk Armyan. SSR*, 2:3–51, 1967. (In Russian).

[30] M.M. Djrbashian. A survey on the theory of orthogonal systems and some open problems. In P. Nevai, editor, *Orthogonal polynomials: Theory and practice*, volume 294 of *Series C: Mathematical and Physical Sciences*, pages 135–146, Boston, 1990. NATO-ASI, Kluwer Academic Publishers.

[31] K. Müller and A. Bultheel. Translation of the Russian paper "Orthogonal systems of rational functions on the unit circle" by M.M. Džrbašian. Technical Report TW253, Department of Computer Science, K.U. Leuven, February 1997.