Accelerating PDE-constrained Optimization with Model Order Reduction (MOR)

KU LEUVEN

Yao Yue Karl Meerbergen Department of Computer Science, KU Leuven

Optimization of Vibration Systems

Parametric Dynamical Systems

$$\begin{cases} \mathcal{L}(\omega,\gamma)x(\omega,\gamma) = f, \\ y(\omega,\gamma) = \ell^* x(\omega,\gamma). \end{cases} \quad (1)$$

- Obtained from the discretization of the underlying PDE.
- ω: frequency; $\gamma \in \mathbb{R}^l$: l parameters.
- $\mathcal{L}(\omega,\gamma)$: a parameterized $n \times n$ matrix.

Objective

Minimize the energy norm of the output by choosing γ:

$$\min_{\gamma} g(\gamma) = \int_{\omega_L}^{\omega_H} |y(\omega,\gamma)|^2 \,\mathrm{d}\omega.$$

Algorithm: Damped BFGS.

Example: Footbridge Damper Optimization

Use four dampers to reduce the vibration of a footbridge.

Goal: Minimize the vibration by tuning the stiffnesses and the damping coefficients of the four dampers. (8 design parameters)

Difficulty

PDE discretization $\Longrightarrow$ Large System Order $\Longrightarrow$

The computations of $g(\gamma)$ and $\nabla g(\gamma)$ are very expensive.

The MOR Framework

Model Order Reduction of Second Order Systems

First consider second order systems without design parameters.

Algorithm. Two-sided SOAR: build the input and output Krylov subspaces for projection:

$$\begin{cases} (K + i\omega C - \omega^2 M)x = f \\ y = \ell^* x \end{cases} \xrightarrow{MOR} \begin{cases} (\widehat{K} + i\omega \widehat{C} - \omega^2 \widehat{M})\widehat{x} = \widehat{f} \\ \widehat{y} = \widehat{\ell}^* \widehat{x} \end{cases} \quad (2)$$

order n $n \gg k$ order k

Moment Matching Properties for two-sided SOAR

Moments: Coefficients in the Taylor expansion.

- the first $2k$ moments of y and $\widehat{y}$ w.r.t. ω match at $\gamma = \gamma^{(0)}$;
- the first $2k-1$ moments of $\frac{\partial y}{\partial \omega}$ and $\frac{\partial \widehat{y}}{\partial \omega}$ match;
- in addition, if we get (2) from (1) by fixing $\gamma = \gamma^{(i)}$, the first k moments $\left.\frac{\partial y}{\partial \gamma_j}\right|_{\gamma=\gamma^{(i)}}$ and $\left.\frac{\partial \widehat{y}}{\partial \gamma_j}\right|_{\gamma=\gamma^{(i)}}$ match. $(1 \le j \le l)$

The MOR Framework

- Generate a two-sided SOAR reduced model for each parameter value accessed by optimization.
- Can approximate both the function value and the gradient for this parameter value: Quasi-Newton methods are suitable.

Extrapolatory MOR

MOR:

$$\mathcal{L}(\omega,\gamma) \longrightarrow \mathcal{L}(\omega,\gamma^{(i)}) \longrightarrow V,\ W \quad \searrow\swarrow \quad \widehat{\mathcal{L}}(\omega,\gamma^{(i)})$$

Extrapolatory MOR:

$$\mathcal{L}(\omega,\gamma) \longrightarrow \mathcal{L}(\omega,\gamma^{(i)}) \longrightarrow V,\ W \quad \searrow\swarrow \quad \widehat{\mathcal{L}}(\omega,\gamma)$$

Valid for the entire parameter space.

Denote the extrapolatory reduced model extrapolated at $\gamma^{(i)}$ by $\widehat{g}^{(i)}$.

Exploit extrapolatory reduced models

Motivation

Due to moment matching properties, we have

$$\widehat{g}^{(i)}(\gamma^{(i)}) \approx g(\gamma^{(i)}), \qquad \nabla \widehat{g}^{(i)}(\gamma^{(i)}) \approx \nabla g(\gamma^{(i)}).$$

So, $\widehat{g}^{(i)}$ should approximate g well around $\gamma^{(i)}$. (for smooth functions)

The Relaxed First Order Condition

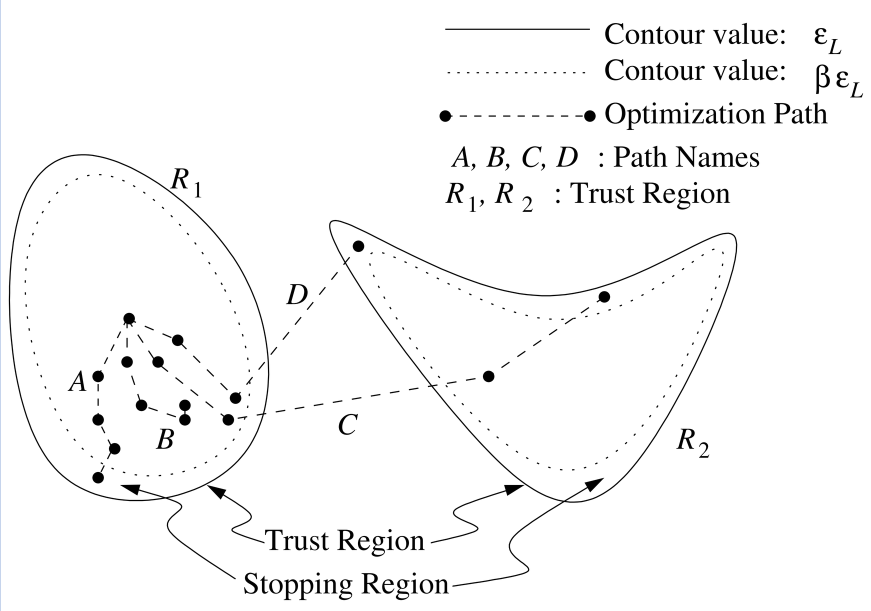
- To safely exploit an extrapolatory reduced model $\widehat{g}^{(i)}(\gamma)$, we use:
  - a heuristic error bound $e^{(i)}(\gamma)$ for $\widehat{g}^{(i)}(\gamma)$. (Based on residual)
  - a heuristic error bound $e_g^{(i)}$ for $\nabla \widehat{g}^{(i)}(\gamma^{(i)})$.
- Good approximation at the extrapolation point: small $e^{(i)}(\gamma^{(i)})$ and $e_g^{(i)}$.
- A reduced model can be refined by enlarging Krylov subspaces.

The basic working procedure for the i-th iteration:

1. Build the i-th reduced model $\widehat{g}^{(i)}$.
2. Formulate the i-th optimization subproblem using $\widehat{g}^{(i)}$ and $e^{(i)}$. ? P1
3. Solve the subproblem to get a candidate for the next iterate $\gamma_{\mathrm{cand}}^{(i+1)}$.
4. Decide whether to accept or update $\gamma_{\mathrm{cand}}^{(i+1)}$ by testing the sufficient decrease condition. ? P2

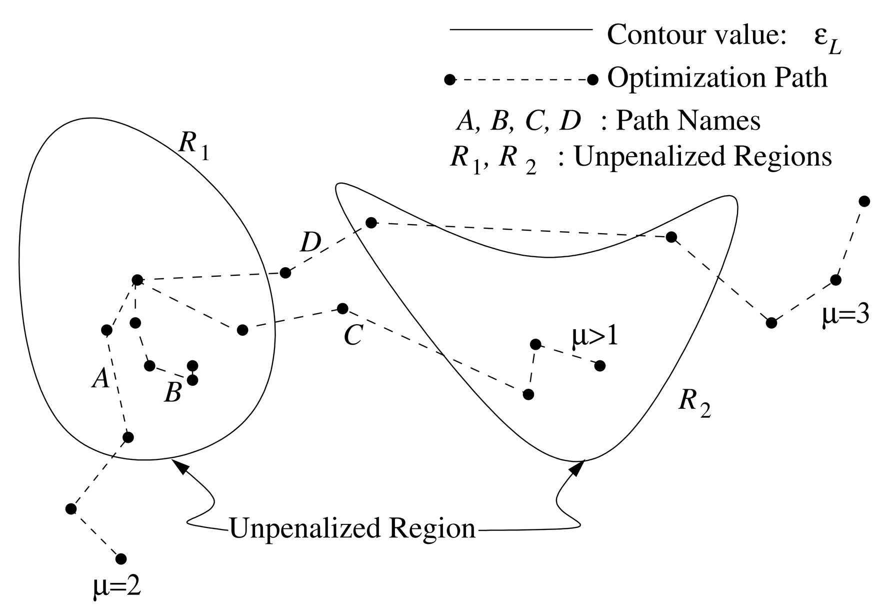
P1: Subproblem Formulation

Two Algorithms



$$\min_{\gamma}\ \widehat{g}^{(i)}(\gamma) \quad \text{s.t.} \quad \frac{e^{(i)}(\gamma)}{\widehat{g}^{(i)}(\gamma)} \le \epsilon_L.$$

Terminate if close to the boundary.



$$\min_{\gamma}\ \widehat{g}^{(i)}(\gamma) + w\left(\frac{e^{(i)}(\gamma)}{\widehat{g}^{(i)}(\gamma)}\right) e^{(i)}(\gamma).$$

Terminate if w is active for μ successive steps. $w \in [0,1]$, continuous.

P2: Convergence Theory

The approximate generalized Cauchy point $\gamma_{\mathrm{AGC}}^{(i)}$

When we use a backtracking-Armijo line search on a descent direction, $\gamma_{\mathrm{AGC}}^{(i)}$ is the first point satisfying both

- the Armijo condition on $\widehat{g}^{(i)}(\gamma)$;
- the constraint of the subproblem.

Theorem for convergence

Under mild conditions, if we accept $\gamma_{\mathrm{cand}}^{(i+1)}$ only when it satisfies

$$\widehat{g}^{(\gamma_{\mathrm{cand}}^{(i+1)})}(\gamma_{\mathrm{cand}}^{(i+1)}) \le \widehat{g}^{(i)}(\gamma_{\mathrm{AGC}}^{(i)})$$

we achieve convergence on g (original model).

- Computationally feasible to check: No evaluation of g.
- When fails, we have several strategies:
  1. Backtrack on $\gamma_{\mathrm{AGC}}^{(i)}$.
  2. Shrink the trust/unpenalized region, and solve the subproblem again.
  3. Refine $\widehat{g}^{(i)}$.
- Sometimes, we can check the condition without generating $\widehat{g}^{(\gamma_{\mathrm{cand}}^{(i+1)})}$: ETR and EP are designed in favor of this case.

Numerical Results for the Footbridge Problem

	Order	Optimum	CPU Time
The MOR Framework	12	24.77751651	879 s
ETR	20	24.78594112	205 s
EP(1)	20	24.7762798	295 s
EP(2)	20	24.7775166	190 s

A single evaluation of g costs 540 s.

yao.yue@cs.kuleuven.be karl.meerbergen@kuleuven.be