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ON THE ADAMJAN-AROV-KREIN APPROXIMATION, IDENTIFICATION AND BALANCED REALIZATION OF A SYSTEM

ABSTRACT

In a 1971 paper, Adamjan, Arov and Krein (AAK) gave an explicit formula for the approximation of an infinite Hankel matrix by a Hankel matrix of lower rank in terms of singular vectors for the Hankel matrix. This has recently received some attention in identification and realization theory where the relation between singular value decomposition (SVD) and balanced realization is now understood. Some properties of the AAK approximation are derived and it is compared with other approximations obtained from singular value decompositions of the system Hankel map.

1. INTRODUCTION : NOTATION AND BASIC RESULTS

Let $\mathcal{L}_p$ and $\mathcal{H}_p$ denote the classical Banach spaces of functions defined on the unit circle in the complex plain, and let H_h denote the Hankel operator with symbol $h \in \mathcal{L}_\infty$, i.e. $H_h f = (I-P)h f$, $\forall f \in \mathcal{H}_2$ with P the orthogonal projection of $\mathcal{L}_2$ onto $\mathcal{H}_2$.

The whole class $h+\mathcal{H}_\infty$ generates the same Hankel operator H_h since H_h only depends upon the coanalytic part of h, i.e. only upon $c_k(h)$, $k \geq 1$, with $c_k(h)$ the Fourier coefficients

$$c_k(h) = \frac{1}{2\pi} \int_0^{2\pi} \zeta^k h(\zeta) d\theta \ , \ \zeta = e^{i\theta} \ , \ k \in \mathbb{Z}.$$

In the basis $\zeta^k = e^{ik\theta}$ for $\mathcal{H}_2$ and $(\mathcal{H}_2)^\perp$ we can work with the matrix representation $(c_{i+j-1}(h))_{i,j\geq 1}$ of H_h, acting as an operator on ℓ_2.

If f is a real function, defined on the set V, then we mean by $\delta[\min_{a \in V} f(a)] = \{F,A\}$ that the minimum F of f(a) for a varying over V is obtained for a = A. If this minimum is unique we write $\delta!$ instead of δ.

We denote by $S(H_h) = \Sigma_1^\infty c_k(h)\zeta^{-k}$ the unique conjugate analytic member in the class $h+\mathcal{H}_\infty$ (which is again a symbol for H_h).

Let $\mathcal{B}$ be the set of all bounded operators from $\mathcal{H}_2$ onto $(\mathcal{H}_2)^\perp$ and $\mathcal{B}^{(k)}$ the subset of $\mathcal{B}$ of bounded operators with a dimension not exceeding k (i.e. the matrix representation has rank at most k). H is the set of all Hankel operators in $\mathcal{B}$ and $H^{(k)}$ similarly all Hankel operators in $\mathcal{B}^{(k)}$.

Concerning the symbols of these classes of Hankel operators we introduce the notation $R^{(k)}$: the set of all conjugate analytic rational functions of degree at most k, multiplicity counted and $F^{(k)} = R^{(k)} + \mathcal{H}_\infty = \{f \in \mathcal{L}_\infty | f=r+h$ with $r \in R^{(k)}$ and $h \in \mathcal{H}_\infty\}$.

We will use a star to indicate the hermitian transpose of a matrix and an upper bar for the complex conjugate. The Adamjan-Arov-Krein result is [1] :

property 1.1

If $H \in \mathcal{H}$ has singular values $s_0(H) \geq s_1(H) \geq \ldots \geq s_{k-1}(H) > s_k(H) = \ldots = s_{k+r}(H)$ and $s_k(H)$ is an eigenvalue of $(H^\star H)^{1/2}$ then there exists $L_k \in \mathcal{H}^{(k)}$ s.t.

$$\exists ! [\min_{L \in \mathcal{H}^{(k+r)}} \| H-L \|] = \{ s_k(H), L_k \}$$

with $L_k = H - H_{\phi_k} \in \mathcal{H}^{(k)}$ and $\phi_k(\zeta) = s_k(H) \frac{\eta_-(\zeta)}{\xi_+(\zeta)}$ where (ξ,η) is an arbitrary pair of singular vectors of H, associated with $s_k(H)$ and $\xi_+(\zeta) = \Sigma_1^\infty \xi_j \zeta^{j-1}$ and $\eta_-(\zeta) = \Sigma_1^\infty \eta_j \zeta^{-j}$.

Moreover

(1) $\eta_-(\zeta)/\xi_+(\zeta)$ is inner (i.e. it is analytic and in modulus 1 a.e. for $|\zeta| = 1$)

(2) $\xi_+(\zeta)$ and $\eta_-(\zeta)$ have the factorizations $\xi_+(\zeta) = b_+(\zeta) a(\zeta) \phi(\zeta)$ and $\eta_-(\zeta) = \zeta\, b_-(\zeta) a(\zeta) \phi(\zeta)$, $|\zeta| = 1$, with
$\phi(\zeta)$ outer (i.e. it is analytic and $\ell n|\phi(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ell n|\phi(e^{i\theta})|d\theta$)
$b_+(\zeta)$ and $b_-(\zeta)$ inner and
$a(\zeta)$ inner and independent of the choice of the pair (ξ,η).

With the Nehari and Kronecker properties, this can be translated into [1].

property 1.2

If $f \in \mathcal{L}_\infty$ and $s_k(H_f)$ an eigenvalue of $(H_f^\star H_f)^{1/2}$, then

$$\exists ! [\min_{h \in F^{(k)}} \| f-h \|_\infty] = \{ s_k(H_f), h_k \}$$

with $h_k = f - \phi_k$ and ϕ_k as in property 1.1.

2. REALIZATION OF LINEAR SYSTEMS

Consider the linear discrete system

$$x_{k+1} = A\, x_k + B\, u_k \qquad x_k \in \mathbb{C}^n,\ u_k, y_k \in \mathbb{C},$$
$$y_k = C\, x_k \qquad A \in \mathbb{C}^{n\times n},\ B \in \mathbb{C}^{n\times 1},\ C \in \mathbb{C}^{1\times n}$$

with final dimensional state space $(n < \infty)$.

Suppose now that $\{u_k\}_{k\in\mathbb{Z}} \in \ell_2$ and that all eigenvalues of A are inside the unit disc, then there exists a unique solution for the system such that $\{x_k\}_{k\in\mathbb{Z}} \in \ell_2^n$. Taking ζ-transforms, this solution is given as $x(\zeta) = (\zeta - A)^{-1} B\, u(\zeta)$. The corresponding $\{y_k\}_{k\in\mathbb{Z}} \in \ell_2$ and the transferfunction

$h(\zeta) = C(\zeta - A)^{-1}B$ is conjugate analytic and uniform bounded. The ζ-transform of the output is given by $y(\zeta) = C(\zeta I - A)^{-1} B\, u(\zeta) \triangleq S\, u(\zeta)$. S is known as the <u>systemoperator</u>. The <u>restricted input/output map</u> is the Hankel operator $H = (I-P)S|_{\mathcal{H}_2} = \mathcal{O}\,\mathcal{C}$ with $\mathcal{O} = [C^\star\ (CA)^\star\ (CA^2)^\star\ \ldots\ ]^\star$ the <u>observability matrix</u> and $\mathcal{C} = [B\ AB\ A^2B\ \ldots]$ the <u>controlability matrix</u>. We suppose the realization <u>minimal</u>, so that rank H = order A = rank $\mathcal{O}$ = rank $\mathcal{C}$. It is possible to chose an equivalent realization $(W^{-1}A\,W, W^{-1}B, CW)$ such that $P_0 \triangleq \mathcal{O}^\star\mathcal{O} = \Sigma = \mathcal{C}\mathcal{C}^\star \triangleq P_c$ where $\Sigma = \mathrm{diag}(s_0(H), s_1(H), \ldots s_{n-1}(H))$ is the diagonal appearing in the singular value decomposition of H. Then, the realization is called <u>balanced</u> [2,3].

We consider now the problem of approximating a system of finite order $n (H \in H^{(n)})$ with a system of lower order $(\hat{H} \in H^{(k)}$ with $k < n)$. This is related to the problem of approximating $r_n \triangleq S(H) \in R^{(n)}$ with a lower order rational function $r_k = S(\hat{H}) \in R^{(k)}$, $k < n$. Several algorithms have been proposed to solve this problem. See[2] for some references. We mention only two possible strategies :

Suppose for simplicity that the singular values of H are all different.

a) Partial minimal realization

One method is to find an approximation $r_k \in F^{(k)}$ as a Padé approximant of r_n, which does not yield a guaranteed stable approximation, so that we have to take the conjugate analytic part r' of the Padé approximant r_k, so that $H_{r'} \in H^{(j)}$ with $j \leqslant k$ and thus will probably not be optimal in $H^{(k)}$ and even not in $H^{(j)}$.

b) Adamjan-Arov-Krein (AAK) approximation

The AAK approximant $\hat{H} = L_k$, given by property 1.1 will be an optimal approximation for H. The theorem only claims that the associated symbol $\psi_k = r_n - \phi_k$ is in $F^{(k)}$, but ψ_k is not rational in general. Thus also here the conjugate analytic part of ψ_k must be taken to obtain $r_k \in R^{(k)}$, but then r_k is <u>not</u> the optimal $\mathcal{L}_\infty$ approximant of r_n in $F^{(k)}$ any more, but it does solve the systemtheoretical problem.

In the next section we will explore the structure of $\psi_k(\zeta)$ somewhat further.

3. SOME PROPERTIES CONCERNING THE AAK APPROXIMATION

<u>property 3.1</u>

Let $H \in H^{(n)}$ and (ξ, η) be a pair of singular vectors of H, associated with $s_k(H)$, the k-th singular value of H, then

(1) $\eta_-(\zeta) \in R^{(j)}$ for some $j \leqslant n$

(2) $\xi_+(\zeta) = \zeta^{-1}\overline{\eta_-(1/\bar{\zeta})}\,\bar{\zeta}_k$ with ζ_k a constant of modulus 1 and $\eta_-(\zeta)$ and $\xi_+(\zeta)$ as in property 1.1.

Let us denote the reciprocal of a polynomial $P(\zeta)$ of degree m by $\hat{P}(\zeta)$ i.e. $\hat{P}(\zeta) = \zeta^m\overline{P(1/\bar{\zeta})}$.

Suppose now $H \in \mathcal{H}^{(n)}$ has symbol $r = \mathcal{S}(H) = \frac{R(\zeta)}{N(\zeta)} \in \mathcal{R}^{(n)}$ with $R(\zeta)$ and $N(\zeta)$ polynomials of degree at most n. If $\eta_-(\zeta)$ is as defined in the previous property, then it can be written as $T(\zeta)/N(\zeta)$ with $N(\zeta)$ the denominator of r and $T(\zeta)$ a polynomial of degree at most n. Thus the AAK approximation $\psi_k(\zeta) = r(\zeta) - s_k(H)\eta_-(\zeta)/\xi_+(\zeta)$ is given by

$$\psi_k(\zeta) = \frac{R(\zeta)}{N(\zeta)} - s_k(H)\,\frac{T(\zeta)\hat{N}(\zeta)}{N(\zeta)\hat{T}(\zeta)}\,\zeta_k = \frac{R(\zeta)\hat{T}(\zeta) - s_k(H)T(\zeta)\hat{N}(\zeta)\zeta_k}{N(\zeta)\hat{T}(\zeta)}\,.$$

If $\alpha \in C$, then we mean by its reciprocal the complex number $1/\bar{\alpha}$. We now formulate

<u>property 3.2</u>

Let $H \in \mathcal{H}^{(n)}$ and $r = \mathcal{S}(H) \in \mathcal{R}^{(n)}$, then the AAK approximation $\psi_k \in \mathcal{F}^{(k)}$ of r is a rational function of degree at most n-1 with all its poles among the zeros of $\xi_+(\zeta)$ (one zero at infinity excluded), which are also the reciprocals of the zeros of $\eta_-(\zeta)$ (again one zero at infinity excluded). Suppose (A,B,C) is a balanced realization of H, a_k the k-th row of A and b_k the k-th element of B $(0 \leqslant k \leqslant n-1!)$, then, if $\xi_+(0) \neq 0$ we find the mentioned zeros of $\eta_-(\zeta)$ in the spectrum of $E = A - b_k^{-1}a_k B$.

The condition $\xi_1 \neq 0$ is no strong restriction but if $\xi_1 = 0$, a minor change in the definition of E is needed.

4. LOWER ORDER APPROXIMATION VIA SVD

Because the AAK approximation is expressed in terms of the singular vectors and values of an infinite Hankel matrix, it is a tempting idea that some of the algorithms computing a lower order approximation via SVD should actually find the AAK approximation. We will mention three proposed conceptual algorithms which give the same approximation, but are definitely different from the AAK approximation. (By conceptual we mean that we formulate the algorithms as if it were possible to compute the SVD of an infinite Hankel matrix $H \in \mathcal{H}^{(n)}$). We look for an approximant in $\mathcal{H}^{(k)}$. We first take an optimal approximation in $\mathcal{B}^{(k)}$, given by Property 1.1, and this in turn is approximated by a Hankel matrix, characterized by a realization triple (A_k,B_k,C_k). This method is followed by Kung [2] and Zeiger-McEwen [4,5] while Moore [3] directly finds the triple (A_k,B_k,C_k) from a balanced realization (A,B,C). Because the balanced realization and a SVD of an infinite Hankel matrix are essentially the same, they all three find the same approximation.

In Kung's paper it is illustrated by an example that the result he obtains with his algorithm is not far from the AAK approximation. This could suggest the errorneous idea that he computes a numerically perturbed AAK approximation. That the nearness to the AAK approximation is only due to the fact that the zeroed singular values of H were very small is illustrated by the following example.

Let $h(\zeta) = \frac{1}{\zeta(\zeta-\varepsilon)}$ with $\varepsilon = 0.1$, then $\{h_k\}_{k \in \mathbb{N}} = \{0,1,\varepsilon,\varepsilon^2,\varepsilon^3,\dots\}$. The eigenvalues of H_h are $\lambda_0 = 1.0568\dots$ and $\lambda_1 = -0.95580\dots$ and all others are zero. The AAK approximation of degree zero is $\lambda_0^2\varepsilon/(\zeta+\lambda_0)$. This gives $\hat{H} = H_{r_k} = 0$, and this is in accordance with property 3.2 which says that $r_k(\zeta)$ is a rational function of degree $n-1 = 1$ and has $k = 0$ poles inside the unit disc.

Similarly for a degree one AAK approximation we find $-\lambda_1^2\varepsilon/(\zeta+\lambda_1)$.

Using the Kung algorithm to find the degree one approximation we get approximately $0.475/(\zeta-0.549)$ with a pole that is definitely different from the AAK pole.

5. ALGORITHMS FOR THE SVD OF AN INFINITE HANKEL MATRIX

It will be clear from the foregoing that computing the SVD of an infinite Hankel matrix $H \in \mathcal{H}^{(n)}$ plays an important role and this is almost the same as the computation of a balanced realization.

Kung [2] proposes to take the N×N leading submatrix of H, such that the Markov parameters h_j for $j > N$ are smaller in magnitude than a preset value and compute the SVD of this submatrix. This could lead to situations where the rank n of H can be very small, while the Markov parameters h_j are slowly decreasing, e.g. $H = H_h$ with $h(\zeta) = (\zeta-\varepsilon)^{-1}$ with $|\varepsilon| \lessapprox 1$. Thus we need the SVD of a large N×N matrix while the first Markov parameters already contain all necessary information.

Moore's technique [3] is to transform an arbitrary realization triple (A,B,C) into a balanced one. This requires more work (two Lyapunov equations and three SVD's) but on matrices of order n×n, where usually $n \ll N$, a saving of computer effort after all.

It is known that the singular values are rather stable with respect to perturbation of H, but the set of singular vectors can be very sensitive, thus the computation of a balanced realization is an ill conditioned problem, and this can be more critical for the Moore method than for the Kung method.

6. CONCLUSION

We have introduced the Adamjan, Arov and Krein approximation result and applied it to the realization problem of a system impuls response. We have compared the AAK result with some heuristic algorithms based on balanced system realization and SVD decomposition.

It has appeared that the AAK realization is not a perturbation of the heuristic realizations. Also, a possible but not very practicable way to compute the AAK realization has been deduced. The work presented has appeared first in Dutch [7]. While working on the English version attention of the authors has been drawn to a recent report of L. Silverman and Maâmar Bettayeb [8] which contains some of the results of the paper.

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