Minimal cubature rules. *Yuan Xu.* Cambridge University Press, 2026 (xvi+259 p.) isbn: 978-1-009-66392-2 (hbk), Cambridge Monographs in Applied and Computational Mathematics, vol 45

To approximate an integral $I[f] = \int_{\mathbb{R}} f(x)w(x)dx$ one selects a set of nodes $\{x_k\}_{k=1}^n$, and constructs the corresponding interpolating polynomial $L_n[f] = \sum_{k=1}^n \ell_k(x) f(x_k)$, and integrates it to obtain an interpolating quadrature formula of the form $I_n[f] = \sum_{k=1}^n w_k f(x_k)$ that approximates $I[f]$. It is obviously exact for all polynomials up to degree $n-1$. If the n nodes are the zeros of a polynomial of degree n orthogonal with respect to the weight w, then this n-point quadrature formula will be exact for all polynomials up to degree $2n-1$. All the nodes are in the interval of integration, and all the quadrature weights are positive. This is called a Gauss quadrature formula, and it is minimal in the sense that it has the minimum number of nodes needed to obtain the degree of exactness $2n-1$. Reducing the number of orthogonality conditions will reduce the degree of exactness. There are close connections with a number of properties of orthogonal polynomials, their recurrence relation, the Christoffel-Darboux relation, and reproducing kernels. Also, there are many well-studied so called classical orthogonal polynomials corresponding to standard weights w on an interval, a half line, or the whole real line.

The purpose of this book is to extend this classical theory as much as possible to the multidimensional case where the discrete approximant of the integral is called a cubature formula. However, there are many hurdles to overcome, starting with a proper notation for multivariate polynomials, an ordering for the monomials $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ that have degree $n = |\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_d$, and more generally an ordering for the polynomials in $\mathcal{P}_n^d = \{P(\mathbf{x}) = \sum_{|\boldsymbol{\alpha}|=n} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}\}$ and in $\Pi_n^d = \{P(\mathbf{x}) = \sum_{k=1}^n P_k(\mathbf{x}), P_k \in \mathcal{P}_k^d\}$. Note that $\dim \Pi_n^d = \binom{n+d}{d}$. So, to fix a cubature formula of degree $2n-2$ or $2n-1$ requires at least $N = \binom{n+d-1}{d}$ simple and "independent" nodes in the sense that they may not be on a hypersurface of degree $n-1$ meaning that they cannot be all zeros of a polynomial of the form $\mathbf{x}^{\boldsymbol{\alpha}} + R(\mathbf{x})$ with $R \in \Pi_{n-1}^d$, $|\boldsymbol{\alpha}| = n$ (for $d = 1$ this would be a monic polynomial of degree n). If this lower bound is reached, it will be called a Gaussian cubature.

A first trail to developing cubature formulas is to design product rules, where scalar quadrature rules in each of the variables are combined to form the cubature (Chapter 2). They are less efficient, but they can be obtained

with classical quadrature rules depending on the domain of integration (simplex, ball, sphere). If the integral has some intrinsic symmetry, then a cubature may reflect that symmetry and become simpler.

Another approach is to generate proper Gaussian cubatures analogously to the scalar case. The first step is to generate multivariate orthogonal polynomials. That means introducing an inner product with respect to a weight function defined on the domain of integration Ω and selecting a unique P_α, with $|\boldsymbol{\alpha}| = n$ and orthogonal to all polynomials preceding according to the ordering defined by $\boldsymbol{\alpha}$ (Chapter 3). It is possible to generalize the recurrence relation and the kernel functions to block versions in the multivariate case, leading to a characterization of the Gauss cubature. As before, the symmetry of Ω may simplify the formulas. Several examples are obtained for particular choices of the weight (Jacobi, Chebyshev, etc.) and the domain (hypercube, ball) (Chapter 4).

Since a Gauss cubature is difficult to construct, if it exists at all, it is still worthwhile to construct cubatures with a certain degree of exactness and a minimal number of nodes. Therefore, lower bounds for the number of nodes have to be obtained (Chapter 5). These allow one to construct minimal cubature rules (Chapter 6) such as a rule using Padua points on a square. Some more advanced examples of (near) minimal cubatures on a square based on orthogonal polynomials and products of Chebyshev/Jacobi weights are derived in a separate chapter (Chapter 7).

On the real line, Chebyshev polynomials are orthogonal with respect to a continuous weight function on $[-1, 1]$, but they are also orthogonal with respect to a discrete inner product because of their close relation to the discrete Fourier transform (DFT). This connection is explored for the multivariate case as well (Chapter 8). The discrete inner product becomes a summation over well-defined lattices, and generalizations of sines and cosines are defined, but the cubature is no longer minimal.

For polynomials satisfying somewhat relaxed orthogonality conditions, the cubature nodes can be considered as a variety of an algebraic ideal. By using some algebraic geometry, the general construction of cubature rules can be reformulated in terms of these varieties and ideals (briefly outlined in Chapter 9).

The latter is a possible way to attack the two fundamental open problems outlined in the final Chapter 10: (1) to obtain sharper lower bounds for the number of nodes required to attain some degree of exactness, and (2) to obtain cubature rules for dimension $d > 2$ (since most of the examples in

this book are for $d = 2$).

In an addendum some ideas are given about cubature rules on a triangle or a simplex (important for triangulation in finite element methods) since most of the previous examples are on the (hyper)cube or the ball. It also includes Gauss-Lobatto versions where nodes on the boundary of the domain can be involved.

The book can be used as a follow-up course after a first introductory course on numerical analysis. Knowledge of Gauss quadrature on the real line is certainly helpful but not necessary, since whatever is needed has been summarized in the first chapter. Every chapter ends with notes and references to the literature where more details and extensions can be found. The list of 184 references (mostly papers) is relevant and up-to-date. There are no exercises though. The book can easily be used for self study, for example to start a PhD project. The ultimate goal is to develop software to compute the integrals, but the approach of the book is mainly theoretical and sidesteps purely numerical aspects of rounding errors and practical implementation concerns. Nonetheless, it is an excellent read for the theoretical background. The proofs are clearly explained, and the careful notation (vectors are consistently denoted in bold, capitals are used for matrices, etc.) helps readability. A short subject index allows one to locate where in the text a topic has been discussed.

I could only spot few typos (sometimes a vector not being bold, or p. 48 rank $A_{m,i} =$ rank r_m^d; the second "rank" should not be there since r_m^d is the rank), but such errors are exceptional and obvious, hence not really harmful.

As a final note, I should mention that this is the first book with a modern state-of-the-art treatment of minimal cubature formulas. The approach is to generalize the idea of Gauss quadrature as much as possible, and when zeros of multivariate orthogonal polynomials are a problem, the next-best thing is to design minimal cubature formulas. This can be an interesting approach for problems of moderately low dimension over a simple domain ($d = 2$ is still an important case). This is quite different from other recent books on cubature which deal with lattice rules and (quasi-)Monte Carlo methods that are especially suitable for very high-dimensional problems.

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