Poles of the topological zeta function for plane curves and Newton polyhedra

Ann Lemahieu and Lise Van Proeyen *

Abstract.— *The local topological zeta function is a rational function associated to a germ of a complex holomorphic function. This function can be computed from an embedded resolution of singularities of the germ. For nondegenerate functions it is also possible to compute it from the Newton polyhedron. Both ways give rise to a set of candidate poles of the topological zeta function, containing all poles.*

For plane curves, Veys showed how to filter the actual poles out of the candidate poles induced by the resolution graph. In this note we show how to determine from the Newton polyhedron of a nondegenerate plane curve which candidate poles are actual poles.

1. Introduction

1.1 The local topological zeta function.– In 1992 Denef and Loeser introduced a new zeta function which they called the topological zeta function because of the topological Euler–Poincaré characteristic turning up in it. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function and let $\pi : X \to \mathbb{C}^n$ be an embedded resolution of $f^{-1}\{0\}$. We denote by $E_i, i \in S$, the irreducible components of $\pi^{-1}(f^{-1}\{0\})$, and by N_i and $\nu_i - 1$ the multiplicities of E_i in the divisor on X of $f \circ \pi$ and $\pi^*(dx_1 \wedge \ldots \wedge dx_n)$, respectively. For $I \subset S$ we denote also $E_I := \cap_{i \in I} E_i$ and $E_I^\circ := E_I \setminus (\cap_{j \notin I} E_j)$. Further we write $\chi(\cdot)$ for the topological Euler–Poincaré characteristic.

The *local topological zeta function associated to* f is the rational function in one complex variable $Z_{top,f}(s) := \sum_{I \subset S} \chi(E_I^\circ \cap h^{-1}\{0\}) \prod_{i \in I} \frac{1}{N_i s + \nu_i}$. Denef and Loeser proved in [DL] that these definitions are independent of the choice of the resolution.

The poles of the local topological zeta function are part of the set $\{-\nu_i/N_i \mid i \in S\}$; therefore this set is called a set of candidate poles. Various conjectures, such as the monodromy conjecture and the holomorphy conjecture, relate the poles to the eigenvalues of the local monodromy of f (see for example [De]). A very remarkable fact is that most of the candidate poles are cancelled in the topological zeta function. In general, it is not known how to see whether a candidate pole is a pole or not. Only for plane curves there exists a complete criterion. In [Ve] Veys showed:

Theorem 1. *Let $f \in \mathbb{C}[x, y]$ be a non-constant polynomial satisfying $f(0) = 0$, and let $\pi : X \to \mathbb{C}^2$ be the minimal embedded resolution of $f^{-1}\{0\}$ in a neighbourhood of 0. Then s_0 is a pole of $Z_{top,f}(s)$ if and only if $s_0 = -\frac{\nu_i}{N_i}$ for some exceptional curve E_i intersecting at least three times other components or $s_0 = -\frac{1}{N_i}$ for some irreducible component E_i of the strict transform of $f = 0$.*

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1.2 The topological zeta function out of the Newton polyhedron.– Let $f \in \mathbb{C}[x_1,\ldots,x_n]$ be a non-constant polynomial satisfying $f(0)=0$. We write $f=\sum_{k\in\mathbb{Z}^n_{\geq 0}} a_k x^k$, where $k=(k_1,\ldots,k_n)$ and $x^k = x_1^{k_1}\cdot\ldots\cdot x_n^{k_n}$. The *support of* f is $\operatorname{supp} f := \{k \in \mathbb{Z}^n_{\geq 0} \mid a_k \neq 0\}$. The *Newton polyhedron* Γ_0 *of* f *at the origin* is the convex hull in $\mathbb{R}^n_{\geq 0}$ of $\bigcup_{k\in\operatorname{supp} f} k+\mathbb{R}^n_{\geq 0}$. A *face* of the Newton polyhedron is the intersection of Γ_0 with a supporting hyperplane. A *facet* is a face of dimension $n-1$. A polynomial $f(x_1,\ldots,x_n)$ is called *nondegenerate with respect to its Newton polyhedron* Γ_0 if for every compact face τ of Γ_0 the polynomials $f_\tau := \sum_{k\in\tau} a_k x^k$ and $\partial f_\tau/\partial x_i, 1\leq i\leq n$, have no common zeroes in $(\mathbb{C}\backslash\{0\})^n$.

If $a_1,\ldots,a_r \in \mathbb{R}^n\backslash\{0\}$, then $\operatorname{cone}(a_1,\ldots,a_r) := \{\sum_{i=1}^r \lambda_i a_i \mid \lambda_i \in \mathbb{R}, \lambda_i > 0\}$. If Δ can be written as $\operatorname{cone}(a_1,\ldots,a_r)$, with $a_1,\ldots,a_r \in \mathbb{Z}^n\backslash\{0\}$ linearly independent over $\mathbb{R}$, then Δ is called a *simplicial cone.* For a simplicial cone Δ spanned by the primitive and linearly independent vectors $a_1,\ldots,a_r \in \mathbb{Z}^n$, the *multiplicity* of Δ, denoted by $\operatorname{mult}(\Delta)$, is the index of the lattice $\mathbb{Z}a_1+\ldots+\mathbb{Z}a_r$ in the group of the points with integral coordinates of the vector space generated by $a_1,\ldots,a_r$. Then $\operatorname{mult}(\Delta)$ is equal to the greatest common divisor of the determinants of the $(r\times r)$-matrices obtained by omitting columns from the matrix A with rows $a_1,\ldots,a_r$.

Let Γ_0 be a Newton polyhedron in $\mathbb{R}^n$. For $a=(a_1,\ldots,a_n)\in\mathbb{R}^n_{\geq 0}$ we put $N(a) := \inf_{x\in\Gamma_0} a\cdot x, \nu(a) := \sum_{i=1}^n a_i$ and $F(a) := \{x\in\Gamma_0 \mid a\cdot x = N(a)\}$. All $F(a), a\neq 0$, are faces of Γ_0. To a face τ of Γ_0 one associates a dual cone $\tau^\circ \subset \mathbb{R}^n$, defined as the closure in $\mathbb{R}^n$ of $\{a\in\mathbb{R}^n_{\geq 0} \mid F(a)=\tau\}$. This is a cone of dimension $n-\dim\tau$ with vertex in the origin. For a facet τ, one has $\tau^\circ = a\mathbb{R}_{\geq 0}$ for some primitive $a\in\mathbb{Z}^n_{\geq 0}$, and then the equation of the hyperplane through τ is $a\cdot x = N(a)$. We also use the notation $N(\tau)$ and $\nu(\tau)$, meaning respectively $N(a)$ and $\nu(a)$ for this associated $a\in\mathbb{Z}^n_{\geq 0}$. The set $\{\tau^\circ \mid \tau \text{ face of } \Gamma_0\}$ defines a subdivision of $\mathbb{R}^n_{\geq 0}$ and is called the *normal fan* to Γ_0. In 1976 Varchenko proved in [Var] that the map from the toric variety corresponding to a regular subdivision of the normal fan to $\mathbb{C}^n$ is an embedded resolution for all polynomials having Γ_0 as Newton polyhedron in the origin and that are nondegenerate with respect to Γ_0. Denef and Loeser used this to provide a formula for the local topological zeta function out of the Newton polyhedron.

Suppose $\Delta = \mathbb{R}_{\geq 0}a_1+\cdots+\mathbb{R}_{\geq 0}a_r$, with $a_1,\ldots,a_r\in\mathbb{Z}^n_{\geq 0}$ linearly independent and primitive. They define $J_\Delta(s) := \frac{\operatorname{mult}(\Delta)}{\prod_{i=1}^r (N(a_i)s+\nu(a_i))}$ and to an arbitrary face τ of Γ_0 they associate the rational function $J_\tau(s) := \sum_{i=1}^k J_{\Delta_i}(s)$, with $\tau^\circ = \cup_{i=1}^k \Delta_i$ a decomposition of τ° into simplicial cones Δ_i of dimension $\ell = \dim\tau^\circ$ satisfying $\dim(\Delta_i\cap\Delta_j) < \ell$ if $i\neq j$.

Theorem 2. [DL, Théorème 5.3] *If* f *is nondegenerate with respect to* Γ_0, *then the local topological zeta function is equal to*

$$Z_{top,f}(s) = \sum_{\tau \text{ vertex of } \Gamma_0} J_\tau(s) + \frac{s}{s+1} \sum_{\substack{\tau \text{ compact} \\ \text{face of } \Gamma_0, \\ \dim\tau\geq 1}} (-1)^{\dim\tau}(\dim\tau)!\operatorname{Vol}(\tau)J_\tau(s).$$

For a face τ of dimension 0, $\operatorname{Vol}(\tau) := 1$. For every other compact face $\operatorname{Vol}(\tau)$ is the *volume* of τ for the volume form ω_τ. This is a volume form on $\operatorname{Aff}(\tau)$, the affine space spanned by τ, such that the parallelepiped spanned by a lattice-basis of $\mathbb{Z}^n\cap\operatorname{Aff}(\tau)$ has volume 1. The product $(\dim\tau)!\operatorname{Vol}(\tau)$ is also called the *normalized volume of* τ. If τ is a simplicial facet, this normalized volume is equal to the multiplicity of the cone spanned by the vertices divided by $N(\tau)$.

Theorem 2 yields another set of candidate poles (containing all poles) of the local topological zeta function, namely -1 together with the rational numbers $-\nu(\tau)/N(\tau)$ for τ a facet of Γ_0. We will say that such a facet *contributes* the candidate pole. In the following section we will give a criterion for a candidate pole of this set to be a pole of the local topological zeta function.

Remark 1. There is also a graphical way to determine the candidate pole contributed by a facet τ. If $(r, \ldots, r)$ is the intersection point of the diagonal of the first quadrant with the affine hyperplane containing τ, then the candidate pole $-\nu(\tau)/N(\tau)$ is equal to $-1/r$.

2. Description of the poles in terms of the Newton polyhedron

We will say that a facet of a 2-dimensional Newton polyhedron is a B_1*-facet* with respect to the variable x (resp. to the variable y) if it has one vertex in the coordinate hyperplane $x = 0$ (resp. $y = 0$) and one vertex at distance one of this hyperplane.

Theorem 3. *Let f be a complex polynomial in two variables. Suppose that f is nondegenerate with respect to its Newton polyhedron Γ_0. Then for a candidate pole $s_0 \neq -1$ contributed by some facet of Γ_0 it holds: s_0 is a pole of $Z_{top,f}$ if and only if s_0 is contributed by a facet of Γ_0 that is no B_1-facet.*

Proof. Suppose first that s_0 is a candidate pole of order 2. Let σ and τ be facets such that $s_0 = \nu(\tau)/N(\tau) = \nu(\sigma)/N(\sigma) \neq 1$, having exactly one point in common. According to Remark 1, the picture should be as in Figure 1. Note that τ or σ might not be compact. The facets τ

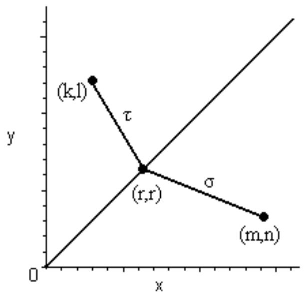

Figure 1

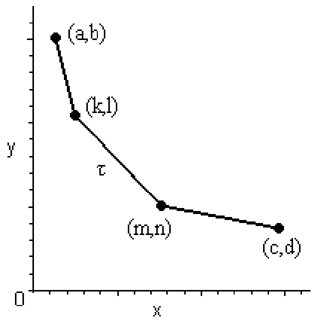

Figure 2

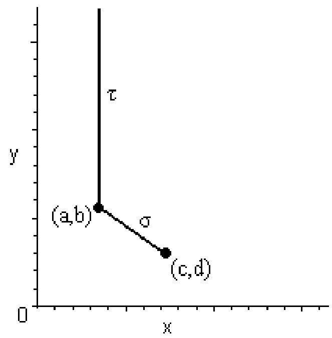

Figure 3

and σ are no B_1-facets, since otherwise r should be equal to 1 and $s_0 = -1$. When looking at the formula for the topological zeta function given in Theorem 2, it is obvious that a candidate pole of order two is also a pole of order two.

Suppose now that s_0 is a candidate pole of order 1 and suppose first that s_0 is contributed by a compact facet τ. Then we fix notation as in Figure 2. Note that $k \neq l$ and $m \neq n$. We sum the contributions to the local topological zeta function coming from the segment τ and the points with coordinates (k, l) and (m, n) (see Theorem 2). If g is the normalized volume of τ (which implies $g = \gcd(m - k, l - n)$), the contribution of τ is

$$-\frac{sg^2}{(s+1)((lm-kn)s+l-n+m-k)},$$

the contribution of the point with coordinates (k, l) is given by

$$\frac{(b-l)(m-k)-(l-n)(k-a)}{((lm-kn)s+l-n+m-k)((bk-al)s+b-l+k-a)}$$

and the contribution of (m, n) is

$$\frac{(l-n)(c-m)-(n-d)(m-k)}{((lm-kn)s+l-n+m-k)((nc-md)s+n-d+c-m)}.$$

Once summed these 3 contributions, we compute the residue Res at $s = -(l-n+m-k)/(lm-kn)$:

$$\text{Res} = \frac{(l-n+m-k)((ml-nk)(ml-nk+k-m+n-l)+g^2(n-m)(k-l))}{(lm-kn)(n-m)(k-l)(ml-nk+k-m+n-l)}.$$

If τ is a B_1-facet, then $k=0, m=1, g=1$ or $n=0, l=1, g=1$. It is easy to calculate that then $\text{Res}=0$. From now on, we suppose τ is no B_1-facet. First we note that $ml-nk+k-m+n-l>0$. This follows from the fact that a candidate pole in dimension two is always bigger than or equal to -1. Since we suppose $s_0 \neq -1$, this leads to $\frac{l-n+m-k}{lm-kn} < 1$.

If τ is intersecting the diagonal of the first quadrant, then there can clearly not exist another facet τ yielding this candidate pole. To show that the candidate pole $s_0 = -\nu(\tau)/N(\tau)$ is a pole of $Z_{top,f}$, it is thus sufficient to prove that $\text{Res} \neq 0$. As in this situation $k<l, k<m, n<m, n<l$, it follows that $\text{Res}=0$ if and only the factor

$$F := (ml-nk)(ml-nk+k-m+n-l)+g^2(n-m)(k-l) = 0.$$

We have $ml-nk>0, k-l<0, n-m<0$ and $ml-nk+k-m+n-l>0$. This leads to $F>0$ and $\text{Res} \neq 0$.

Suppose now that τ is lying above the diagonal in the first quadrant. Then we have $k<m<n<l$. We prove now that $\text{Res}<0$. It is easy to see that this is equivalent to $F>0$. We observe that

$$\begin{aligned} & (ml-nk)(ml-nk+k-m+n-l)-(l-k)(n-m)(m-k)^2 \\ = \ & lm(m-k)(m-k-1)+k(n-m)(m-k)^2+kn(m-k) \\ & +(l-n)(lm(m-1)-nk^2)+kn(l-n). \end{aligned}$$

All terms in this summand are greater or equal than 0 and we find that the total expression is equal to 0 exactly when $k=0$ and $m=1$ or equivalently, when τ is a B_1-facet. If τ is not a B_1-facet, then we find $F>0$ because $g^2 \leq (m-k)^2$.

If τ is a facet lying under the diagonal, then one can permute n with k and m with l. We get the same conclusion: the residue is strictly negative unless the facet is a B_1-facet.

Now we are left with the case that s_0 is a candidate pole of order 1, contributed by a facet τ that is not compact. Suppose $\text{Aff}(\tau) \leftrightarrow x=a$ and (a,b), (c,d) and σ are as in Figure 3. (The line segment σ might not be compact, in that case $d=b$.) Then the candidate pole $s_0=-1/a$ only turns up in the term of the local topological zeta function originating from the point (a,b). As τ induces a candidate pole of order 1, one has that $a \neq b$. This term is equal to

$$\frac{c-a}{(as+1)((bc-ad)s+b-d+c-a)}.$$

The residue in s_0 is $\frac{1}{(a-b)}$. As in the case of a compact facet, when τ intersects the diagonal of the first quadrant, then s_0 is only contributed by τ and thus s_0 is a pole. If τ is lying completely above the diagonal, then $\frac{1}{(a-b)} < 0$. Also for non-compact facets parallel with the x-axis one gets the same conclusion.

Hence, contributions coming from different facets (compact or not compact) do not cancel each other. This ends the proof. □

Remark 2. Notice that the computed residues do not depend on the neighbour segments of τ, although they are used in the computation of the residue.

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