RECURSIVE RELATIONS FOR BLOCK HANKEL AND TOEPLITZ SYSTEMS

PART I : DIRECT RECURSIONS

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Part 1 : direct recursions

Abstract

A set of formulas is given for the relations that exist between the first or last block row or column of the inverse of a block Hankel or Toeplitz matrix. This is related to making arbitrary steps in a matrix Padé table.

1. Introduction

In [1] a number of recursive algorithms for the matrix Padé table are given. These algorithms compute rows, sawteeth, diagonal and staircases in the table. In this report we give formulas to make arbitrary steps in the table.

We start with introducing some notation :

$I_k = (\delta_{ij})$ is the $k \times k$ unit matrix and $J_k = (\delta_{k-i,j})$ the $k \times k$ anti-unit matrix. By $A \otimes B$ we denote the Kronecker product : $A \otimes B = (a_{ij}B)$.

e_k, $k=0,1,\ldots$ are unit column vectors of approriate dimensions. We shall also need cutting operators : $\lrcorner = [\, I_n \,|\, 0 \,] \otimes I_p$ and $\urcorner = [\, 0 \,|\, I_n \,] \otimes I_p$. Each block will be supposted to have size $p \times p$.

A superscript ^{t} is used to indicate the block transpose.

If $\{t_k\}_{-\infty}^{\infty}$ is a sequence of $p \times p$ matrices, then we define

$$T_n = \begin{bmatrix} t_0 & \cdots & t_{-n} \\ \vdots & \ddots & \vdots \\ t_n & \cdots & t_0 \end{bmatrix} \quad , \quad \downarrow T_n = \begin{bmatrix} t_1 & \cdots & t_{-n+1} \\ \vdots & \ddots & \vdots \\ t_{n+1} & \cdots & t_1 \end{bmatrix}$$

and

$$\uparrow T_n = \begin{bmatrix} t_{-1} & \cdots & t_{-n-1} \\ \vdots & \ddots & \vdots \\ t_{n-1} & \cdots & t_{-1} \end{bmatrix}$$

and for a similar sequence $\{h_n\}_{-\infty}^{\infty}$ we have

$$H_n = \begin{bmatrix} h_0 & \cdots & h_n \\ \vdots & \iddots & \vdots \\ h_n & \cdots & h_{2n} \end{bmatrix} \quad , \quad {}_{\downarrow}H_n = \begin{bmatrix} h_1 & \cdots & h_{n+1} \\ \vdots & \iddots & \vdots \\ h_{n+1} & \cdots & h_{2n+2} \end{bmatrix} \quad \text{and}$$

$${}_{\uparrow}H_n = \begin{bmatrix} h_{-1} & \cdots & h_{n-1} \\ \vdots & \iddots & \vdots \\ h_{n-1} & \cdots & h_{2n-2} \end{bmatrix} .$$

2. Matrix Padé approximants

The notion of matrix Padé approximant has been given e.g. in [1]. We don't repeat everything here but recall only that the right denominator $Q^{[m/n]}(x)$ of the (m,n) approximant satisfies

$$T^{[m/n]}\, Q^{[m/n]} = e_0 \otimes p_m^{[m/n]} \qquad (1)$$

or

$$T^{[m+1/n]}\, Q^{[m/n]} = e_n \otimes r_0^{[m/n]} \qquad (2)$$

$$, \quad T^{[i/j]} = \begin{bmatrix} f_i & \cdots & f_{i-j} \\ \vdots & & \vdots \\ f_{i+j} & \cdots & f_i \end{bmatrix} \quad \text{with } f_k$$

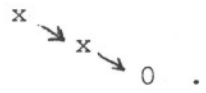
coefficients of the given formal power series $\sum_0^\infty f_k z^k$.

Negatively indexed f_k are supposed to be zero. $Q^{[m/n]} = [\, q_n^{[m/n]} \ldots q_0^{[m/n]} \,]^t$ is the block vector with the coefficients of $Q^{[m/n]}(z)$ in the order of descending powers of z.

(1) or (2) have to be completed with a normalizing condition for $Q^{[m/n]}(z)$. We choose it to be $q_0^{[m/n]} = I$.

In [1] recursive algorithms are given which use e.g. relations between the denominator vectors $Q^{[m/n-1]}$, $Q^{[m/n]}$ and $Q^{[m/n+1]}$. They are the solutions of $T_{n-1}Q_{n-1} = e_{n-1} \otimes r_{n-1}$, $T_nQ_n = e_n \otimes r_n$ and $T_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$ respectively with $r_k = r^{[m/k]}$, $T_k = T^{[m/k]}$ and $Q_k = Q^{[m/k]}$ for k = n-1,n,n+1. These denominators are related to three successive Padé approximants on a row of the Padé table. Therefore we can represent the relation as $x \to x \to 0$. The crosses represent known quantities on the row and the circle is the quantity computed from it on the same row. A similar recursion exists for three successive elements on a diagonal of the Padé table. This relation is connected with the solution

of three nested Hankel matrices. We represent this relation as



In this report we shall give formulas for other steps that we can imagine in a Padé table, like



We have to remark that all these recursions can only work if we suppose a certain normality condition for the Padé table, so that all the matrices we need to invert are nonsingular. In what follows we shall suppose that this condition is satisfied whereever needed.

3. How to use the formulas

For each possible step, we shall give the Hankel and Toeplitz formulations. The proofs are left behind since they do not differ essentially from those given in [1]. The elements of Q_n are $[q_n^0 \ldots q_n^n]^t$, those of P_n are $[p_n^0 \ldots p_n^n]^t$ etc. The elements of a Toeplitz matrix are denoted by t_k : e.g. $T_n = (t_{i-j})_0^n$ and those of a Hankel matrix by h_k : $H_n = (h_{i+j})_0^n$.

Following the ! sign will be given the systems of which the solution is known. After the question mark ? we give the system to be solved. The indexing of h_k or t_k will always refer to the latter system. E.g. if we have

$$?T_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

then $T_{n+1} = (t_{i-j})_{ij=0}^{n+1}$ etc.

The index$_n$ of T_n or H_n will always refer to the dimension of the matrix. The capitals T and S are reserved for Toeplitz matrices and H and K for Hankel matrices. For the solutions of the systems we reserve Q and P, and for the non-zero righthand side blocks r,s,v,w.

There is a certain redundancy in the formulas. Indeed, if we know a recursion for Q_k, being the solution of $T_kQ_k = e_k \otimes r_k$, then we immediately know the recursion for the solution of a number of other problems :

Indeed, suppose we have as for F1 :

a) $!T_{n-1}Q_{n-1} = e_{n-1} \otimes r_{n-1}$

$T_n \; Q_n = e_n \otimes r_n$ with normalization $q_k^k = I$

$?T_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$

$$Q_{n+1} = \begin{bmatrix} 0 \\ Q_n \end{bmatrix} + \begin{bmatrix} Q_{n-1} \\ 0 \\ 0 \end{bmatrix} a_{n+1} + \begin{bmatrix} Q_n \\ 0 \end{bmatrix} b_{n+1}$$

$$r_{n+1} = r_{n-1}a_{n+1} + r_n b_{n+1}$$

$$a_{n+1} = -s_{n-1}^{-1} s_n$$

$$b_{n-1} = -s_n^{-1}(s'_n + s'_{n-1}a_{n+1}) = -s_n^{-1}(s'_n s_n^{-1} + s'_{n-1}s_{n-1}^{-1})s_n$$

with

$$\begin{bmatrix} s'_{n-k} \\ s_{n-k} \end{bmatrix} = \begin{bmatrix} t_{k-1} & \cdots & t_{2k-n-1} \\ t_k & \cdots & t_{2k-n} \end{bmatrix} Q_{n-k}, \quad k=0,1.$$

If $T_k Q_k = e_k \otimes r_k$, then $[T_k(J \otimes I)][(J \otimes I)Q_k] = e_k \otimes r_k$.

Thus it is easily verified that if $H_k = T_k(J \otimes I)$, then with

$P_k = (J \otimes I)Q_k$, we have

b) $!H_{n-1}P_{n-1} = e_{n-1} \otimes r_{n-1}$

$H_n \; P_n = e_n \otimes r_n$ with $p_0^k = I$

$?H_{n+1}P_{n+1} = e_{n+1} \otimes r_{n+1}$

where

$$P_{n+1} = \begin{bmatrix} P_n \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ P_{n-1} \end{bmatrix} a_{n+1} + \begin{bmatrix} 0 \\ P_n \end{bmatrix} b_{n+1}$$

$$r_{n+1} = r_{n-1}a_{n+1} + r_n b_{n+1}$$

with a_{n+1} and b_{n+1} as before and where

$$\begin{bmatrix} s'_{n-k} \\ s_{n-k} \end{bmatrix} = \begin{bmatrix} h_{2k} & \cdots & h_{n+k} \\ h_{2k+1} & \cdots & h_{n+k+1} \end{bmatrix} P_{n-k}, \quad k=0,1$$

i.e. replace t_i by h_{i+n+1} and arrange in opposite order.

With the $\uparrow$ operation we define $\underline{T}_k = \uparrow T_k$ so that we have for the same Q_k that

a') $$!\underline{T}_{n-1} Q_{n-1} = e_0 \otimes s_{n-1}$$
$$\underline{T}_n \; Q_n = e_0 \otimes s_n \qquad \text{with } q_k^k = I.$$
$$?\underline{T}_{n+1} Q_{n+1} = e_0 \otimes s_{n+1}$$

Because the Q_k vectors are the same they obey the same recursion as given in a) but the indices of t-elements are augmented with 1 : I.e.

$$\begin{bmatrix} s'_{n-k} \\ s_{n-k} \end{bmatrix} = \begin{bmatrix} \underline{t}_k & \cdots & \underline{t}_{2k-n} \\ \underline{t}_{k+1} & \cdots & \underline{t}_{2k-n+1} \end{bmatrix} Q_{n-k}, \quad k=0,1.$$

Similarly if we define $\underline{H}_k = \uparrow H_k$, then we have

b') $$!\underline{H}_{n-1} P_{n-1} = e_0 \otimes s_{n-1}$$
$$\underline{H}_n \; P_n = e_0 \otimes s_n \qquad \text{with } p_0^k = I$$
$$?\underline{H}_{n+1} P_{n+1} = e_0 \otimes s_{n+1}$$

with the same recursion as in b) but with

$$\begin{bmatrix} s'_{n-k} \\ s_{n-k} \end{bmatrix} = \begin{bmatrix} \underline{h}_{2k+1} & \cdots & \underline{h}_{n+k+1} \\ \underline{h}_{2k+2} & \cdots & \underline{h}_{n+k+2} \end{bmatrix} P_{n-k}, \quad k=0,1.$$

Now we can apply $(J \otimes I)$ from the left to the problems b') and a'), thus we obtain with $\tilde{\underline{T}}_k = (J \otimes I)\underline{H}_k$ and $\tilde{\underline{H}}_k = (J \otimes I)\underline{T}_k$.

c) $$!\tilde{\underline{T}}_{n-1} P_{n-1} = e_{n-1} \otimes s_{n-1}$$
$$\tilde{\underline{T}}_n \; P_n = e_n \otimes s_n \qquad \text{with } p_0^k = I.$$
$$?\tilde{\underline{T}}_{n+1} P_{n+1} = e_{n+1} \otimes s_{n+1}$$

We have again the same recursion as in b) but with

$$\begin{bmatrix} s'_{n-k} \\ s_{n-k} \end{bmatrix} = \begin{bmatrix} \tilde{\underline{t}}_{n-2k} & \cdots & \tilde{\underline{t}}_{-k} \\ \tilde{\underline{t}}_{n-2k-1} & \cdots & \tilde{\underline{t}}_{-k-1} \end{bmatrix} P_{n-k}, \quad k=0,1$$

i.e. replace $\underline{h}_i$ by $\tilde{\underline{t}}_{n+1-i}$

and

d) $!\tilde{H}_{n-1}Q_{n-1} = e_{n-1} \otimes s_{n-1}$

$\tilde{H}_n \ Q_n = e_n \otimes s_n$ with $q_k^k = I$

$?\tilde{H}_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$

The same recursion as in a) with

$$\begin{bmatrix} s'_k \\ s_k \end{bmatrix} = \begin{bmatrix} \tilde{h}_{k+1} & \cdots & \tilde{h}_{2k+1} \\ \tilde{h}_k & \cdots & \tilde{h}_{2k} \end{bmatrix} Q_k \ , \ k = n-1,n.$$

i.e. replace $\underline{t}_i$ by $\tilde{h}_{n+1-i}$.

Like we derived a') and b') from a) and b), we can derive c') and d') from c) and d). This gives with the notation $\tilde{\underline{T}}_k = \uparrow\tilde{T}_k$ and $\tilde{\underline{H}}_k = \uparrow\tilde{H}_k$.

c') $!\tilde{\underline{T}}_{n-1}P_{n-1} = e_0 \otimes r_{n-1}$

$\tilde{\underline{T}}_n \ P_n = e_0 \otimes r_n$ with $p_0^k = I$

$?\tilde{\underline{T}}_{n+1}P_{n+1} = e_0 \otimes r_{n+1}$

The same formula as in b) with

$$\begin{bmatrix} s'_{n-k} \\ s_{n-k} \end{bmatrix} = \begin{bmatrix} \tilde{\underline{t}}_{n-2k+1} & \cdots & \tilde{\underline{t}}_{1-k} \\ \tilde{\underline{t}}_{n-2k} & \cdots & \tilde{\underline{t}}_{-k} \end{bmatrix} P_{n-k} \ , \ k = 0,1.$$

d') $!\tilde{\underline{H}}_{n-1}Q_{n-1} = e_0 \otimes r_{n-1}$

$\tilde{\underline{H}}_n \ Q_n = e_0 \otimes r_n$, $q_k^k = I$

$?\tilde{\underline{H}}_{n+1}Q_{n+1} = e_0 \otimes r_{n+1}$

The same formula as in a) with

$$\begin{bmatrix} s'_k \\ s_k \end{bmatrix} = \begin{bmatrix} \tilde{\underline{h}}_{k+2} & \cdots & \tilde{\underline{h}}_{2k+2} \\ \tilde{\underline{h}}_{k+1} & \cdots & \tilde{\underline{h}}_{2k+1} \end{bmatrix} Q_k \ , \ k = n-1,n.$$

In the following we shall give the formulations (a), (b), (c) and (d), (a'), (b'), (c') and (d') are left to the reader since their formulation is very simple to derive.

The normalization for (a) and (d) is always $q_k^k = I$, and for (b) and (c) $q_0^k = I$.

If another normalization for Q_{n+1} is desired, then we have to add an additional normalization step. E.g. if in problem (a) we had $q_0^k = I$, $k = n-1,n,n+1$ then it is easily seen that

$$Q_{n+1} = \left(\begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} + \begin{bmatrix} Q_{n-1} \\ 0 \\ 0 \end{bmatrix} a_{n+1} + \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} b_{n+1} \right) \Lambda_{n+1}$$

with $\Lambda_{n+1} = (a_{n+1} + b_{n+1})^{-1}$ and a_{n+1} and b_{n+1} as before. Since the first coefficient in the expression between brackets is $(0 + Ia_{n+1} + Ib_{n+1})$.

Here is a scheme that classifies the possible steps. It gives the number of the formulas to be used. E.g. to compute the step $x \to x^{\nearrow 0}$, use formula F4.

F21	F17	F6	F4
F14	X	X	F2
F21	F17	F6	F4

F22	F12	F8
F18	X	F5
F18	X	F5
F22	F12	F8

	F23	F11	F1
F15	F19	X	F9
F16	X	F7	F3
F13	F20	F10	

F13	F20	F10	
F16	X	F7	F3
F15	F19	X	F9
	F23	F11	F1

References

1. A. Bultheel : Recursive algorithms for the matrix Padé table. Math. Comp. 35 (151), 875-892, 1980.

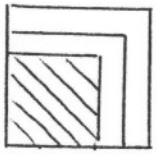
F 1

a

$$!T_{n-1}\,Q_{n-1} = e_{n-1} \otimes r_{n-1}$$

$$T_n\,Q_n = e_n \otimes r_n$$

$$?T_{n+1}\,Q_{n+1} = e_{n+1} \otimes r_{n+1}$$



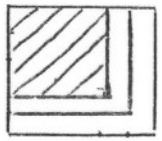
$$T_n = \lceil T_{n+1} \rfloor^t$$

$$T_{n-1} = \lceil T_n \rfloor^t$$

$$\begin{bmatrix} s'_{n-k} \\ s_{n-k} \end{bmatrix} = \begin{bmatrix} t_{k-1} \cdots t_{2k-n-1} \\ t_k \quad \cdots t_{2k-n} \end{bmatrix} Q_{n-k} \quad , \; k = 0,1$$

$$Q_{n+1} = \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} + \begin{bmatrix} Q_{n-1} \\ 0 \\ 0 \end{bmatrix} a_{n+1} + \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} b_{n+1}$$

$$\begin{bmatrix} s'_k \\ s_k \end{bmatrix} = \begin{bmatrix} h_{k+1} \cdots h_{2k+1} \\ h_k \quad \cdots h_{2k} \end{bmatrix} Q_k \quad , \; k = n-1, n$$



$$H_n = \lfloor H_{n+1} \rfloor^t$$

$$H_{n-1} = \lfloor H_n \rfloor^t$$

$$!H_{n-1}\,Q_{n-1} = e_{n-1} \otimes s_{n-1}$$

$$H_n\,Q_n = e_n \otimes s_n$$

$$?H_{n+1}\,Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

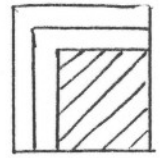
d

b

$$!H_{n-1}\,Q_{n-1} = e_{n-1} \otimes r_{n-1}$$

$$H_n\,Q_n = e_n \otimes r_n$$

$$?H_{n+1}\,Q_{n+1} = e_{n+1} \otimes r_{n+1}$$



$$H_n = \lceil H_{n+1} \rceil^t$$

$$H_{n-1} = \lceil H_n \rceil^t$$

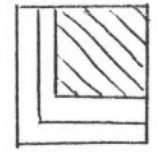
$$\begin{bmatrix} s'_{n-k} \\ s_{n-k} \end{bmatrix} = \begin{bmatrix} h_{2k} \cdots h_{n+k} \\ h_{2k+1} \cdots h_{n+k+1} \end{bmatrix} Q_{n-k} \quad , \; k = 0,1$$

$$a_{n+1} = -s_{n-1}^{-1}\, s_n$$

$$b_{n+1} = -s_n^{-1}(s'_n + s'_{n-1} a_{n+1})$$

$$Q_{n+1} = \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ Q_{n-1} \end{bmatrix} a_{n+1} + \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} b_{n+1}$$

$$\begin{bmatrix} s'_k \\ s_k \end{bmatrix} = \begin{bmatrix} t_{2k-n} \cdots t_{k-n} \\ t_{2k-n-1} \cdots t_{k-n-1} \end{bmatrix} Q_k \quad , \; k = n-1, n$$



$$T_n = \lfloor T_{n+1} \rceil^t$$

$$T_{n-1} = \lfloor T_n \rceil^t$$

$$!T_{n-1}\,Q_{n-1} = e_{n-1} \otimes s_{n-1}$$

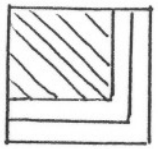
$$T_n\,Q_n = e_n \otimes s_n$$

$$?T_{n+1}\,Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

c

a F 2 b

$$!T_{n-1}Q_{n-1} = e_{n-1} \otimes r_{n-1}$$
$$T_n\ Q_n = e_n \otimes r_n$$
$$?T_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$



$$T_n = \lrcorner T_{n+1} \lrcorner^t$$
$$T_{n-1} = \lrcorner T_n \lrcorner^t$$

$$\begin{bmatrix} s_k \\ r_k \end{bmatrix} = \begin{bmatrix} t_{-1} \cdots t_{-k-1} \\ t_k \cdots t_0 \end{bmatrix} Q_k \ , \ k = n-1, n$$

x→x→0

$$\begin{bmatrix} s_{n-k} \\ r_{n-k} \end{bmatrix} = \begin{bmatrix} h_k \cdots h_n \\ h_{n+1} \cdots h_{2n+1-k} \end{bmatrix} Q_{n-k} \ , \ k = 0, 1$$

$$Q_{n+1} = \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} + \begin{bmatrix} 0 \\ Q_{n-1} \\ 0 \end{bmatrix} a_{n+1} + \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} b_{n+1}$$

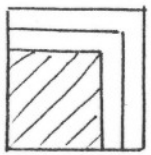
$$a_{n+1} = -s_{n-1}^{-1} s_n$$
$$b_{n+1} = -r_n^{-1} r_{n-1} a_{n+1}$$

$$Q_{n+1} = \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ Q_{n-1} \\ 0 \end{bmatrix} a_{n+1} + \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} b_{n+1}$$

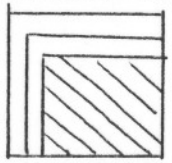
$$\begin{bmatrix} s_{n-k} \\ r_{n-k} \end{bmatrix} = \begin{bmatrix} h_{n+1} \cdots h_{2n+1-k} \\ h_k \cdots h_n \end{bmatrix} Q_{n-k} \ , \ k = 0, 1$$

x→x→0

$$\begin{bmatrix} s_k \\ r_k \end{bmatrix} = \begin{bmatrix} t_k \cdots t_0 \\ t_{-1} \cdots t_{-k-1} \end{bmatrix} Q_k \ , \ k = n-1, n$$



$$H_n = \ulcorner H_{n+1} \lrcorner^t$$
$$H_{n-1} = \ulcorner H_n \lrcorner^t$$



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d c

a

$$!T_{n-1}Q_{n-1} = e_{n+1} \otimes r_{n-1}$$

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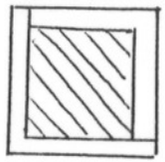


$$T_n = \lrcorner T_{n+1} \urcorner^t$$

$$T_{n-1} = \urcorner T_n \lrcorner^t$$

$$\left[\begin{array}{c|c} r_{n-1} & r_n \\ \hline r'_{n-1} & r'_n \end{array}\right] = \left[\begin{array}{ccc} t_{n-1} & \cdots & t_{-1} \\ t_n & \cdots & t_0 \end{array}\right] \left[\begin{array}{c|c} Q_{n-1} & \\ \hline 0 & Q_n \end{array}\right]$$

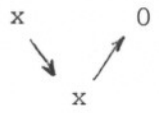


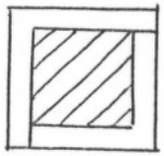
$$Q_{n+1} = \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} + \begin{bmatrix} Q_{n-1} \\ 0 \\ 0 \end{bmatrix} a_{n+1} + \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} b_{n+1}$$

$$a_{n+1} = -r_{n-1}^{-1}(r'_{n-1}r_{n-1}^{-1} - r'_n r_n^{-1})^{-1} r_n$$

$$b_{n+1} = r_n^{-1}(r'_{n-1}r_{n-1}^{-1} - r'_n r_n^{-1})^{-1} r_n$$

$$\left[\begin{array}{c|c} r_{n-1} & r_n \\ \hline r'_{n-1} & r'_n \end{array}\right] = \left[\begin{array}{ccc} h_1 & \cdots & h_{n+1} \\ h_0 & \cdots & h_n \end{array}\right] \left[\begin{array}{c|c} Q_{n-1} & \\ \hline 0 & Q_n \end{array}\right]$$





$$H_n = \urcorner H_{n+1} \urcorner^t$$

$$H_{n-1} = \lrcorner H_n \lrcorner^t$$

$$!H_{n-1}Q_{n-1} = e_{n-1} \otimes s_{n-1}$$

$$H_n\,Q_n = e_n \otimes s_n$$

$$?H_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

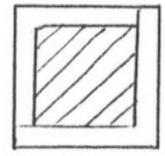
d

b

$$!H_{n-1}Q_{n-1} = e_{n-1} \otimes r_{n-1}$$

$$H_n\,Q_n = e_n \otimes r_n$$

$$?H_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$



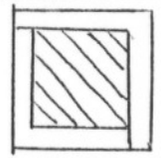
$$H_n = \lrcorner H_{n+1} \lrcorner^t$$

$$H_{n-1} = \urcorner H_n \urcorner^t$$

$$\left[\begin{array}{c|c} r_{n-1} & r_n \\ \hline r'_{n-1} & r'_n \end{array}\right] = \left[\begin{array}{ccc} h_n & \cdots & h_{2n} \\ h_{n+1} & \cdots & h_{2n+1} \end{array}\right] \left[\begin{array}{c|c} 0 & \\ \hline Q_{n-1} & Q_n \end{array}\right]$$

$$Q_{n+1} = \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ Q_{n-1} \end{bmatrix} a_{n+1} + \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} b_{n+1}$$

$$\left[\begin{array}{c|c} r_{n-1} & r_n \\ \hline r'_{n-1} & r_n \end{array}\right] = \left[\begin{array}{ccc} t_0 & \cdots & t_{-n} \\ t_{-1} & \cdots & t_{-(n+1)} \end{array}\right] \left[\begin{array}{c|c} 0 & \\ \hline Q_{n-1} & Q_n \end{array}\right]$$



$$T_n = \urcorner T_{n+1} \lrcorner^t$$

$$T_{n-1} = \lrcorner T_n \urcorner^t$$

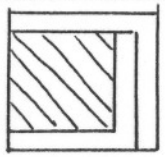
c

a

$$!T_{n-1}Q_{n-1} = e_{n-1} \otimes r_{n-1}$$

$$T_n\ Q_n = e_n \otimes r_n$$

$$?T_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$



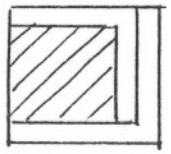
$$T_n = \lceil T_{n+1} \rfloor^t$$

$$T_{n-1} = \lfloor T_n \rfloor^t$$

$$\begin{bmatrix} s'_k \\ s_k \end{bmatrix} = \begin{bmatrix} t_{-1} \cdots t_{-(k+1)} \\ t_0 \cdots t_{-k} \end{bmatrix} Q_k \ , \ k = n-1,n$$

$$Q_{n+1} = \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} + \begin{bmatrix} 0 \\ Q_{n-1} \\ 0 \end{bmatrix} a_{n+1} + \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} b_{n+1}$$

$$\begin{bmatrix} s'_k \\ s_k \end{bmatrix} = \begin{bmatrix} h_{n+1} \cdots h_{n+k+1} \\ h_n \cdots h_{n+k} \end{bmatrix} Q_k \ , \ k = n-1,n$$



$$H_n = \lfloor H_{n+1} \rfloor^t$$

$$H_{n-1} = \lceil H_n \rfloor^t$$

$$!H_{n-1}Q_{n-1} = e_{n-1} \otimes s_{n-1}$$

$$H_n\ Q_n = e_n \otimes s_n$$

$$?H_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

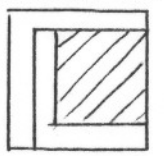
d

b

$$!H_{n-1}Q_{n-1} = e_{n-1} \otimes r_{n-1}$$

$$H_n\ Q_n = e_n \otimes r_n$$

$$?H_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$



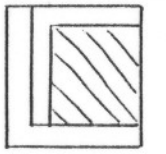
$$H_n = \lceil H_{n+1} \rceil^t$$

$$H_{n-1} = \lfloor H_n \rceil^t$$

$$\begin{bmatrix} s'_k \\ s_k \end{bmatrix} = \begin{bmatrix} h_{n-k} \cdots h_n \\ h_{n+1-k} \cdots h_{n+1} \end{bmatrix} Q_k \ , \ k = n-1,n$$

$$Q_{n+1} = \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ Q_{n-1} \\ 0 \end{bmatrix} a_{n+1} + \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} b_{n+1}$$

$$\begin{bmatrix} s'_k \\ s_k \end{bmatrix} = \begin{bmatrix} t_k \cdots t_0 \\ t_{k-1} \cdots t_{-1} \end{bmatrix} Q_k \ , \ k = n-1,n$$



$$T_n = \lfloor T_{n+1} \rceil^t$$

$$T_{n-1} = \lceil T_n \rceil^t$$

$$!T_{n-1}Q_{n-1} = e_{n-1} \otimes s_{n-1}$$

$$T_n\ Q_n = e_n \otimes s_n$$

$$?T_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

c

$$x \to x \nearrow 0$$

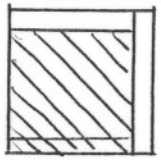
$$a_{n+1} = -s_{n-1}^{-1}\ s_n$$

$$b_{n+1} = -s_n^{-1}(s'_n + s'_{n-1}a_{n+1})$$

$$x \to x \searrow 0$$

a

$$!S_n\ P_n = e_n \otimes v_n$$
$$T_n\ Q_n = e_n \otimes r_n$$
$$?T_{n+1}\ Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

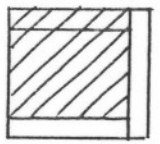


$$S_n = \downarrow T_n$$
$$T_n = \lrcorner T_{n+1} \lrcorner^t$$

$$w_n = [\, t_0 \ldots t_{-n} \,]\, P_n$$
$$s_n = [\, t_{-1} \ldots t_{-(n+1)} \,]\, Q_n$$

$$Q_{n+1} = \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} + \begin{bmatrix} P_n \\ \\ 0 \end{bmatrix} a_{n+1}$$

$$w_n = [\, h_n \ldots h_{2n} \,]\, P_n$$
$$s_n = [\, h_{n+1} \ldots h_{2n+1} \,]\, Q_n$$



$$K_n = \uparrow H_n$$
$$H_n = \urcorner H_{n+1} \lrcorner^t$$

$$!K_n\ P_n = e_n \otimes w_n$$
$$H_n\ Q_n = e_n \otimes s_n$$
$$?H_{n+1}\ Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

d

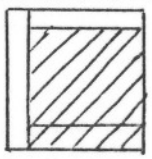
$$x \to 0 \qquad x \quad 0$$
$$\uparrow \qquad\quad \downarrow \nearrow$$
$$x \qquad x$$

$$a_{n+1} = -w_n^{-1}\, s_n$$

$$x \qquad x$$
$$\downarrow \qquad\quad \uparrow \searrow$$
$$x \to 0 \qquad x \quad 0$$

b

$$!K_n\ P_n = e_n \otimes v_n$$
$$H_n\ Q_n = e_n \otimes r_n$$
$$?H_{n+1}\ Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

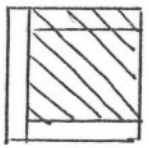


$$K_n = \downarrow H_n$$
$$H_n = \lrcorner H_{n+1} \urcorner^t$$

$$w_n = [\, h_1 \ldots h_{n+1} \,]\, P_n$$
$$s_n = [\, h_0 \ldots h_n \,]\, Q_n$$

$$Q_{n+1} = \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ P_n \end{bmatrix} a_{n+1}$$

$$w_n = [\, t_{n-1} \ldots t_{-1} \,]\, P_n$$
$$s_n = [\, t_n \ldots t_0 \,]\, Q_n$$



$$S_n = \uparrow H_n$$
$$H_n = \urcorner H_{n+1} \urcorner^t$$

$$!S_n\ P_n = e_n \otimes w_n$$
$$T_n\ Q_n = e_n \otimes s_n$$
$$?T_{n+1}\ Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

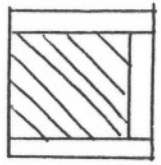
c

a

$$!S_n\,P_n = e_n \otimes v_n$$

$$S_{n+1}P_{n+1} = e_{n+1} \otimes v_{n+1}$$

$$?T_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$



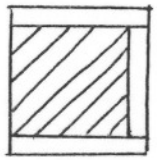
$$S_n = \lrcorner S_{n+1} \lrcorner^t$$

$$S_{n+1} = \downarrow T_{n+1}$$

$$w_k = [\,t_0 \ldots t_{-k}\,]\,P_k \quad , \; k = n, n+1$$

$$Q_{n+1} = P_{n+1} + \begin{bmatrix} P_n \\ \\ 0 \end{bmatrix} a_{n+1}$$

$$w_k = [\,h_n \ldots h_{n+k}\,]\,P_k \quad k = n, n+1$$



$$K_n = \rceil K_{n+1} \lrcorner^t$$

$$K_{n+1} = \uparrow H_{n+1}$$

$$!K_n\,P_n = e_n \otimes w_n$$

$$K_{n+1}P_{n+1} = e_{n+1} \otimes w_{n+1}$$

$$?H_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

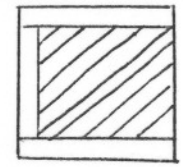
d

b

$$!K_n\,P_n = e_n \otimes v_n$$

$$K_{n+1}P_{n+1} = e_{n+1} \otimes v_{n+1}$$

$$?H_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$



$$K_n = \lrcorner K_{n+1} \urcorner^t$$

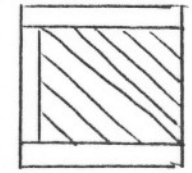
$$K_{n+1} = \downarrow H_{n+1}$$

$$w_k = [\,h_{n+1-k} \ldots h_{n+1}\,]\,P_k \quad , \; k = n, n+1$$

$$a_{n+1} = -w_n^{-1}\, w_{n+1}$$

$$Q_{n+1} = P_{n+1} + \begin{bmatrix} 0 \\ \\ P_n \end{bmatrix} a_{n+1}$$

$$w_k = [\,t_{k-1} \ldots t_{-1}\,]\,P_k \quad , \; k = n, n+1$$



$$S_n = \rceil S_{n+1} \urcorner^t$$

$$S_{n+1} = \uparrow T_{n+1}$$

$$!S_n\,P_n = e_n \otimes w_n$$

$$S_{n+1}P_{n+1} = e_{n+1} \otimes w_{n+1}$$

$$?T_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

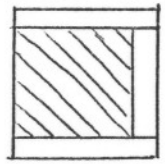
c

a

$$!S_{n+1}P_{n+1} = e_{n+1} \otimes v_{n+1}$$

$$T_n\ Q_n = e_n \otimes r_n$$

$$?T_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$



$$S_{n+1} = \uparrow T_{n+1}$$

$$T_n = \lrcorner T_{n+1} \lrcorner^t$$

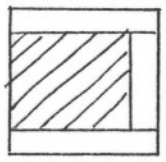
$$r_n = [t_n \ldots t_0]\, Q_n$$

$$v_{n+1} = [t_n \ldots t_{-1}]\, P_{n+1}$$

$$Q_{n+1} = P_{n+1} + \begin{bmatrix} Q_n \\ 0 \end{bmatrix} a_{n+1}$$

$$r_n = [h_0 \ldots h_n]\, Q_n$$

$$v_{n+1} = [h_0 \ldots h_{n+1}]\, P_{n+1}$$



$$K_{n+1} = \downarrow H_{n+1}$$

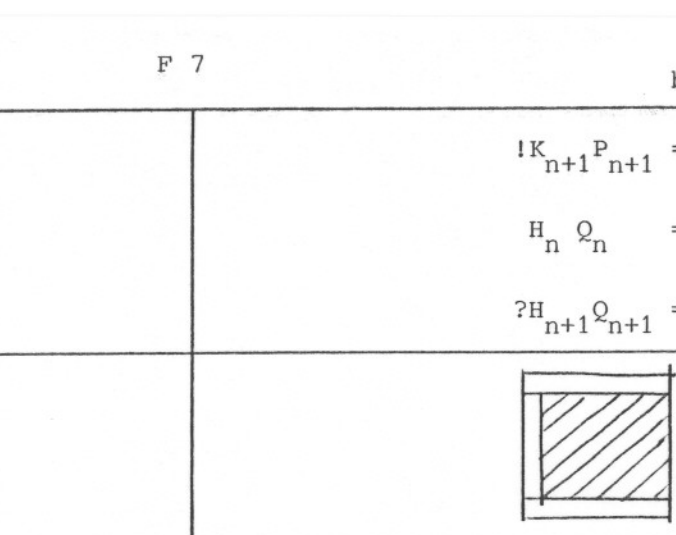
$$H_n = \urcorner H_{n+1} \llcorner^t$$

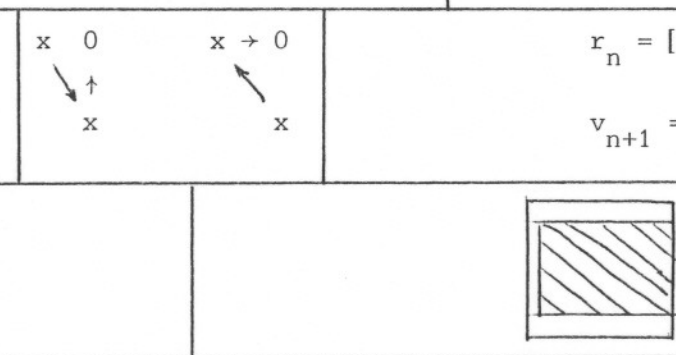
$$!K_{n+1}P_{n+1} = e_{n+1} \otimes w_{n+1}$$

$$H_n\ Q_n = e_n \otimes s_n$$

$$?H_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

d





$$a_{n+1} = -r_n^{-1}\, v_{n+1}$$

b

$$!K_{n+1}P_{n+1} = e_{n+1} \otimes v_{n+1}$$

$$H_n\ Q_n = e_n \otimes r_n$$

$$?H_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

$$K_{n+1} = \uparrow H_{n+1}$$

$$H_n = \lrcorner H_{n+1} \urcorner^t$$

$$r_n = [h_{n+1} \ldots h_{2n+1}]\, Q_n$$

$$v_{n+1} = [h_n \ldots h_{2n+1}]\, P_{n+1}$$

$$Q_{n+1} = P_{n+1} + \begin{bmatrix} 0 \\ Q_n \end{bmatrix} a_{n+1}$$

$$r_n = [t_{-1} \ldots t_{-(n+1)}]\, Q_n$$

$$v_{n+1} = [t_0 \ldots t_{-(n+1)}]\, P_{n+1}$$

$$S_{n+1} = \downarrow T_{n+1}$$

$$T_n = \urcorner T_{n+1} \urcorner^t$$

$$!S_{n+1}P_{n+1} = e_{n+1} \otimes w_{n+1}$$

$$T_n\ Q_n = e_n \otimes s_n$$

$$?T_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

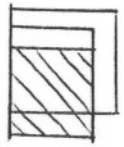
c

a

$$!T_n\ Q_n = e_n \otimes r_n$$

$$S_n\ P_n = e_n \otimes v_n$$

$$?T_{n+1}\ Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

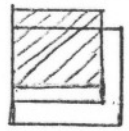


$$T_n = \lceil T_{n+1} \rfloor^t$$

$$S_n = \downarrow T_n$$

$$\begin{bmatrix} w'_n & s'_n \\ w_n & s_n \end{bmatrix} = \begin{bmatrix} t_0 \cdots t_{-(n+1)} \\ t_1 \cdots t_{-n} \end{bmatrix} \begin{bmatrix} P_n & 0 \\ 0 & Q_n \end{bmatrix}$$

$$Q_{n+1} = \begin{bmatrix} 0 \\ Q_n \end{bmatrix} + \begin{bmatrix} P_n \\ 0 \end{bmatrix} a_{n+1} + \begin{bmatrix} Q_n \\ 0 \end{bmatrix} b_{n+1}$$



$$H_n = \lfloor H_{n+1} \rfloor^t$$

$$K_n = \uparrow H_n$$

$$\begin{bmatrix} w'_n & s'_n \\ w_n & s_n \end{bmatrix} = \begin{bmatrix} h_n \cdots h_{2n+1} \\ h_{n-1} \cdots h_{2n} \end{bmatrix} \begin{bmatrix} P_n & 0 \\ 0 & Q_n \end{bmatrix}$$

$$!H_n\ Q_n = e_n \otimes s_n$$

$$K_n\ P_n = e_n \otimes w_n$$

$$?H_{n+1}\ Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

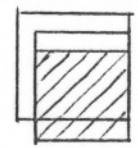
d

b

$$!H_n\ Q_n = e_n \otimes r_n$$

$$K_n\ P_n = e_n \otimes v_n$$

$$?H_{n+1}\ Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

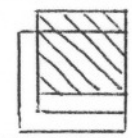


$$H_n = \lceil H_{n+1} \rceil^t$$

$$K_n = \downarrow H_n$$

$$\begin{bmatrix} w'_n & s'_n \\ w_n & s_n \end{bmatrix} = \begin{bmatrix} h_0 \cdots h_{n+1} \\ h_1 \cdots h_{n+2} \end{bmatrix} \begin{bmatrix} 0 & Q_n \\ P_n & 0 \end{bmatrix}$$

$$Q_{n+1} = \begin{bmatrix} Q_n \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ P_n \end{bmatrix} a_{n+1} + \begin{bmatrix} 0 \\ Q_n \end{bmatrix} b_{n+1}$$



$$T_n = \lfloor T_{n+1} \rceil^t$$

$$S_n = \uparrow T_n$$

$$!T_n\ Q_n = e_n \otimes s_n$$

$$S_n\ P_n = e_n \otimes w_n$$

$$?T_{n+1}\ Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

c

$$\begin{array}{c} 0 \\ \nearrow \\ x \\ \uparrow \\ x \end{array}$$

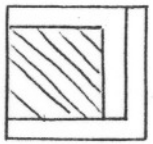
$$a_{n+1} = -w_n^{-1}\ s_n$$

$$b_{n+1} = -s_n^{-1}(s'_n + w'_n a_{n+1})$$

$$\begin{array}{c} x \\ \downarrow \\ x \\ \searrow \\ 0 \end{array}$$

a

$$!T_{n-1}Q_{n-1} = e_{n-1} \otimes r_{n-1}$$
$$T_n\ Q_n = e_n \otimes r_n$$
$$?T_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

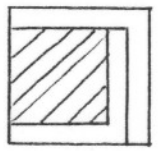


$$T_n = \lrcorner T_{n+1} \lrcorner^t$$
$$T_{n-1} = \urcorner T_n \lrcorner^t$$

$$\begin{bmatrix} s_{n-k} \\ r_{n-k} \end{bmatrix} = \begin{bmatrix} t_{k-1} \cdots t_{2k-n-1} \\ t_n \cdots t_k \end{bmatrix} Q_{n-k} \quad , \ k=0,1$$

$$Q_{n+1} = \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} + \begin{bmatrix} Q_{n-1} \\ 0 \\ 0 \end{bmatrix} a_{n+1} + \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} b_{n+1}$$

$$\begin{bmatrix} s_k \\ r_k \end{bmatrix} = \begin{bmatrix} h_{k+1} \cdots h_{2k+1} \\ h_0 \cdots h_k \end{bmatrix} Q_k \quad , \ k=0,1$$

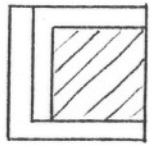


$$H_n = \lrcorner H_{n+1} \urcorner^t$$
$$H_{n-1} = \urcorner H_n \urcorner^t$$

$$!H_{n-1}Q_{n-1} = e_{n-1} \otimes s_{n-1}$$
$$H_n\ Q_n = e_n \otimes s_n$$
$$?H_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

b

$$!H_{n-1}Q_{n-1} = e_{n-1} \otimes r_{n-1}$$
$$H_n\ Q_n = e_n \otimes r_n$$
$$?H_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

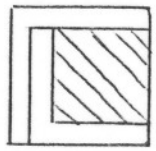


$$H_n = \lrcorner H_{n+1} \urcorner^t$$
$$H_{n-1} = \urcorner H_n \urcorner^t$$

$$\begin{bmatrix} s_{n-k} \\ r_{n-k} \end{bmatrix} = \begin{bmatrix} h_{2k} \cdots h_{n+k} \\ h_{n+k+1} \cdots h_{2n+1} \end{bmatrix} Q_{n-k} \quad , \ k=0,1$$

$$Q_{n+1} = \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ Q_{n-1} \end{bmatrix} a_{n+1} + \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} b_{n+1}$$

$$\begin{bmatrix} s_{n-k} \\ r_{n-k} \end{bmatrix} = \begin{bmatrix} t_{n-2k} \cdots t_{-k} \\ t_{-k-1} \cdots t_{-(n+1)} \end{bmatrix} Q_{n-k} \quad , \ k = 0,1$$



$$T_n = \urcorner T_{n+1} \urcorner^t$$
$$T_{n-1} = \lrcorner T_n \urcorner^t$$

$$!T_{n-1}Q_{n-1} = e_{n-1} \otimes s_{n-1}$$
$$T_n\ Q_n = e_n \otimes s_n$$
$$?T_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

x → 0 ↗ x

$$a_{n+1} = -s_{n-1}^{-1}\ s_n$$
$$b_{n+1} = -r_n^{-1}\ r_{n-1}\ a_{n+1}$$

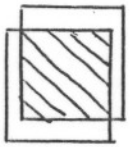
x ↘ x → 0

d

c

a

$$!S_n\ P_n = e_n \otimes v_n$$
$$S_{n+1} P_{n+1} = e_{n+1} \otimes v_{n+1}$$
$$?T_{n+1} Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

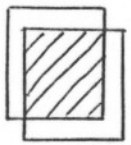


$$S_n = \lceil T_{n+1} \rfloor^t$$
$$S_{n+1} = \downarrow\downarrow T_{n+1}$$

$$\begin{bmatrix} w'_{n+k} \\ w_{n+k} \end{bmatrix} = \begin{bmatrix} t_{k-1} \cdots t_{-n-1} \\ t_k \cdots t_{-n} \end{bmatrix} P_{n+k} \qquad k = 0,1$$

$$Q_{n+1} = (P_{n+1} + \begin{bmatrix} 0 \\ \\ P_n \end{bmatrix} a_{n+1} + \begin{bmatrix} P_n \\ \\ 0 \end{bmatrix} b_{n+1}) \Lambda_{n+1}$$

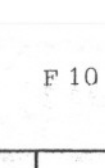
$$\begin{bmatrix} w'_{n+k} \\ \\ w_{n+k} \end{bmatrix} = \begin{bmatrix} h_{n-k+1} \cdots h_{2n+1} \\ \\ h_{n-k} \cdots h_{2n} \end{bmatrix} P_{n+k} \qquad k = 0,1$$



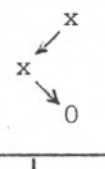
$$K_n = \lfloor H_{n+1} \rfloor^t$$
$$K_{n+1} = \uparrow\uparrow H_{n+1}$$

$$!K_n\ P_n = e_n \otimes w_n$$
$$K_{n+1} P_{n+1} = e_{n+1} \otimes w_{n+1}$$
$$?H_{n+1} Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

d

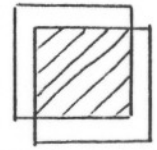


$$a_{n+1} = -w_n^{-1} w_{n+1}$$
$$b_{n+1} = -w_n^{-1} (w'_{n+1} + w'_n a_{n+1})$$
$$\Lambda_{n+1} = (I + a_{n+1})^{-1}$$



b

$$!K_n\ P_n = e_n \otimes v_n$$
$$K_{n+1} P_{n+1} = e_{n+1} \otimes v_{n+1}$$
$$?H_{n+1} Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

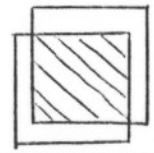


$$K_n = \lceil H_{n+1} \rceil^t$$
$$K_{n+1} = \downarrow\downarrow H_{n+1}$$

$$\begin{bmatrix} w'_k \\ w_k \end{bmatrix} = \begin{bmatrix} h_0 \cdots h_k \\ h_1 \cdots h_{k+1} \end{bmatrix} P_k \qquad h = n, n+1$$

$$Q_{n+1} = (P_{n+1} + \begin{bmatrix} P_n \\ \\ 0 \end{bmatrix} a_{n+1} + \begin{bmatrix} 0 \\ \\ P_n \end{bmatrix} b_{n+1}) \Lambda_{n+1}$$

$$\begin{bmatrix} w'_k \\ \\ w_k \end{bmatrix} = \begin{bmatrix} t_n \cdots t_{n-k} \\ \\ t_{n-1} \cdots t_{n-k-1} \end{bmatrix} P_k \qquad k = n, n+1$$



$$S_n = \lfloor T_{n+1} \rceil^t$$
$$S_{n+1} = \uparrow\uparrow T_{n+1}$$

$$!S_n\ P_n = e_n \otimes w_n$$
$$S_{n+1} P_{n+1} = e_{n+1} \otimes w_{n+1}$$
$$?T_{n+1} Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

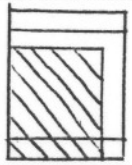
c

a

$$!S_n\ P_n = e_n \otimes v_n$$

$$S_{n+1} P_{n+1} = e_{n+1} \otimes v_{n+1}$$

$$?T_{n+1} Q_{n+1} = e_{n+1} \otimes r_{n+1}$$



$$S_{n+1} = \downarrow T_{n+1}$$

$$S_n = \urcorner S_{n+1} \llcorner^t$$

$$w_{n+k} = [\, t_{1-k} \cdots t_{-(n+2k-1)}]\, P_{n+k} \quad , \ k = 0,1$$

$$b_{n+1} = -(p_0^{n+1})^{-1} p_0^n \ a_{n+1}$$

0
↑
x
x ↗

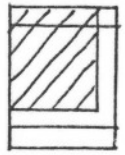
$$Q_{n+1} = \urcorner \left\{ \begin{bmatrix} 0 \\ \\ P_{n+1} \end{bmatrix} + \begin{bmatrix} P_n \\ 0 \\ 0 \end{bmatrix} a_{n+1} + \begin{bmatrix} P_{n+1} \\ \\ 0 \end{bmatrix} b_{n+1} \right\}$$

$$a_{n+1} = -w_n^{-1}\, w_{n+1}$$

$$w_k = [\, h_{k-1} \cdots h_{2k-1}]\, P_k \quad , \ k = n, n+1$$

$$b_{n+1} = -(p_0^{n+1})^{-1} p_0^n \ a_{n+1}$$

x ↘
x
↓
0



$$K_{n+1} = \uparrow H_{n+1}$$

$$K_n = \llcorner K_{n+1} \llcorner^t$$

$$!K_n\ P_n = e_n \otimes w_n$$

$$K_{n+1} P_{n+1} = e_{n+1} \otimes w_{n+1}$$

$$?H_{n+1} Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

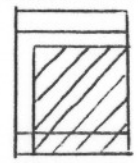
d

b

$$!K_n\ P_n = e_n \otimes v_n$$

$$K_{n+1} P_{n+1} = e_{n+1} \otimes v_{n+1}$$

$$?H_{n+1} Q_{n+1} = e_{n+1} \otimes r_{n+1}$$



$$K_{n+1} = \downarrow H_{n+1}$$

$$K_n = \urcorner K_{n+1} \urcorner^t$$

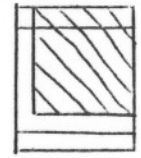
$$w_{n+k} = [\, h_{2(1-k)} \cdots h_{n+2-k}]\, P_{n+k} \quad , \ k = 0,1$$

$$b_{n+1} = -(p_n^{n+1})^{-1} p_n^n \ a_{n+1}$$

$$Q_{n+1} = \llcorner \left\{ \begin{bmatrix} P_{n+1} \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ P_n \end{bmatrix} a_{n+1} + \begin{bmatrix} 0 \\ \\ P_{n+1} \end{bmatrix} b_{n+1} \right\}$$

$$w_{n+k} = [\, t_{n-2+2k} \cdots t_{k-2}]\, P_{n+k} \quad , k = 0,1$$

$$b_{n+1} = -(p_n^{n+1})^{-1} p_n^n \ a_{n+1}$$



$$S_{n+1} = \uparrow T_{n+1}$$

$$S_n = \llcorner S_{n+1} \urcorner^t$$

$$!S_n\ P_n = e_n \otimes w_n$$

$$S_{n+1} P_{n+1} = e_{n+1} \otimes w_{n+1}$$

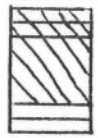
$$?T_{n+1} Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

c

$!S_{-n}\ P_n = e_n \otimes v_{-n}$

$S_n\ P_n = e_n \otimes v_n$

$?T_n\ Q_n = e_n \otimes r_n$



$S_n = \uparrow T_n$

$S_{-n} = \uparrow S_n$

$[v_n v_{-n}] = [t_{n-1} \ldots t_{-2}] \begin{bmatrix} P_n & 0 \\ 0 & P_{-n} \end{bmatrix}$

$\Lambda_{n+1} = (I + (p^n_{-n-1} - p^n_{n-1}) a_{n+1})^{-1}$

x
↓
x
↓
0

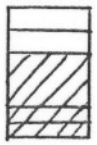
$Q_n = \lrcorner \left(\begin{bmatrix} P_n \\ 0 \end{bmatrix} + \left(\begin{bmatrix} 0 \\ P_{-n} \end{bmatrix} - \begin{bmatrix} 0 \\ P_n \end{bmatrix} \right) a_{n+1} \right) \Lambda_{n+1}$

$a_{n+1} = -v_{-n}\ v_n$

$[v_n v_{-n}] = [h_0 \ldots h_{n+1}] \begin{bmatrix} P_n & 0 \\ 0 & P_{-n} \end{bmatrix}$

$\Lambda_{n+1} = (I + (p^n_{-n-1} - p^n_{n-1}) a_{n+1})^{-1}$

0
↑
x
↑
x



$K_n = \downarrow H_n$

$K_{-n} = \downarrow K_n$

$!K_{-n}\ P_n = e_n \otimes w_{-n}$

$K_n\ P_n = e_n \otimes w_n$

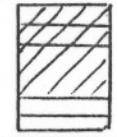
$?H_n\ Q_n = e_n \otimes s_n$

d

$!K_{-n}\ P_n = e_n \otimes v_{-n}$

$K_n\ P_n = e_n \otimes v_n$

$?H_n\ Q_n = e_n \otimes r_n$



$K_n = \uparrow H_n$

$K_{-n} = \uparrow K_n$

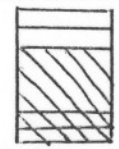
$[v_n v_{-n}] = [h_{n-1} \ldots h_{2n}] \begin{bmatrix} 0 & P_{-n} \\ P_n & 0 \end{bmatrix}$

$\Lambda_{n+1} = (I + (p^n_{-1} - p^n_1) a_{n+1})^{-1}$

$Q_n = \urcorner \left(\begin{bmatrix} 0 \\ P_n \end{bmatrix} + \left(\begin{bmatrix} P_{-n} \\ 0 \end{bmatrix} - \begin{bmatrix} P_n \\ 0 \end{bmatrix} \right) a_{n+1} \right) \Lambda_{n+1}$

$[v_n v_{-n}] = [t_1 \ldots t_{-n}] \begin{bmatrix} 0 & P_{-n} \\ P_n & 0 \end{bmatrix}$

$\Lambda_{n+1} = (I + (p^n_{-1} - p^n_1) a_{n+1})^{-1}$



$S_n = \downarrow T_n$

$S_{-n} = \downarrow S_n$

$!S_{-n}\ P_n = e_n \otimes w_{-n}$

$S_n\ P_n = e_n \otimes w_n$

$?T_n\ Q_n = e_n \otimes s_n$

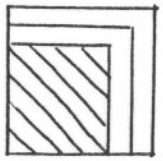
c

a

$$!T_{n+1}Q_{n+1} = e_{n+1}\otimes r_{n+1}$$

$$T_n\ Q_n = e_n\otimes r_n$$

$$?T_{n-1}Q_{n-1} = e_n\otimes r_{n-1}$$

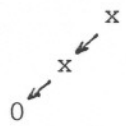


$$T_n = \urcorner T_{n+1} \llcorner^t$$

$$T_{n-1} = \urcorner T_n \llcorner^t$$

$$a_{n+1} = -(q_n^{n+1} - q_{n-1}^n)^{-1}$$

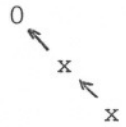
$$\Lambda_{n+1} = (q_{n-1}^{n+1} - q_{n-2}^n + q_{n-1}^n a_{n+1})^{-1}$$

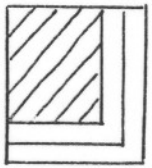


$$Q_{n-1} = \llcorner \llcorner (Q_{n+1} - \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} + \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} a_{n+1})\Lambda_{n+1}$$

$$a_{n+1} = -(q_n^{n+1} - q_{n-1}^n)^{-1}$$

$$\Lambda_{n+1} = (q_{n-1}^{n+1} - q_{n-2}^n + q_{n-1}^n a_{n+1})^{-1}$$





$$H_n = \llcorner H_{n+1} \llcorner^t$$

$$H_{n-1} = \llcorner H_n \llcorner^t$$

$$!H_{n+1}Q_{n+1} = e_{n+1}\otimes s_{n+1}$$

$$H_n\ Q_n = e_n\otimes s_n$$

$$?H_{n-1}Q_{n-1} = e_{n-1}\otimes s_{n-1}$$

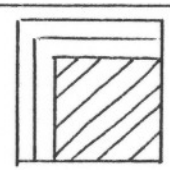
d

b

$$!H_{n+1}Q_{n+1} = e_{n+1}\otimes r_{n+1}$$

$$H_n\ Q_n = e_n\otimes r_n$$

$$?H_{n-1}Q_{n-1} = e_{n-1}\otimes r_{n-1}$$



$$H_n = \urcorner H_{n+1} \urcorner^t$$

$$H_{n-1} = \urcorner H_n \urcorner^t$$

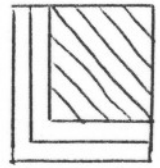
$$a_{n+1} = -(q_1^{n+1} - q_1^n)^{-1}$$

$$\Lambda_{n+1} = (q_2^{n+1} - q_2^n + q_1^n a_{n+1})^{-1}$$

$$Q_{n-1} = \urcorner \urcorner (Q_{n+1} - \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} a_{n+1})\Lambda_{n+1}$$

$$a_{n+1} = -(q_1^{n+1} - q_1^n)^{-1}$$

$$\Lambda_{n+1} = (q_2^{n+1} - q_2^n + q_1^n a_{n+1})^{-1}$$



$$T_n = \llcorner T_{n+1} \urcorner^t$$

$$T_{n-1} = \llcorner T_n \urcorner^t$$

$$!T_{n+1}Q_{n+1} = e_{n+1}\otimes s_{n+1}$$

$$T_n\ Q_n = e_n\otimes s_n$$

$$?T_{n-1}Q_{n-1} = e_{n-1}\otimes s_{n-1}$$

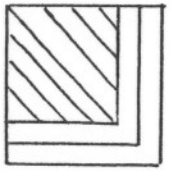
c

a

$$!T_n\,Q_n = e_n \otimes r_n$$

$$T_{n+1} Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

$$?T_{n-1} Q_{n-1} = e_{n-1} \otimes r_{n-1}$$



$$T_n = \lrcorner T_{n+1} \lrcorner^t$$

$$T_{n-1} = \lrcorner T_n \lrcorner^t$$

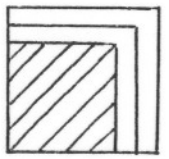
$$a_{n+1} = -(q_0^n)^{-1} q_0^{n+1}$$

$$\Lambda_{n+1} = [\,(q_n^{n+1} - q_{n-1}^n) + a_{n+1}\,]^{-1}$$

$$Q_{n-1} = \urcorner \lrcorner\, (Q_{n+1} - \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} + \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} a_{n+1}) \Lambda_{n+1}$$

$$a_{n+1} = -(q_0^n)^{-1} q_0^{n+1}$$

$$\Lambda_{n+1} = [\,(q_n^{n+1} - q_{n-1}^n) + a_{n+1}\,]^{-1}$$



$$H_n = \urcorner H_{n+1} \lrcorner^t$$

$$H_{n-1} = \urcorner H_n \lrcorner^t$$

$$!H_n\,Q_n = e_n \otimes r_n$$

$$H_{n+1} Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

$$?H_{n-1} Q_{n-1} = e_{n-1} \otimes r_{n-1}$$

d

$$0 \leftarrow x \leftarrow x$$

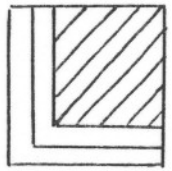
$$0 \leftarrow x \leftarrow x$$

b

$$!H_n\,Q_n = e_n \otimes r_n$$

$$H_{n+1} Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

$$?H_{n-1} Q_{n-1} = e_{n-1} \otimes r_{n-1}$$



$$H_n = \lrcorner H_{n+1} \urcorner^t$$

$$H_{n-1} = \lrcorner H_n \urcorner^t$$

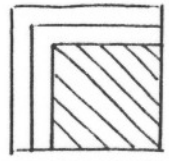
$$a_{n+1} = -(q_n^n)^{-1} q_{n+1}^{n+1}$$

$$\Lambda_{n+1} = [\,(q_1^{n+1} - q_1^n) + a_{n+1}\,]^{-1}$$

$$Q_{n-1} = \lrcorner \urcorner\, (Q_{n+1} - \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} a_{n+1}) \Lambda_{n+1}$$

$$a_{n+1} = -(q_n^n)^{-1} q_{n+1}^{n+1}$$

$$\Lambda_{n+1} = [\,(q_1^{n+1} - q_1^n) + a_{n+1}\,]^{-1}$$



$$T_n = \urcorner T_{n+1} \urcorner^t$$

$$T_{n-1} = \urcorner T_n \urcorner^t$$

$$!T_n\,Q_n = e_n \otimes s_n$$

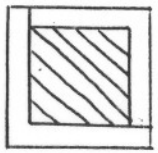
$$T_{n+1} Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

$$?T_{n-1} Q_{n-1} = e_{n-1} \otimes s_{n-1}$$

c

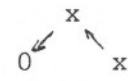
a

$$!T_{n+1}Q_{n+1} = e_{n+1}\otimes r_{n+1}$$
$$T_n\ Q_n = e_n\otimes r_n$$
$$?T_{n-1}Q_{n-1} = e_{n-1}\otimes r_{n-1}$$



$$T_n = \lrcorner T_{n+1} \urcorner^t$$
$$T_{n-1} = \urcorner T_n \lrcorner^t$$

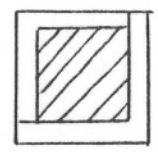
$$a_{n+1} = q_{n-1}^n - q_n^{n+1}$$
$$\Lambda_{n+1} = (q_{n-1}^{n+1} - q_{n-2}^n + q_{n-1}^n a_{n+1})^{-1}$$



$$Q_{n-1} = \lrcorner\lrcorner (Q_{n+1} - \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} + \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} a_{n+1})\Lambda_{n+1}$$

b

$$!H_{n+1}Q_{n+1} = e_{n+1}\otimes r_{n+1}$$
$$H_n\ Q_n = e_n\otimes r_n$$
$$?H_{n-1}Q_{n-1} = e_{n-1}\otimes r_{n-1}$$



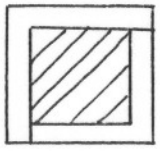
$$H_n = \lrcorner H_{n+1} \lrcorner^t$$
$$H_{n-1} = \urcorner H_n \urcorner^t$$

$$a_{n+1} = q_1^n - q_1^{n+1}$$
$$\Lambda_{n+1} = (q_2^{n+1} - q_2^n + q_1^n a_{n+1})^{-1}$$

$$Q_{n-1} = \urcorner\urcorner (Q_{n+1} - \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} a_{n+1})\Lambda_{n+1}$$

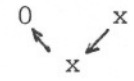
d

$$!H_{n+1}Q_{n+1} = e_{n+1}\otimes s_{n+1}$$
$$H_n\ Q_n = e_n\otimes s_n$$
$$?H_{n-1}Q_{n-1} = e_{n-1}\otimes s_{n-1}$$



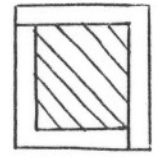
$$H_n = \urcorner H_{n+1} \urcorner^t$$
$$H_{n-1} = \lrcorner H_n \lrcorner^t$$

$$a_{n+1} = q_{n-1}^n - q_n^{n+1}$$
$$\Lambda_{n+1} = (q_{n-1}^{n+1} - q_{n-2}^n + q_{n-1}^n a_{n+1})^{-1}$$



c

$$!T_{n+1}Q_{n+1} = e_{n+1}\otimes s_{n+1}$$
$$T_n\ Q_n = e_n\otimes s_n$$
$$?T_{n-1}Q_{n-1} = e_{n-1}\otimes s_{n-1}$$

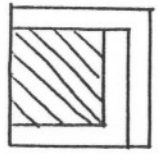


$$T_n = \urcorner T_{n+1} \lrcorner^t$$
$$T_{n-1} = \lrcorner T_n \urcorner^t$$

$$a_{n+1} = q_1^n - q_1^{n+1}$$
$$\Lambda_{n+1} = (q_2^{n+1} - q_2^n + q_1^n a_{n+1})^{-1}$$

a

$$!T_{n+1}Q_{n+1} = e_{n+1}\otimes r_{n+1}$$
$$T_n\ Q_n = e_n\otimes r_n$$
$$?T_{n-1}Q_{n-1} = e_{n-1}\otimes r_{n-1}$$

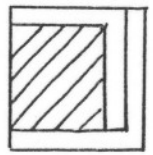


$$T_n = \urcorner\, T_{n+1}\, \llcorner^t$$
$$T_{n-1} = \llcorner T_n\, \llcorner^t$$

$$a_{n+1} = -(q_0^n)^{-1} q_0^{n+1}$$
$$\Lambda_{n+1} = (q_n^{n+1} - q_{n-1}^n + a_{n+1})^{-1}$$

$$Q_{n-1} = \urcorner\, \llcorner\, (Q_{n+1} - \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} + \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} a_{n+1})\Lambda_{n+1}$$

$$a_{n+1} = -(q_0^n)^{-1} q_0^{n+1}$$
$$\Lambda_{n+1} = (q_n^{n+1} - q_{n-1}^n + a_{n+1})^{-1}$$



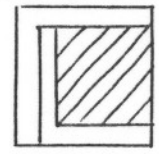
$$H_n = \llcorner H_{n+1}\, \llcorner^t$$
$$H_{n-1} = \urcorner H_n\, \llcorner^t$$

$$!H_{n+1}Q_{n+1} = e_{n+1}\otimes s_{n+1}$$
$$H_n\ Q_n = e_n\otimes s_n$$
$$?H_{n-1}Q_{n-1} = e_{n-1}\otimes s_{n-1}$$

d

b

$$!H_{n+1}Q_{n+1} = e_{n+1}\otimes r_{n+1}$$
$$H_n\ Q_n = e_n\otimes r_n$$
$$?H_{n-1}Q_{n-1} = e_{n-1}\otimes r_{n-1}$$

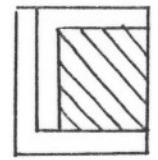


$$H_n = \urcorner\, H_{n+1}\, \urcorner^t$$
$$H_{n-1} = \llcorner H_n\, \urcorner^t$$

$$a_{n+1} = -(q_n^n)^{-1} (q_{n+1}^{n+1})$$
$$\Lambda_{n+1} = (q_1^{n+1} - q_1^n + a_{n+1})^{-1}$$

$$Q_{n-1} = \llcorner\, \urcorner\, (Q_{n+1} - \begin{bmatrix} Q_n \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ Q_n \end{bmatrix} a_{n+1})\Lambda_{n+1}$$

$$a_{n+1} = -(q_n^n)^{-1} (q_{n+1}^{n+1})$$
$$\Lambda_{n+1} = (q_1^{n+1} - q_1^n + a_{n+1})^{-1}$$



$$T_n = \llcorner\, T_{n+1}\, \urcorner^t$$
$$T_{n-1} = \urcorner\, T_n\ \urcorner^t$$

$$!T_{n+1}Q_{n+1} = e_{n+1}\otimes s_{n+1}$$
$$T_n\ Q_n = e_n\otimes s_n$$
$$?T_{n-1}Q_{n-1} = e_{n-1}\otimes s_{n-1}$$

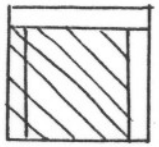
c

a

$!S_n\ P_n = e_n \otimes v_n$

$T_{n+1} Q_{n+1} = e_{n+1} \otimes r_{n+1}$

$?T_n\ Q_n = e_n \otimes r_n$



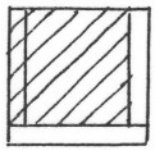
$S_n = \urcorner\, T_{n+1} \urcorner^t$

$T_n = \urcorner\, T_{n+1} \lrcorner^t$

$\Lambda_{n+1} = (p^n_{n-1} - q^{n+1}_n)^{-1}$

$Q_n = \lrcorner\, (\begin{bmatrix} 0 \\ \\ P_n \end{bmatrix} - Q_{n+1}) \Lambda_{n+1}$

$\Lambda_{n+1} = (p^n_{n-1} - q^{n+1}_n)^{-1}$



$K_n = \lrcorner\, H_{n+1} \urcorner^t$

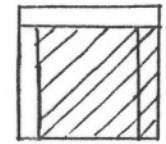
$H_n = \lrcorner\, H_{n+1} \lrcorner^t$

b

$!K_n\ P_n = e_n \otimes v_n$

$H_{n+1} Q_{n+1} = e_{n+1} \otimes r_{n+1}$

$?H_n\ Q_n = e_n \otimes r_n$



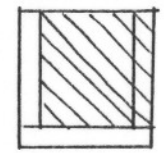
$K_n = \urcorner\, H_{n+1} \lrcorner^t$

$H_n = \urcorner\, H_{n+1} \urcorner^t$

$\Lambda_{n+1} = (p^n_1 - q^{n+1}_1)^{-1}$

$Q_n = \urcorner\, (\begin{bmatrix} P_n \\ \\ 0 \end{bmatrix} - Q_{n+1}) \Lambda_{n+1}$

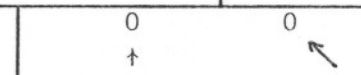
$\Lambda_{n+1} (p^n_1 - q^{n+1}_1)^{-1}$



$S_n = \lrcorner\, T_{n+1} \lrcorner^t$

$T_n = \urcorner\, T_{n+1} \urcorner^t$





$!K_n\ P_n = e_n \otimes w_n$

$H_{n+1} Q_{n+1} = e_{n+1} \otimes r_{n+1}$

$?H_n\ Q_n = e_n \otimes r_n$

d

$!S_n\ P_n = e_n \otimes w_n$

$T_{n+1} Q_{n+1} = e_{n+1} \otimes r_{n+1}$

$?T_n\ Q_n = e_n \otimes r_n$

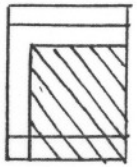
c

a

$$!S_{n+1}P_{n+1} = e_{n+1} \otimes v_{n+1}$$

$$T_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

$$?T_n\ Q_n \quad = e_n \otimes r_n$$



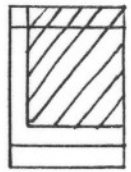
$$T_n = \urcorner T_{n+1} \urcorner^t$$

$$S_{n+1} = \uparrow T_{n+1}$$

$$\Lambda_{n+1} = (p_n^{n+1} - q_n^{n+1})^{-1}$$

$$Q_n = \llcorner (P_{n+1} - Q_{n+1}) \Lambda_{n+1}$$

$$\Lambda_{n+1} = (p_n^{n+1} - q_n^{n+1})^{-1}$$



$$H_n = \llcorner H_{n+1} \urcorner^t$$

$$K_{n+1} = \downarrow H_{n+1}$$

$$!K_{n+1}P_{n+1} = e_{n+1} \otimes w_{n+1}$$

$$H_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

$$?H_n\ Q_n \quad = e_n \otimes s_n$$

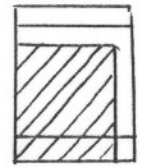
d

b

$$!K_{n+1}P_{n+1} = e_{n+1} \otimes v_{n+1}$$

$$H_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

$$?H_n\ Q_n \quad = e_n \otimes r_n$$



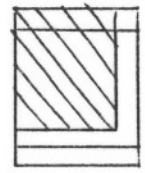
$$H_n = \urcorner H_{n+1} \llcorner^t$$

$$K_{n+1} = \uparrow H_{n+1}$$

$$\Lambda_{n+1} = (p_1^{n+1} - q_1^{n+1})^{-1}$$

$$Q_n = \urcorner (P_{n+1} - Q_{n+1}) \Lambda_{n+1}$$

$$\Lambda_{n+1} = (p_1^{n+1} - q_1^{n+1})^{-1}$$



$$T_n = \llcorner T_{n+1} \llcorner^t$$

$$S_{n+1} = \downarrow T_{n+1}$$

$$!S_{n+1}P_{n+1} = e_{n+1} \otimes w_{n+1}$$

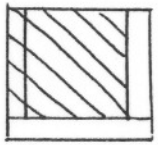
$$T_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

$$?T_n\ Q_n \quad = e_n \otimes s_n$$

c

a

$$!S_n\ P_n = e_n \otimes v_n$$
$$T_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$
$$?T_n\ Q_n = e_n \otimes r_n$$

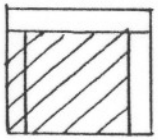


$$T_n = \lrcorner T_{n+1} \lrcorner^t$$
$$S_n = \lrcorner T_{n+1} \urcorner^t$$

$$\Lambda_{n+1} = (p^n_{n-1} - q^{n+1}_n)^{-1}$$

$$Q_n = \lrcorner \left(\begin{bmatrix} 0 \\ P_n \end{bmatrix} - Q_{n+1} \right) \Lambda_{n+1}$$

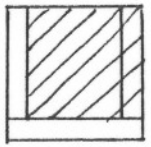
$$\Lambda_{n+1} = (p^n_{n-1} - q^{n+1}_n)^{-1}$$



$$H_n = \urcorner H_{n+1} \lrcorner^t$$
$$K_n = \urcorner H_{n+1} \urcorner^t$$

b

$$!K_n\ P_n = e_n \otimes v_n$$
$$H_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$
$$?H_n\ Q_n = e_n \otimes r_n$$

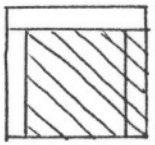


$$H_n = \lrcorner H_{n+1} \urcorner^t$$
$$K_n = \lrcorner H_{n+1} \lrcorner^t$$

$$\Lambda_{n+1} = (p^n_1 - q^{n+1}_1)^{-1}$$

$$Q_n = \urcorner \left(\begin{bmatrix} P_n \\ 0 \end{bmatrix} - Q_{n+1} \right) \Lambda_{n+1}$$

$$\Lambda_{n+1} = (p^n_1 - q^{n+1}_1)^{-1}$$



$$T_n = \urcorner T_{n+1} \urcorner^t$$
$$S_n = \urcorner T_{n+1} \lrcorner^t$$

d

$$!K_n\ P_n = e_n \otimes v_n$$
$$H_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$
$$?H_n\ Q_n = e_n \otimes s_n$$

c

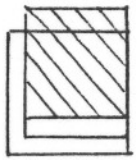
$$!S_n\ P_n = e_n \otimes v_n$$
$$T_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$
$$?T_n\ Q_n = e_n \otimes s_n$$

a

$!S_n P_n = e_n \otimes v_n$

$S_{n+1} P_{n+1} = e_{n+1} \otimes v_{n+1}$

$?T_n Q_n = e_n \otimes r_n$



$T_n = \uparrow S_n$

$S_n = \lrcorner S_{n+1} \urcorner^t$

$w_{n+k} = [t_k \ldots t_{-n}] P_{n+k}$ $\quad k=0,1$

$b_{n+1} = -(p_0^n)^{-1} p_0^{n+1}$

$$\begin{array}{ccc} 0 & & \\ \uparrow & & \\ x & & \\ & \nwarrow & x \end{array}$$

$Q_n = \urcorner \{P_{n+1} + \begin{bmatrix} 0 \\ P_n \end{bmatrix} a_{n+1} + \begin{bmatrix} P_n \\ 0 \end{bmatrix} b_{n+1}\} \Lambda_{n+1}$

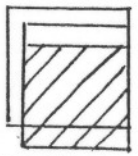
$a_{n+1} = -w_n^{-1} w_{n+1}$

$\Lambda_{n+1} = (I + a_{n+1})^{-1}$

$w_k = [h_{2n-1-k} \ldots h_{2n-1}] P_k$ $\quad k=n,n+1$

$b_n = -(p_0^n)^{-1} p_0^{n+1}$

$$\begin{array}{ccc} & & x \\ & \swarrow & \\ x & & \\ \downarrow & & \\ 0 & & \end{array}$$



$H_n = \downarrow K_n$

$K_n = \urcorner K_{n+1} \urcorner^t$

$!K_n P_n = e_n \otimes w_n$

$K_{n+1} P_{n+1} = e_{n+1} \otimes w_{n+1}$

$?H_n Q_n = e_n \otimes s_n$

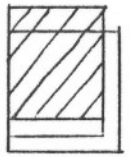
d

b

$!K_n P_n = e_n \otimes v_n$

$K_{n+1} P_{n+1} = e_{n+1} \otimes v_{n+1}$

$?H_n Q_n = e_n \otimes r_n$



$H_n = \uparrow K_n$

$K_n = \lrcorner K_{n+1} \lrcorner^t$

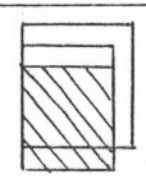
$w_k = [h_0 \ldots h_k] P_k$ $\quad k=n,n+1$

$b_{n+1} = -(p_n^n)^{-1} p_{n+1}^{n+1}$

$Q_n = \lrcorner (P_{n+1} + \begin{bmatrix} P_n \\ 0 \end{bmatrix} a_{n+1} + \begin{bmatrix} 0 \\ P_n \end{bmatrix} b_{n+1}) \Lambda_{n+1}$

$w_k = [t_{n-1} \ldots t_{n-1-k}] P_k$ $\quad k=n,n+1$

$b_{n+1} = -(p_n^n)^{-1} p_{n+1}^{n+1}$



$T_n = \downarrow S_n$

$S_n = \urcorner S_{n+1} \lrcorner^t$

$!S_n P_n = e_n \otimes w_n$

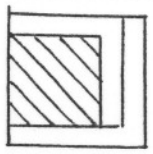
$S_{n+1} P_{n+1} = e_{n+1} \otimes w_{n+1}$

$?T_n Q_n = e_n \otimes s_n$

c

a

$$!T_n\,Q_n = e_n \otimes r_n$$
$$T_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$
$$?T_{n-1}Q_{n-1} = e_{n-1} \otimes r_{n-1}$$

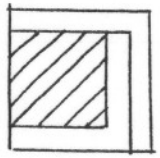


$$T_n = \lrcorner T_{n+1} \lrcorner^t$$
$$T_{n-1} = \urcorner T_n \lrcorner^t$$

$$a_{n+1} = -(q_n^{n+1} - q_{n-1}^n)$$
$$\Lambda_{n+1} = [\,(q_{n-1}^{n+1} - q_{n-2}^n) + q_{n-1}^n a_{n+1}\,]^{-1}$$

$$Q_{n-1} = \lrcorner \lrcorner\,(Q_{n+1} - \begin{bmatrix} 0 \\ Q_n \end{bmatrix} + \begin{bmatrix} Q_n \\ 0 \end{bmatrix} a_{n+1})\,\Lambda_{n+1}$$

$$a_{n+1} = -(q_n^{n+1} - q_{n-1}^n)$$
$$\Lambda_{n+1} = [\,(q_{n-1}^{n+1} - q_{n-2}^n) + q_{n-1}^n a_{n+1}\,]^{-1}$$



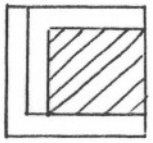
$$H_n = \urcorner H_{n+1} \lrcorner^t$$
$$H_{n-1} = \lrcorner H_n \lrcorner^t$$

$$!H_n\,Q_n = e_n \otimes s_n$$
$$H_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$
$$?H_{n-1}Q_{n-1} = e_{n-1} \otimes s_{n-1}$$

d

b

$$!H_n\,Q_n = e_n \otimes r_n$$
$$H_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$
$$?H_{n-1}Q_{n-1} = e_{n-1} \otimes r_{n-1}$$



$$H_n = \lrcorner H_{n+1} \urcorner^t$$
$$H_{n-1} = \urcorner H_n \urcorner^t$$

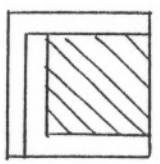
$$0 \swarrow x \leftarrow x$$

$$a_{n+1} = -(q_1^{n+1} - q_1^n)$$
$$\Lambda_{n+1} = [\,(q_2^{n+1} - q_2^n) + q_1^n a_{n+1}\,]^{-1}$$

$$Q_{n-1} = \urcorner \urcorner\,\{Q_{n+1} - \begin{bmatrix} Q_n \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ Q_n \end{bmatrix} a_{n+1})\,\Lambda_{n+1}$$

$$0 \nwarrow x \leftarrow x$$

$$a_{n+1} = -(q_1^{n+1} - q_1^n)$$
$$\Lambda_{n+1} = [\,(q_2^{n+1} - q_2^n) + q_1^n a_{n+1}\,]^{-1}$$



$$T_n = \urcorner T_{n+1} \urcorner^t$$
$$T_{n-1} = \lrcorner T_n \urcorner^t$$

$$!T_n\,Q_n = e_n \otimes s_n$$
$$T_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$
$$?T_{n-1}Q_{n-1} = e_{n-1} \otimes s_{n-1}$$

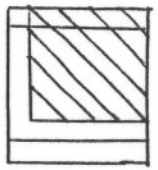
c

a

$$!S_n\ P_n = e_n \otimes v_n$$

$$T_n\ Q_n = e_n \otimes r_n$$

$$?T_{n-1}\ Q_{n-1} = e_{n-1} \otimes r_{n-1}$$



$$S_n = \downarrow T_n$$

$$T_{n-1} = \lrcorner T_n \urcorner^t$$

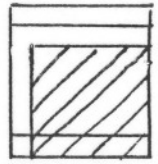
$$a_{n+1} = -(q_0^n)^{-1}[(q_1^n)(q_0^n)^{-1} - (p_1^n)(p_0^n)^{-1}]^{-1} q_0^n$$

$$b_{n+1} = (p_0^n)^{-1}[(q_1^n)(q_0^n)^{-1} - (p_1^n)(p_0^n)^{-1}]^{-1} q_0^n$$

$$Q_{n-1} = \urcorner \urcorner \left(\begin{bmatrix} 0 \\ Q_n \end{bmatrix} + \begin{bmatrix} Q_n \\ 0 \end{bmatrix} a_{n+1} + \begin{bmatrix} P_n \\ 0 \end{bmatrix} b_{n+1} \right)$$

$$a_{n+1} = -(q_0^n)^{-1}[(q_1^n)(q_0^n)^{-1} - (p_1^n)(p_0^n)^{-1}]^{-1} q_0^n$$

$$b_{n+1} = (p_0^n)^{-1}[(q_1^n)(q_0^n)^{-1} - (p_1^n)(p_0^n)^{-1}]^{-1} q_0^n$$



$$K_n = \uparrow H_n$$

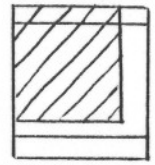
$$H_{n-1} = \urcorner H_n \urcorner^t$$

b

$$!K_n\ P_n = e_n \otimes v_n$$

$$H_n\ Q_n = e_n \otimes r_n$$

$$?H_{n-1}\ Q_{n-1} = e_{n-1} \otimes r_{n-1}$$



$$K_n = \downarrow H_n$$

$$H_{n-1} = \lrcorner H_n \lrcorner^t$$

$$a_{n+1} = -(q_n^n)^{-1}[(q_{n-1}^n)(q_n^n)^{-1} - (p_{n-1}^n)(p_n^n)^{-1}]^{-1} q_n^n$$

$$b_{n+1} = (p_n^n)^{-1}[(q_{n-1}^n)(q_n^n)^{-1} - (p_{n-1}^n)(p_n^n)^{-1}]^{-1} q_n^n$$

$$Q_{n-1} = \lrcorner \lrcorner \left(\begin{bmatrix} Q_n \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ Q_n \end{bmatrix} a_{n+1} + \begin{bmatrix} 0 \\ P_n \end{bmatrix} b_{n+1} \right)$$

$$a_{n+1} = -(q_n^n)^{-1}[(q_{n-1}^n)(q_n^n)^{-1} - (p_{n-1}^n)(p_n^n)^{-1}]^{-1} q_n^n$$

$$b_{n+1} = (p_n^n)^{-1}[(q_{n-1}^n)(q_n^n)^{-1} - (p_{n-1}^n)(p_n^n)^{-1}]^{-1} q_n^n$$

$$K_n = \uparrow H_n$$

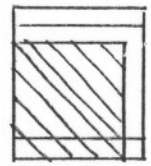
$$H_{n-1} = \urcorner H_n \urcorner^t$$

d

$$!K_n\ P_n = e_n \otimes w_n$$

$$H_n\ Q_n = e_n \otimes s_n$$

$$?H_{n-1}\ Q_{n-1} = e_{n-1} \otimes s_{n-1}$$



$$S_n = \uparrow T_n$$

$$T_{n-1} = \urcorner T_n \lrcorner^t$$

c

$$!S_n\ P_n = e_n \otimes w_n$$

$$T_n\ Q_n = e_n \otimes s_n$$

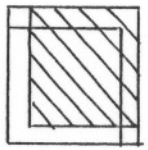
$$?T_{n-1}\ Q_{n-1} = e_{n-1} \otimes s_{n-1}$$

a

$$!S_n\ P_n = e_n \otimes v_n$$

$$T_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

$$?T_n\ Q_n = e_n \otimes r_n$$

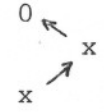


$$T_n = \lrcorner T_{n+1} \urcorner^t$$

$$S_n = \urcorner T_{n+1} \lrcorner^t$$

$$a_{n+1} = -(q_1^{n+1} - p_1^n (p_0^n)^{-1} q_0^{n+1})^{-1}\ q_0^{n+1}$$

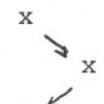
$$b_{n+1} = -(p_0^n)^{-1} (q_0^{n+1}) a_{n+1}$$

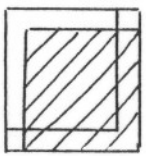


$$Q_n = \urcorner \urcorner \left(\begin{bmatrix} 0 \\ \\ Q_{n+1} \end{bmatrix} + \begin{bmatrix} Q_{n+1} \\ \\ 0 \end{bmatrix} a_{n+1} + \begin{bmatrix} P_n \\ 0 \\ 0 \end{bmatrix} b_{n+1} \right)$$

$$a_{n+1} = -(q_1^{n+1} - p_1^n (p_0^n)^{-1} q_0^{n+1})^{-1}\ q_0^{n+1}$$

$$b_{n+1} = -(p_0^n)^{-1} (q_0^{n+1}) a_{n+1}$$





$$H_n = \urcorner H_{n+1} \urcorner^t$$

$$K_n = \lrcorner H_{n+1} \lrcorner^t$$

$$!K_n\ P_n = e_n \otimes w_n$$

$$H_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

$$?H_n\ Q_n = e_n \otimes s_n$$

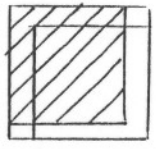
d

b

$$!K_n\ P_n = e_n \otimes v_n$$

$$H_{n+1}Q_{n+1} = e_{n+1} \otimes r_{n+1}$$

$$?H_n\ Q_n = e_n \otimes r_n$$



$$H_n = \lrcorner H_{n+1} \lrcorner^t$$

$$K_n = \urcorner H_{n+1} \urcorner^t$$

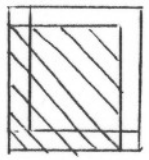
$$a_{n+1} = -(q_n^{n+1} - p_{n-1}^n (p_n^n)^{-1} q_{n+1}^{n+1})^{-1}\ q_{n+1}^{n+1}$$

$$b_{n+1} = -(p_n^n)^{-1} (q_{n+1}^{n+1}) a_{n+1}$$

$$Q_n = \lrcorner \lrcorner \left(\begin{bmatrix} Q_{n+1} \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ Q_{n+1} \end{bmatrix} a_{n+1} + \begin{bmatrix} 0 \\ 0 \\ P_n \end{bmatrix} b_{n+1} \right)$$

$$a_{n+1} = -(q_n^{n+1} - p_{n-1}^n (p_n^n)^{-1} q_{n+1}^{n+1})^{-1}\ q_{n+1}^{n+1}$$

$$b_{n+1} = -(p_n^n)^{-1} (q_{n+1}^{n+1}) a_{n+1}$$



$$T_n = \urcorner T_{n+1} \lrcorner^t$$

$$S_n = \lrcorner T_{n+1} \urcorner^t$$

$$!S_n\ P_n = e_n \otimes w_n$$

$$T_{n+1}Q_{n+1} = e_{n+1} \otimes s_{n+1}$$

$$?T_n\ Q_n = e_n \otimes s_n$$

c