# Construction of Minimal Cubature Formulae for the Square and the Triangle, using Invariant Theory

*Ronald Cools*
*Ann Haegemans*

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Department of Computer Science, K.U.Leuven

*ABSTRACT*

The knots and weights of a cubature formula are determined by a system of nonlinear equations. The number of equations and unknowns can be reduced by imposing some structure on the formula.

We are concerned with the construction of cubature formulae which are invariant under rotations. Using invariant theory, we obtain a smaller system of algebraically independent equations.

This is used to construct cubature formulae for the square and the triangle. New results are:

— a 24-point formula of degree 11 for the square,

— a 33-point formula of degree 13 for the square,

— a 15 point formula of degree 8 for the triangle and

— a 22 point formula of degree 10 for the triangle.

## 1. Introduction

Invariant theory is a beautiful nineteenth century technique.
Recently it's used to study error-correcting codes [16] and to construct cubature formulae for 2 and 3 dimensional regions [1], [4] and [7]. These cubature formulae were all invariant with respect to reflection groups. In order to obtain cubature formulae with less knots, we will use a subgroup of a reflection group.

## 2. Invariant theory

Let $G$ be a group of linear transformations acting on a $n$-dimensional vector space $V$ over a field $K$.

**Definition 1:** An invariant of $G$ is a polynomial $\phi(\vec{x})$ which is unchanged by every linear transformation in $G$. Or, without words :

$$\phi(g(\vec{x})) = \phi(\vec{x}), \forall g \in G, \forall \vec{x} \in V.$$

**Definition 2:** Polynomials $\psi_1(\vec{x}), \ldots, \psi_r(\vec{x})$ are called algebraically dependent if there is a polynomial $\psi(\vec{x})$ in $r$ variables with complex coefficients, not all zero, such that $\psi(\psi_1(\vec{x}), \ldots, \psi_r(\vec{x})) \equiv 0$.

Otherwise the polynomials $\psi_1(\vec{x}), \ldots, \psi_r(\vec{x})$ are algebraically independent.

The following theorems are useful.

**Theorem 1 :** *Any $n+1$ polynomials in $n$ variables are algebraically dependent.*

**Proof :** See [9].

□

**Theorem 2 :** *There always exist $n$ algebraically independent invariants of $G$ in $n$ variables.*

**Proof :** See [2].

□

**Definition 3 :** The $G$-orbit containing the point $\vec{a} \in V$ is the set of points $g(\vec{a})$, where $\vec{a}$ is fixed and $g$ runs through all elements of the group $G$.

We need some kind of basis to describe all invariants as functions of the elements of this basis.

**Definition 4 :** Let $\phi_1(\vec{x}), \ldots, \phi_l(\vec{x})$ be invariants of $G$. $\phi_1(\vec{x}), \ldots, \phi_l(\vec{x})$ form an integrity basis for the invariants of $G \Leftrightarrow$ any invariant of $G$ is a polynomial in $\phi_1(\vec{x}), \ldots, \phi_l(\vec{x})$. Each polynomial $\phi_i(\vec{x})$ is called a basic invariant of $G$.

From theorem 1 we know that if $l > n$ then there exist polynomial equations, called syzygies, relating $\phi_1(\vec{x}), \ldots, \phi_l(\vec{x})$.

Let $\mathbb{C}[f_1, \ldots, f_n]$ be the set of all polynomials in $f_1, \ldots, f_n$ with complex coefficients.
Let $\mathbb{P}(G)$ be the vector space of invariants of the group $G$, $\mathbb{P}_d(G)$ the vector space of invariants of $G$ of degree $\leqslant d$ and $\mathbb{P}_d$ the vector space of all polynomials in $n$ variables of degree $\leqslant d$.

**Definition 5 :** A good integrity basis for $\mathbb{P}(G)$ consists of homogeneous invariants $\phi_1(\vec{x}), \ldots, \phi_l(\vec{x})$ $(l \geqslant n)$ where $\phi_1(\vec{x}), \ldots, \phi_n(\vec{x})$ are algebraically independent and

$$\mathbb{P}(G) = \mathbb{C}[\phi_1(\vec{x}), \ldots, \phi_n(\vec{x})] \quad \text{if} \quad l = n$$

or

$$\mathbb{P} = \mathbb{C}[\phi_1(\vec{x}), \ldots, \phi_n(\vec{x})] \oplus \phi_{n+1}(\vec{x}) \,.\, \mathbb{C}[\phi_1(\vec{x}), \ldots, \phi_n(\vec{x})]$$
$$\oplus \cdots \oplus \phi_l(\vec{x}) \,.\, \mathbb{C}[\phi_1(\vec{x}), \ldots, \phi_n(\vec{x})] \quad \text{if} \quad l > n.$$

From Sloane [16] we know what the syzygies are for a good integrity basis $\phi_1, \ldots, \phi_l$ :

— If $l = n$, there are no syzygies.

— If $l > n$, there are $(l-n)^2$ syzygies expressing the products $\phi_i \,.\, \phi_j$ $(i > n,\ j > n)$ in terms of $\phi_1, \ldots, \phi_n$ if the product is not commutative. Thus if the product is commutative, there are $(l-n)(l-n+1)/2$ syzygies expressing the products $\phi_i \,.\, \phi_j$ $(i > n\ ,\ j \geqslant i)$ in terms of $\phi_1, \ldots, \phi_n$ .

For each integer $i \geqslant 0$, the homogeneous invariants of degree $i$ form a finite dimensional vector space over $\mathbb{R}$ of dimension $c_i$. A useful formula for the $c_i$'s is given by the following.

**Theorem 3** *(Molien's formula)*

*Let $\omega_1(\sigma), \ldots, \omega_n(\sigma)$ be the eigenvalues of $\sigma \in G$. Then*

$$\sum_{i=0}^{\infty} c_i t^i = \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1-\omega_1(\sigma)t) \cdots (1-\omega_n(\sigma)t)} \,.$$

**Proof** See [12].

□

If $\phi_1(\vec{x}), \ldots, \phi_l(\vec{x})$ form a good integrity basis for the invariants of $G$ and $d_i = \deg \phi_i(\vec{x})$, then (see

Sloane [16])

$$\sum_{i=0}^{\infty} c_i t^i = \frac{1}{\prod_{i=1}^{n} (1-t^{d_i})} \qquad \text{if} \quad l = n$$

or

$$\sum_{i=0}^{\infty} c_i t^i = \frac{1 + \sum_{j=n+1}^{l} t^{d_j}}{\prod_{i=1}^{n} (1-t^{d_i})} \qquad \text{if} \quad l > n.$$

**Example 1:**

Let $V = \mathbb{R}^2$ and $G$ the group of rotation symmetries of a square.
Using matrices to describe linear transformations, we can write

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

The following polynomials are invariants of $G$ : $\phi_1 = r^2$, $\phi_2 = r^4 \cos 4\theta$ and $\phi_3 = r^4 \sin 4\theta$ where $x = r \cos \theta$ and $y = r \sin \theta$.

The invariants $\phi_1, \phi_2, \phi_3$ must be algebraically dependent according to theorem 1.
Indeed : $\phi_3^2 - \phi_1^4 + \phi_2^2 \equiv 0$.

The Molien-series for this group is

$$\sum_{i=0}^{\infty} c_i t^i = \frac{1}{4} \left( \frac{1}{1+t^2-2t} + \frac{2}{1+t^2} + \frac{1}{1+t^2+2t} \right)$$

$$= \frac{1+t^4}{(1-t^2)(1-t^4)} .$$

The polynomials $\phi_1, \phi_2, \phi_3$ form a good integrity basis of $G$.
The single syzygie is $\phi_3^2 = \phi_1^4 - \phi_2^2$.

## 3. Invariant cubature formulae

Consider an integral

$$I[f] = \int_{\Omega} w(\vec{x}) f(\vec{x}) d\vec{x} \tag{1}$$

over an arbitrary region $\Omega \in \mathbb{R}^n$ with $w(\vec{x}) > 0$.

**Definition 6 :** A cubature formula is an approximation for the integral $I[f]$, of the form

$$Q[f] = \sum_{i=0}^{N} w_i \, f(\vec{x}^{(i)}). \tag{2}$$

The non-zero constants $w_i$ are called weights and the $\vec{x}^{(i)}$ are called knots.
A cubature formula has degree $d$ if $I[f] = Q[f] \quad \forall f \in \mathbb{P}_d$.

**Definition 7 :** A cubature formula is said to be invariant with respect to a group $G$ if the domain of integration $\Omega$ and the weight function $w(\vec{x})$ remain unchanged under a transformation $g \in G$, and if the set of knots is a union of $G$-orbits. The knots of one and the same orbit have the same coefficient $w_i$.

The usefulness of invariant theory, in the context of constructing cubature formulae, is highlighted by the following result due to Sobolev :

**Theorem 4 :** *Let the formula (2) be invariant with respect to G. The cubature formula (2) has degree d* $\Leftrightarrow I[f] = Q[f] \; \forall f \in \mathbb{P}_d(G)$.

Proof : See [17].

□

The knots and weights of a cubature formula are the solution of the system of nonlinear equations

$$Q[f_i] = I[f_i] \quad i = 1, \ldots, \dim \mathbb{P}_d$$

where the $f_i$ form a basis for the vector space $\mathbb{P}_d$ and the knots and weights are the unknowns. The number of equations and unknowns can be reduced by imposing some structure additional on the formula . Sobelev's theorem suggests that we look for invariant formulae, that is, solutions of the equations

$$Q[\phi_i] = I[\phi_i] \quad i = 1, \ldots, \dim \mathbb{P}_d(G)$$

where the $\phi_i$ form a basis for $\mathbb{P}_d(G)$ . This was done in [1], [4] and [7] when $G$ is a reflection group, for which there exist no syzygies relating the basic invariants.
In the following sections, $G$ will be the group of rotation-symmetries of a square or the group of rotation-symmetries of a triangle.

## 4. Cubature formulae for the square

We want to construct cubature formulae for $C_2$ : the square $\{(x,y) : -1 \leqslant x,y \leqslant 1\}$ with weight function $w(x,y) = 1$. The symmetry-group of a square contains 4 reflections and 4 rotations. It is known as the dihedral group $D_4$ of order 8. The rotations form a subgroup $R_4$.

In example 1 we found a good integrity basis for $\mathbb{P}(R_4)$ : $\phi_1 = r^2$, $\phi_2 = r^4 \cos 4\theta$, $\phi_3 = r^4 \sin 4\theta$.

The system of nonlinear equations we must solve, is described by

**Theorem 5 :** *An invariant cubature formula with respect to $R_4$, of degree $d = 2k-1$ is a solution of the system of nonlinear equations*

$$Q[\phi_1^r \phi_2^s \phi_3^t] = I[\phi_1^r \phi_2^s \phi_3^t]$$

*with* $2r + 4s + 4t \leqslant d$, $r,s \in \mathbb{N}, t \in \{0,1\}$.

*The number of equations* $= \dim \mathbb{P}_d(R_4) = \left\lfloor \frac{k^2+1}{2} \right\rfloor$.

**Proof :**

The number of linear independent invariants of degree $i$, $c_i$, can be found using the Molien-serie :

$$\sum_{i=0}^{\infty} c_i t^i = \frac{1+t^4}{(1-t^2)(1-t^4)}$$

$$\Rightarrow 1 + t^4 = (1-t^2-t^4+t^6) \sum_{i=0}^{\infty} c_i t^i$$

$$c_i = 0 \quad \text{when } i \text{ is odd}$$

$$\Rightarrow c_0 = 1,\ c_2 = 1,\ c_4 = 3,\ c_i = c_{i-2}+c_{i-4}-c_{i-6},\ i \geqslant 6$$

$$\Rightarrow c_i = 2\left\lfloor \frac{i}{4} \right\rfloor + 1 \quad \text{when } i \text{ is even.}$$

Then, $\dim \mathbb{P}_d(R_4) = \sum_{i=0}^{d} c_i = \left\lfloor \frac{k^2+1}{2} \right\rfloor$.

We know that $\phi_1$, $\phi_2$ and $\phi_3$ form a good integrity basis for $\mathbb{P}(R_4)$.
The single syzygie is $\phi_3^2 = \phi_1^4 - \phi_2^2$.
Thus the "monomials" $\phi_1^r.\phi_2^s.\phi_3^t$ with $r,s \in \mathbb{N}$, $t \in \{0,1\}$ and $2r+4s+4t \leqslant d$ form a basis for $\mathbb{P}_d(R_4)$.

□

Each orbit in an invariant cubature formula introduces a number of unknowns in the nonlinear equations and gives a number of knots in the cubature formula. The different types of orbits are described in table 1.

| type | point | number of unknowns | number of points in a $R_4$-orbit | unknowns |
|---|---|---|---|---|
| 1 | $(a,b)$ | 3 | 4 | $a$, $b$, weight |
| 2 | $(a,a)$ | 2 | 4 | $a$, weight |
| 3 | $(a,0)$ | 2 | 4 | $a$, weight |
| 4 | $(0,0)$ | 1 | 1 | weight |

Table 1 : different types of $R_4$-orbits.

Let $F_i$ be the number of orbits of type $i$ in an invariant cubature formula.

We will look for cubature formulae with the number of unknowns equal to the number of equations, thus

$$3F_1 + 2(F_2+F_3) + F_4 = \dim \mathbb{P}_d(R_4).$$

In order to obtain cubature formulae with the smallest number of knots, we use as many orbits with 3 unknowns as possible, thus

$$F_2+F_3+F_4 \leqslant 1.$$

In table 2 we give the structure of the minimal formulae, the number of knots, the place where the formulae can be found and the symmetry-group of the formulae.

| $d$ | $k$ | $\dim \mathbb{P}_d(R_4)$ | $F_1$ | $F_2+F_3$ | $F_4$ | number of knots | reference | $G$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 1 | 1 | [18] | $D_4$ |
| 3 | 2 | 2 | 0 | 1 | 0 | 4 | [18] | $D_4$ |
| 5 | 3 | 5 | 1 | 1 | 0 | 8 | [18] | $D_4$ |
| 7 | 4 | 8 | 2 | 1 | 0 | 12 | [8] | $D_4$ |
| 9 | 5 | 13 | 4 | 0 | 1 | 17 | [13] | $R_4$ |
| 11 | 6 | 18 | 6 | 0 | 0 | 24 | Table 5 | $R_4$ |
| 13 | 7 | 25 | 8 | 0 | 1 | 33 | Table 6 | $R_4$ |
| 15 | 8 | 32 | 10 | 1 | 0 | 44 | [18] | $D_4$ |

Table 2 : cubature formulae invariant w.r.t. $R_4$.

## New results for the square :

— *degree 11 with 24 knots :* The formula has all knots inside the region and all weights are positive. The number of knots is equal to Möller's lower bound [13]. The structure of the formula can be

seen in figure 1.
Mantel [11] told us that $R_4$ was a possible structure for a nonfortuitous minimal cubature formula of degree 11 for a $R_4$ invariant region. However, he could not guarantee the existence of the formula.
On the other hand, Möller and Schmid [14] believed that the formula didn't exist.

— *degree 13 with 33 knots :* The formula has all knots inside the region and all weights are positive. The structure of the formula can be seen in figure 2. Cubature formulae with 34, 35 and 36 knots are known [3], but they all have knots outside the region. A 37-point formula with all knots inside the region was found by Rabinowitz and Richter [15].

## 5. Cubature formulae for the triangle

We want to construct cubature formulae for $T_2$ : the triangle $\{(x,y) : x,y \geqslant 0,\ x+y \leqslant 1\}$ with weight function $w(x,y) = 1$.

It is easier to apply invariant theory to construct cubature formulae for $\Delta$ : the triangle $\{(x,y) : x \leqslant \frac{1}{2},\ \sqrt{3}\,y-x \leqslant 1,\ -\sqrt{3}\,y-x \leqslant 1\}$. The symmetry-group of this triangle contains 3 reflections and 3 rotations. It is known as the dihedral group $D_3$ of order 6.
The rotations form a subgroup $R_3$.

Cubature formulae for $\Delta$ can be transformed into cubature formulae for $T_2$ of the same degree, by the following affine transformation

$$\Delta \to T_2 : \begin{cases} \dfrac{1-2x}{3} \to x \\ \dfrac{1+x+\sqrt{3}\,y}{3} \to y. \end{cases}$$

Analogous as in example 1, it can be shown that $\phi_1 = r^2$, $\phi_2 = r^3 \cos 3\theta$, $\phi_3 = r^3 \sin 3\theta$, with $x = r\cos\theta$ and $y = r \sin\theta$, form a good integrity basis for the invariants of $R_3$.
The single syzygie is $\phi_2^2 + \phi_3^2 - \phi_1^3 = 0$.

The system of nonlinear equations we must solve, is described by

**Theorem 6 :** *An invariant cubature formula with respect to $R_3$, of degree d is a solution of the system of nonlinear equations*

$$Q[\phi_1^r\phi_2^s\phi_3^t] = I[\phi_1^r\phi_2^s\phi_3^t]$$

*with* $2r+3s+3t \leqslant d$, $\quad r,s \in \mathbb{N},\ t \in \{0,1\}$.
*The number of equations* $= \dim \mathbb{P}_d(R_3) = \left[\dfrac{d^2+3d+6}{6}\right]$.

**Proof :**

The number of linear independent invariants of degree $i$, $c_i$, can be found using the Molien-serie :

$$\sum_{i=0}^{\infty} c_i t^i = \frac{1+t^3}{(1-t^2)(1-t^3)}$$

$$\Rightarrow 1+t^3 = (1-t^2-t^3+t^5) \sum_{i=0}^{\infty} c_i t^i$$

$$\Rightarrow c_0 = 1,\ c_1 = 0,\ c_2 = 1,\ c_3 = 2,\ c_4 = 1$$

$$c_i = c_{i-2} + c_{i-3} - c_{i-5}, \quad i \geqslant 5$$

$$\Rightarrow c_i = \frac{i+1}{3} + \frac{4\sqrt{3}}{9} \sin\left(\frac{2(i+1)\pi}{3}\right).$$

Then,

$$\dim \mathbb{P}_d(R_3) = \sum_{i=0}^{d} c_i = \frac{(d+1)(d+2)}{6} + \frac{2}{3} \text{ if } d \bmod 3 = 0$$

$$= \frac{(d+1)(d+2)}{6} \text{ if } d \bmod 3 \neq 0$$

$$= \left\lfloor \frac{(d+1)(d+2)+4}{6} \right\rfloor.$$

We know that $\phi_1$, $\phi_2$ and $\phi_3$ form a good integrity basis for $\mathbb{P}(R_3)$.
The syzygie is $\phi_3^2 = \phi_1^3 - \phi_2^2$.
Thus the "monomials" $\phi_1^r \phi_2^s \phi_3^t$ with $r,s \in \mathbb{N}$, $t \in \{0,1\}$ and $2r+3s+3t \leqslant d$ form a basis for $\mathbb{P}_d(R_3)$.

□

The different types of orbits are described in table 3.

| type | point | number of unknowns | number of points in a $R_3$-orbit | unknowns |
|---|---|---|---|---|
| 1 | $(a,b)$ | 3 | 3 | $a$, $b$, weight |
| 2 | $(a,0)$ | 2 | 3 | $a$, weight |
| 3 | $(0,0)$ | 1 | 1 | weight |

Table 3 : different types of $R_3$-orbits.

Let $F_i$ be the number of orbits of type $i$ in an invariant cubature formula.
The number of equations is equal to the number of unknowns if

$$3F_1+2F_2+F_3 = \dim \mathbb{P}_d(R_3).$$

In order to obtain cubature formulae with the smallest number of knots we must choose the $F_i$ so that

$$F_2 + F_3 \leqslant 1.$$

In table 4 we give the structure of the minimal formulae, the number of knots, the place where the formulae can be found and the symmetry-group of the formulae.

| $d$ | $\dim \mathbb{P}_d(R_3)$ | $F_1$ | $F_2$ | $F_3$ | number of knots | $G$ | reference |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 1 | 1 | $D_3$ | [18] |
| 2 | 2 | 0 | 1 | 0 | 3 | $D_3$ | [5] |
| 3 | 4 | 1 | 0 | 1 | 4 | $D_3$ | [18] |
| 4 | 5 | 1 | 1 | 0 | 6 | $D_3$ | [5] |
| 5 | 7 | 2 | 0 | 1 | 7 | $D_3$ | [18] |
| 6 | 10 | 3 | 0 | 1 | 10 | | |
| 7 | 12 | 4 | 0 | 0 | 12 | $R_3$ | [6] |
| 8 | 15 | 5 | 0 | 0 | 15 | $R_3$ | Table 7 |
| 9 | 19 | 6 | 0 | 1 | 19 | $D_3$ | [10] |
| 10 | 22 | 7 | 0 | 1 | 22 | $R_3$ | Table 8 |

Table 4 : cubature formulae invariant w.r.t. $R_3$.

## New results for the triangle

— *degree 8 with 15 knots* : The number of knots is equal to Stroud's lower bound [18] and thus all weights are positive. The formula has 3 knots outside the region.

— *degree 10 with 22 knots* : The number of knots is equal to Stroud's lower bound [18] and thus all weights are positive. The formula has 3 knots outside the region.

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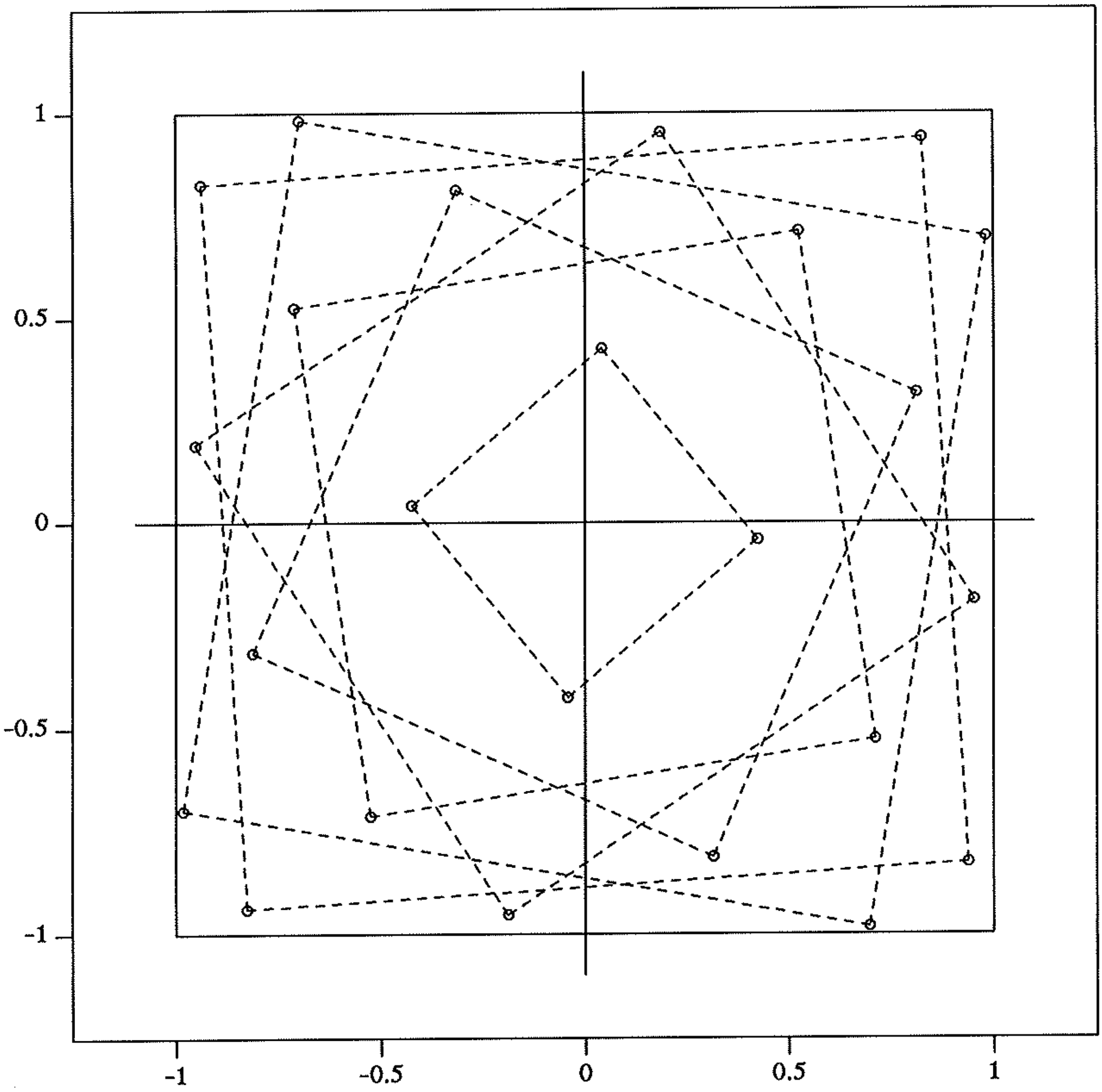


*Fig. 1: 24-point formula of degree 11 for $C_2$.*

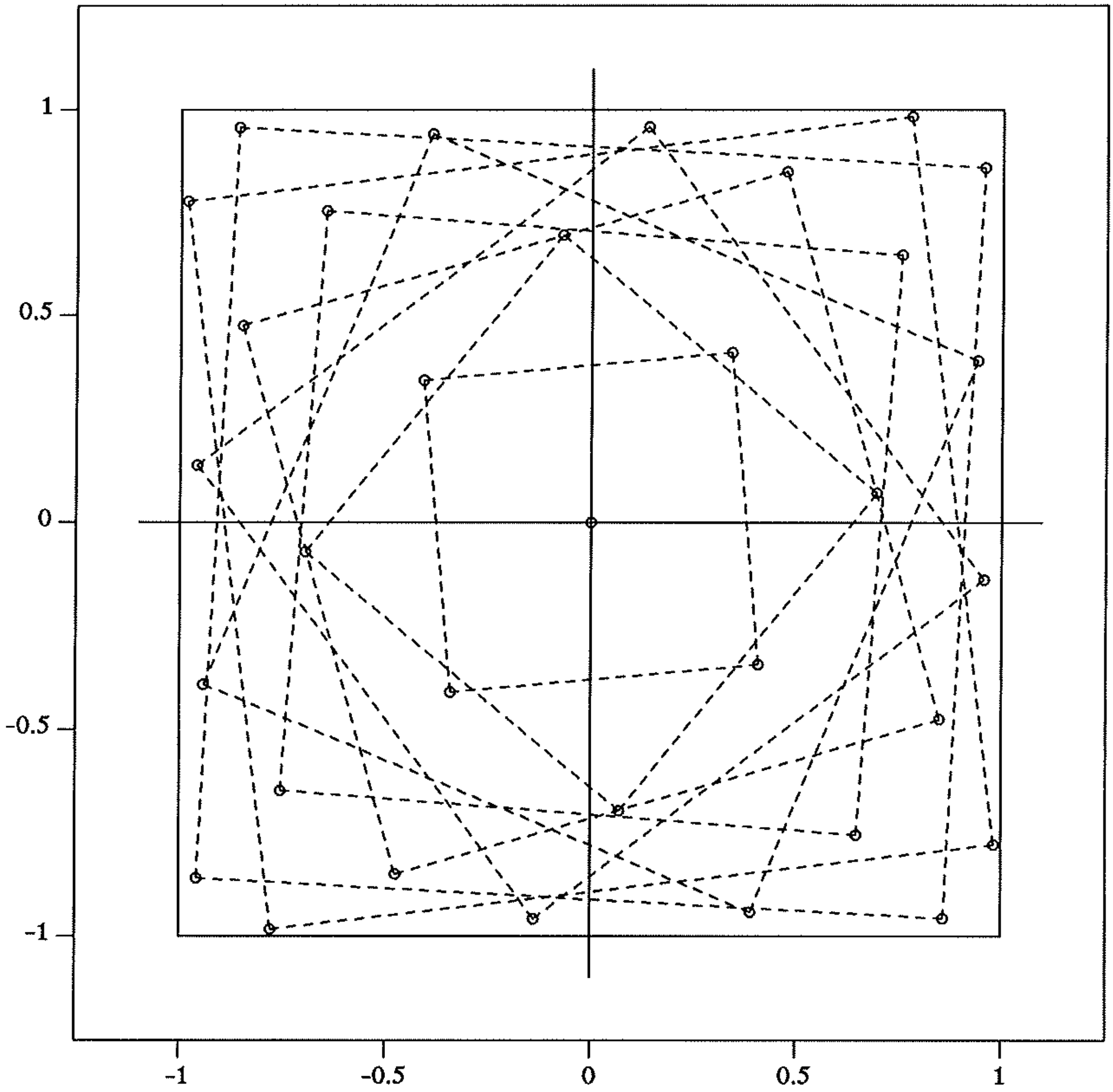


*Fig. 2 : 33-point formula of degree 13 for* $C_2$.

Table 5: 24-point formula of degree 11 for $C_2$

| $i$ | $w_i$ | $x_i$ | $y_i$ |
|---|---|---|---|
| 1 | 0.48020763350723814563D-01 | 0.98263922354085547295D+00 | 0.69807610454956756478D+00 |
| 2 | 0.66071329164550595674D-01 | 0.82577583590296393730D+00 | 0.93948638281673690721D+00 |
| 3 | 0.97386777358668164196D-01 | 0.18858613871864195460D+00 | 0.95353952820153201585D+00 |
| 4 | 0.21173634999894860050D+00 | 0.81252054830481310049D+00 | 0.31562343291525419599D+00 |
| 5 | 0.22562606172886338740D+00 | 0.52532025036454776234D+00 | 0.71200191307533630655D+00 |
| 6 | 0.35115871839824543766D+00 | 0.41658071912022368274D-01 | 0.42484724884866925062D+00 |

Table 6: 33-point formula of degree 13 for $C_2$

| $i$ | $w_i$ | $x_i$ | $y_i$ |
|---|---|---|---|
| 1 | 0.29991838864499131666D-01 | 0.77880971155441942252D+00 | 0.98348668243987226379D+00 |
| 2 | 0.38174421317083669640D-01 | 0.95729769978630736566D+00 | 0.85955600564163892859D+00 |
| 3 | 0.60424923817749980681D-01 | 0.13818345986246535375D+00 | 0.95892517028753485754D+00 |
| 4 | 0.77492738533105339358D-01 | 0.94132722587292523695D+00 | 0.39073621612946100068D+00 |
| 5 | 0.11884466730059560108D+00 | 0.47580862521827590507D+00 | 0.85007667369974857597D+00 |
| 6 | 0.12976355037000271129D+00 | 0.75580535657208143627D+00 | 0.64782163718701073204D+00 |
| 7 | 0.21334158145718938943D+00 | 0.69625007849174941396D+00 | 0.70741508996444936217D-01 |
| 8 | 0.25687074948196783651D+00 | 0.34271655604040678941D+00 | 0.40930456169403884330D+00 |
| 9 | 0.30038211543122536139D+00 | 0.00000000000000000000D+00 | 0.00000000000000000000D+00 |

Table 7: 15-point formula of degree 8 for $T_2$

| $i$ | $w_i$ | $x_i$ | $y_i$ |
|---|---|---|---|
| 1 | 0.16058343856681218798D-09 | 0.34579201116826902882D+00 | 0.36231682215692616667D+01 |
| 2 | 0.26530624434780379347D-01 | 0.65101993458939166328D-01 | 0.87016510156356306078D+00 |
| 3 | 0.29285717640165892159D-01 | 0.65177530364879570754D+00 | 0.31347788752373300717D+00 |
| 4 | 0.43909556791220782402D-01 | 0.31325121067172530696D+00 | 0.63062143431895614010D+00 |
| 5 | 0.66940767639916174192D-01 | 0.51334692063945414949D+00 | 0.28104124731511039057D+00 |

Table 8: 22-point formula of degree 10 for $T_2$

| $i$ | $w_i$ | $x_i$ | $y_i$ |
|---|---|---|---|
| 1 | 0.15319130036758557631D-06 | 0.58469201683584513031D-01 | -0.54887778772527519316D+00 |
| 2 | 0.13260526227928785221D-01 | 0.50849285064031410705D-01 | 0.90799059794957813439D+00 |
| 3 | 0.15646439344539042136D-01 | 0.51586732419949674487D+00 | 0.46312452842927062902D+00 |
| 4 | 0.21704258224807323311D-01 | 0.24311033191739048230D+00 | 0.72180595182371959467D+00 |
| 5 | 0.21797613600129922367D-01 | 0.75397765920922660134D+00 | 0.20647569839132397633D+00 |
| 6 | 0.38587913508193459468D-01 | 0.42209207910846960294D+00 | 0.12689533413411127327D+00 |
| 7 | 0.39699584282594413022D-01 | 0.19823878846663354068D+00 | 0.62124412566393319745D+00 |
| 8 | 0.47910534861520060665D-01 | 0.33333333333333333333D+00 | 0.33333333333333333333D+00 |