Analytical Formulas for the Directivity of General Antenna Arrays.

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Abstract—**Some formulas for rotationally symmetric patterns, as well as an optimal computation of directivities have already been published. The previous results are generalized in this paper for completely arbitrary element patterns.**

Antenna Arrays; Directivity; analytical expressions, indoor propagation

I. Introduction

The directivity function, the ratio between the power density in a certain direction and the average power density, is defined in [1] and [2],

$$D(\theta,\phi)=\frac{\left|\overrightarrow{F}(\theta,\phi)\right|^2}{\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\overrightarrow{F}(\theta,\phi)\overrightarrow{F}^*(\theta,\phi)\sin\theta d\theta d\phi}=\left|\overrightarrow{F'}(\theta,\phi)\right|^2 D \qquad (1)$$

$\overrightarrow{F}(\theta,\phi)$ is the complex far-field pattern (* denotes complex conjugate),

$\overrightarrow{F'}(\theta,\phi)=\frac{\overrightarrow{F}(\theta,\phi)}{F_{MAX}}$ is the normalised field of the array (electric or magnetic) and D is the directivity in the direction of maximum radiation. For an array of N elements, the far-field pattern can be written as the product of an element pattern and an array factor. The array factor depends on the (complex) excitation coefficients (a_i) and the positions of the elements as well as on the frequency. The maximum value occurs if all amplitudes add up in phase, hence (n=N-1) $f_{MAX}=\sum_{i=0}^{n}\left|a_i\right|$. For the most common arrays this occurs within the far-field pattern corresponding with real angles θ and ϕ.

II. Computations

A. Previous derivations

Using [3, 8.411.1], we have shown that the denominator of (1) for vertical Hertzian dipoles could be expressed as:

$$N_D=2\sum_{i=0}^{n}\sum_{j=i+1}^{n}\mathrm{Re}(a_i a_j^*)\left[f_1\left(k_0\left|\overrightarrow{p_i}-\overrightarrow{p_j}\right|\right)-f_2\left(k_0\left|\overrightarrow{p_i}-\overrightarrow{p_j}\right|\right)k_0^2(z_i-z_j)^2\right] +\frac{2}{3}\sum_{i=0}^{n}\left|a_i\right|^2 \qquad (2)$$

$$\overrightarrow{p_i}-\overrightarrow{p_j}=(\overrightarrow{r_i}-\overrightarrow{r_j})-(\overrightarrow{r_i}-\overrightarrow{r_j}).\overrightarrow{i_z}\overrightarrow{i_z}$$

p, denoting the projection of the array element centres on a plane (in this case the x-y plane). The smooth distance functions f_1 and f_2 could be written as spherical Bessel functions as follows:

$$f_1(x)=j_0(x)-\frac{j_1(x)}{x}$$
$$f_2(x)=-\frac{j_2(x)}{x^2} \qquad (3)$$

These have a zero derivative for small intermediate distances between antenna elements and look like Fig. 1:

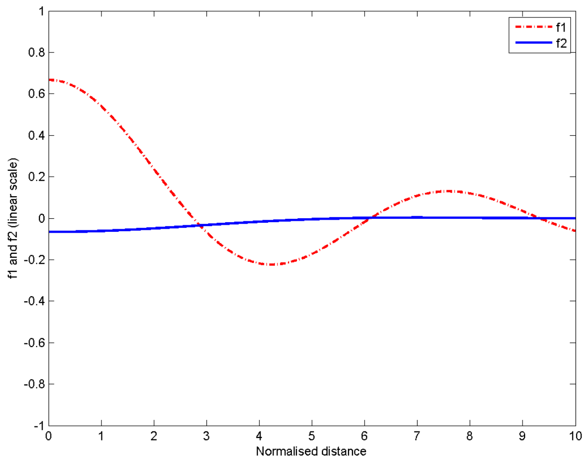

Fig. 1: distance functions for Hertzian dipoles.

For the isotropic radiators the function f_1 is obviously the known sinc function. Those functions can be expressed in terms of trigonometric functions as follows:

$$f_1(x) = \operatorname{sinc} x - \frac{\operatorname{sinc} x - \cos x}{x^2}$$
$$f_2(x) = \frac{x \sin x + 3\cos x - 3\operatorname{sinc} x}{x^4} \quad (4)$$

With limits for d→0 equal to:

$$\lim_{x\to 0} f_1(x) = 2/3$$
$$\lim_{x\to 0} f_2(x) = -1/15 \quad (5)$$

B. Generalisation

The first generalization concerned patterns that were only dependent of θ. In [5], we found that, for a $\cos^m\theta$ pattern, it was possible to write the denominator as:

$$N_D = \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j^* \sum_{k=0}^{m} \frac{(-1)^{m-k} m!(2(m-k))!}{k!(m-k)!^2 (2k_0 |\overline{p_i} - \overline{p_j}|)^{m-k}} j_{m-k}(k_0 |\overline{p_i} - \overline{p_j}|) \quad (6)$$

This can be more explicitly written as:

$$N_D = \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j^* [j_0(\alpha_{ij}) - C_m^1 \frac{j_1(\alpha_{ij})}{\alpha_{ij}} + 3C_m^2 \frac{j_2(\alpha_{ij})}{\alpha_{ij}^2} - \ldots + (-1)^m 1.3\ldots(2m-1)\frac{j_m(\alpha_{ij})}{\alpha_{ij}^m}] \quad (7)$$

with $\alpha_{ij} = k_0 |\overline{p_i} - \overline{p_j}|$, only dependent on i-j for linear equidistant arrays.
As a special case, for m=2 also a simple formula is obtained:

$$N_D = \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j^* [j_0(\alpha_{ij}) - \frac{2 j_1(\alpha_{ij})}{\alpha_{ij}} + \frac{3 j_2(\alpha_{ij})}{\alpha_{ij}^2}] \quad (8)$$

For a linear equidistant array, the directivities can be computed in a fast way, by making use of the power coefficients:

$$c_l = \sum_{i=0}^{n-l} a_i a_{i+l}^* \quad (9)$$

$$N_D = 2\sum_{l=0}^{n} \operatorname{Re}(c_l e^{-jl(\overline{k_{MAX}} - \overline{k'_{MAX}}).\overline{d}}) F(lk_o d) + c_0 F(0)$$

(10)

where F is the function of the spherical Bessel functions and k'_{max} being the direction of the beam steered maximum (the unsteered one being at k_{max}. For practical applications (corporate feeding of the array) the coefficients vary little with frequency and hence the c_l□s can be computed in advance. In that case the geometric layout looks like Fig. 2.

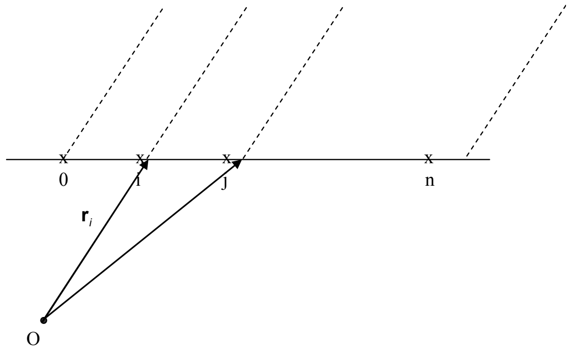

Fig. 2: linear array.

To program this efficiently all c_l □s are computed in advance in one vector, destroying the excitation coefficients. Until the (E(N/2)-1)th (E is the integer just lower than or equal to N/2) calculation of c_l the new element is added in the vector after the end of vector a (c_l is added in a_{n+1}) (Fig. 3 a). The total number of memory elements is thus N+E(N/2)-1=E((3N-2)/2). After the (E(N/2)-1)th coefficient c_l, the next coefficient c_l can be placed directly in element a_l, because those excitation coefficients are not longer required (Fig. 3 b). At this stage the original excitation coefficients start to be destroyed in the computation. After the buffering the first elements can be copied in place. This means that a_{n+1}=c_1 is put back at memory location a_1, a_{n+2} on a_2 etc (Fig. 3 c).

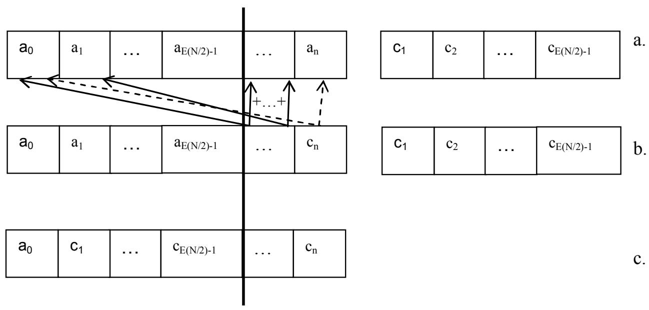

Fig. 3: Efficient computation of the coefficients cl for equidistant arrays

An example for an 8 element uniform array is given in Fig. 4.

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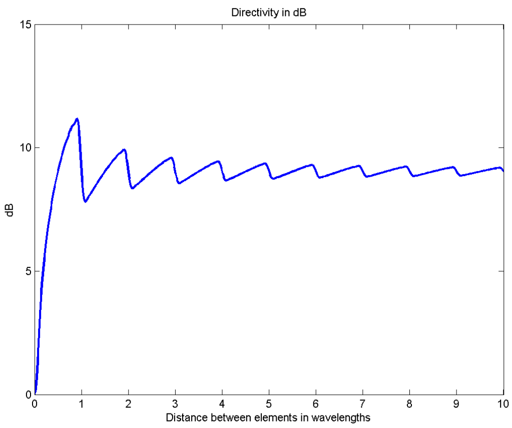

Fig. 4: Efficient computation of the directivity of an 8 element linear array

An example for an 8 element -20 dB Chebychev array is given in Fig. 4.

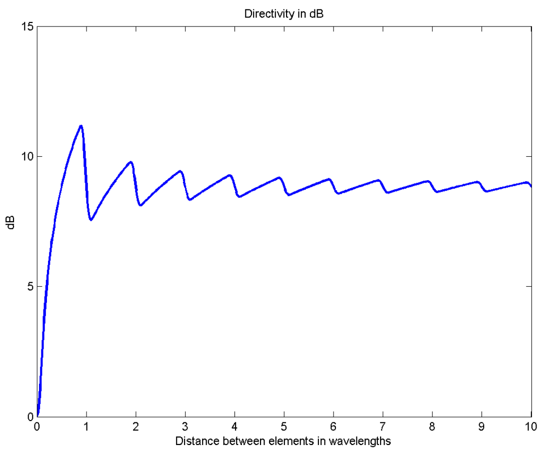

Fig. 5: Efficient computation of the directivity of an 8 element -20 dB Chebychev linear array

We notice the expected behaviour in Fig. 6 for a simple linear 8 element array, and notice that in some cases (when the distance is close to 0.9 wavelengths), the directivity of the uniform array might be less than that of a -20 dB sidelobe Chebychev array. We can also see that the directivity of the array is not equal to the sum of the element directivity and the isotropic directivity, even if the difference is usually less than 1 dB.

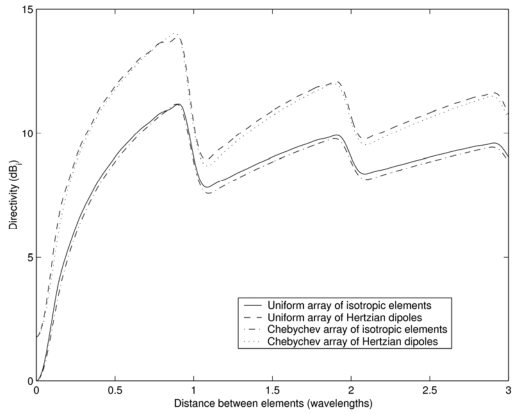

Fig. 6: distance functions for Hertzian dipoles and isotropic elements.

This formula can be further generalised for completely arbitrary patterns, by computing multiple derivatives of the simple sinc function with respect to both $\alpha(\theta)$ and $\beta\ (\varphi)$.

$$\left[\operatorname{sinc}\left(\sqrt{\alpha^2+\beta^2}\right)\right] \tag{11}$$

III. APPLICATIONS

If we are only interested in the far-field of an antenna - true in most cases of indoor wireless computations not too close to the antennas (but those cases guarantee a sufficient signal strength and a sufficient direct to multipath ratio) - the computational time is reduced in proportion to the number of array elements while retaining high accuracy. This is illustrated in one example of a room at ISM frequencies (Fig. 7)

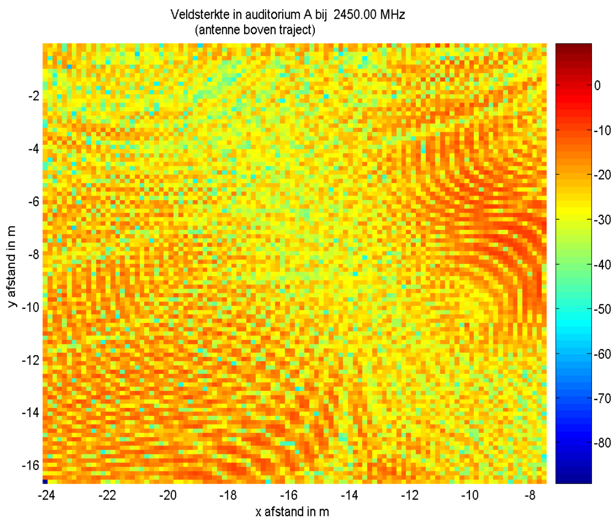

Fig. 7: Field strength in a large auditorium at 2.45 GHz.

The difference between a computation with the array factor and the directivity (Fig. 8) and one with the separate elements

(Fig. 9) is not very large, but the first takes much less computational time (about equal to the number of elements).

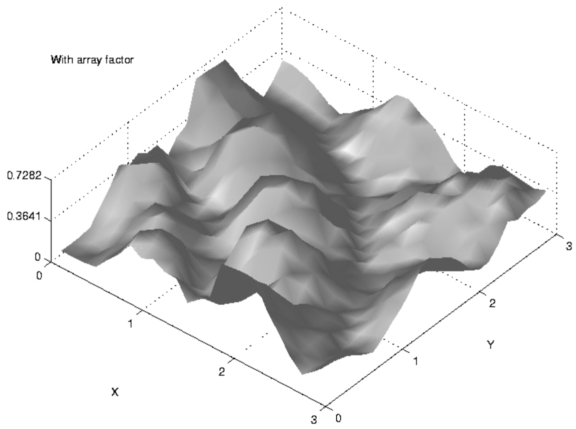

Fig. 8: Field strength in a 3x3x3 cubiculum at 2.45 GHz with array factor.

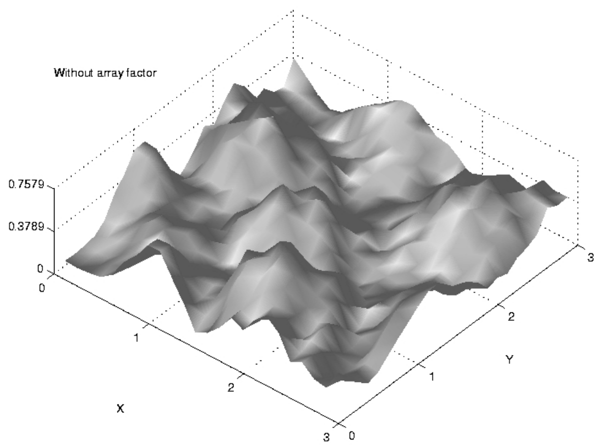

Fig. 9: Field strength in a 3x3x3 cubiculum at 2.45 GHz with separate elements.

IV. CONCLUSIONS

Directivities of arbitrary arrays can be easily computed in function of only spherical Bessel functions and the knowledge of the excitation coefficients.

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