# Quadrature on the half line and two-point Padé approximants to Stieltjes functions. II Convergence*

A. Bultheel† C. Díaz-Mendoza‡ P. González-Vera‡ R. Orive‡

**Abstract**

Let $\alpha$ be a distribution function on $[a,b]$ $(0 \leq a < b \leq +\infty)$ such that the moment $c_k = \int_a^b x^k d\alpha(x)$ exist for all the integers $k$. The main course of the paper is to give convergence results both for sequences of Two-point Padé approximants to the Cauchy transform of the distribution $\alpha$ (Stieltjes function) and for quadrature formulas based on exactly integrating Laurent polynomials.
The case when $d\alpha(x) = k(x)dx$, $k(x)$ being a possibly complex function is also considered and estimates of the rate of convergence are given.

**Keywords:** two-point Padé approximation, Stieltjes function, orthogonal Laurent polynomials, quadrature formula.
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## 1 Introduction

A Stieltjes function $F_\alpha$ is the Cauchy transform of a distribution $\alpha$ on $[a,b]$ with $0 \leq a < b \leq +\infty$. In the first part of this paper [3], it is explained how the two-point-Padé approximants (2PA) and two-point Padé-type approximants (2PTA) to a Stieltjes function are related to numerical quadrature formulas for $I_\alpha(f) = \int_a^b f d\alpha$. Two-point Padé-type approximants approximate the function by satisfying a number of interpolation conditions in two points (0 and $\infty$). The number of conditions can be equally distributed between 0 and $\infty$ or not. If all the conditions are taken in one of the points, then one obtains ordinary (i.e., one-point) Padé-type approximants. The 2PA's are 2PTA's where all the degrees of freedom are used to satisfy a maximum number of interpolation conditions. If all the interpolation conditions are at 0 or $\infty$, then one obtains ordinary Padé approximants at the origin or at infinity. For more details and definitions, see Section 2 of Part I of this paper [3] and the references given there. The 2PTA's are related to quadrature formulas of interpolatory type and the 2PA are related to Gaussian quadrature formulas which are also of interpolatory type but which have a maximal domain of validity.

In [3] it was shown that classical properties for Padé approximants to Stieltjes functions also apply in the two-point case. In particular it was shown that one can enclose a Stieltjes function by its 2PA's, so that the latter reveal monotonicity in their convergence behaviour.

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†Department of Computer Science, K.U.Leuven, Belgium
‡Department Análisis Math., Univ. La Laguna, Tenerife, Spain

In this sequel to Part I, we show that the 2PA's and 2PTA's do converge to the Stieltjes function and we also give an estimate for the rate of convergence. Because of the intimate relation between these rational approximants for Stieltjes functions and corresponding quadrature formulas, we also obtain convergence results for the latter.

Because the boundary point $b = +\infty$ gives extra complications, almost all the results in this paper are for the case of a finite interval $[a, b]$ with $0 \leq a < b < +\infty$. The case of the whole half-line will be discussed in a forthcomming paper.

After some preliminary definitions and observations in Section 2, we give in the three subsequent sections the announced convergence properties.

In Section 3 we show the convergence for quadrature formulas of Gaussian type and for the corresponding 2PA's. If $p(n)$ is the number of interpolation conditions at the origin and $q(n) + 1 = 2n - p(n)$ the number of interpolation conditions at $\infty$ for the 2PA's, then it is assumed that $q(n) \to \infty$. This condition allows us to give convergence results without imposing a Carlemann-type condition for the moments of $\alpha$. It is shown that the Gaussian quadrature formulas converge for continuous functions on $[a, b]$ (Theorem 3.1) and that the 2PA's converge uniformly in a compact subset of the cut plane $\hat{\mathbb{C}} - [a, b] = \mathbb{C} \cup \{\infty\} - [a, b]$ (Theorem 3.2). If $p(n)$ as well as $q(n)$ go to $\infty$, then we prove the convergence of the Gaussian quadrature formulas for arbitrary Riemann-Stieltjes integrable functions (Theorem 3.4).

In Section 4, we prove the convergence for the interpolatory quadrature formulas and 2PTA's. In Section 3, $\alpha$ was supposed to be positive, so that the nodes of the quadrature formulas are zeros of Laurent polynomials orthogonal with respect to $\alpha$. All these zeros are simple and in $(a, b)$. In Section 4 we allow the distribution $\alpha$ to be complex. Hence the zeros of orthogonal Laurent polynomials are in general not feasible as nodes for the quadrature formulas. Therefore the nodes are chosen to be zeros of Laurent polynomials orthogonal with respect to a positive distribution $\sigma$. The domain of validity for the quadrature formulas is spanned by $\{x^{-p(n)}, \ldots, x^{q(n)}\}$ with $p(n) + q(n) = n$. Suppose that $\alpha$ and $\sigma$ are absolutely continuous, i.e., $d\alpha(x) = \alpha'(x)dx$ and $d\sigma(x) = \sigma'(x)dx$. Then if $\alpha'/\sqrt{\sigma'} \in L_2$, i.e., if $|\alpha'|^2/\sigma'$ is Riemann integrable over $[a, b]$, and if $q(n) \to \infty$, convergence of the quadrature formulas is ensured for continuous functions (Remark 4.1), for Riemann-Stieltjes integrable functions $f \in L_1(d\sigma)$ (Corollary 4.5), and for Riemann-Stieltjes integrable functions $f \in L_1(d\alpha)$ (Theorem 4.4). A corresponding convergence result for 2PTA's is given in Corollary 4.2.

Finally in Section 5, the rate of convergence of 2PA's and 2PTA's is discussed. As a consequence, we can also estimate the rate of convergence of the quadrature formulas. The latter can be obtained for Gaussian or interpolatory-type quadrature formulas when integrating functions which are analytic in a neighbourhood of the interval $[a, b]$ and where it is now assumed that $0 \leq \lim p(n)/n \leq 1$. It is then shown that the convergence of the 2PA's (Theorem 5.2), 2PTA's (Theorem 5.4), Gaussian quadrature formulas (Corollary 5.3) and interpolatory quadrature formulas (Corollary 5.5) is geometrical in all cases.

# 2 Preliminaries

Throughout this section we assume that $\alpha$ is a distribution function on $[a, b]$ ($0 \leq a < b \leq +\infty$) i.e. a non decreasing, bounded function with infinitely many points of increase on $[a, b]$. Let us suppose in addition the the moments

$$c_k = \int_a^b x^k d\alpha(x) \tag{2.1}$$

exist for all $k \in \mathbb{Z}$.
For any pair $(p, q)$ of integers, with $p \leq q$ we denote by $\Lambda_{p,q}$ the subspace of Laurent polynomials, i.e. functions of the form

$$L(z) = \sum_{j=p}^{q} a_j z^j, \quad a_j \in \mathbb{C}; \tag{2.2}$$

$\Lambda$ will denote the space all Laurent polynomials (L-polynomials), $\Lambda_{0,n} = \Pi_n$ the subspace of polynomials of degree $n$ at most, and $\Pi$ the space of all polynomials.
For the given distribution $\alpha$, we consider Riemann-Stieltjes integrals of the form

$$I_\alpha(f) = \int_a^b f(x) d\alpha(x) \tag{2.3}$$

and linear quadrature formulas (q.f.) with nodes $\{x_j : j = 1, \ldots, n\}$ and weights $\{A_j : j = 1, 2, \ldots, n\}$ given by:

$$I_n(f) = \sum_{j=1}^{n} A_j f(x_j). \tag{2.4}$$

They are intended to be estimates for $I_\alpha(f)$. Such q.f.'s are related to rational approximants to the Cauchy transform of $\alpha$. To see this we first introduce some definitions.
Let $F_\alpha(z) \equiv F(z) = \int_a^b (z - x)^{-1} d\alpha(x)$ be the Cauchy transform of $\alpha$, which represents an analytic function in the cut plane $\hat{\mathbb{C}} - [a, b]$. If $0 < a < b < +\infty$, $F_\alpha$ admits the Taylor expansions

$$L^0(z) = -\sum_{j=0}^{\infty} c_{-(j+1)} z^j \quad \text{and} \quad L^\infty(z) = \sum_{j=1}^{\infty} c_{j-1} z^{-j} \tag{2.5}$$

around $z = 0$ and $z = \infty$ respectively.

When $a = 0$ (respectively $b = +\infty$) the expansion $L^0$ (respectively $L^\infty$) in (2.5) should be understood in the asymptotic sense for $z \in \mathbb{R}$ and $z \to 0^-$ (respectively for $z \to -\infty$). See [13] for further details.

Let $p$ and $q$ be nonnegative integers and assume that a quadrature formula $I_n(f)$ of the form (2.4) has been provided which is exact in $\Lambda_{-p,q}$, i.e. for which

$$I_n(f) = I_\alpha(f), \quad \forall f \in \Lambda_{-p,q}. \tag{2.6}$$

We also say that $\Lambda_{-p,q}$ is "a domain of validity" for the quadrature formula $I_n(f)$ if (2.6) is satisfied. With the coefficients $\{A_j\}$ and nodes $\{x_j\}$ we define the rational function

$$F_n(z) = \sum_{j=1}^{n} \frac{A_j}{z - x_j} \tag{2.7}$$

with poles at $\{x_j : j = 1, \ldots, n\}$ and corresponding residuals $\{A_j : j = 1, 2, \ldots, n\}$. Then, it is easily proved that

$$\begin{aligned} L^0(z) - F_n(z) &= \sum_{j=p}^{\infty} \tilde{c}_j z^j &&= O(z^p) \\ L^\infty(z) - F_n(z) &= \sum_{j=q+2}^{\infty} c_j^* z^{-j} &&= O\left(\tfrac{1}{z^{q+2}}\right) \end{aligned} \tag{2.8}$$

The same remark made for (2.5) should be applied in (2.8). Thus $F_n(z)$ does not only stand for the rational function (2.7) but also for its (formal) power series expansion at 0 or at $\infty$ as required by the context. Conversely, any rational function as given by (2.7) and satisfying (2.8) provides a quadrature formula of type (2.4) being exact in $\Lambda_{-p,q}$.

Recalling the notation and definitions of two-point Padé(-type) approximants from the first part of this paper [3], we can easily make the following observations.

If $p+q=n-1$, then $F_n$ represents a two-point Padé-type approximant (2PTA) to the pair $(L^0, L^\infty)$ of order $p$. The notation for such an approximant is $(p/n)_{(L^0,L^\infty)}$, but, for simplicity, we shall also write $(p/n)_{F_\alpha}$ i.e.

$$F_n(z) \equiv (p/n)_{(L^0,L^\infty)}(z) = (p/n)_{F_\alpha}(z).$$

If $p+q=2n-1$, then $F_n$ coincides with the two-point Padé approximant (2PA) to $(L^0, L^\infty)$ of order $p$, denoted as $[p/n]_{(L^0,L^\infty)}$ or as $[p/n]_{F_\alpha}$, which in this paper, is stripped to $[p/n]$ most of the time (not to be confused with one-point Padé approximants which in the literature are often represented by this notation). If we take $p=0$ ($q=2n-1$) we have

$$L^\infty(z) - F_n(z) = O\left(\frac{1}{z^{2n+1}}\right)$$

and thus, the one-point Padé approximant of type $(n-1,n)$ to the formal power series $L^\infty$ (i.e. the PA of $F_\alpha$ at infinity) is obtained. However, taking $q=0$ ($p=2n-1$) we have $L^\infty(z) - F_n(z) = O(1/z^2)$ and $L^0(z) - F_n(z) = O(z^{2n-1})$. So, we can see that the one-point Padé approximant of type $(n-1,n)$ to $L^0$ (i.e. the PA of $F_\alpha$ at the origin) is not covered. This is due to the restriction $q \geq 0$. In order to get one-point Padé approximant at the origin, we should allow that $q \geq -1$. In order to include both cases $p+q=n-1$ (2PTA) and $p+q=2n-1$ (2PA), we assume in what follows that $q \geq -1$.
Concerning the existence of domains of validity, we have the following

**Lemma 2.1** *Given $n$ distinct nonzero nodes $x_1, \ldots, x_n$ on $[a,b]$, there exist unique weights $A_1, \ldots, A_n$ so that*

$$I_n(g) = \sum_{j=1}^{n} A_j g(x_j) = I_\alpha(g) \quad , \quad \forall g \in \Lambda_{-p,q}$$

*with $p$ and $q$ arbitrary integers satisfying $p \geq 0$, $q \geq -1$ and $p+q=n-1$.*

The q.f. of the previous lemma is described as follows

$$I_n(g) = I_\alpha(L_n),$$

where $L_n = L_n[g] \in \Lambda_{-p,q}$ is the unique Laurent polynomial in $\Lambda_{-p,q}$ satisfying the Lagrange interpolation conditions

$$L_n(x_j) = g(x_j), \quad j = 1, 2, \ldots, n.$$

Thus

$$L_n(x) = \sum_{j=1}^{n} L_{n,j}(x) g(x_j), \tag{2.9}$$

where $L_{n,j} \in \Lambda_{-p,q}$ satisfies the interpolation conditions

$$L_{n,j}(x_k) = \delta_{j,k}, \quad j,k = 1, 2, \ldots, n.$$

If we set

$$V_n(x) = x^{-p} \prod_{j=1}^{n} (x - x_j) \in \Lambda_{-p,q+1},$$

then

$$L_{n,j}(x) = \frac{V_n(x)}{(x - x_j)V_n'(x_j)} \in \Lambda_{-p,q}.$$

Therefore, one has

$$A_i = I_\alpha(L_{n,j}) = I_\alpha \left( \frac{V_n(x)}{(x - x_j)V_n'(x_j)} \right), \quad j = 1, 2, \ldots, n. \tag{2.10}$$

As in the classical situation (i.e. polynomial quadrature) we have the following

**Lemma 2.2** *Let $p$ and $q$ be arbitrary integers satisfying $p \geq 0$, $q \geq -1$, and $p + q = 2n - 1$. Then, there cannot exist a quadrature formula like (2.4) which is exact in $\Lambda_{-(p+1),q}$ or $\Lambda_{-p,q+1}$.*

**Proof.** Define the nodal polynomial $\omega_n(x) = \prod_{j=1}^{n}(x - x_j)$. Then, taking $f(x) = x^{-p}\omega_n^2(x) \in \Lambda_{-p,q+1}$ and $g(x) = x^{-p-1}\omega_n^2(x) \in \Lambda_{-(p+1),q}$, we see that $I_\alpha(f) > 0$ and $I_\alpha(g) > 0$ but $I_n(f) = I_n(g) = 0$. □

**Lemma 2.3** *Let $p \geq 0$ and $q \geq -1$ be arbitrary integers such that $p + q = 2n - 1$. Then, there exist $n$ distinct nodes $\{x_1, \ldots, x_n\} \subset (a, b)$ and positive weights $\{A_1, \ldots, A_n\}$ so that*

$$I_n(f) = I_\alpha(f), \quad \forall f \in \Lambda_{-p,q}.$$

**Proof.** Define the distribution $d\tilde{\alpha}(x) = x^{-p}d\alpha(x)$ and let $Q_n$ denote the $n$th orthogonal polynomial for $d\tilde{\alpha}(x)$. Let $x_1, \ldots, x_n$ be its zeros which, as is well known, are simple and in $(a, b)$. It is well known that there exist positive real numbers $\tilde{A}_1, \ldots, \tilde{A}_n$, such that

$$\int_a^b P(x)d\tilde{\alpha}(x) = \sum_{j=1}^{n} \tilde{A}_j P(x_j), \quad \forall P \in \Pi_{2n-1}.$$

Take $f \in \Lambda_{-p,g}$, then $f(x) = x^{-p}P(x)$, with $P \in \Pi_{2n-1}$ (recall that $p + q = 2n - 1$). Thus, with $A_j = x_j^p \tilde{A}_j > 0$ it holds that

$$I_\alpha(f) = \int_a^b f(x)d\alpha(x) = \int_a^b \frac{P(x)}{x^p} d\alpha(x) = \int_a^b P(x)d\tilde{\alpha}(x) = \sum_{j=1}^{n} \tilde{A}_j P(x_j) = \sum_{j=1}^{n} A_j f(x_j).$$

This proves the lemma. □

**Remark 2.1** In [3] it was proved that the q.f. obtained in Lemma 2.3 is the unique one with domain of validity $\Lambda_{-p,q}$ $(p + q = 2n - 1)$. By Lemma 2.2, such a domain will be called "maximal" and the corresponding quadrature "Gaussian" in $\Lambda_{-p,q}$. Furthermore, the rational function $F_n(z) = \sum_{j=1}^{n} A_j/(z - x_j)$ represents a $[p/n]$ 2PA to the pair $(L^0, L^\infty)$ or equivalently to the function $F_\alpha(z)$. Observe that $0 \leq p \leq 2n$ and that $p = 2n$ corresponds to $q = -1$.

# 3 Convergence of Gaussian quadrature formulas and 2PA

Throughout this section we assume that $0 \leq a < b < +\infty$. Our aim is to study the convergence of quadrature formulas $I_n(f) = \sum_{j=1}^{n} A_{n,j} f(x_{n,j})$ as $n \to \infty$. For each $n$, $I_n$ is assumed to be exact in $\Lambda_{-p(n),q(n)}$ with $p(n)$ and $q(n)$ integers satisfying $p(n) \geq 0$, $q(n) \geq -1$, and $p(n) + q(n) = 2n - 1$.

It is known [1] that if for the moments $c_k$ of $d\alpha$, at least one of the Carleman-type conditions $\sum_{k=1}^{\infty} (c_k)^{-\frac{1}{2k}} = \infty$ or $\sum_{k=1}^{\infty} (c_{-k})^{-\frac{1}{2k}} = \infty$ is satisfied, then the moment problem is determinate. An easy estimate shows that if $0 < a < b < +\infty$, then $(c_k)^{\frac{1}{2k}}$ is bounded and hence $\sum_{k=1}^{\infty} (c_k)^{-\frac{1}{2k}} = \infty$. However, since we allow $a$ to be zero, this can not be guaranteed. Our approach will not require a Carlemann-type condition. We require instead that $\lim_{n\to\infty} q(n) = \infty$. Then the polynomials will be dense in $C[a,b]$, which will be essential in our proof. Otherwise, if $q(n)$ remains bounded, as far as we have been able to see, a convergence result without a Carlemann-type condition does not seem feasible.

As a consequence, the convergence of the corresponding sequence of 2PA's $\{[p(n)/n]_{F_\alpha}\}$ for the Cauchy thansform $F_\alpha$ of $\alpha$ shall be dealt with.

So, we restrict our attention to the case where $\lim_{n\to\infty} q(n) = \infty$. Observe that in this case, $I_n$ is exact in $\Pi_{q(n)}$ for each $n \in \mathbb{N}$, and hence that $\{I_n\}$ converges for any polynomial. Since in addition, we have that (take $f \equiv 1$)

$$\sum_{j=1}^{n} |A_{n,j}| = \sum_{j=1}^{n} A_{n,j} = c_0 = \int_a^b d\alpha(x),$$

the classical Polya Theorem [7], or in a more general context, the Banach-Steinhaus theorem [14], allows us to ensure that the Gaussian q.f.'s converge for all functions continuous in $[a,b]$.

**Theorem 3.1** *Let $0 \leq a < b < +\infty$. Let $\{p(n)\}$ be a sequence of integers such that $0 \leq p(n) \leq 2n$ and $\lim_{n\to\infty}[2n - p(n)] = \infty$. Then for the Gaussian q.f. $I_n(f)$, exact in $\Lambda_{-p(n),2n-p(n)-1}$ we have*

$$\lim_{n\to\infty} I_n(f) = I_\alpha(f), \quad \forall f \in C[a,b].$$

Thus, we have

**Theorem 3.2** *Let $0 \leq a < b < +\infty$. Let $F_\alpha$ be the Cauchy transform of the distribution function $\alpha$ on $[a,b]$ and let $(L^0, L^\infty)$ be the series (2.5) associated with $F_\alpha$. Let $\{p(n)\}$ be a sequence of integers such that $0 \leq p(n) \leq 2n$ and $\lim_{n\to\infty}[2n - p(n)] = \infty$. Let $[p(n)/n] = [p(n)/n]_{(L^0,L^\infty)}$ denote a 2PA to the pair $(L^0, L^\infty)$. Then, the sequence $\{[p(n)/n]\}_{n\in\mathbb{N}}$ converges to the function $F_\alpha(z)$ uniformly on compact subsets of $\hat{\mathbb{C}} - [a,b]$.*

**Proof.** For each $z \in \mathbb{C} - [a,b]$ consider the function (in the variable $x$)

$$f_z(x) = \frac{1}{z - x}$$

which is clearly continuous in $[a,b]$. Then, by Theorem 3.1, we have

$$\lim_{n\to\infty} I_n(f_z) = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{A_{n,j}}{z - x_{n,j}} = \lim_{n\to\infty} [p(n)/n](z) = I_\alpha(f_z) = F_\alpha(z)$$

and the pointwise convergence is proved.
Let $K$ be a compact subset of $\hat{\mathbb{C}} - [a,b]$. Since $[p(n)/n](z) = \sum_{j=1}^{n} A_{n,j}/(z - x_{n,j})$, we have

$$\max_{z\in K}\{|[p(n)/n](z)|\} = \|[p(n)/n]\|_{L_\infty(d\alpha)} \leq \frac{c_0}{\delta(K)},$$

where

$$\delta(K) = \min_{z\in K}\{|z-a|, |z-b|\} > 0.$$

Therefore $\{[p(n)/n]\}_{n\in\mathbb{N}}$ represents a normal family in $\mathbb{C} - [a,b]$, and the Stieltjes-Vitali theorem [11] guarantees the uniform convergence. □

**Remark 3.1** If we arrange the 2PA's for the series $(L^0, L^\infty)$ in a triangle array (the 2PA table [6]) of the form

$$\begin{array}{ccccc} & & [0/2] & \dots \\ & [0/1] & [1/2] & \dots \\ [0/0] & [1/1] & [2/2] & \dots \\ & [2/1] & [3/2] & \dots \\ & & [4/2] & \dots \end{array}$$

then the sequence $\{[n/n]\}_{n\in\mathbb{N}}$ is called the main row and corresponds to the so-called "balanced situation" in which the orders of interpolation at the origin and at infinity are the same. Parallel to it one finds the 2PA's $[n+k/n]$ on the $k$th row. We can also consider diagonals of the form $[k/n]$ with $k \geq 0$ fixed. The uniform convergence of all these sequences is guaranteed by Theorem 3.2. If $a > 0$, then also the diagonals $[2n-k/n]$ converge for fixed since the moment problem $\int_a^b x^n d\alpha(x) = c_n$, $n = 0, -1, -2, \dots$ is determinate and the Carleman condition $\sum_{\nu=1}^{\infty}(c_{-\nu})^{-\frac{1}{2\nu}} = \infty$ is satisfied. However, if $a = 0$, we cannot ensure the convergence of the corresponding sequence of q.f.'s for polynomials and hence it is most likely that a Carlemann-type condition will be required in order to guarantee the determinacy of the strong moment problem, and as a result the convergence of the sequence of 2PA's. (For further details in this respect, see the recent paper by López-Lagomasino and Finkelshtein [16]).

To end this section, let us point out that from Theorem 3.1 and the positivity of the weights $\{A_{n,j}\}$, the convergence of the quadrature process $\{I_n(f)\}_{n\in\mathbb{N}}$ may be extended to integrands $f$ being bounded Riemann-Stieltjes integrable function w.r.t. $d\alpha$. (In what follows we write $f \in S(d\alpha)$). Moreover, if in addition to the requirement $\lim_{n\to\infty} q(n) = \infty$ we impose that $\lim_{n\to\infty} p(n) = \infty$, the family of Gaussian q.f.'s on $\Lambda_{-p(n),q(n)}$ (with $p(n) + q(n) = 2n - 1$) converges for any function $F$ of the form $F(x) = x^{-m}f(x)$, with $f \in S(d\alpha)$ and $m$ a fixed integer. To prove this, we first need the following known result (see [22, p. 11]).

**Lemma 3.3** *Assume $f \in S(d\alpha)$ and define*

$$M = \max\{|\inf_{x\in[a,b]} f(x)|, |\sup_{x\in[a,b]} f(x)|\}.$$

*Then for every $\epsilon > 0$, there exist polynomials $\rho$ and $P$ such that*

$$-M - \epsilon \leq \rho(x) \leq f(x) \leq P(x) \leq M + \epsilon, \quad \forall x \in [a,b],$$

*and*

$$\int_a^b [P(x) - \rho(x)]d\alpha(x) < \epsilon.$$

**Theorem 3.4** *Let $\{p(n)\}_{n\in\mathbb{N}}$ be a sequence of integers such that $0 \le p(n) \le 2n$ and $\lim_{n\to\infty} p(n) = \lim_{n\to\infty} 2n - p(n) = \infty$.*
*For $n = 1, 2, \ldots$, let $I_n(F) = \sum_{j=1}^{n} A_{n,j} F(x_{n,j})$ denote the $n$-point Gaussian quadrature formula for $I_\alpha(F)$ which is exact in $\Lambda_{-p(n),2n-p(n)-1}$. Then, if $F(x) = x^{-m} f(x)$ with $f \in S(d\alpha)$ and $m$ a fixed integer, it holds that*

$$\lim_{n\to\infty} I_n(F) = I_\alpha(F) = \int_a^b F(x) d\alpha(x). \tag{3.1}$$

**Proof.** Let $\epsilon > 0$ be given. By Lemma 3.3, these exist polynomials $\rho(x)$ and $P(x)$ and a positive constant $M$ such that

$$-M - \epsilon \le \rho(x) \le f(x) \le P(x) \le M + \epsilon, \quad \forall x \in [a, b]. \tag{3.2}$$

Since $P(x) - \rho(x) \le (M + \epsilon) - (-M - \epsilon) = 2(M + \epsilon)$, one has

$$\int_a^b [P(x) - \rho(x)]^2 d\alpha(x) \le \int_a^b 2(M + \epsilon)[P(x) - \rho(x)] d\alpha(x) \le 2\epsilon(M + \epsilon).$$

Now, define $R(x) = x^{-m} P(x)$ and $r(x) = x^{-m}\rho(x)$. $R(x)$ and $r(x)$ are L-polynomials and by (3.2)

$$r(x) \le F(x) \le R(x), \quad \forall x \in [a, b] - \{0\}. \tag{3.3}$$

Since, $\lim_{n\to\infty} p(n) = \lim_{n\to\infty}[2n - p(n)] = \infty$, there exists an $n_0 \in \mathbb{N}$ such that for any natural $n \ge n_0$

$$r \in \Lambda_{-p(n),q(n)} \quad \text{and} \quad R \in \Lambda_{-p(n),q(n)},$$

where $q(n) = 2n - p(n) - 1$. Hence, by Lemma 2.3,

$$\int_a^b r(x) d\alpha(x) = \sum_{j=1}^{n} A_{n,j} r(x_{n,j}) \quad \text{and} \quad \int_a^b R(x) d\alpha(x) = \sum_{j=1}^{n} A_{n,j} R(x_{n,j}).$$

Now, making use of the Schwarz inequality, we can write

$$\begin{aligned} I_n(R) - I_n(r) &= I_\alpha(R - r) = \int_a^b \frac{P(x) - \rho(x)}{x^m} d\alpha(x) \\ &\le \left[\int_a^b [P(x) - \rho(x)]^2 d\alpha(x)\right]^{1/2} \left[\int_a^b x^{-2m} d\alpha(x)\right]^{1/2} \\ &\le \sqrt{2\epsilon(M + \epsilon) c_{-2m}}. \end{aligned}$$

By (3.3) we then have,

$$I_n(R) - I_n(F) \le I_n(R) - I_n(r) \le \sqrt{2\epsilon(M + \epsilon) c_{-2m}}.$$

Hence, for all $n \ge n_0$

$$\begin{aligned} |I_\alpha(F) - I_n(F)| &\le |I_\alpha(F) - I_\alpha(R) + I_\alpha(R) - I_n(F)| \\ &\le |I_\alpha(F - R)| + |I_\alpha(R) - I_n(F)| \\ &\le I_\alpha\Big(R(x) - F(x)\Big) + I_n(R - F) \\ &\le I_\alpha\Big(R(x) - r(x)\Big) + I_n(R - r) \\ &\le 2\sqrt{2\epsilon(M + \epsilon) c_{-2m}}. \end{aligned}$$

This proves the theorem. □

# 4 Convergence of interpolatory quadrature formulas and 2PTA's

In this section we are concerned with a general (possibly complex) measure $\alpha$ such that $\int_a^b d|\alpha| < +\infty$ and supp $(\alpha) \subset [a,b]$, with $0 \leq a < b < +\infty$.
In this case, general results about the location of the zeros of the orthogonal polynomials w.r.t. the complex varying measures $d\alpha_n(x) = x^{-k(n)}d\alpha(x)$ cannot be given, and therefore, Gaussian quadrature (and related 2PA) are in general not feasible.
Thus, we will make use of interpolatory quadrature formulas (and hence 2PTA). So let $\{p(n)\}$ and $\{q(n)\}$ be two sequences of integers such that for each $n \in \mathbb{N}$: $p(n) \geq 0$, $q(n) \geq -1$, and $p(n) + q(n) = n - 1$.
We will find out how to choose the triangular table $\mathbb{X} = \{x_{n,j} : j = 1, \ldots, n, n = 1, 2, \ldots\}$ of points $x_{n,j}$ contained in $[a,b]$, such that $x_{n,j} \neq x_{n,k}$ for $j \neq k$, so that the sequence of formulas

$$I_n(f) = \sum_{j=1}^{n} A_{n,j} f(x_{n,j}) \tag{4.1}$$

with weights $\{A_{n,j} : j = 1, \ldots, n\}$ uniquely determined by

$$I_n(f) = I_\alpha(f) \quad \forall f \in \Lambda_{-p(n),q(n)}, \tag{4.2}$$

will converge for functions $f$ from a class that is as large as possible. Initially the nodes $\{x_{n,j} : j = 1, \ldots, n\}$ will be taken as the zeros of the $n$th orthogonal polynomial $Q_n$ w.r.t.

$$d\sigma_n(x) = \frac{d\sigma(x)}{x^{2p(n)}} \tag{4.3}$$

where $\sigma$ is a distribution on $[a,b]$, with $\sigma' > 0$ a.e.
To make the distinction between different quadratures, we now denote by $I_n^\alpha$ the interpolatory q.f. (4.2), exact for $I_\alpha(f)$ in $\Lambda_{-p(n),q(n)}$, and by $I_n^\sigma$ the corresponding Gaussian q.f. which is exact in $\Lambda_{-2p(n),2q(n)+1}$ for $I_\sigma(f)$, i.e.

$$I_n^\sigma(f) := \sum_{j=1}^{n} \lambda_{n,j} f(x_{n,j}) = I_\sigma(f) := \int_a^b f(x) d\sigma(x), \quad \forall f \in \Lambda_{-2p(n),2q(n)+1}. \tag{4.4}$$

Recall that $\lambda_{n,j} > 0$, $j = 1, 2, \ldots, n$.

Let us also suppose that $\lim_{n\to\infty} q(n) = \infty$, in order to guarantee that the sequence $\{I_n^\sigma\}_{n\in\mathbb{N}}$ converges for every polynomial, without the Carlemann condition being required.

Assume in addition that $d\alpha(x) = \alpha'(x)dx$, $d\sigma(x) = \sigma'(x)dx$, and that $\alpha'/\sqrt{\sigma'} \in L_2[a,b]$, i.e., there exists a positive constant $K_1$ such that

$$\int_a^b \frac{|\alpha'(x)|^2}{\sigma'(x)} dx = K_1^2 < +\infty. \tag{4.5}$$

Under these conditions, the following theorem holds

**Theorem 4.1** *Let $f$ be continuous on $[a,b]$, i.e., $f \in C[a,b]$ and define the polynomial minimax errors*

$$\rho_m(f) = \min_{P\in\Pi_m} \|f - P\|_{L_\infty[a,b]}, \quad m = 0, 1, \ldots$$

*Then there exists a positive constant $K$ such that*

$$|I_\alpha(f) - I_n^\alpha(f)| \leq K \rho_{q(n)}(f), \quad \forall f \in C[a,b]. \tag{4.6}$$

**Proof.** Let $\{x_{n,j}, j = 1, \ldots, n\}$ be the nodes of the q.f. $I_n^\alpha$. By $L_n[f]$ we denote the Laurent polynomial from $\Lambda_{-p(n),q(n)}$ which interpolates the function $f$ in these nodes and by $P_n[f]$ we denote the Lagrange interpolating polynomial at the same nodes.

We have already seen that $I_n^\alpha(f) = I_\alpha(L_n[f])$. Clearly this is equivalent with $I_n^\alpha(f) = I_{\alpha_n}(P_n[\tilde{f}])$, where $\tilde{f}(x) = x^{p(n)}f(x)$ and $d\alpha_n(x) = x^{-p(n)}d\alpha(x)$.

Thus, taking (4.5) and the Cauchy-Schwarz inequality into account, we have

$$|I_\alpha(f) - I_n^\alpha(f)| = |I_{\alpha_n}(\tilde{f} - P_n[\tilde{f}])| \leq K_1 \|\tilde{f} - P_n[\tilde{f}]\|_{L_2(d\sigma_n)},$$

where $d\sigma_n(x) = x^{-2p(n)}\sigma'(x)dx$.

Now, let $\{\pi_n\}_{n\in\mathbb{N}}$ be the sequence of minimax polynomials for $f$. It is well known that

$$\lim_{n\to\infty} \|f - \pi_n\|_{L_\infty[a,b]} = \lim_{n\to\infty} \rho_n(f) = 0.$$

Since $p(n) + q(n) = n - 1$, it is clear that we have $P_n[\tilde{f}_n] = \tilde{f}_n$ if $\tilde{f}_n(x) = x^{p(n)}\pi_{q(n)}(x)$ and therefore,

$$\begin{aligned} \|\tilde{f} - P_n[\tilde{f}]\|_{L_2(d\sigma_n)} &= \|\tilde{f} - \tilde{f}_n + \tilde{f}_n - P_n[\tilde{f}]\|_{L_2(d\sigma_n)} \\ &\leq \|\tilde{f} - \tilde{f}_n\|_{L_2(d\sigma_n)} + \|P_n[\tilde{f} - \tilde{f}_n]\|_{L_2(d\sigma_n)} \\ &\leq K_2\rho_{q(n)}(f) + \|P_n[\tilde{f} - \tilde{f}_n]\|_{L_2(d\sigma_n)}, \end{aligned}$$

where $K_2 = \{\int_a^b d\sigma(x)\}^{1/2} < +\infty$.

Finally, from the orthogonality property of the polynomial $Q_n(x) = \prod_{j=1}^n (x - x_{n,j})$ w.r.t. $d\sigma_n$, one has

$$\begin{aligned} \|P_n[\tilde{f} - \tilde{f}_n]\|^2_{L_2(d\sigma_n)} &= I_n^\sigma\left([f - \pi_{q(n)}]^2\right) \\ &= \sum_{j=1}^n \lambda_{n,j}[f(x_{n,j}) - \pi_{q(n)}(x_{n,j})]^2 \leq \rho^2_{q(n)}(f)\sum_{j=1}^n \lambda_{n,j} \\ &= \rho^2_{q(n)}(f)\int_a^b d\sigma(x). \end{aligned}$$

Therefore, setting $K = 2K_1K_2$, (4.6) follows. □

**Remark 4.1** Since $\lim_{n\to\infty} q(n) = \infty$, the Weierstrass approximation theorem enables us to ensure that

$$\lim_{n\to\infty} I_n^\alpha(f) = I_\alpha(f), \quad \forall f \in C[a,b].$$

Furthermore, from (4.4) and making use of the Jackson Theorems (see e.g. [4]), we can deduce estimates for the error in the interpolatory q.f. for a continuous integrand $f$ in terms of its modulus of continuity $\omega(f;\delta)$.

Clearly (4.6) implies that

$$|I_\alpha(f) - I_n^\alpha(f)| = O\left(\omega(f; \tfrac{\pi}{q(n)+1})\right).$$

Therefore, if $f \in \mathrm{Lip}_\beta[a,b]$ with $0 < \beta \leq 1$, we have that

$$|I_\alpha(f) - I_n^\alpha(f)| = O((q(n)+1)^{-\beta}).$$

Certainly, a more rapid decrease of the error can be achieved if $f$ is $r$ times continuously differentiable, i.e. if $f \in C^{(r)}[a,b]$ for $r \geq 1$ [4].

**Corollary 4.2** *With the notation and the conditions of Theorem 4.1, the sequence of 2PTA's for the Cauchy transform $F_\alpha$, which are given by*

$$(p(n)/n)_{L^0,L^\infty}(z) = \sum_{j=1}^{n} \frac{A_{n,j}}{z - x_{n,j}}, \quad n = 1, 2, \ldots$$

*converges uniformly to $F_\alpha(f)$ in any compact subset of $\hat{\mathbb{C}} - [a, b]$.*

**Proof.** The fact that $\lim_{n\to\infty} I_n^\alpha(f) = I_\alpha(f)$ for any $f \in C[a, b]$, implies by Polya's Theorem that the sequence $S_n = \sum_{j=1}^{n} |A_{n,j}|$ is bounded. Therefore, following the same arguments as in Theorem 3.2, the conclusion can be easily obtained. □

We now want to prove convergence of the q.f.'s for a class of functions $\mathcal{F}_\sigma[a, b] \subset L_2(d\sigma)$ which, for a fixed integer $m$, is defined by

$$\mathcal{F}_\sigma[a, b] = \{F : [a, b] \mapsto \mathbb{R} : F(x) = x^{-m} f(x) \text{ for some integer } m \geq 0, \quad f \in S(d\sigma)\}.$$

Clearly, if $a > 0$, then $F$ itself will be in $S(d\sigma)$. If $a = 0$ then $I_\sigma(F)$ still exists as an improper Riemann-Stieltjes integral. This follows from the fact that $f$ is bounded and the assumption that the moments $\int_a^b x^m d\sigma(x)$ exist for all integer $m$. Note that $\mathcal{F}_\sigma[a, b]$ is closed under multiplication and

$$\Lambda \subset \mathcal{F}_\sigma[a, b] \subset L_2(d\sigma).$$

Thus $F \in \mathcal{F}_\sigma[a, b]$ implies that $F^2 \in \mathcal{F}_\sigma[a, b]$ and hence, by (4.5), $F$ will also be in $\mathcal{F}_\alpha[a, b]$.

Let us consider a non decreasing sequence $\{p(n)\}$ of integers such that $0 \leq p(n) \leq n$ and $\lim_{n\to\infty} p(n) = \lim_{n\to\infty} q(n) = \infty$ where $q(n) = n - p(n)$. For each $n \geq 0$, set

$$\mathcal{R}_n = \Lambda_{-p(n),q(n)}.$$

Thus, we have a sequence of nested subspaces of Laurent polynomials. Indeed, $\dim(\mathcal{R}_n) = n + 1$ and $\mathcal{R}_n \subset \mathcal{R}_{n+1}$. Clearly $\{x^j : j = -p(n), \ldots, q(n)\}$ represents a basis for $\mathcal{R}_n$. By applying the Gram-Schmidt orthogonalization process in $L_2(d\sigma)$ to this basis we can obtain an orthogonal basis, say $\{R_j : j = 0, 1, \ldots, n\}$ with $R_j \in \mathcal{R}_j - \mathcal{R}_{j-1}$. We then have

**Lemma 4.3** *$R_n$ has exactly $n$ distinct zeros on $(a, b)$.*

**Proof.** Since $R_n \in \mathcal{R}_n$, there is a $Q_n \in \Pi_n$ of exact degree $n$ such that $R_n(z) = z^{-p(n)} Q_n(z)$. Furthermore, we have

$$I_\sigma(x^j R_n) = 0, \quad -p(n-1) \leq j \leq q(n-1).$$

Thus

$$\int_a^b x^j \frac{Q_n(x)}{x^{p(n)}} d\sigma(x) = 0, \quad -p(n-1) \leq j \leq q(n-1).$$

This implies that

$$\int_a^b x^{j+p(n-1)} \frac{Q_n(x)}{x^{p(n)+p(n-1)}} d\sigma(x) = 0, \quad -p(n-1) \leq j \leq q(n-1),$$

or equivalently

$$\int_a^b x^k Q_n(x) \frac{d\sigma(x)}{x^{p(n)+p(n-1)}} = 0, \quad k = 0, 1, \ldots, n-1.$$

In other words, $Q_n(x)$ coincides, up to a multiplicative factor, with the $n$th orthogonal polynomial w.r.t. $x^{-p(n)-p(n-1)} d\sigma(x)$ and the proof follows. □

**Remark 4.2** Let $\{t_{n,j}\}_{j=1}^{n}$ be the zeros of $R_n(x)$. Then a q.f. $\tilde{I}_n^{\sigma}(f) = \sum_{j=1}^{n} B_{n,j} f(t_{n,j})$ can be constructed such that $B_{n,j} > 0$ and

$$\tilde{I}_n^{\sigma}(f) = I_{\sigma}(f), \quad \forall f \in \Lambda_{-p(n)-p(n-1),q(n)+q(n-1)}.$$

By Theorem 3.2 one immediately obtains

$$\lim_{n\to\infty} \tilde{I}_n^{\sigma}(f) = I_{\sigma}(f), \quad \forall f \in S(d\sigma).$$

It should also be noted that in general $\tilde{I}_n^{\sigma}(f)$ is not valid in $\mathcal{R}_{2n-1}$ except when $p(2n-1) = p(n) + p(n-1)$. This was the basic assumption handled in the previous work [3]. The particular case $p(n) = \left\lfloor \frac{n+1}{2} \right\rfloor$, where $\lfloor x \rfloor$ denotes the integer part of the real number $x$, has been studied by Jones, Njåstad and Thron [12] (see also, [10], [2], [18]).

For every $n \geq 1$ and every function $F$ defined at the points $\{t_{n,j} : j = 1, 2, \ldots, n\}$, let us consider its interpolating L-polynomial in $\mathcal{R}_{n-1}$ which can be written as

$$L_n^F(x) = L_n[F](x) = \sum_{j=1}^{n} F(t_{n,j}) L_{n,j}(x) \in \mathcal{R}_{n-1}.$$

Recall that $L_{n,j} \in \mathcal{R}_{n-1}$, $j = 1, \ldots, n$ and $L_{n,j}(t_{n,k}) = \delta_{jk}$ so that $L_n^F(t_{n,j}) = f(t_{n,j})$, $j = 1, 2, \ldots, n$.

**Theorem 4.4** *Let $F \in \mathcal{F}_{\sigma}[a,b]$. Then the sequence of Laurent polynomials $L_n^F = L_n[F] \in \mathcal{R}_{n-1}$ interpolating in the zeros of the nth orthogonal basis function $R_n \in \mathcal{R}_n - \mathcal{R}_{n-1}$ converges to $F$ in $L_2(d\sigma)$-norm, i.e.*

$$\lim_{n\to\infty} \|L_n^F - F\|_{L_2(d\sigma)}^2 = \lim_{n\to\infty} \int_a^b |L_n^F(x) - F(x)|^2 d\sigma(x) = 0.$$

**Proof.** Note that

$$\int_a^b |L_n^F(x) - F(x)|^2 d\sigma(x) = \int_a^b [F(x)]^2 d\sigma(x) + \int_a^b [L_n^F(x)]^2 d\sigma(x) - 2\int_a^b F(x) L_n^F(x) d\sigma(x).$$

Now,

$$L_n^F \in \mathcal{R}_{n-1} = \Lambda_{-p(n-1),q(n-1)}$$

and therefore

$$(L_n^F)^2 \in \Lambda_{-2p(n-1),2q(n-1)} \subset \Lambda_{-p(n)-p(n-1),q(n)+q(n-1)},$$

where we used the assumption $p(n) \leq p(n+1)$. Hence

$$\tilde{I}_{\sigma}\left([L_n^F]^2\right) = \sum_{j=1}^{n} B_{n,j} [L_n^F(t_{n,j})]^2 = \tilde{I}_n^{\sigma}(F^2).$$

Furthermore,

$$\left| \int_a^b F(x) L_n^F(x) d\sigma(x) \right|^2 \leq \left\{ \int_a^b [F(x)]^2 d\sigma(x) \right\} \left\{ \int_a^b [L_n^F(x)]^2 d\sigma(x) \right\} = I_{\sigma}(F^2) \tilde{I}_n^{\sigma}(F^2).$$

Since

$$\lim_{n\to\infty} \tilde{I}_n^{\sigma}(F^2) = I_{\sigma}(F^2),$$

it follows that

$$\limsup_{n\to\infty}\int_a^b |F(x)-L_n^F(x)|^2 d\sigma(x) \le 4\int_a^b [F(x)]^2 d\sigma(x).$$

Given $\epsilon>0$, we know by Lemma 3.3, that there exists a positive constant $M$ and a polynomial $q(x)$ such that $-M-\epsilon \le f(x) \le q(x) \le M+\epsilon$, $\forall x\in[a,b]$ and

$$\int_a^b |f(x)-q(x)| d\sigma(x) < \epsilon.$$

Thus, for any integer $k>1$, one has

$$\int_a^b |f(x)-q(x)|^k d\sigma(x) = \int_a^b |f(x)-q(x)|^{k-1}|f(x)-q(x)| d\sigma(x) \le 2^{k-1}(M+\epsilon)^{k-1}\epsilon.$$

Define $Q(x)=x^{-m}q(x)$, then (take $k=4$)

$$\begin{aligned}\int_a^b |F(x)-Q(x)|^2 d\sigma(x) &= \int_a^b |f(x)-q(x)|^2 \frac{d\sigma(x)}{x^{2m}} \\ &\le \left\{\int_a^b |f(x)-q(x)|^4 d\sigma(x)\right\}^{1/2}\left\{\int_a^b \frac{d\sigma(x)}{x^{4m}}\right\}^{1/2} \\ &\le \sqrt{\epsilon}[2(M+\epsilon)]^{3/2}\sqrt{d_{-4m}}.\end{aligned}$$

where we used the notation

$$d_k = \int_a^b x^k d\sigma(x), \quad k=0,\pm1,\pm2,\ldots$$

to denote the moments corresponding to the distribution $\sigma$.
Now, for sufficiently large $n$, we have $L_n^Q = L_n[Q] = Q$ and consequently

$$F - L_n^F = F - Q + Q - L_n^F = F - Q - L_n^{F-Q}.$$

Therefore, and because $F-Q\in\mathcal{F}_\sigma[a,b]$,

$$\begin{aligned}\limsup_{n\to\infty}\int_a^b |F(x)-L_n^F(x)|^2 d\sigma(x) &= \limsup_{n\to\infty}\int_a^b |(F-Q)(x)-L_n^{F-Q}(x)|^2 d\sigma(x) \\ &\le 4\int_a^b [F(x)-Q(x)]^2 d\sigma(x) \\ &\le 4\sqrt{\epsilon}[2(M+\epsilon)]^{3/2}\sqrt{d_{-4m}}.\end{aligned}$$

This implies that

$$\limsup_{n\to\infty}\int_a^b [F(x)-L_n^F(x)]^2 d\sigma(x) = 0$$

and the proof follows. □

**Remark 4.3** When taking $p(n)=0$, $\forall n\ge 0$, the classical Erdős-Turán Theorem [7] is recovered. On the other hand the choice $p(n)=\left\lfloor\frac{n+1}{2}\right\rfloor$ was considered earlier by Njåstad in [17].

Recall that if $\alpha$ and $\sigma$ satisfy (4.5), then $F \in \mathcal{F}_\sigma[a,b]$ implies $F \in \mathcal{F}_\alpha[a,b]$. Thus one has

**Corollary 4.5** *Assume that the distributions $\alpha$ and $\sigma$ satisfy (4.5). Then for the q.f. $I_n^\alpha$ of (4.1) one has*

$$\lim_{n\to\infty} I_n^\alpha(F) = I_\alpha(F), \quad \forall F \in \mathcal{F}_\sigma[a,b].$$

**Proof.** If $L_n^F$ is as in the previous theorem, then by the Cauchy-Schwarz inequality and (4.5) one has

$$|I_\alpha(F) - I_n^\alpha(F)| = |I_\alpha(F - L_n^F)| \le \|F - L_n^F\|_{L_2(d\sigma)} \cdot K_1$$

with $K_1$ as in (4.5). Now the result follows by Theorem 4.4. □

As we have already seen from Polya's Theorem and Theorem 4.1 we have that $S_n = \sum_{j=1}^n |A_{n,j}| \le M$, where $M$ is an absolute constant. Similar to the polynomial case (see e.g. [20]), we can deduce that $\{S_n\}$ is a convergent sequence, that is,

$$\lim_{n\to\infty} S_n = \int_a^b |\alpha'(x)|dx = \int_a^b d|\alpha|.$$

More precisely, we prove the following result.

**Theorem 4.6** *Let $\{p(n)\}_1^\infty$ be a nondecreasing sequence of integers such that $0 \le p(n) \le n$. Let $q(n) = n - p(n)$ and assume that $\lim_{n\to\infty} p(n) = \lim_{n\to\infty} q(n) = \infty$. Let $\sigma(x)$ be a distribution function on $[a,b]$ $(0 \le a < b < +\infty)$ such that $d\sigma(x) = \sigma'(x)dx$, $\sigma'(x) > 0$ a.e. on $[a,b]$. Let $\alpha(x)$ be a complex distribution function such that $d\alpha(x) = \alpha'(x)dx$. Let $\{x_{n,j} : j = 1,2,\dots,n\}$ be the zeros of the nth orthogonal polynomial w.r.t. $x^{-p(n)-p(n-1)}d\sigma(x)$. For each $n = 1,2,\dots$ consider $I_n(f) = \sum_{j=1}^n A_{n,j} f(x_{n,j})$, the interpolating quadrature formula exact in $\mathcal{R}_{n-1} = \Lambda_{-p(n-1),q(n-1)}$. If (4.5) holds, then one has*

$$\lim_{n\to\infty} \sum_{j=1}^n |A_{n,j}| f(x_{n,j}) = \int_a^b f(x)|\alpha'(x)|dx, \quad \forall f \in \mathcal{F}_\alpha[a,b].$$

**Proof.** Since

$$\Lambda = \bigcup_{n=0}^\infty \mathcal{R}_n = \bigcup_{n=0}^\infty \Lambda_{-p(n),q(n)},$$

we consider the sequence $\{R_n\}_0^\infty$ as defined in Lemma 4.3 which represents an orthogonal basis for $\Lambda$ with respect to the inner product in $L_2(d\sigma)$. Set $R_j(x) = x^{-p(j)}Q_j(x) \in \Lambda_{-p(j),q(j)}$ with $Q_j \in \Pi_j$. In order to display the dependency of the coefficients $A_{n,j}$ on $\alpha'$ we write $A_{n,j}(\alpha')$. We have

$$A_{n,j}(\alpha') = \frac{x_{n,j}^{p(n-1)}}{Q_n'(x_{n,j})} \int_a^b \frac{Q_n(x)}{x - x_{n,j}} \frac{\alpha'(x)dx}{x^{p(n-1)}}.$$

Since by (4.5) $\alpha'/\sigma' \in L_2(d\sigma)$ we can consider the Fourier expansion of this function with respect to the orthonormal basis $\{R_j\}_0^\infty$ i.e.

$$\frac{\alpha'}{\sigma'}(x) \sim \sum_{k=0}^\infty b_k R_k(x), \quad b_k = \int_a^b R_k(x)\sigma'(x)dx, \quad k = 0,1,\dots . \tag{4.7}$$

Therefore, one has by orthogonality

$$A_{n,j}(\alpha) = \sum_{k=0}^{n-1} b_k \left( \frac{x_{n,j}^{p(n-1)}}{Q_n'(x_{n,j})} \int_a^b \frac{Q_n(x)}{(x - x_{n,j})x^{p(n-1)}} R_k(x)\sigma'(x)dx \right).$$

As in Remark 4.2, let $\{B_{n,j} : j = 1, \ldots, n\}$ be the positive weights of the corresponding Gaussian q.f. exact in $\Lambda_{-p(n)-p(n-1),q(n)+q(n-1)}$. That is

$$I_n^\sigma(f) = \sum_{j=1}^{n} B_{n,j} f(x_{n,j}) = I_\sigma(f), \quad \forall f \in \Lambda_{-p(n)-p(n-1),q(n)+q(n-1)}.$$

Recall that for $k = 0, 1, \ldots, n-1$:

$$\frac{Q_n(x)}{(x - x_{n,j})x^{p(n-1)}} R_k(x) \in \Lambda_{-p(n-1)-p(k),q(n-1)+q(k)} \subset \Lambda_{-p(n)-p(n-1),q(n)+q(n-1)}.$$

Hence, we get

$$A_{n,j}(\alpha') = B_{n,j} \sum_{k=1}^{n-1} b_k R_k(x_{n,j}) = B_{n,j} S_{n-1}^{\alpha'/\sigma'}(x_{n,j}) \tag{4.8}$$

where $S_{n-1}^{\alpha'/\sigma'}(x)$ denotes the $(n-1)$th partial sum of the Fourier expansion (4.7).
For a given $\epsilon > 0$, one first has to find a natural number $n_0 = n_0(\epsilon)$ so that for any $n \geq n_0$

$$\left| \int_b^a f(x)|\alpha'(x)|dx - \sum_{j=1}^{n} |A_{n,j}(\alpha')| f(x_{n,j}) \right| < \epsilon, \tag{4.9}$$

where $f$ is an arbitrary function in $S(d|\alpha'|)$.
Let $\omega(x)$ be an $L_1$-integrable function on $[a, b]$ which will be characterized later. We have

$$\begin{aligned}
Y_n &= \left| \int_a^b f(x)|\alpha'(x)|dx - \sum_{j=1}^{n} |A_{n,j}(\alpha')| f(x_{n,j}) \right| \\
&\leq \left| \int_a^b f(x)|\alpha'(x)|dx - \int_a^b f(x)|\omega(x)|dx \right| \\
&\quad + \left| \sum_{j=1}^{n} |A_{n,j}(\alpha')| f(x_{n,j}) - \sum_{j=1}^{n} |A_{n,j}(\omega)| f(x_{n,j}) \right| \\
&\quad + \left| \sum_{j=1}^{n} |A_{n,j}(\omega)| f(x_{n,j}) - \int_a^b f(x)|\omega(x)|dx \right|.
\end{aligned}$$

Since $f$ is a bounded function on $[a, b]$, there exists a positive constant $M(f)$ so that

$$\begin{aligned}
Y_n &\leq M(f) \left\{ \int_a^b |\alpha'(x) - \omega(x)|dx + \sum_{j=1}^{n} |A_{n,j}(\alpha' - \omega)| \right\} + \\
&\quad + \left| \sum_{j=1}^{n} |A_{n,j}(\omega)| f(x_{n,j}) - \int_a^b f(x)|\omega(x)|dx \right| \\
&\leq M(f) \left\{ \int_a^b \left| \frac{\alpha'(x)}{\sigma'(x)} - \frac{\omega(x)}{\sigma'(x)} \right| \sigma'(x)dx + \sum_{j=1}^{n} B_{n,j} \left| S_{n-1}^{\frac{\alpha'}{\sigma'} - \frac{\omega}{\sigma'}}(x_{n,j}) \right| \right\} \\
&\quad + \left| \sum_{j=1} B_{n,j} |S_{n-1}^{\omega/\sigma'}(x_{n,j})| f(x_{n,j}) - \int_a^b f(x)|\omega(x)|dx \right|.
\end{aligned}$$

Set $\tilde{\alpha}' = \alpha'/\sigma'$ and $\tilde{\omega} = \omega/\sigma'$.
Making use of the Schwarz inequality, it follows that

$$\begin{aligned} Y_n \;\leq\; & M(f)\left\{ \|\tilde{\alpha}' - \tilde{\omega}\|_{L_2(d\sigma)} \cdot d_0^{1/2} + d_0^{1/2} \left( \sum_{j=1}^{n} B_{n,j} \left| S_{n-1}^{\tilde{\alpha}'-\tilde{\omega}}(x_{n,j}) \right|^2 \right)^{1/2} \right. \\ & \left. + \left| \sum_{j=1}^{n} B_{n,j} \left| S_n^{\tilde{\omega}}(x_{n,j}) \right| f(x_{n,j}) - \int_a^b f(x)|\omega(x)|dx \right| \right\}, \end{aligned}$$

where $d_0 = \int_a^b d\sigma(x)$.

Since $\tilde{\alpha}' \in L_2(d\sigma)$ and taking into account that the polynomials are dense in $L_2(d\sigma)$, there must exist an $R \in \Pi$ such that

$$\|\tilde{\alpha}' - R\|_{L_2(d\sigma)} < \epsilon. \tag{4.10}$$

On the other hand,

$$\left| S_{n-1}^{\tilde{\alpha}'-\tilde{\omega}} \right|^2 \in \Lambda_{-p(n)-p(n-1),q(n)+q(n-1)}.$$

Hence

$$\left( \sum_{j=1} B_{n,j} \left| S_{n-1}^{\tilde{\alpha}'-\tilde{\omega}}(x_{n,j}) \right|^2 \right)^{1/2} = \left\| S_{n-1}^{\tilde{\alpha}'-\tilde{\omega}} \right\|_{L_2(d\sigma)} \leq \|\tilde{\alpha}' - \tilde{\omega}\|_{L_2(d\sigma)},$$

where in the last step we have used Bessel's inequality [4].
With the L-polynomial $R$ of (4.10), we now fix $\omega$ by setting $\tilde{\omega} = R$, i.e, $\omega = R\sigma'$. Thus there exists an $n_1$ such that

$$\forall n \geq n_1 : \quad S_{n-1}^{\tilde{\omega}}(x) = S_{n-1}^{R}(x) = R(x).$$

Therefore,

$$Y_n \leq 2M(f) d_0^{1/2} \epsilon + \left| \sum_{j=1}^{n} B_{n,j} |R(x_{n,j})| f(x_{n,j}) - \int_a^b f(x)|R(x)|\sigma'(x)dx \right|. \tag{4.11}$$

Note that $|R|f \in \mathcal{F}_\sigma[a,b]$. So, by Theorem 3.4, there exists a nonnegative integer $n_2$, such that

$$\forall n \geq n_2 : \quad \left| \sum_{j=1}^{n} B_{n,j} |R(x_{n,j})| f(x_{n,j}) - \int_a^b |R(x)| f(x)\sigma'(x)dx \right| < \epsilon. \tag{4.12}$$

Thus, by (4.11) and (4.12), the inequality (4.9) follows. This proves the theorem when $f \in S(d|\alpha'|)$. It remains to extend this to all $f \in \mathcal{F}_\alpha[a,b]$, i.e. functions of the form $f(x) = x^{-m} g(x)$, with $g \in S(d\alpha)$ a Riemann integrable function on $[a,b]$. We can do this by making use of Lemma 3.3 and proceeding as in Theorem 3.4. □

# 5 Analytic integrands

The aim of this section is to obtain estimates for the error $R_n(f) = I_\alpha(f) - I_n(f)$ of the quadrature formula $I_n(f) = \sum_{j=1}^{n} A_{n,j} f(x_{n,j})$ of either Gaussian or of interpolatory type

in terms of the error of the related 2PA's (or 2PTA's) for the case when $f$ is an analytic function in a neighbourhood of the interval $[a,b]$ (again we will assume $0 \leq a < b < +\infty$). Moreover, for the "simple case" $0 < a < b < +\infty$, asymptotics for the rate of convergence, i.e. $\lim_{n\to\infty} |R_n(f)|^{1/n}$ will be deduced in terms of the rate of convergence given by $\lim_{n\to\infty} |E_n(z)|^{1/n}$ where $E_n(z) = F_\alpha(z) - F_n(z)$ with $F_n(z) = [p(n)/n]_{(L^0,L^\infty)}$ or $F_n(z) = (p(n)/n)_{(L^0,L^\infty)}$ and $0 \leq p(n) \leq 2n$ or $0 \leq p(n) \leq n$ respectively. So let $f$ be an analytic function in a neighbourhood $U$ of $[a,b]$ and let $\Gamma$ be a smooth contour surrounding $[a,b]$ such that $\Gamma$ and its interior is included in $U$. Then by Cauchy's formula and Fubini's Theorem, we have that

$$I_\alpha(f) = \frac{1}{2\pi i}\int_\Gamma f(z)F_\alpha(z)dz \tag{5.1}$$

and

$$\begin{aligned} I_n^\alpha(f) = \sum_{j=1}^n A_{n,j}f(x_{n,j}) &= \sum_{j=1}^n \mathrm{Res}(F_n; x_{n,j})f(x_{n,j}) \\ &= \sum_{j=1}^n \mathrm{Res}(F_n f; x_{n,j}) = \frac{1}{2\pi i}\int_\Gamma f(z)F_n(z)dz. \end{aligned} \tag{5.2}$$

Therefore

$$R_n(f) = I_\alpha(f) - I_n^\alpha(f) = \frac{1}{2\pi i}\int_\Gamma f(z)E_n(z)dz, \tag{5.3}$$

and hence $|R_n(f)| \leq C\|f\|_{L_\infty(\Gamma)}\|E_n\|_{L_\infty(\Gamma)}$, or equivalently, $|R_n(f)| = O(\|E_n\|_{L_\infty(\Gamma)})$.

In order to obtain an estimate for the rate of convergence, suppose now that $0 < a < b < +\infty$. In this case we will see that estimates of $\lim_{n\to\infty} |E_n(z)|^{1/n}$ can be obtained by studying the $n$th root asymptotics of orthogonal polynomials w.r.t. varying measures. Most of the results of this section are essentially particular cases of the corresponding results in [21, § 6.1] and [9] for the general multipoint situation. However, to make this paper self-contained, we present here a simplified treatment for the two-point case.

First assume that $\alpha$ is a positive measure supported on $[a,b]$ and denote by $Q_n$ the $n$th orthogonal polynomial w.r.t. $d\alpha_n(x) = x^{-p(n)}d\alpha(x)$ where $\lim_{n\to\infty} p(n)/2n = \theta$ with $0 \leq \theta \leq 1$. Also, denote by $\tilde{Q}_n$ the corresponding orthonormal polynomial, thus $\tilde{Q}_n = Q_n/\|Q_n\|_{L_2(d\alpha_n)}$. Note that here $d\alpha_n = \phi_n d\alpha$ where $\phi_n(x) = x^{-p(n)}$ is a family of continuous and nonnegative functions on $[a,b]$. Thus, if for any measure $\mu$ with compact support in $\mathbb{C}$, we denote by $V^\mu$ the logarithmic potential i.e.

$$V^\mu(z) = -\int \log|z-t|d\mu(t),$$

then

$$\lim_{n\to\infty}\frac{1}{n}\log\phi_n = -2\theta\log x = 2\theta V^{\delta_0}(x),$$

uniformly on $[a,b]$ where $\delta_0$ denotes the Dirac measure corresponding to the point $x = 0$. For a measure $\nu$ supported on $\mathbb{C} - [a,b]$, there exists a measure $\hat{\nu}$ supported on $[a,b]$ such that $V^{\hat{\nu}-\nu}(z) = w$ is a constant in $[a,b]$. This $\hat{\nu}$ is called the balayage of $\nu$ onto $[a,b]$ or the equilibrium distribution of $[a,b]$ in the presence of the external field $-V^\nu$. In addition,

$$V^{\hat{\nu}-\nu}(z) = w - G_{[a,b]}(\nu; z), \quad z \in \mathbb{C} - [a,b] \tag{5.4}$$

where $G_{[a,b]}(\nu;z)$ is the Green potential of the measure $\nu$ in $[a,b]$, that is

$$G_{[a,b]}(\nu;z) = \int g_{[a,b]}(z,t)d\nu(t),$$

where as usual, $g_{[a,b]}(z,t)$ is the Green function with singularity in $t \in \hat{\mathbb{C}} - [a,b]$. For more details see e.g. Stahl and Totik [21] or Landkof [15]. In the remainder we shall however need the logarithmic potential for a measure $\nu$ supported in $\hat{\mathbb{C}}$. In this case, the definition of logarithmic potential can be extended if we make use of the following "spherical normalization" of the definition [19]

$$V^{\nu}(z) = -\int_{|t|\leq 1} \log|z-t|d\nu(t) - \int_{|t|>1} \log|1-z/t|d\nu(t). \tag{5.5}$$

Under these conditions, we have the following Lemma (see [21, § 3.2] and [8]).

**Lemma 5.1** *Let $\{p(n)\}_1^{\infty}$ be a sequence of nonnegative integers satisfying*

$$\lim_{n\to\infty} p(n)/2n = \theta \quad \textit{with} \quad 0 \leq \theta \leq 1.$$

*Then a) and b) below will hold.*

*a) Let $\alpha$ be a positive measure such that* $\operatorname{supp}(\alpha) \subset [a,b]$ *with $0 < a < b < +\infty$; let $\tilde{Q}_n$ be the nth orthonormal polynomial w.r.t. $d\alpha_n(x) = x^{-p(n)}d\alpha(x)$. Then we have*

$$\liminf_{n\to\infty} |\tilde{Q}_n(z)|^{1/n} \geq \exp\{G_{[a,b]}(\nu;z) - V^{\nu}(z)\} \tag{5.6}$$

*uniformly in compact subsets of $\mathbb{C} - [a,b]$ where $\nu = \theta\delta_0 + (1-\theta)\delta_{\infty}$ and $G_{[a,b]}(\nu;z)$ denotes the Green potential of the measure $\nu$ in $[a,b]$. Thus*

$$G_{[a,b]}(\nu;z) \equiv \theta g_{[a,b]}(z,0) + (1-\theta)g_{[a,b]}(z,\infty). \tag{5.7}$$

*b) If in addition, $\alpha$ is such that $\alpha' > 0$ a.e. in $[a,b]$ (and hence $\alpha$ is regular in the nth root asymptotic sense) then*

$$\lim_{n\to\infty} |\tilde{Q}_n(z)|^{1/n} = \exp\{G_{[a,b]}(\nu;z) - V^{\nu}(z)\}, \quad z \in \mathbb{C} - [a,b] \tag{5.8}$$

$$\lim_{n\to\infty} |Q_n(z)|^{1/n} = \exp\{-V^{\hat{\nu}}(z)\}, \quad z \in \mathbb{C} - [a,b] \tag{5.9}$$

*and*

$$\limsup_{n\to\infty} |Q_n(z)|^{1/n} \leq \exp\{-V^{\hat{\nu}}(z)\}, \quad z \in \mathbb{C}, \tag{5.10}$$

*where the convergence is uniform in compact subsets of the respective domains in (5.8)–(5.10) and $\hat{\nu}$ denotes the balayage of $\nu$ from $\hat{\mathbb{C}} - [a,b]$ onto $[a,b]$.*

We are now in a position to prove the following

**Theorem 5.2** *Let $\{p(n)\}$ be a sequence of integers satisfying*

$$0 \leq p(n) \leq 2n \quad \textit{and} \quad \lim_{n\to\infty} p(n)/2n = \theta \quad \textit{with} \quad 0 \leq \theta \leq 1$$

*and let $G_{[a,b]}(\nu;z)$ be given by (5.7) where the parameter $\theta$ appears explicitly.*
*Then a) and b) below will hold.*

*a) If $\alpha$ is a positive measure on $[a,b]$ with $0 < a < b < +\infty$, then the sequence of 2PA's $[p(n)/n]$ for the Cauchy transform $F_\alpha$ converges geometrically to $F_\alpha$ in compact subsets of $\mathbb{C} - [a,b]$ with degree of convergence given by*

$$\limsup_{n\to\infty} |E_n(z)|^{1/2n} \leq \exp\{-G_{[a,b]}(\nu; z)\} < 1, \tag{5.11}$$

*where $E_n = F_\alpha - [p(n)/n]$.*

*b) If in addition, $\alpha' > 0$ a.e. on $[a,b]$, then the asymptotic rate of convergence given by (5.11) is exact. i.e.,*

$$\lim_{n\to\infty} |E_n(z)|^{1/2n} = \exp\{-G_{[a,b]}(\nu; z)\} < 1. \tag{5.12}$$

**Proof.** a) It is easily seen that the error $E_n(z)$ for the 2PA's $[p(n)/n]$ where $0 \leq p(n) \leq 2n$, admits the following integral expression

$$E_n(z) = F_\alpha(z) - [p(n)/n](z) = \frac{z^{p(n)}}{[Q_n(z)]^2} \int_a^b \frac{[Q_n(x)]^2}{x^{p(n)}(z-x)} d\alpha(x). \tag{5.13}$$

It is clear that in (5.13) we can replace the orthogonal $Q_n$ by the orthonormal $\tilde{Q}_n$. Now, since the function $g(x) = x^{-p(n)} Q_n^2(x)$ is nonnegative in $[a,b]$, we have for $z$ in a compact subset of $\mathbb{C} - [a,b]$:

$$0 < C_1(z) = \frac{1}{\max_{x\in[a,b]} |z-x|} \leq \frac{1}{|z-x|} \leq \frac{1}{\min_{x\in[a,b]} |z-x|} = C_2(z) < \infty.$$

Thus we have that

$$\lim_{n\to\infty} \left| \int \frac{[\tilde{Q}_n(x)]^2 d\alpha(x)}{x^{p(n)}(z-x)} \right|^{1/2n} = \lim_{n\to\infty} \left\{ \int \frac{[\tilde{Q}_n(x)]^2 d\alpha(x)}{x^{p(n)}} \right\}^{1/2n} = \lim_{n\to\infty} \|\tilde{Q}_n\|^{1/n}_{L_2(d\alpha_n)} = 1.$$

From (5.13), (5.6), and (5.5) it follows that

$$\limsup_{n\to\infty} |E_n(z)|^{1/2n} = \limsup_{n\to\infty} \left| \frac{z^{p(n)}}{[\tilde{Q}_n(z)]^2} \right|^{1/2n} \leq \exp\{-G_{[a,b]}(\nu; z)\}.$$

This proves a).

b) This is clear by applying (5.8) instead of (5.6). □

**Corollary 5.3** *If $\alpha$ is a positive measure on $[a,b]$, $f$ an analytic function in a neighbourhood of $[a,b]$ and $\{I_n^\alpha(f)\}_{n=1}^\infty$ is the sequence of Gaussian q.f. in $\Lambda_{-p(n),q(n)}$ where $\{p(n)\}$ and $\{q(n)\}$ are sequences of nonnegative integers such that $p(n)+q(n) = 2n-1$ and $\lim_{n\to\infty} p(n)/2n = \theta$, $(0 \leq \theta \leq 1)$. Then we have*

$$\limsup_{n\to\infty} |R_n(f)|^{1/2n} = \limsup_{n\to\infty} |I_\alpha(f) - I_n^\alpha(f)|^{1/2n} \leq \exp(-\tau) < 1 \tag{5.14}$$

*where $\tau = \sup\{\rho > 0 : f$ is analytic in the closed domain $G_{[a,b]}(\nu; z) \leq \rho\}$.*

**Proof.** The family $G_{[a,b]}(\nu;z)=\rho$, $(\rho>0)$ consists of equipotential curves for the Green potential $G_{[a,b]}(\nu;\cdot)$. For $\rho=0$, one obtains the interval $[a,b]$ and for any $\rho>0$ the curve $\Gamma_\rho=\{z\in\mathbb{C}: G_{[a,b]}(\nu;z)=\rho\}$ is a closed contour surrounding $[a,b]$. Thus the proof follows as an immediate consequence of Theoren 5.2 and formula (5.3). □

Let us now study the case of a general (possibly complex) measure $\alpha$. In Section 4, we constructed sequences of interpolating q.f. based on the zeros of orthogonal polynomials w.r.t. varying positive measures of the form $d\sigma_n(x)=x^{-2p(n)}d\sigma(x)$ or $d\sigma_n(x)=x^{-p(n)-p(n-1)}d\sigma(x)$ where $\sigma$ was a given positive measure and $0\le p(n)\le n$. Define $q(n)=n-1-p(n)$ and suppose that $\lim_{n\to\infty}p(n)/n=\theta$, $(0\le\theta\le 1)$. Denote by $F_n(z)=(p(n)/n)_{F_\alpha}(z)$ the related 2PTA's to $F_\alpha$. Then

**Theorem 5.4** *For the error of the 2PTA's $F_n(z)$ defined above, we have*

$$\limsup_{n\to\infty}|E_n(z)|^{1/n}=\limsup_{n\to\infty}|F_\alpha(z)-F_n(z)|^{1/n}\le\exp\{-G_{[a,b]}(\nu;z)\}<1$$

*where $\nu=\theta\delta_0+(1-\theta)\delta_\infty$.*

**Proof.** Recall that for 2PTA's, instead of (5.13), we have the following integral representation for the error $E_n$

$$E_n(z)=\frac{z^{p(n)}}{Q_n(z)}\int\frac{Q_n(x)}{x^{p(n)}(z-x)}d\alpha(x). \tag{5.15}$$

Now, by applying (5.9) and (5.10) to the polynomial $Q_n(x)$ orthonormal w.r.t. $d\sigma_n(x)=x^{-2p(n)}d\sigma(x)$, (5.15) gives

$$\begin{aligned}\limsup_{n\to\infty}|E_n(z)|^{1/n} &\le \limsup_{n\to\infty}\left|\frac{z^{p(n)}}{Q_n(z)}\right|^{1/n}\limsup_{n\to\infty}\left\|\frac{Q_n(x)}{x^{p(n)}}\right\|^{1/n}_{L_\infty([a,b])}\\ &\le \frac{\limsup|z^{p(n)}/Q_n(z)|^{1/n}}{\liminf\left\{\min_{x\in[a,b]}\frac{x^{p(n)}}{|Q_n(x)|}\right\}^{1/n}}\le\exp\left\{V^{\hat\nu-\nu}(z)-\min_{x\in[a,b]}V^{\hat\nu-\nu}(x)\right\}.\end{aligned}$$

But since $V^{\hat\nu-\nu}=w$ is constant in $[a,b]$, one has from (5.4)

$$\limsup_{n\to\infty}|E_n(z)|^{1/n}\le\exp\left\{V^{\hat\nu-\nu}(z)-w\right\}=\exp\left\{-G_{[a,b]}(\nu;z)\right\}<1.$$

□

In a similar way to Corollary 5.3, it follows that

**Corollary 5.5** *Let $\sigma$ be a given positive measure on $[a,b]$ with $0<a<b<+\infty$, let $\{x_{n,j}: j=1,\dots,n\}$ be the zeros of the nth orthogonal polynomial w.r.t. $d\sigma_n(x)=x^{-2p(n)}d\sigma(x)$, and let $\{p(n)\}$ be a sequence of nonnegative integers such that $0\le p(n)\le n$ and $0\le\lim_{n\to\infty}p(n)/n=\theta\le 1$. Let*

$$I_n^\alpha(f)=\sum_{j=1}^n A_{n,j}f(x_{n,j})$$

*be the interpolatory q.f. in $\Lambda_{-p(n),q(n)}$ $(q(n)=n-1-p(n))$. Assume that $f$ is an analytic function in a neighbourhood of $[a,b]$. Then with $R_n(f)=I_\alpha(f)-I_n^\alpha(f)$, we have*

$$\limsup_{n\to\infty}|R_n(f)|^{1/n}\le\exp(-\tau)<1 \tag{5.16}$$

*where $\tau$ is defined as in Corollary 5.3.*

**Remark 5.1** The proof of part b) in Lemma 5.1 is based upon the fact that if $\sigma' > 0$ a.e. in $[a,b]$ and $Q_n$ is the $n$th orthogonal polynomial w.r.t. $d\sigma_n(x) = x^{-2p(n)}d\sigma(x)$, and $0 \leq \lim_{n\to\infty} p(n)/n = \theta \leq 1$, then for the sequence of the so called counting measures of zeros associated with $\{Q_n\}$, i.e.,

$$\chi_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{x_{n,j}}$$

one has (see e.g. [21])

$$\chi_n \xrightarrow[n]{*} \hat{\nu} = \theta\hat{\delta}_0 + (1-\theta)\hat{\delta}_\infty, \tag{5.17}$$

where $\xrightarrow[n]{*}$ denotes convergence in the weak star topology and $\hat{\delta}_0$ and $\hat{\delta}_\infty$ are the respective balayages of $\delta_0$ and $\delta_\infty$ from $\hat{\mathbb{C}} - [a,b]$ onto $[a,b]$. It is clear that (take into account that $0 \notin [a,b]$)

$$g_{[a,b]}(z,0) = g_{[b^{-1},a^{-1}]}(z^{-1},\infty).$$

**Remark 5.2** In [5], we proved that the limit distribution (5.17) of zeros $x_{n,j}$ could be easily obtained from a numerical and computational point of view, by taking for instance the polynomial $Q_n$ given by

$$Q_n(x) = x^{p(n)}T_{p(n)}^{[b^{-1},a^{-1}]}(x^{-1})T_{n-p(n)}^{[a,b]}(x) \tag{5.18}$$

where $T_m^{[c,d]}$ is the $m$th Chebyshev polynomial of the first kind for the interval $[c,d]$. Thus, with the choice (5.18) we obtain the same asymptotic rate of convergence but with the advantage that the computation of zeros of varying orthogonal polynomials is avoided and consequently the stability of the quadrature process can be improved.

# References


[1] E. Alden. A survey of weak and strong moment problems with generalizations. Technical report, Dept. Math. Umeå, 1985.

[2] C.M. Bonan-Hamada, W.B. Jones, W.J. Thron, and A. Magnus. Para-orthogonal Laurent polynomials and associated sequences of rational functions. *Numer. Algorithms*, 3:67–74, 1992.

[3] A. Bultheel, P. González-Vera, and R. Orive. Quadrature on the half line and two-point Padé approximants to Stieltjes functions. Part I: Algebraic aspects. *J. Comput. Appl. Math.*, 65:57–72, 1996.

[4] E.W. Cheney. *Introduction to approximation theory.* McGraw Hill, 1966.

[5] C. Díaz-Mendoza, P. González-Vera, and R. Orive. On convergence of two-point Padé-type approximants. *Numer. Math.*, 72:295–312, 1996.

[6] A. Draux. *Polynômes orthogonaux formels – applications*, volume 974 of *Lecture Notes in Math.* Springer, Berlin, 1983.

[7] G. Freud. *Orthogonal polynomials.* Pergamon Press, Oxford, 1971.

[8] A.A. Gonchar and E.A. Rakhmanov. Equilibrium measure and the distribution of zeros of extremal polynomials. *Math. USSR-Sb.*, 53:119–130, 1986.

[9] P. González-Vera, M. Jiménez-Páiz, G. López-Lagomasino, and R. Orive. On the convergence of quadrature formulas connected with multipoint Padé-type approximants. *J. Math. Anal. Appl.*, 1996. To appear.

[10] P. González-Vera and O. Njåstad. Convergence of two-point Padé approximants to series of Stieltjes. *J. Comput. Appl. Math.*, 32:97–105, 1990.

[11] E. Hille. *Analytic function theory*, volume 2. Ginn and Co., Boston, 1962.

[12] W.B. Jones, O. Njåstad, and W.J. Thron. Two-point Padé expansions for a family of analytic functions. *J. Comput. Appl. Math.*, 9:105–124, 1983.

[13] W.B. Jones, W.J. Thron, and H. Waadeland. A strong Stieltjes moment problem. *Trans. Amer. Math. Soc.*, 206:503–528, 1980.

[14] V.J. Krylov. *Approximate calculation of integrals.* MacMillan, 1962.

[15] N.S. Landkof. *Foundations of modern potential theory.* Springer, New York, 1972.

[16] G. López-Lagomasino and A.M. Finkelshtein. Rate of convergence of two-point Padé approximants and logarithmic asymptotics of Laurent-type orthogonal polynomials. *Constr. Approx.*, 11:255–286, 1995.

[17] O. Njåstad. Laurent polynomials and Gaussian quadrature. In T. Håvie, editor, *Romberg seminar on quadrature, interpolation, extrapolation and rational approximants*, Det Kongelige Norske Videnskabers Selskab Skrifter (No. 2), pages 102–120, 1989.

[18] O. Njåstad and W.J. Thron. The theory of sequences of orthogonal $L$-polynomials. In H. Waadeland and H. Wallin, editors, *Padé Approximants and Continued Fractions*, Det Kongelige Norske Videnskabers Selskab Skrifter (No. 1), pages 54–91, 1983.

[19] E.A. Rakhmanov and A.A. Gonchar. Equilibrium distributions and degree of rational approximation of analytic functions. *Math. USSR-Sb.*, 62:305–348, 1989.

[20] I.H. Sloan and W.E. Smith. Properties of interpolatory product rules. *SIAM J. Numer. Anal.*, 19:427–442, 1982.

[21] H. Stahl and V. Totik. *General orthogonal polynomials.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1992.

[22] G. Szegő. *Orthogonal polynomials*, volume 33 of *Amer. Math. Soc. Colloq. Publ.* Amer. Math. Soc., Providence, Rhode Island, 3rd edition, 1967. First edition 1939.