Polygusa Version 0.4

A computerprogram to calculate Igusa's Local Zeta Function and the Topological Zeta Function for non-degenerated polynomials

Kathleen Hoornaert and Davy Loots
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This program was written in Maple V Release 5.1. One can download the source code from the web [HL00]. The calculation of the convex hull of a set of points and a set of directions of recession is based on the program "cdd" of K. Fukuda [Fuk97].

See [DL92, Théorème 5.3] for the definition of the topological zeta function and the formula on which the calculation for the topological zeta function in this program is based.

```
> restart;read "Igusa.m";
```

We calculate some examples to illustrate some theorems and conjectures from the thesis.
The syntax is the following:

Zetafunction(polynomial, the list of variables, set of options);

The options have to be chosen from the following ones (if one chooses no option at all, the program expects one chooses "igusa"):

newtoninfo: to print
- the points of the support of *f*,
- the vertices of the Newton polyhedron of *f*,
- the number of facets and proper faces,
- the defining system of inequalities of the Newton polyhedron,
- a picture of the Newton polyhedron and a picture of the cones associated to its faces if the dimension is 2 or 3.

igusa: to calculate the global Igusa's local zeta function Z_f of the polynomial *f*.

compactigusa: to calculate the local Igusa's local zeta function $Z_{f,0}$ of the polynomial *f*.

topol: to calculate the global topological zeta function of the polynomial *f*. For the definition of the parameter *d*; see [DL92, Théorème 5.3]. The standard *d* is 1.

compacttopol: to calculate the local topological zeta function of the polynomial *f*.

nocheck: the condition of non-degeneracy is not checked. This can be useful if this verification takes too long. Attention: if the polynomial is degenerated, the result is invalid. By changing the coefficients, one obtains a non-degenerated polynomial and for this

polynomial the result holds. This option can be useful in combination with topol or compacttopol, because then the actual coefficients do not matter.

For igusa and compactigusa the coefficients are important for the numbers N_τ.

poleinfo: to get all the (real) candidate poles of the chosen zeta function, their expected orders and the faces of maximal pure contribution to those candidate poles.

allpurefaces: to get also the other faces of pure contribution to the candidate (real) poles.

faceinfo: to print for each face τ the appropriate information for the chosen zeta function:

- dimension,
- number of the face τ,
- the points of the support of f_τ,
- the directions of recession of τ,
- the generators of Δ_τ, the cone associated to the face τ,
- a partition of Δ_τ into simplicial cones,
- the multiplicities of the sets of generators of the cones in the partition above,

for igusa and compactigusa also

- the associated integer points if the number of integer points is <20,
- f_τ, N_τ, S_τ and L_τ

for topol and compacttopol also

- J_τ and Vol(τ)*(dim τ!).

poincare: to print the Poincaré series of f,

$P(t)=\Sigma_{e \text{ in } N}\ p^{(-n\,e)} N_e\, t^e$,

and some numbers $N_e = \#\{x \text{ in } (Z_{p^e})^n \mid f(x)=0 \bmod p^e\}$.

(In this case the program will add the option "igusa".)

monodromy: to calculate the zeta function and characteristic polynomial of the monodromy in the origin [Var76].

monodromyinfo: to print the P^i and ζ^i from the same article.

allinfo: is equal to newtoninfo, monodromy, monodromyinfo, poleinfo, allpurefaces, faceinfo, poincare.

One can also use the command

timeanalysis_Zetafunction(polynomial, the list of variables, set of options);

Then one gets in addition an overview of the execution time of certain procedures. (The execution time is in seconds.) It can happen that two consecutive calculations of the same command give a different calculation time, because some parts in the program are random (e.g., the sequence of the faces, the partition into simplicial cones,...)

The examples below were calculated on a Pentium III 600 MHz, with 256 MB RAM, running Windows 98 SE.

Some further remarks:

All calculated zeta functions should be already simplified. So, the actual poles (and their orders) of a chosen zeta function are equal to the zeros (and their orders) of the denominator.

Ex. 1: $(y^2+z^3x)^2(zy+x^3)^3+x^{18}+y^{18}+z^{18}$

newtoninfo + igusa

This polynomial is degenerated over C with respect to all the faces of its Newton polyhedron. In this case, the formula from Theorem 2.3.2 can not be used.

```
> Zetafunction((y^2+z^3*x)^2*(z*y+x^3)^3+x^18+y^18+z^18,
  [x,y,z], newtoninfo, igusa);
```

Supportpoints = [[0, 7, 3], [3, 6, 2], [6, 5, 1], [9, 4, 0], [1, 5, 6], [4, 4, 5], [7, 3, 4], [10, 2, 3], [2, 3, 9], [5, 2, 8], [8, 1, 7], [11, 0, 6], [18, 0, 0], [0, 18, 0], [0, 0, 18]]

Vertices = {[0, 7, 3], [9, 4, 0], [2, 3, 9], [11, 0, 6], [18, 0, 0], [0, 18, 0], [0, 0, 18]}

Number of facets = 8

Number of proper faces = 29

Defining system of inequalities of the Newton polyhedron :

Facet 1 : 9*x+15*y+7*z-126 >= 0

Facet 2 : 12*x-216+27*y+14*z >= 0

Facet 3 : z >= 0

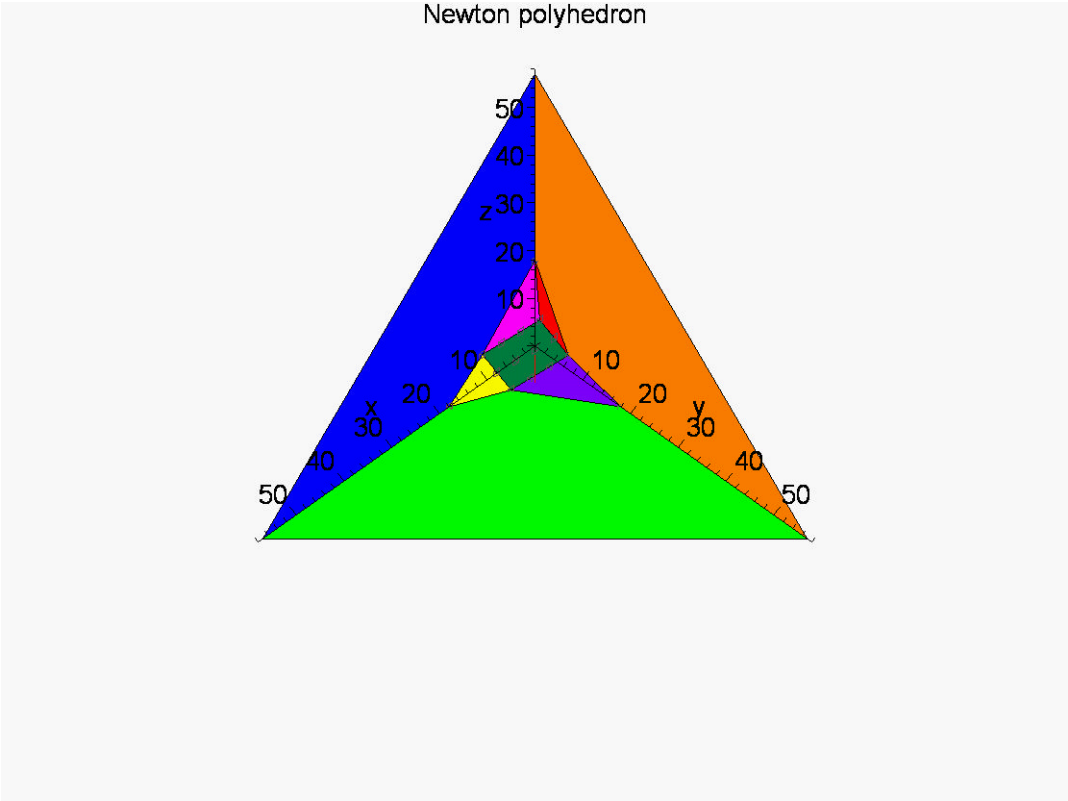
Facet 4 : y >= 0

Facet 5 : 12*x+25*y+11*z-198 >= 0

Facet 6 : 14*x+9*y-162+33*z >= 0

Facet 7 : x >= 0

Facet 8 : x-17+2*y+z >= 0




```
The number of different ftau is 27. If you wish to calculate
the Igusa's local zeta function for a specific prime, enter
the prime. If you wish to enter an abstract number of solutions
for each face, enter 'p'
```

Enter prime number, or 'p', end with semicolon ;p;

$$y^7 z^3 + 3 y^6 z^2 x^3 + 3 y^5 z x^6 + y^4 x^9 + 2 y^5 z^6 x + 6 y^4 z^5 x^4 + 6 y^3 z^4 x^7 + 2 y^2 z^3 x^{10} + z^9 x^2 y^3 + 3 z^8 x^5 y^2 + 3 z^7 x^8 y + z^6 x^{11}$$

is degenerated over C, the formula for the zeta function is invalid.

The system of equations

$$y^7 z^3 + 3 y^6 z^2 x^3 + 3 y^5 z x^6 + y^4 x^9 + 2 y^5 z^6 x + 6 y^4 z^5 x^4 + 6 y^3 z^4 x^7 + 2 y^2 z^3 x^{10} + z^9 x^2 y^3 + 3 z^8 x^5 y^2 + 3 z^7 x^8 y + z^6 x^{11} = 0$$

$$9 y^6 z^2 x^2 + 18 y^5 z x^5 + 9 y^4 x^8 + 2 y^5 z^6 + 24 y^4 z^5 x^3 + 42 y^3 z^4 x^6 + 20 y^2 z^3 x^9 + 2 z^9 x y^3 + 15 z^8 x^4 y^2 + 24 z^7 x^7 y + 11 z^6 x^{10} = 0$$

$$7 y^6 z^3 + 18 y^5 z^2 x^3 + 15 y^4 z x^6 + 4 y^3 x^9 + 10 y^4 z^6 x + 24 y^3 z^5 x^4 + 18 y^2 z^4 x^7 + 4 y z^3 x^{10} + 3 z^9 x^2 y^2 + 6 z^8 x^5 y + 3 z^7 x^8 = 0$$

$$3 y^7 z^2 + 6 y^6 z x^3 + 3 y^5 x^6 + 12 y^5 z^5 x + 30 y^4 z^4 x^4 + 24 y^3 z^3 x^7 + 6 y^2 z^2 x^{10} + 9 z^8 x^2 y^3 + 24 z^7 x^5 y^2 + 21 z^6 x^8 y + 6 z^5 x^{11} = 0$$

has as solution in C\{0}:

$$\{x = x,\ helper_1 = \frac{1}{x},\ helper_2 = \frac{1}{y},\ y = y,\ z = -\frac{x^3}{y},\ helper_3 = -\frac{y}{x^3}\},$$

$$\{helper_3 = \frac{1}{z}, helper_2 = \frac{1}{y}, y = y, z = z, x = -\frac{y^2}{z^3}, helper_1 = -\frac{z^3}{y^2}\}$$

```
Warning, f is degenerated!
```

Ex. 2: $x^4 y^3 z^2 + x^7 y^4 z + z^5$

igusa + poleinfo + p=3 + timeanalysis

See also Example 2.4.1 and Example 3.5.1.
In this example all expected orders of the poles are equal to the actual orders.

```
> timeanalysis_Zetafunction(x^4*y^3*z^2+x^7*y^4*z+z^5, [x,y,z],
  igusa, poleinfo);
```

The candidate real poles of Igusa's local zeta function of

$$x^4 y^3 z^2 + x^7 y^4 z + z^5$$

in function of s are:

```
-7/20 with expected order: 1
The face of maximal pure contribution to this candidate pole is face 3, the
face tau_0.
(So, the responsible faces for this order are all the faces therein)
```

supportpoints $= \{[4, 3, 2], [0, 0, 5]\}$, *directions of recession* $= \{[0, 1, 0]\}$,

dimension $= 2$

cone $= \{[3, 0, 4]\}$

partition of the associated cone into simplicial cones $= [\{[3, 0, 4]\}]$

```
-2/5 with expected order:2
The face(s) of maximal pure contribution to this candidate pole is/are
(so, the responsible faces for this order are all the faces therein)
face 10
```

supportpoints $= \{[7, 4, 1], [4, 3, 2]\}$, *directions of recession* $= \{\ \}$, *dimension* $= 1$

cone $= \{[1, 0, 3], [0, 1, 1]\}$

partition of the associated cone into simplicial cones $= [\{[1, 0, 3], [0, 1, 1]\}]$

```
-1 with expected order:1
The face(s) of maximal pure contribution to this candidate pole is/are
(so, the responsible faces for this order are all the faces therein)
face 5
```

supportpoints $= \{[7, 4, 1]\}$, *directions of recession* $= \{[1, 0, 0], [0, 1, 0]\}$,

dimension $= 2$

cone $= \{[0, 0, 1]\}$

partition of the associated cone into simplicial cones $= [\{[0, 0, 1]\}]$

```
The number of different ftau is 6. If you wish to calculate
the Igusa's local zeta function for a specific prime, enter
the prime. If you wish to enter an abstract number of solutions
for each face, enter 'p'
```

```
Enter prime number, or 'p', end with semicolon ;3;
```

$$x^4 y^3 z^2 + x^7 y^4 z + z^5$$

is non-degenerated over F_3

with respect to all the faces of its Newton polyhedron

Igusa's local zeta function of

$$x^4 y^3 z^2 + x^7 y^4 z + z^5$$

in function of t

$$-4\,(-486\,t^{16} - 59049\,t^5 + 59049\,t - 729\,t^{11} - 708588 + 162\,t^{17} + 18\,t^{23} + 4374\,t^9 - 54\,t^{21} - 3\,t^{26} + 2\,t^{31} + 9\,t^{25} - 3\,t^{30} - 54\,t^{22} - 1458\,t^{14} - 13122\,t^8 + 243\,t^{20} + 486\,t^{15} + 19683\,t^6) \Big/ ((-3+t)\,(-9+t^5)^2\,(t^5+9)\,(-2187+t^{20}))$$

in function of s

$$4\,(3^s)^5\,(486\,(3^s)^{15} + 59049\,(3^s)^{26} - 59049\,(3^s)^{30} + 729\,(3^s)^{20} + 708588\,(3^s)^{31} - 162\,(3^s)^{14} - 18\,(3^s)^8 - 4374\,(3^s)^{22} + 54\,(3^s)^{10} + 3\,(3^s)^5 - 2 - 9\,(3^s)^6 + 3\,3^s + 54\,(3^s)^9 + 1458\,(3^s)^{17} + 13122\,(3^s)^{23} - 243\,(3^s)^{11} - 486\,(3^s)^{16} - 19683\,(3^s)^{25}) \Big/ (3\,3^s - 1)\,(9\,(3^s)^5 - 1)^2\,(1 + 9\,(3^s)^5)\,(2187\,(3^s)^{20} - 1))$$

function	depth	calls	time	time%	bytes	bytes%
Zetafunction	1	1	2.016	95.45	10698716	98.42
Newtonpoly	1	1	.096	4.55	157844	1.45
Ntau	1	3	0.000	0.00	13636	.13
total:	3	5	2.112	100.00	10870196	100.00

Ex. 3: $x^2 z + y^2 z + u^3$

igusa + poleinfo + allpurefaces + faceinfo + poincare + p==3 mod 4
There are 33 proper faces. We omitted the information about most of the faces to save pages of output.

This example illustrates Theorem 4.1.11, Theorem 5.3.6 and Conjecture 0.3.3 (see also Example 4.2.2).
The largest real (besides -1) candidate pole $-1/t_0$ of Igusa's local zeta function is -4/3 and has expected order 2. However the actual order of -4/3 is 1 for p == 3 mod 4. Note that there is a vertex of τ_0 contained in $\{0, 1, 2\}^4$. So, the condition of Theorem 4.1.11 is not satisfied. The face τ_0 is unstable for p == 3 mod 4. So, by Theorem 5.3.6, it is not surprising that $-1/t_0$ is not a pole of Igusa's local zeta function of the expected order.
The conditions of Conjecture 0.3.3 are not satisfied: not all faces of pure contribution are unstable with respect to the same variable. Face 6 has e_3 as direction of recession.

We suppose that p == 3 mod 4 in the example below.
The numbers of solutions in (F_p\{0})^4 are:

$x^2 z + y^2 z + u^3, x^2 z + u^3, y^2 z + u^3 \ : (p-1)^3$

$x^2 z + y^2 z \ : 0$

```
> Zetafunction(x^2*z+y^2*z+u^3, [x,y,z,u], igusa, poleinfo,
  allpurefaces, faceinfo, poincare);
```

The candidate real poles of Igusa's local zeta function of

$$x^2 z + y^2 z + u^3$$

in function of s are:

```
-1 with expected order: 1
(If all N_tau are 0, then the expected order of -1 is 0)

-4/3 with expected order: 2
The face of maximal pure contribution to this candidate pole is face 12, the
face tau_0.
(So, the responsible faces for this order are all the faces therein)
```

supportpoints $= \{[2, 0, 1, 0], [0, 2, 1, 0], [0, 0, 0, 3]\}$, *directions of recession* $= \{\ \}$,

dimension $= 2$

$$cone = \{[3, 3, 0, 2], [0, 0, 3, 1]\}$$

partition of the associated cone into simplicial cones $= [\{[3, 3, 0, 2], [0, 0, 3, 1]\}]$

```
The other faces of pure contribution to this candidate pole are:
face 4
```

supportpoints $= \{[2, 0, 1, 0], [0, 2, 1, 0], [0, 0, 0, 3]\}$,

directions of recession $= \{[0, 1, 0, 0], [1, 0, 0, 0]\}$, *dimension* $= 3$

$$cone = \{[0, 0, 3, 1]\}$$

partition of the associated cone into simplicial cones $= [\{[0, 0, 3, 1]\}]$

```
face 6
```

supportpoints $= \{[2, 0, 1, 0], [0, 2, 1, 0], [0, 0, 0, 3]\}$,

directions of recession $= \{[0, 0, 1, 0]\}$, *dimension* $= 3$

$$cone = \{[3, 3, 0, 2]\}$$

partition of the associated cone into simplicial cones $= [\{[3, 3, 0, 2]\}]$

```
The number of different ftau is 7. If you wish to calculate
the Igusa's local zeta function for a specific prime, enter
the prime. If you wish to enter an abstract number of solutions
for each face, enter 'p'
```

```
Enter prime number, or 'p', end with semicolon ;p;
```

$$x^2 z + y^2 z + u^3$$

is non-degenerated over F_p, for p>>0,

with respect to all the faces of its Newton polyhedron

```
Give number of solutions in (F_p\{0})^4 of :
```

$$x^2 z + y^2 z + u^3$$

Enter number or, if you don't know it, write, N1

```
Enter number, end with semicolon ;(p-1)^3;
Give number of solutions in (F_p\{0})^4 of :
```

$$x^2 z + y^2 z$$

Enter number or, if you don't know it, write, N2

Enter number, end with semicolon ;0;

Give number of solutions in (F_p\{0})^4 of :

$$x^2 z + u^3$$

Enter number or, if you don't know it, write, N3

Enter number, end with semicolon ;(p-1)^3;

Give number of solutions in (F_p\{0})^4 of :

$$y^2 z + u^3$$

Enter number or, if you don't know it, write, N4

Enter number, end with semicolon ;(p-1)^3;

Face 1: dimension =3

supportpoints $= \{[0, 2, 1, 0], [0, 0, 0, 3]\}$,

directions of recession $= \{[0, 0, 0, 1], [0, 1, 0, 0], [0, 0, 1, 0]\}$

Associated cone $= \{[1, 0, 0, 0]\}$

Partition into simplicial cones $= [\{[1, 0, 0, 0]\}]$

Associated multiplicities (in order) $= [1]$

Associated integer points (in order) $= [\{[0, 0, 0, 0]\}]$

f-tau $= y^2 z + u^3$, *N-tau* $= (p-1)^3$, *S-tau* $= \dfrac{1}{p-1}$, *S-tau in s* $= \dfrac{1}{p-1}$,

L-tau $= \dfrac{(p-1)^4 - \dfrac{p\,(p-1)^3\,(1-t)}{p-t}}{p^4}$, *L-tau in s* $= \dfrac{(p-1)^3\,(p^2\,p^s - 2\,p\,p^s + 1)}{p^4\,(p\,p^s - 1)}$

Face 32: dimension =0

supportpoints $= \{[0, 0, 0, 3]\}$, *directions of recession* $= \{\ \}$

Associated cone $= \{[0, 1, 0, 0], [1, 0, 0, 0], [0, 0, 1, 0], [3, 3, 0, 2], [0, 0, 3, 1]\}$

Partition into simplicial cones $= [\{[0, 1, 0, 0], [1, 0, 0, 0], [3, 3, 0, 2], [0, 0, 3, 1]\},$

$\{[0, 1, 0, 0], [1, 0, 0, 0], [0, 0, 3, 1]\},$

$\{[0, 1, 0, 0], [1, 0, 0, 0], [0, 0, 1, 0], [0, 0, 3, 1]\}]$

Associated multiplicities (in order) $= [6, 1, 1]$

Associated integer points (in order) $= [$

$\{[1, 1, 2, 1], [1, 1, 1, 1], [0, 0, 0, 0], [2, 2, 2, 2], [2, 2, 0, 1], [3, 3, 1, 2]\},$

$\{[0, 0, 0, 0]\}, \{[0, 0, 0, 0]\}]$

f-tau $= u^3$, *N-tau* $= 0$, *S-tau* $= \dfrac{t^3\,(2\,p^6\,t^3 - p^5\,t^3 - p^4\,t^3 + p^9 - p^8 + p^{10} - t^6)}{(p-1)^3\,(p^8 - t^6)\,(p^4 - t^3)}$,

S-tau in s $= \dfrac{2\,(p^s)^3\,p^6 - (p^s)^3\,p^5 - p^4\,(p^s)^3 + (p^s)^6\,p^9 - p^8\,(p^s)^6 + (p^s)^6\,p^{10} - 1}{(p-1)^3\,(p^4\,(p^s)^3 - 1)^2\,(p^4\,(p^s)^3 + 1)}$,

L-tau $= \dfrac{(p-1)^4}{p^4}$, *L-tau in s* $= \dfrac{(p-1)^4}{p^4}$

Face 33: dimension =0

supportpoints $= \{[0, 2, 1, 0]\}$, *directions of recession* $= \{\ \}$

Associated cone $= \{[0, 0, 0, 1], [1, 0, 0, 0], [3, 3, 0, 2], [0, 0, 3, 1]\}$

Partition into simplicial cones $= [\{[0, 0, 0, 1], [1, 0, 0, 0], [3, 3, 0, 2], [0, 0, 3, 1]\}]$

Associated multiplicities (in order) $= [9]$

Associated integer points (in order) $= [\{[0, 0, 1, 1], [0, 0, 2, 1], [1, 1, 0, 1], [1, 1, 1, 1], [2, 2, 0, 2], [0, 0, 0, 0], [2, 2, 2, 2], [1, 1, 2, 2], [2, 2, 1, 2]\}]$

f-tau $= y^2 z$, *N-tau* $= 0$, *S-tau* $= \dfrac{(p^2 t^5 + 2 p^3 t^4 + p^4 t^3 + 2 p^6 t^2 + t^6 + p^8 + p^7 t)\, t^3}{(p-1)^2 (p^8 - t^6)(p^4 - t^3)}$,

S-tau in s $= \dfrac{p^2 p^s + 2 (p^s)^2 p^3 + p^4 (p^s)^3 + 2 (p^s)^4 p^6 + 1 + p^8 (p^s)^6 + (p^s)^5 p^7}{(p-1)^2 (p^4 (p^s)^3 - 1)^2 (p^4 (p^s)^3 + 1)}$,

L-tau $= \dfrac{(p-1)^4}{p^4}$, *L-tau in s* $= \dfrac{(p-1)^4}{p^4}$

Igusa's local zeta function of

$x^2 z + y^2 z + u^3$

in function of t

$$\frac{(p-1)(t+p^2)(p^6 - t p^4 - p^3 t + p^3 t^2 + t^2 p^2 - t^3)}{(p-t)(p^4 - t^3)(p^4 + t^3)}$$

in function of s

$$\frac{(p^s)^3 (p-1)(p^2 p^s + 1)((p^s)^3 p^6 - (p^s)^2 p^4 + p^s p^3 - (p^s)^2 p^3 + p^2 p^s - 1)}{(p^4 (p^s)^3 + 1)(p^4 (p^s)^3 - 1)(p\, p^s - 1)}$$

The Poincaré series of

$x^2 z + y^2 z + u^3$

is equal to:

$$\frac{-t^6 + p\, t^5 - t^5 + p^4 t^3 - p^3 t^3 - p^5 t^2 + p^6 t^2 + p^9}{(p-t)(p^4 - t^3)(p^4 + t^3)}$$

Now we are going to calculate some numbers N_e,

where N_e = #{x in Z_p^n | f(x)=0 mod p^e}

Give the e from which you want to start calculating,

e must be a positive integer >0, end with semicolon ';'100;

Give the e uptill which you want to calculate,

e must be a positive integer >0, end with semicolon ';'102;

$N_100 = (p^{34} + p^{35} + p^{33} - p - 1)\, p^{265}$

$N_101 = (p^{35} + p^{36} + p^{34} - p^2 - 1)\, p^{267}$

$N_102 = (p^{35} + p^{36} + p^{34} - p - 1)\, p^{270}$

Ex. 4: $x^7 y^3 z + x^4 y^8 + x^8 y^4 + z^7$ and $x^7 y^3 z + x^4 y^8 + x^8 y^4$

igusa + p = 3 , p = 5 and p = 31

This example illustrates Theorem 5.3.8 and the remark at the end of Section 0.3 (see also Remark 2 after Theorem 5.2.1).
Examine the first polynomial. The expected order of the candidate pole -1/3 is equal to 1 and the face of maximal contribution to -1/3 is unstable. Although the condition "for every face μ of maximal pure contribution to -1/3, one has that the face μ is the only face of $\Gamma(f_\mu)$ of maximal pure contribution to -1/3" is not satisfied, -1/3 is not an actual pole of the Igusa zeta function of the first polynomial (for p=3, p=5 and p=31). Maybe, this condition might be omitted in Theorem 5.3.8. On the other hand, -1/3 is a pole of order 1 for the Igusa zeta function of the second polynomial (for p=3, p=5 and p=31).

```
> timeanalysis_Zetafunction(x^7*y^3*z+x^4*y^8+x^8*y^4+z^7,
  [x,y,z], igusa);
The number of different ftau is 12. If you wish to calculate
the Igusa's local zeta function for a specific prime, enter
the prime. If you wish to enter an abstract number of solutions
for each face, enter 'p'
Enter prime number, or 'p', end with semicolon ;3;
```

$$x^7 y^3 z + x^4 y^8 + x^8 y^4 + z^7$$

is non-degenerated over F_3

with respect to all the faces of its Newton polyhedron

Igusa's local zeta function of

$$x^7 y^3 z + x^4 y^8 + x^8 y^4 + z^7$$

```
function           depth    calls     time    time%         bytes   bytes%
-------------------------------------------------------------------------
Zetafunction           1        1   91.498    99.96     154303660    99.82
Newtonpoly             1        1     .032      .03        244184      .16
Ntau                   1        8     .009      .01         34068      .02
-------------------------------------------------------------------------
total:                 3       10   91.539   100.00     154581912   100.00
```

To save pages of output, we only print the denominator:

$$(3\,3^s - 1)\,(27\,(3^s)^7 - 1)\,(177147\,(3^s)^{28} - 1)\,(1 + 7976644307687250986 3361\,(3^s)^{154})$$
$$(1 + 282429536481\,(3^s)^{77})\,(282429536481\,(3^s)^{77} - 1)$$

```
> timeanalysis_Zetafunction(x^7*y^3*z+x^4*y^8+x^8*y^4+z^7,
  [x,y,z], igusa);
The number of different ftau is 12. If you wish to calculate
the Igusa's local zeta function for a specific prime, enter
the prime. If you wish to enter an abstract number of solutions
for each face, enter 'p'
Enter prime number, or 'p', end with semicolon ;5;
```

$$x^7 y^3 z + x^4 y^8 + x^8 y^4 + z^7$$

is non-degenerated over F_5

with respect to all the faces of its Newton polyhedron

Igusa's local zeta function of

$$x^7 y^3 z + x^4 y^8 + x^8 y^4 + z^7$$

```
function           depth    calls     time    time%         bytes   bytes%
-------------------------------------------------------------------------
Zetafunction           1        1  105.670    99.93     163875560    99.74
Newtonpoly             1        1     .041      .04        237248      .14
Ntau                   1        8     .035      .03        192020      .12
-------------------------------------------------------------------------
total:                 3       10  105.746   100.00     164304828   100.00
```

To save pages of output, we only print the denominator:

$$(5\,5^s - 1)\,(125\,(5^s)^7 - 1)\,(1 + 355271367880050092935562133789 0625\,(5^s)^{154})$$
$$(1 + 59604644775390625\,(5^s)^{77})\,(-1 + 59604644775390625\,(5^s)^{77})\,(48828125\,(5^s)^{28} - 1)$$

```
> timeanalysis_Zetafunction(x^7*y^3*z+x^4*y^8+x^8*y^4+z^7,
  [x,y,z], igusa);
The number of different ftau is 12. If you wish to calculate
the Igusa's local zeta function for a specific prime, enter
the prime. If you wish to enter an abstract number of solutions
for each face, enter 'p'
Enter prime number, or 'p', end with semicolon ;31;
```

$$x^7 y^3 z + x^4 y^8 + x^8 y^4 + z^7$$

is non-degenerated over F_31

with respect to all the faces of its Newton polyhedron

Igusa's local zeta function of

$$x^7 y^3 z + x^4 y^8 + x^8 y^4 + z^7$$

```
function           depth    calls     time    time%         bytes   bytes%
-------------------------------------------------------------------------
Zetafunction           1        1  342.344    92.23    1735640864    94.71
Ntau                   1        8   28.801     7.76      96605832     5.27
Newtonpoly             1        1     .038      .01        243676      .01
-------------------------------------------------------------------------
total:                 3       10  371.183   100.00    1832490372   100.00
```

To save pages of output, we only print the denominator:

$$(31\,31^s - 1)\,(1 +$$
$$3849118698863268035973518447376511314241334958623873860848322273530086 41$$
$$(31^s)^{154})\,(1 + 62041266096552768818830045157315712 1\,(31^s)^{77})$$
$$(62041266096552768818830045157315712 1\,(31^s)^{77} - 1)\,(29791\,(31^s)^7 - 1)$$
$$(25408476896404831\,(31^s)^{28} - 1)$$

```
> timeanalysis_Zetafunction(x^7*y^3*z+x^4*y^8+x^8*y^4, [x,y,z],
  igusa);
The number of different ftau is 7. If you wish to calculate
the Igusa's local zeta function for a specific prime, enter
```

```
the prime. If you wish to enter an abstract number of solutions
for each face, enter 'p'
```

Enter prime number, or 'p', end with semicolon ;3;

$$x^4 y^8 + x^8 y^4 + x^7 y^3 z$$

is non-degenerated over F_3

with respect to all the faces of its Newton polyhedron

Igusa's local zeta function of

$$x^4 y^8 + x^8 y^4 + x^7 y^3 z$$

in function of t

$$-4\,(2\,t^{41} - 6\,t^{37} - 3\,t^{36} + 9\,t^{35} + 9\,t^{33} - 27\,t^{32} + 18\,t^{31} - 54\,t^{27} + 54\,t^{24} - 243\,t^{20} + 243\,t^{19} + 243\,t^{17} - 729\,t^{16} + 486\,t^{14} - 1458\,t^{10} - 729\,t^{8} + 2187\,t^{7} + 2187\,t^{5} - 2187\,t^{4} - 13122) \Big/ ((-3 + t^3)(-3 + t^4)(-9 + t^{11})(t^{11} + 9)(t^{22} + 81)(-3 + t))$$

in function of s

$$4\,(3^s)^{11}(-2 + 6\,(3^s)^4 + 3\,(3^s)^5 - 9\,(3^s)^6 - 9\,(3^s)^8 + 27\,(3^s)^9 - 18\,(3^s)^{10} + 54\,(3^s)^{14} - 54\,(3^s)^{17} + 243\,(3^s)^{21} - 243\,(3^s)^{22} - 243\,(3^s)^{24} + 729\,(3^s)^{25} - 486\,(3^s)^{27} + 1458\,(3^s)^{31} + 729\,(3^s)^{33} - 2187\,(3^s)^{34} - 2187\,(3^s)^{36} + 2187\,(3^s)^{37} + 13122\,(3^s)^{41}) \Big/ ((3\,(3^s)^3 - 1)(3\,(3^s)^4 - 1)(9\,(3^s)^{11} - 1)(1 + 9\,(3^s)^{11})(1 + 81\,(3^s)^{22})(3\,3^s - 1))$$

```
function           depth    calls     time    time%         bytes   bytes%
-------------------------------------------------------------------------
Zetafunction           1        1    2.533    98.83      13320292    98.70
Newtonpoly             1        1     .030     1.17        158564     1.17
Ntau                   1        4    0.000     0.00         16412      .12
-------------------------------------------------------------------------
total:                 3        6    2.563   100.00      13495268   100.00
```

> timeanalysis_Zetafunction(x^7*y^3*z+x^4*y^8+x^8*y^4, [x,y,z], igusa);

```
The number of different ftau is 7. If you wish to calculate
the Igusa's local zeta function for a specific prime, enter
the prime. If you wish to enter an abstract number of solutions
for each face, enter 'p'
```

Enter prime number, or 'p', end with semicolon ;5;

$$x^4 y^8 + x^8 y^4 + x^7 y^3 z$$

is non-degenerated over F_5

with respect to all the faces of its Newton polyhedron

Igusa's local zeta function of

$$x^4 y^8 + x^8 y^4 + x^7 y^3 z$$

in function of t

$$-16\,(4\,t^{41} - 20\,t^{37} - 5\,t^{36} + 25\,t^{35} + 25\,t^{33} - 125\,t^{32} + 100\,t^{31} - 500\,t^{27} + 500\,t^{24} - 3125\,t^{20} + 3125\,t^{19} + 3125\,t^{17} - 15625\,t^{16} + 12500\,t^{14} - 62500\,t^{10} - 15625\,t^{8} + 78125\,t^{7} + 78125\,t^{5} - 78125\,t^{4} - 1562500) \Big/ ((-25 + t^{11})(t^{11} + 25)(t^{22} + 625)(-5 + t^3)(-5 + t^4)(-5 + t))$$

in function of s

$$16\,(5^s)^{11}\,(-4+20\,(5^s)^4+5\,(5^s)^5-25\,(5^s)^6-25\,(5^s)^8+125\,(5^s)^9-100\,(5^s)^{10}$$
$$+500\,(5^s)^{14}-500\,(5^s)^{17}+3125\,(5^s)^{21}-3125\,(5^s)^{22}-3125\,(5^s)^{24}+15625\,(5^s)^{25}$$
$$-12500\,(5^s)^{27}+62500\,(5^s)^{31}+15625\,(5^s)^{33}-78125\,(5^s)^{34}-78125\,(5^s)^{36}$$
$$+78125\,(5^s)^{37}+1562500\,(5^s)^{41})\Big/($$
$$(25\,(5^s)^{11}-1)\,(1+25\,(5^s)^{11})\,(1+625\,(5^s)^{22})\,(5\,(5^s)^3-1)\,(5\,(5^s)^4-1)\,(5\,5^s-1))$$

function	depth	calls	time	time%	bytes	bytes%
Zetafunction	1	1	2.551	98.23	13290928	98.18
Newtonpoly	1	1	.030	1.16	158016	1.17
Ntau	1	4	.016	.62	87744	.65
total:	3	6	2.597	100.00	13536688	100.00

```
> timeanalysis_Zetafunction(x^7*y^3*z+x^4*y^8+x^8*y^4, [x,y,z],
  igusa);
The number of different ftau is 7. If you wish to calculate
the Igusa's local zeta function for a specific prime, enter
the prime. If you wish to enter an abstract number of solutions
for each face, enter 'p'
Enter prime number, or 'p', end with semicolon ;31;
```

$$x^4y^8+x^8y^4+x^7y^3z$$

is non-degenerated over F_31

with respect to all the faces of its Newton polyhedron

Igusa's local zeta function of

$$x^4y^8+x^8y^4+x^7y^3z$$

in function of t

$$-900\,(30\,t^{41}-930\,t^{37}-31\,t^{36}+961\,t^{35}+961\,t^{33}-29791\,t^{32}+28830\,t^{31}-893730\,t^{27}$$
$$+893730\,t^{24}-28629151\,t^{20}+28629151\,t^{19}+28629151\,t^{17}-887503681\,t^{16}$$
$$+858874530\,t^{14}-26625110430\,t^{10}-887503681\,t^8+27512614111\,t^7+27512614111\,t^5$$
$$-27512614111\,t^4-25586731123230)\Big/($$
$$(-961+t^{11})\,(t^{11}+961)\,(t^{22}+923521)\,(-31+t^3)\,(-31+t^4)\,(-31+t))$$

in function of s

$$900\,(31^s)^{11}\,(-30+930\,(31^s)^4+31\,(31^s)^5-961\,(31^s)^6-961\,(31^s)^8+29791\,(31^s)^9$$
$$-28830\,(31^s)^{10}+893730\,(31^s)^{14}-893730\,(31^s)^{17}+28629151\,(31^s)^{21}$$
$$-28629151\,(31^s)^{22}-28629151\,(31^s)^{24}+887503681\,(31^s)^{25}-858874530\,(31^s)^{27}$$
$$+26625110430\,(31^s)^{31}+887503681\,(31^s)^{33}-27512614111\,(31^s)^{34}$$
$$-27512614111\,(31^s)^{36}+27512614111\,(31^s)^{37}+25586731123230\,(31^s)^{41})\Big/($$

$$(961\,(31^s)^{11}-1)\,(1+961\,(31^s)^{11})\,(1+923521\,(31^s)^{22})\,(31\,(31^s)^3-1)\,(31\,(31^s)^4-1)$$
$$(31\,31^s-1))$$

function	depth	calls	time	time%	bytes	bytes%
Zetafunction	1	1	24.106	65.16	78593736	64.63
Ntau	1	4	12.864	34.77	42853436	35.24
Newtonpoly	1	1	.026	.07	158872	.13
total:	3	6	36.996	100.00	121606044	100.00

Ex. 5: $x^7 y^3 z + x^4 y^8 + x^8 y^4 + z^7$

topol

```
> timeanalysis_Zetafunction(x^7*y^3*z+x^4*y^8+x^8*y^4+z^7,
  [x,y,z], topol);
```

Now we are going to calculate the topological zeta function of

$$x^7 y^3 z + x^4 y^8 + x^8 y^4 + z^7$$

Enter natural number for the parameter d (the standard d is 1)

```
Enter natural number for the parameter d, end with semicolon;1;
```

$$x^7 y^3 z + x^4 y^8 + x^8 y^4 + z^7$$

is non-degenerated with respect to all the faces of its global Newton polyhedron.

The global topological zeta function of

$$x^7 y^3 z + x^4 y^8 + x^8 y^4 + z^7$$

with d=1

in function of s is

$$-\frac{1}{4}\,\frac{482944\,s^4+538412\,s^3+186837\,s^2+11634\,s-3168}{(s+1)\,(77\,s+24)\,(28\,s+11)\,(7\,s+3)}$$

function	depth	calls	time	time%	bytes	bytes%
Zetafunction	1	1	5.342	99.18	17908664	98.68
Newtonpoly	1	1	.044	.82	240144	1.32
Ntau	0	0	0.000	0.00	0	0.00
total:	2	2	5.386	100.00	18148808	100.00

Ex. 6: $x^3 y^4 z + x^{10} y^{15} + x^{17} y^{12} + x^8 y^{19} + xy z^9 + x^2 y z^2$

newtoninfo +igusa + poleinfo + allpurefaces + p=5

This example illustrates Conjecture 0.3.3.
The candidate pole -4/5 has expected order 2, but is not an actual pole. Note that all faces of pure contribution to -4/5 are unstable with respect to the same variable z.

```
> Zetafunction(x^3*y^4*z+x^10*y^15+x^17*y^12+x^8*y^19+xy*z^9+x^
```

```
2*y*z^2, [x,y,z], newtoninfo, poleinfo, allpurefaces, igusa);
```

Supportpoints = [[3, 4, 1], [10, 15, 0], [17, 12, 0], [8, 19, 0], [0, 0, 9], [2, 1, 2]]

Vertices = { [2, 1, 2], [0, 0, 9], [8, 19, 0], [10, 15, 0], [17, 12, 0], [3, 4, 1] }

Number of facets = 11

Number of proper faces = 33

Defining system of inequalities of the Newton polyhedron :

Facet 1 : x-8+5*z >= 0

Facet 2 : 7*x+2*z-18 >= 0

Facet 3 : x-4+z >= 0

Facet 4 : y-7+3*z >= 0

Facet 5 : 2*x-35+y+25*z >= 0

Facet 6 : z >= 0

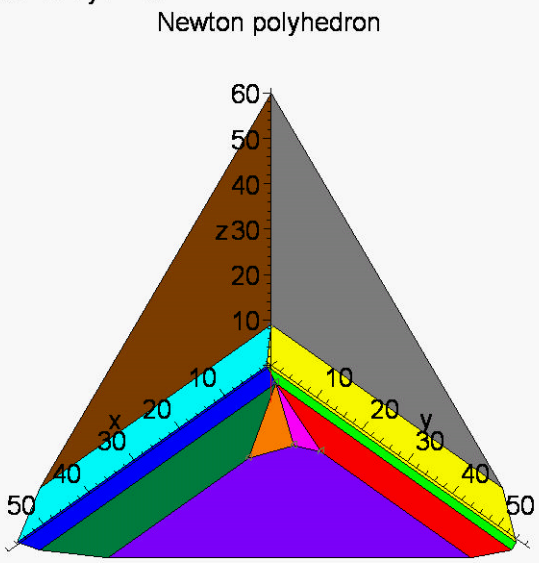
Facet 7 : 3*x-135+7*y+98*z >= 0

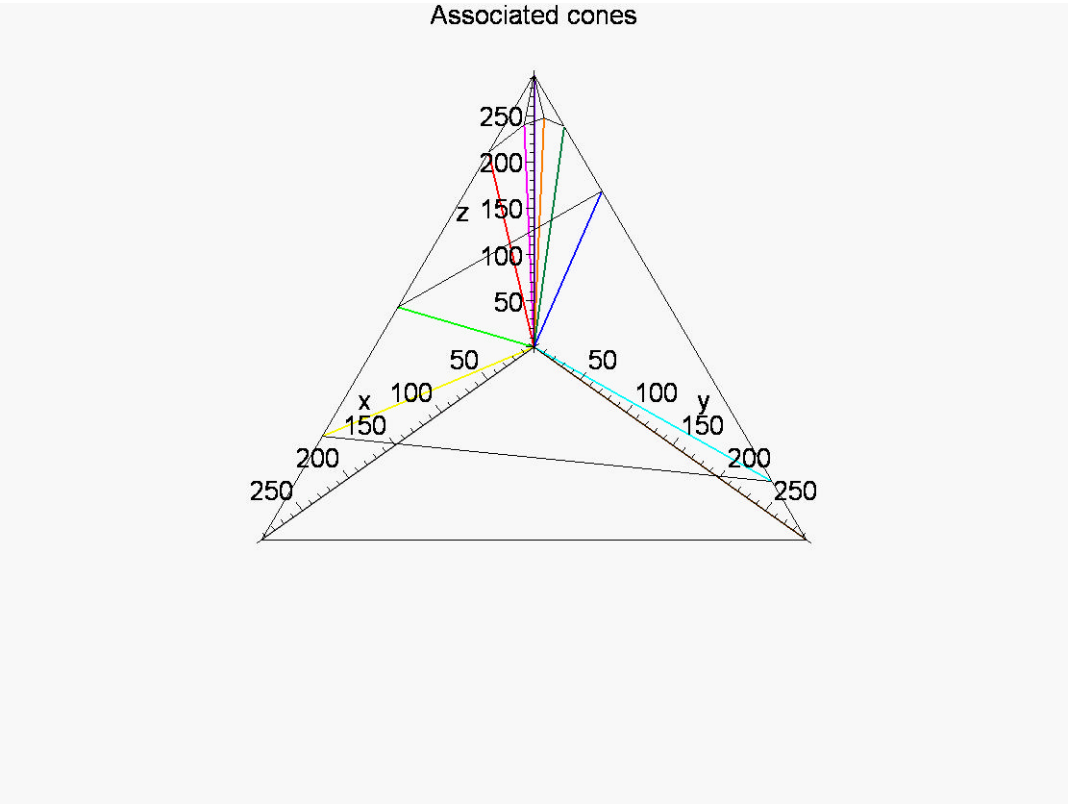
Facet 8 : y-12+8*z >= 0

Facet 9 : 7*y+z-9 >= 0

Facet 10 : x >= 0

Facet 11 : y >= 0





The candidate real poles of Igusa's local zeta function of

$$x^3 y^4 z + x^{10} y^{15} + x^{17} y^{12} + x^8 y^{19} + xy z^9 + x^2 y z^2$$

in function of s are:

```
-1 with expected order: 1
(If all N_tau are 0, then the expected order of -1 is 0)

-1/2 with expected order: 2
The face of maximal pure contribution to this candidate pole is face 25, the
face tau_0.
(So, the responsible faces for this order are all the faces therein)
```

supportpoints $= \{[2, 1, 2]\}$, *directions of recession* $= \{[0, 1, 0]\}$, *dimension* $= 1$

$$\textit{cone} = \{[7, 0, 2], [1, 0, 1]\}$$

partition of the associated cone into simplicial cones $= [\{[7, 0, 2], [1, 0, 1]\}]$

```
The other faces of pure contribution to this candidate pole are:
face 2
```

supportpoints $= \{[2, 1, 2], [0, 0, 9]\}$, *directions of recession* $= \{[0, 1, 0]\}$,

dimension $= 2$

$$\textit{cone} = \{[7, 0, 2]\}$$

partition of the associated cone into simplicial cones $= [\{[7, 0, 2]\}]$

```
face 3
```

supportpoints $= \{[2, 1, 2], [3, 4, 1]\}$, *directions of recession* $= \{[0, 1, 0]\}$,

dimension $= 2$

$$\textit{cone} = \{[1, 0, 1]\}$$

partition of the associated cone into simplicial cones $= [\{[1, 0, 1]\}]$

```
-4/7 with expected order:1
The face(s) of maximal pure contribution to this candidate pole is/are
```

```
(so, the responsible faces for this order are all the faces therein)
face 4
```
$$\textit{supportpoints} = \{[2, 1, 2], [3, 4, 1]\},\ \textit{directions of recession} = \{[1, 0, 0]\},$$
$$\textit{dimension} = 2$$
$$\textit{cone} = \{[0, 1, 3]\}$$
$$\textit{partition of the associated cone into simplicial cones} = [\{[0, 1, 3]\}]$$
```
-3/4 with expected order:1
The face(s) of maximal pure contribution to this candidate pole is/are
(so, the responsible faces for this order are all the faces therein)
face 1
```
$$\textit{supportpoints} = \{[8, 19, 0], [3, 4, 1]\},\ \textit{directions of recession} = \{[0, 1, 0]\},$$
$$\textit{dimension} = 2$$
$$\textit{cone} = \{[1, 0, 5]\}$$
$$\textit{partition of the associated cone into simplicial cones} = [\{[1, 0, 5]\}]$$
```
face 8
```
$$\textit{supportpoints} = \{[17, 12, 0], [3, 4, 1]\},\ \textit{directions of recession} = \{[1, 0, 0]\},$$
$$\textit{dimension} = 2$$
$$\textit{cone} = \{[0, 1, 8]\}$$
$$\textit{partition of the associated cone into simplicial cones} = [\{[0, 1, 8]\}]$$
```
-4/5 with expected order:2
The face(s) of maximal pure contribution to this candidate pole is/are
(so, the responsible faces for this order are all the faces therein)
face 15
```
$$\textit{supportpoints} = \{[10, 15, 0], [3, 4, 1]\},\ \textit{directions of recession} = \{\ \},\ \textit{dimension} = 1$$
$$\textit{cone} = \{[2, 1, 25], [3, 7, 98]\}$$
$$\textit{partition of the associated cone into simplicial cones} = [\{[2, 1, 25], [3, 7, 98]\}]$$
```
The other faces of pure contribution to this candidate pole are:
face 5
```
$$\textit{supportpoints} = \{[8, 19, 0], [10, 15, 0], [3, 4, 1]\},\ \textit{directions of recession} = \{\ \},$$
$$\textit{dimension} = 2$$
$$\textit{cone} = \{[2, 1, 25]\}$$
$$\textit{partition of the associated cone into simplicial cones} = [\{[2, 1, 25]\}]$$
```
face 7
```
$$\textit{supportpoints} = \{[10, 15, 0], [17, 12, 0], [3, 4, 1]\},\ \textit{directions of recession} = \{\ \},$$
$$\textit{dimension} = 2$$
$$\textit{cone} = \{[3, 7, 98]\}$$
$$\textit{partition of the associated cone into simplicial cones} = [\{[3, 7, 98]\}]$$
```
-8/9 with expected order:1
The face(s) of maximal pure contribution to this candidate pole is/are
(so, the responsible faces for this order are all the faces therein)
face 9
```
$$\textit{supportpoints} = \{[2, 1, 2], [0, 0, 9]\},\ \textit{directions of recession} = \{[1, 0, 0]\},$$
$$\textit{dimension} = 2$$

$$cone = \{[0, 7, 1]\}$$

$$partition\ of\ the\ associated\ cone\ into\ simplicial\ cones = [\{[0, 7, 1]\}]$$

```
The number of different ftau is 17. If you wish to calculate
the Igusa's local zeta function for a specific prime, enter
the prime. If you wish to enter an abstract number of solutions
for each face, enter 'p'
```

Enter prime number, or 'p', end with semicolon ;5;

$$x^3 y^4 z + x^{10} y^{15} + x^{17} y^{12} + x^8 y^{19} + xy z^9 + x^2 y z^2$$

is non-degenerated over F_5

with respect to all the faces of its Newton polyhedron

Igusa's local zeta function of

$$x^3 y^4 z + x^{10} y^{15} + x^{17} y^{12} + x^8 y^{19} + xy z^9 + x^2 y z^2$$

in function of t

$$\frac{2}{5}(3822509765625\, t^{12} + 125885009765625\, t^7 - 3051757812500000\, t^3 + 25272369384765625\, t + 6220703125\, t^{18} - 122412109375\, t^{16} + 15181884765625\, t^{10} - 610351562500000\, t^6 + 2799072265625\, t^{13} - 2994537353515625\, t^5 - 500000\, t^{24} + 1263828125\, t^{23} - 641953125\, t^{22} - 2916015625\, t^{20} + 4091015625\, t^{21} - 23837890625\, t^{17} + 1669921875\, t^{19} + 25453125\, t^{26} - 3\, t^{38} + 1875\, t^{31} - 3475\, t^{32} - 85\, t^{34} - 1578759765625\, t^{14} - 135898590087890625 + 196875000000\, t^{15} - 75\, t^{35} + 37421875\, t^{25} - 217437744140625\, t^8 + 3\, t^{39} - 5035400390625\, t^{11} + 1023125\, t^{29} + 1875\, t^{28} - 21875\, t^{27} - 1026125\, t^{30} + 3147125244140625\, t^4 - 45416259765625\, t^9) \Big/ ((t^2 - 5)^2 (t^2 + 5)(-625 + t^7)(-5 + t)(t^4 + 5\, t^2 + 25)(t^{12} + 125\, t^6 + 15625)(-390625 + t^9))$$

in function of s

$$\frac{2}{5}(-3 + 3\, 5^s + 75\, (5^s)^4 + 2916015625\, (5^s)^{19} - 196875000000\, (5^s)^{24} - 15181884765625\, (5^s)^{29} - 37421875\, (5^s)^{14} - 125885009765625\, (5^s)^{32} - 25272369384765625\, (5^s)^{38} + 610351562500000\, (5^s)^{33} - 1875\, (5^s)^{11} + 500000\, (5^s)^{15} + 641953125\, (5^s)^{17} - 4091015625\, (5^s)^{18} + 3051757812500000\, (5^s)^{36} - 3147125244140625\, (5^s)^{35} + 135898590087890625\, (5^s)^{39} - 1263828125\, (5^s)^{16} + 45416259765625\, (5^s)^{30} - 2799072265625\, (5^s)^{26} + 23837890625\, (5^s)^{22} - 1669921875\, (5^s)^{20} - 3822509765625\, (5^s)^{27} + 122412109375\, (5^s)^{23} + 85\, (5^s)^5 + 3475\, (5^s)^7 - 1875\, (5^s)^8 + 1026125\, (5^s)^9 + 21875\, (5^s)^{12} - 1023125\, (5^s)^{10}$$

$$
\begin{vmatrix}
 & -25453125\,(5^s)^{13} + 1578759765625\,(5^s)^{25} + 217437744140625\,(5^s)^{31} \\
 & +2994537353515625\,(5^s)^{34} + 5035400390625\,(5^s)^{28} - 6220703125\,(5^s)^{21} \Big/ (\\
 & (-1+5\,(5^s)^2)^2\,(1+5\,(5^s)^2)\,(625\,(5^s)^7-1)\,(5\,5^s-1)\,(1+5\,(5^s)^2+25\,(5^s)^4) \\
 & (1+125\,(5^s)^6+15625\,(5^s)^{12})\,(390625\,(5^s)^9-1))
\end{vmatrix}
$$