OPTEC
K.U.Leuven Optimization in Engineering Center

Multigrid of the second kind for the optimal control of time-periodic PDEs

Dirk Abbeloos, Moritz Diehl, Michael Hinze and Stefan Vandewalle
Department of Computer Science, K.U. Leuven

Abstract

We present a *multigrid method of the second kind* to optimize time-periodic, parabolic, partial differential equations (PDEs). We consider a *quadratic tracking objective* with a *linear PDE constraint*. The first order optimality conditions, given by a coupled system of boundary value problems, can be rewritten as an integral equation of the second kind, which is solved by multigrid of the second kind. The evaluation of the integral operator consists of solving sequentially a boundary value problem for respectively the state and the adjoints.

Motivation

Time-periodic PDEs are used to model a variety of **industrial applications**, relevant to OPTEC, e.g. *solar powerplants*, *biochemical reactors* and *cyclic controlled chemical reactors*.

General problem

Find **optimal control** $u \in U$ which minimizes the objective function $J : Y \times U \to R$ and satisfies the **parabolic PDE-constraint** $e\,(y,u) = F$, i.e.,

$$\mathbb{GP}\begin{cases}\min_{y,u}\ J(y,u)\\ \text{s.t.}\qquad e(y,u)=F\end{cases}$$

where J is **the compromise** between the cost of the control and its benefit, e.g. *the error between the state* $y(u)$ *and a prescribed target state* z. The **system model** and the **periodicity constraint** are included in $e : Y \times U \to P^*$.

Model problem

Tracking a prescribed state z in the L^2-norm, with an L^2-cost on the distributed control $u \in U$.

$$\mathbb{MP}_1\begin{cases}\min_{y,u}\ \frac{c_1}{2}\|y-z\|^2_{L^2(\Omega^T)}+\frac{c_2}{2}\|y\,(T)-z_T\|^2_{L^2(\Omega)}+\frac{\alpha}{2}\|u\|^2_U & \\ \qquad\qquad \partial_t y-\triangle y=u+f & \text{in } \Omega\times(0,T)\\ \text{s.t.}\qquad y\,(0)-y\,(T)=0 & \text{on } \Omega\\ \qquad\qquad\qquad y=0 & \text{on } \partial\Omega\times(0,T)\,,\end{cases}$$

where α is a **regularization parameter** for tuning the control effort and the approximation error.

Optimality conditions

Define the **Lagrangian** and introduce the **adjoint** $p \in P^{**}$ as,

$$\mathcal{L}\,(y,u,p)=J_1(y)+\alpha J_2(u)-\langle p,e(y,u)\rangle_{P^{**},P^*}$$

The **first order conditions (FOC)** are then given by,

$$\mathcal{L}_u=0\Rightarrow\qquad \alpha\ J_2'(u)-e_u^*(y,u)p=0 \tag{1}$$

$$\mathcal{L}_y=0\Rightarrow\qquad J_1'(y)-e_y^*(y,u)p=0 \tag{2}$$

$$\mathcal{L}_p=0\Rightarrow\qquad e(y,u)=0. \tag{3}$$

The state (3) and the adjoint (2) equations are both **boundary value problems** given by a parabolic PDE.

For the model problem:

$$\mathcal{L}_y=0\Rightarrow\begin{cases}-\partial_t p-\triangle p=c_1\,(y-z) & \text{in } \Omega\times(0,T)\\ p\,(T)-p\,(0)=c_2\,(y(T)-z_T) & \text{on } \Omega\\ p=0 & \text{on } \partial\Omega\times(0,T)\,.\end{cases}$$

Fredholm integral equation of the 2^{nd} kind

Assume that $e(y,u)=Ay+Bu$ models a *linear problem*.
(3) $\Rightarrow y(u)$, (2) $\Rightarrow p(y,u)$ and substituting in (1), results in,

$$\left(\alpha\ J_2'\right)u+\left(B^*A^{-*}J_1'\ A^{-1}B\right)u+\left(B^*A^{-*}J_1'A^{-1}F+B^*A^{-*}J_1'\ z\right)=0 \tag{4}$$

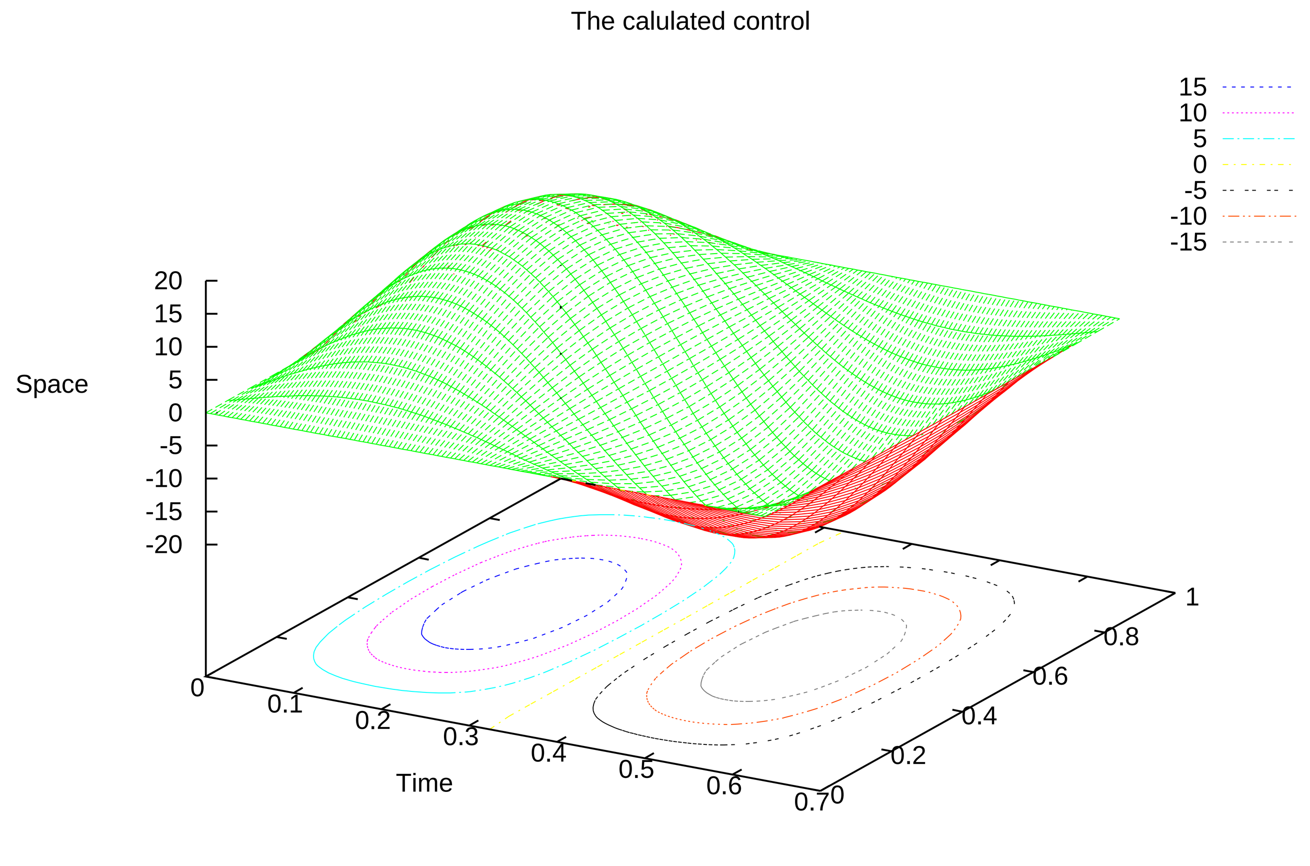
Numerical Algorithm

Indirect discretization of (4), by discretizing the FOC with **dg(0)cg(1)**-method on a *time-space grid*.

Apply **multigrid of the second kind** on (4).

- Evaluate the kernel of (4) for given u by
  1. solving (3) for y with a *space-time multigrid* (1^{st} kind)
  2. solving (2) for p with a *space-time multigrid* (1^{st} kind)
- *Semi-coarsening* (1^{st} kind) and *full coarsening* (2^{nd} kind).

Numerical results



α	0.1	0.05	0.01	0.005
ρ	$2.4\ 10^{-05}$	$1.7\ 10^{-04}$	$5.3\ 10^{-03}$	$2.5\ 10^{-02}$

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