

Katholieke Universiteit Leuven
Faculteit der Wetenschappen
Instituut voor Theoretische Fysica

de Sitter space and supergravity in various dimensions

Geert Smet

Promotoren:
Prof. Dr. Walter Troost
Prof. Dr. Antoine Van Proeyen

Proefschrift ingediend tot
het behalen van de graad van
Doctor in de Wetenschappen

(2006)

Dankwoord

Een thesis schrijf je niet alleen, wil het cliché. En dat is waar gebleken. Heel wat mensen hebben op een of andere manier bijgedragen tot de totstandkoming van dit werk en verdienen dan ook een plaats in deze thesis.

- Mijn promotoren, Walter Troost en Antoine Van Proeyen, om mij de kans te geven aan een doctoraat te beginnen, mij vrij te laten zelf mijn onderzoeksonderwerpen te kiezen, en mij af en toe te voorzien van welgemeende goede raad. Toine wil ik ook nog bedanken voor het nalezen van de gepubliceerde artikels waarop dit werk gebaseerd is.

- De leden van de examencommissie, Désiré Bollé, Frederik Denef, Mark Fannes, Raymond Gastmans, Mario Trigiante, Walter Troost en Antoine Van Proeyen, voor het lezen van deze thesis en de vele nuttige opmerkingen.

- Mijn ouders, om mij de kans te geven te studeren, en altijd in mij te geloven.

- Mijn bureaugenoten, Yves Demasure en Bert Cosemans. Ik heb veel geleerd van jullie beiden en er was altijd een zeer aangename werksfeer. Ik heb jullie tijdens het laatste jaar van mijn doctoraat dan ook hard gemist. Bert Cosemans wil ik ook nog bedanken voor de prettige samenwerking die uiteindelijk ook een artikel heeft opgeleverd.

- Joris Van den Bergh, voor een vruchtbare samenwerking, vele discussies over fysica, en geregelde avondlijke uitstapjes naar de Oude Markt en omgeving.

- Marco Baiesi, Alessio Celi and Tassilo Ott, for the many enjoyable evenings, playing games, or discussing in pubs.

- Everybody that played in the infamous ITF soccer team over the years. There are too many people to mention, I do not want to take the risk of forgetting anyone, but I am sure you guys know who you are. We did not win a lot, but it was always fun.

- De snaartheoreten uit Brussel, Paul Koerber, Stijn Nevens en Alexander Wijns, voor de leuke momenten tijdens de middagpauze van de wekelijkse seminaries en op verschillende buitenlandse conferenties.

- De belastingbetalende inwoners van België, voor het onrechtstreeks financieren van dit werk.

Bedankt allen.

I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me. [Memoirs of the Life, Writings, and Discoveries of Sir Isaac Newton (1855) by Sir David Brewster (Volume II. Ch. 27)]

Contents

1 Introduction **3**
- 1.1 Elementary particles, superstrings and M-theory . . . 3
- 1.2 The AdS/CFT correspondence and holography . . . 5
- 1.3 Cosmology . . . 6
- 1.4 de Sitter space in supergravity . . . 7
- 1.5 Thesis outline . . . 8

2 Maximally symmetric spaces and cosmology **11**
- 2.1 Maximally symmetric spaces . . . 11
- 2.2 Geometry and causal structure of Minkowski space . . . 13
- 2.3 Geometry and causal structure of de Sitter space . . . 15
- 2.4 Geometry and causal structure of Anti-de Sitter space . . . 20
- 2.5 Cosmology . . . 23
  - 2.5.1 The Robertson-Walker metric . . . 23
  - 2.5.2 The Friedmann equations . . . 24
  - 2.5.3 The concordance model: Λ-CDM . . . 27
  - 2.5.4 Inflation . . . 30

3 Holography **35**
- 3.1 The AdS/CFT correspondence . . . 35
  - 3.1.1 The Maldacena conjecture for $\mathcal{N} = 4$ SYM . . . 35
  - 3.1.2 AdS/CFT: a concrete proposal . . . 37
  - 3.1.3 Intermezzo: stability of AdS . . . 39
  - 3.1.4 AdS/CFT and gauged supergravity . . . 42
- 3.2 A dS/CFT correspondence ? . . . 44
  - 3.2.1 dS/CFT correspondence: a few proposals . . . 44
  - 3.2.2 Stability of dS space . . . 45
- 3.3 Holography in flat space . . . 46

4 de Sitter space in supergravity **48**
- 4.1 de Sitter vacua in 4D supergravity . . . 48
  - 4.1.1 Gauged supergravity in 4D: general features . . . 48
  - 4.1.2 $\mathcal{N} = 8$ gauged supergravity . . . 51

4.1.3 $\mathcal{N} = 2$ gauged supergravity . . . 54
4.1.4 $\mathcal{N} = 4$ gauged supergravity . . . 70
4.2 de Sitter vacua in 5D supergravity . . . 74
4.2.1 $\mathcal{N} = 8$ gauged supergravity . . . 75
4.2.2 $\mathcal{N} = 2$ gauged supergravity . . . 76
4.2.3 $\mathcal{N} = 4$ gauged supergravity . . . 90
4.3 de Sitter in 10D and 11D supergravity . . . 98
4.3.1 The theories and their potentials . . . 98
4.3.2 $dS \times H^{p,q}$ solutions . . . 99
4.3.3 The Maldacena/Nunez no go theorem . . . 100
4.4 Summary and open problems . . . 102

5 Flux compactifications and orientifolds **104**
5.1 Flux compactifications in string theory: some basic ideas . . . 104
5.2 Calabi-Yau orientifolds and special geometry . . . 106
5.2.1 Type IIB on Calabi-Yau manifolds and orientifolds . . . 107
5.2.2 The O3/O7 orientifold truncation . . . 110
5.2.3 The Kähler potential . . . 114
5.2.4 Surviving isometries of special quaternionic-Kähler manifolds . . . 116
5.2.5 Truncation of homogeneous very special quaternionic-Kähler manifolds . . . 121
5.2.6 Truncation of symmetric special quaternionic-Kähler manifolds . . . 124
5.2.7 Dual Kähler spaces . . . 129
5.3 Summary and open problems . . . 133

A Conventions **136**

B Samenvatting **137**
B.1 Elementaire deeltjes, supersnaren en M-theorie . . . 137
B.2 De AdS/CFT correspondentie en holografie . . . 139
B.3 Kosmologie . . . 141
B.4 de Sitter ruimten in supergravitatie . . . 142
B.5 Overzicht van de thesis . . . 143

Chapter 1

Introduction

1.1 Elementary particles, superstrings and M-theory

Fellow humans, cars, trees, the air we breathe, amazingly everything we see around us is made of just a few different elementary particles that interact with each other in only 4 different ways. These 4 ways of interacting can be thought of as the result of 4 fundamental forces acting on the particles. Each particle is a source of force fields, and will exert forces on all the other particles thereby inducing interactions. The forces we are most familiar with from everyday life are gravity and electromagnetism, but at the subatomic level two other forces become important, called the weak and strong force. While at very large (length) scales, the scale of galaxies or even the scale of everyday life, gravity is the dominating force, at the energy scales that are currently accessible in particle accelerators, gravity is so weak compared to the other 3 forces that its effects can be ignored. All current particle accelerator experiments can be described with stunning accuracy by a theory called the Standard Model, which is a relativistic quantum field theory of the strong, weak and electromagnetic force. In these theories, the interactions between the matter particles, called fermions, are described by exchanging force particles called bosons, and the particles move according to the counterintuitive rules of (relativistic) quantum mechanics. While the Standard Model has great success in predicting the outcomes of current particle experiments, it nevertheless comes with some mathematical and conceptual problems. The main problem of the Standard Model is that some physical quantities cannot be determined by the theory itself, but are parameters that have to be determined by experiment and put in the theory by hand. The Standard Model contains 29 of these parameters. Another problem, which is a problem of most quantum field theories, is the frequent appearance of infinities in calculations of physical quantities. In order to remove them and be able to make some meaningful predictions, people have invented a procedure called renormalization. Even though this procedure works well in quantum field theories of the weak, strong and electromagnetic force, which are actually special quantum field theories called gauge theories, it still is a somewhat unsatisfactory procedure from the strictly mathematical point of view. Moreover, renormalization turns out to fail in a quantum field theory of gravity. Although we do not need a quantum

theory of gravity to explain the current experiments, at higher and higher energies, when we will be probing smaller and smaller distance, (quantum) gravity will become more and more important*. We therefore know already that at some point, the Standard Model will have to be replaced by a more complete theory that also contains quantum gravity.
The most promising candidate for such a theory is superstring theory. According to this theory, the elementary particles are actually tiny vibrating strings. Superstring theory has several appealing properties, which make the theory very compelling. For instance, instead of having several different elementary particles, superstring theory has only two different strings, open and closed strings, and all elementary particles are just strings vibrating in various ways. The graviton, the force particle of gravity appears naturally as one of the massless excitations of a closed string. Gauge bosons like the force particles of the Standard Model on the other hand can be both vibrating open and vibrating closed strings. Because a quantum theory of only open strings turns out not be consistent, superstring theory naturally leads to a unification of gravity with the Standard Model forces. Strings interact with each other by joining and splitting, of which the various forces are just different manifestations. Superstring theory also has a beautiful symmetry, called supersymmetry, which is a symmetry between the fermions (elementary matter particles) and the bosons (force particles) in the theory. Moreover, superstring theory does not have the infinity problems of quantum field theories (of particles), it is finite order by order in perturbation theory.
One of the less appealing features of superstring theory is that there are actually 5 different superstring theories, called type IIA, type IIB, type I, SO(32) heterotic and $E_8 \times E_8$ heterotic. Only type I is a theory of open and closed strings, all others only contain closed strings. For consistency, the gauge group of type I has to be SO(32), while there are two consistent heterotic theories, either with gauge group SO(32) or with gauge group $E_8 \times E_8$. The type II theories on the other hand do not have a gauge group. The superstring theories are most easy to describe when the strings move in 10-dimensional flat Minkowski space, in which case the type II theories have the maximal amount of 32 supersymmetries, while the other superstring theories have 16 supersymmetries.
All the superstring theories are only defined perturbatively, as a (divergent) series expansion in the string coupling constant, which governs the strength of string interactions. However, in the mid 90s a lot was learned about their nonperturbative behaviour. For instance, it was found that the type II and type I superstring theories have nonperturbative objects, called D-branes, which are hypersurfaces on which open strings end. Besides closed strings, the complete type II theories also turns out to contain open strings, but while the closed strings can move freely through spacetime, the open strings are constrained to end on D-branes. Moreover all kind of relations between the different superstring theories were found. T-duality showed us that a superstring theory compactified

*The force of gravity falls off as the inverse square of the distance and therefore becomes weaker and weaker with distance. It dominates at large distances because the weak and strong force fall of a lot quicker with distance, and because there are both negative and positive electromagnetic charges and these tend to cancel each other out at large distances, while there is only one kind of gravitational charge, namely mass or energy.

on a circle with a small radius can be equivalent (dual) to a string theory compactified on a circle with a large radius. For instance, type IIA is T-dual to type IIB, and the SO(32) heterotic is T-dual to the $E_8 \times E_8$ heterotic superstring. Similarly to T-duality, mirror symmetry taught us that type IIA string theory compactified on a 6-dimensional Calabi-Yau manifold is dual to type IIB string theory compactified on a 6-dimensional mirror Calabi-Yau manifold. Finally S-duality learned us that the strong coupling limit of a superstring theory can be dual to the weak coupling limit of another superstring theory. For example, the SO(32) heterotic string is S-dual to the type I string, with S-duality mapping the fundamental string of the heterotic theory to the D1-string (= D-brane with 1 spatial dimension) of the type I theory. Furthermore, type IIB is selfdual under S-duality, while type IIA and the $E_8 \times E_8$ heterotic superstring have an even weirder strong coupling behaviour, they are S-dual to 11-dimensional theories of which little is known. All these dualities together finally led to the conclusion that the 5 different string theories are just the result of perturbation expansions around different vacua of one more fundamental theory, called M-theory. Unfortunately, a completely satisfactory definition of this theory has not yet been found.

1.2 The AdS/CFT correspondence and holography

A few years after the discovery of dualities between the 5 superstring theories, another amazing duality in superstring theory was found, called the Anti-de Sitter/conformal field theory (AdS/CFT) correspondence. In its original form the AdS/CFT correspondence was a conjectured equivalence between type IIB string theory in $AdS_5 \times S^5$ and $4D$ $\mathcal{N} = 4$ Super Yang Mills theory. What is truly remarkable about this conjecture is that it claims that a quantum theory containing gravity (string theory) can equivalently be described by an ordinary gauge theory (not much different from those used in the Standard Model). Moreover according to the correspondence, a quantum theory of gravity in 5 noncompact dimensions can be described by a quantum theory 'without' gravity in 4 dimensions. The AdS/CFT correspondence is therefore a concrete realization of the so called holographic principle, which basically states that the maximal amount of information that can be stored in a given region of spacetime is determined by the area of the boundary of that region (and not by the volume of that region as one would naively expect). The holographic principle arose from work on black holes by Hawking and others which showed that black holes act like thermodynamic objects when quantum effects are taken into account. A measure for the amount of information in a black hole is its entropy, which turns out to be proportional to the area of the black hole horizon. Observing that cramming more and more information in a small region would eventually lead to the formation of a black hole*, then leads to the holographic principle. Many physicists believe that any good quantum theory of gravity will obey this holographic principle.

*We expect the formation of black holes on the basis of general relativity, the classical theory of gravity. There is of course the possibility that quantum gravity somehow prevents the formation of black holes, in which case the holographic principle need not hold.

The AdS/CFT correspondence can be used to give a nonperturbative definition of superstring theory in an AdS background and is also another example of a strong coupling/weak coupling duality. In the strong coupling limit of the CFT, the string coupling constant becomes weak and string perturbation theory can be used, and vice versa, when the string coupling becomes strong, the CFT becomes weakly coupled. Because of this, the AdS/CFT correspondence is difficult to prove, but it has been tested in several different ways, leaving little doubt that the conjecture is correct. Most of the tests have been done in the so called supergravity approximation. Besides a number of massless states, superstring theory contains an infinite tower of massive states. At sufficiently low energies, the massive states can be ignored, and an effective theory for the massless fields can be written down. Restricting to theories with at most two derivatives and working at tree level, i.e., not taking into account quantum corrections in the string coupling constant, the effective theory turns out to be a supergravity theory, which are (locally) supersymmetric versions of Einstein's general relativity theory. Since at present it is not known how to quantize superstring theory in $AdS_5 \times S^5$, most of the calculations are done in the limit where the low energy supergravity theory is valid.
Anti-de Sitter space is a maximally symmetric solution of the Einstein equations with negative cosmological constant. Since many physicists expect quantum gravity to be holographic, similar correspondences have been proposed for the maximally symmetric spaces with zero or positive cosmological constant, namely Minkowski space and de Sitter (dS) space respectively. However, these correspondences are a lot more speculative, since there are no string theory derivations of these correspondences, unlike the AdS/CFT correspondence, which has a very clear and natural string theory origin. We will look at the AdS/CFT correspondence and some other holographic proposals in more detail in chapter 3.

1.3 Cosmology

One of the main problems of superstring theory and high-energy physics in general has been the lack of experimental input in the last 20 years. Besides evidence for small neutrino* masses and some hints of gauge coupling unification†, the only other clues we got from nature, in our search for a more fundamental theory than the Standard Model, came from cosmology. In the late nineties, several research groups observing supernovae discovered that our universe seemed to be expanding at an accelerating rate, much to everybody's surprise. Since the acceptance of the Big Bang theory, it was always assumed that gravity would slow down the expansion of the universe. The fact that the expansion of the universe is not decelerating, but instead accelerating means that there is another force in nature which opposes gravity at large distances. The currently most popular

*Neutrino's are fermionic particles which interact very weakly with other particles. They only interact via the weak and gravitational force. Billions of neutrino's pass through your body every second, but only one or two will be stopped by your body during your entire lifetime!

†Gauge coupling unification is the idea that the weak, strong and electromagnetic force are just different manifestations of one, more fundamental, (gauge) force.

way to explain this observation is to conjecture the existence of a strange form of energy that permeates space and has negative pressure, called dark energy. The simplest form of dark energy, and a form that fits current observations very well, is a positive cosmological constant.
Besides dark energy, current observations also support the existence of dark matter in the universe, matter that does not emit electromagnetic radiation, but has a measurable gravitational effect on visible matter such as stars. Most of the dark matter must necessarily come in a form we are not familiar with today. In fact, according to the current standard model of cosmology, the Λ-CDM model, which we will present in chapter 2, our current universe is mostly made of stuff we are not familiar with. While everything we see around us, is made of a small amount of elementary particles, this form of matter only accounts for 5% of the matter/energy content of the universe. Approximately 25% comes in the form of dark matter and a whopping 70% in the form of dark energy.
At present, no theory can explain the above bizarre properties of the universe. We do not understand why there is so much dark energy*, nor why there is so much dark matter. We don't even know the nature of dark matter. However, a good dark matter candidate is a weakly interacting particle that is often predicted by supersymmetric extensions of the Standard Model. Supersymmetry also improves other problems of the Standard Model, like the gauge hierarchy problem and gauge coupling unification†, and is a very natural symmetry from the conceptual point of view ('equality' between bosons and fermions). There are therefore lots of reasons to believe that supersymmetry and superstring theory, which implies supersymmetry and is the best candidate for a quantum theory of gravity, are steps in the right direction. We might already know it for sure in just a few years, when the Large Hadron Collider (LHC) is fully operational. Although mainly built to detect the Higgs particle, a massive boson predicted by the Standard Model, there is the exciting possibility it will detect the first supersymmetric partner particles. Time will tell.

1.4 de Sitter space in supergravity

The discovery that the expansion of the universe is accelerating led to a vigorous and still ongoing search for dS spaces in superstring theory and its low-energy cousin supergravity, as well as for an analog of the AdS/CFT correspondence for dS space. While holography for spaces with a positive cosmological constant remains clouded in mystery, by now several ways to construct dS vacua in superstring theory have been proposed and several supergravity theories with dS vacua have been constructed. In this thesis we will restrict ourselves to supergravity, even though we draw some of our motivation from the possibly underlying superstring theory. We will give a detailed account of the search for

*Actually, most quantum field theories predict a cosmological constant that is even a lot bigger than observed. This is the famous cosmological constant problem and is one of the biggest unsolved problems in high-energy physics, but that's an entirely different story altogether. We will not attempt to solve it in this thesis and will therefore ignore this problem completely!

†Explaining these problems would lead us to far astray. See, e.g., [1] if you would like to know more.

dS vacua in supergravity in chapter 4, focusing mainly on supergravity theories in 4 and 5 dimensions, since these are the most relevant dimensions for cosmological and holographical applications.
In order to study vacuum solutions of a supergravity theory, only the scalar fields in the theory are important. The relevant part of any supergravity action is therefore

$$S = \int d^D x \sqrt{-g} \left[\frac{1}{2\kappa^2} R - \frac{1}{2} a_{ij}(\phi) g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j - V(\phi) \right] , \tag{1.4.1}$$

where κ^2 is the gravitational coupling constant, R is the scalar curvature of D-dimensional space-time and ϕ^i are a number of scalar fields, labeled by an index i. The scalar fields can be thought of as coordinates of a Riemannian manifold, with a_{ij} the positive definite metric. Both the scalar metric a_{ij} and the scalar potential V are constrained by supersymmetry. When there are at least 8 supersymmetries in the theory, the scalar potential is completely determined by the gauge group and supersymmetry. Moreover, in 4D and 5D supergravity theories with more than 8 supersymmetries the scalar manifold is always a coset manifold and is also completely determined by supersymmetry, while in theories with exactly 8 supersymmetries, the scalar manifold has an interesting geometric structure, called 'special geometry'. Knowledge of the scalar manifold is necessary to calculate the scalar potential. Since theories with exactly 8 supersymmetries will turn out to play a special role in our search for dS vacua, and special geometry will also be important in chapter 5, we give a detailed summary of special geometry in 4D and 5D supergravity in chapter 4.

1.5 Thesis outline

Inspired by the holographic principle and the recent observations in cosmology, the main topic of this thesis is the search for dS vacua in supergravity. The thesis is structured as follows. In chapters 2 and 3, we give the background in cosmology and holography necessary to appreciate the physical context in which the results of chapters 4 and 5 can or should be seen, and we review some important results concerning stability in dS and AdS spaces. We then cover the main topic of the thesis in chapter 4, where we study dS vacua in various supergravity theories. This chapter contains both own work [100, 117] and related results by other authors. Finally, the fifth chapter mainly contains our own work on orientifold truncations in 4D $\mathcal{N} = 2$ supergravity [134]. Contact is made with the main theme of the thesis by noting that orientifolds are often a crucial ingredient in flux compactifications, which is the most popular method to construct dS vacua in superstring theory. After this birds-eye view of the thesis, we will now give a more detailed chapter by chapter description of the content.
In chapter 2, we start by introducing some useful concepts of (pseudo-)Riemannian geometry. We give the definitions of a Killing vector, homogeneous space, isotropy, maximally symmetric space, and mention a few interesting theorems. Next, we investigate the 3 Lorentzian maximally symmetric spaces in great detail, comparing dS space with

AdS space and the more familiar Minkowski space, using several coordinate systems to uncover their properties. Subsequently, we introduce the basic concepts of cosmology, Robertson-Walker metric, Hubble law, Friedmann equations, equation of state, etc, and we show how dS space fits into this theoretical framework. Then, we summarize the results of current observations, and we present the current standard model of cosmology, the Λ-CDM model. For completeness, we also give a short review of the idea of inflation, since this is another area of cosmology where dS space plays a crucial role.

In chapter 3, we start by reviewing the prime example of a holographic correspondence, the AdS/CFT correspondence. We summarize the original arguments of Maldacena that lead to the conjectured equivalence between superstring theory on $AdS_5 \times S^5$ and 4D $\mathcal{N} = 4$ Super Yang-Mills theory and discuss several generalized AdS/CFT correspondences. We also show that a lot can already be learned by studying the low-energy supergravity theory of the superstring theory, and in this context we also study the stability (under scalar fluctuations) of AdS spaces. Next, we summarize a few proposals for a dS/CFT correspondence, and study the stability (under scalar fluctuations) of dS space. Reading this section will make it clear that the situation is a lot more unclear than in the AdS case. Finally, for completeness we also briefly review the status of holography in flat space. Also here the current situation is far from clear.

Chapter 4 contains a detailed study of dS vacuum solutions in $D = 4$ and $D = 5$ supergravity. For reasons that will be explained in the chapter, we focus on supergravity theories with 8, 16 or 32 supersymmetries. As we will discuss, most of the dS vacua that are known, share some intriguing features. They are unstable under small scalar fluctuations and the masses of the scalar fluctuations are all quantized. The only known examples of stable dS vacua in supergravity theories with at least 8 supersymmetries occur in theories with exactly 8 supersymmetries and we will discuss all known examples. Since theories with 8 supersymmetries are special in this respect, we will devote a great deal of space to them and give a detailed presentation of the special geometry involved. After we have studied 4D and 5D supergravity in great detail, we summarize the possible 10D and 11D supergravity theories and review how some of the known 4D and 5D dS vacua can be embedded in 10D or 11D. In this context, we also discuss a well known no go theorem, excluding compactifications from 10D or 11D to 4D or 5D dS space under certain reasonable assumptions. One way to evade the no go theorem is to allow for the presence of negative tension objects such as the orientifold planes in superstring theory and this brings us to the final chapter.

In chapter 5, we first briefly discuss some of the merits of flux compactifications in superstring theory and give some pointers to the literature. This section's goal is to sketch the context, not to give an introduction to this huge subject. After we have made the context clear in the first section, we give a detailed presentation of our work on orientifold truncations in the second section. We study in detail truncations of the hyperscalar manifold of 4D $\mathcal{N} = 2$ supergravity that arise when O3 and/or O7 orientifold planes are present in Calabi-Yau compactifications of IIB string theory. The truncations break half of the supersymmetries (4 of the 8), and therefore reduce the original quaternionic-Kähler hyperscalar manifolds to Kähler manifolds. For the symmetric special quaternionic-Kähler manifolds

we give a complete overview of the resulting Kähler spaces. In case the quaternionic-Kähler space is 'very special', the truncation naturally leads to a map between two Kähler spaces of equal dimension, as was first noted by Ferrara et al. in [148], and the Kähler spaces are called dual. We give other examples of dual Kähler spaces, discovered in [134], and show that they also lead to flat or no-scale $\mathcal{N} = 1$ supergravity potentials, just like in [148].
Both the fourth and the fifth chapter are concluded by a summary and a selection of interesting open problems. The thesis also contains 2 appendices. In appendix A, we describe the conventions we use, and in appendix B, we give a summary of the thesis in Dutch.

Chapter 2

Maximally symmetric spaces and cosmology

In this chapter, we first introduce the concept of a maximally symmetric space, and then study the 3 maximally symmetric Lorentzian spaces in some more detail, focusing mainly on properties that are important in holographic correspondences. We then go on and introduce the basics of cosmology, in which maximally symmetric spaces also play a role in several different ways. We give a survey of the current standard model of cosmology and briefly discuss inflation, highlighting the role dS space plays in cosmology today.

2.1 Maximally symmetric spaces

Before we can introduce the maximally symmetric spaces, we have to review some basic concepts and facts of (pseudo-)Riemannian geometry, which will also play an important role in later chapters. The presentation here is mostly based on [2].
Given a metric $g_{\mu\nu}(x)$ an isometry of g is a transformation $x \to x'$ that satisfies

$$g_{\mu\nu}(x) = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x') \,. \tag{2.1.1}$$

In the special case of an infinitesimal coordinate transformation

$$x'^{\mu} = x^{\mu} + \epsilon\, k^{\mu}(x) \qquad \text{with } |\epsilon| \ll 1 \,, \tag{2.1.2}$$

equation (2.1.1) leads to

$$k_{\sigma;\rho} + k_{\rho;\sigma} = 0 \,, \tag{2.1.3}$$

with $k_{\sigma} \equiv g_{\mu\sigma} k^{\mu}$ and $k_{\sigma;\rho}$ the covariant derivative of k_{σ}, defined as

$$k_{\sigma;\rho} \equiv \frac{\partial k_{\sigma}}{\partial x^{\rho}} - \Gamma^{\lambda}_{\sigma\rho} k_{\lambda} \,, \tag{2.1.4}$$

$$\Gamma^{\lambda}_{\sigma\rho} \equiv \frac{1}{2} g^{\mu\lambda} \left(\frac{\partial g_{\rho\mu}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\rho}} - \frac{\partial g_{\rho\sigma}}{\partial x^{\mu}} \right) . \tag{2.1.5}$$

The four-vector field $k_\sigma(x)$ is called a Killing vector of the metric $g_{\mu\nu}(x)$. Any linear combination of Killing vectors (with constant coefficients) is again a Killing vector, so the space of vector fields spanned by the Killing vectors determines all infinitesimal isometries of a metric. Equation (2.1.3) is quite restrictive. Using properties of the curvature tensor $R^\mu{}_{\nu\rho\sigma}$, one can prove that (2.1.3) implies

$$k_{\mu;\rho;\sigma} = R^\lambda{}_{\sigma\rho\mu} k_\lambda \,. \tag{2.1.6}$$

This means that all the derivatives of k_σ at some point X will be linear combinations of $k_\sigma(X)$ and $k_{\sigma;\rho}(X)$. We can thus write any Killing vector $k_\sigma(x)$ as

$$k_\sigma(x) = A^\lambda_\sigma(x;X) k_\lambda(X) + B^{\lambda\nu}_\sigma(x;X) k_{\lambda;\nu}(X) \,, \tag{2.1.7}$$

where A^λ_σ and $B^{\lambda\nu}_\sigma$ are functions that depend on the metric and on X, but not on the initial values $k_\lambda(X)$ and $k_{\lambda;\nu}(X)$, and are thus the same for all Killing vectors.
A set of Killing vectors $k^n_\sigma(x)$ is said to be independent if they do not satisfy any linear relations of the form

$$\sum_n c_n k^n_\sigma(x) = 0 \,, \tag{2.1.8}$$

with constant coefficients c_n. Equation (2.1.7) teaches us that there can be at most $D(D+1)/2$ independent Killing vectors in D dimensions. A metric is said to be maximally symmetric if it admits the maximum number of $D(D+1)/2$ independent Killing vectors.
Very important in cosmology and supergravity are the concepts of a homogeneous and isotropic space, which we will therefore introduce next. A metric space is called homogeneous if there exist infinitesimal isometries that carry any given point X into any other point in its immediate neighborhood. This means the metric must admit Killing vectors that at any given point take all possible values. A metric is said to be isotropic about a given point X if there exist infinitesimal isometries that leave the point X fixed, $k^\lambda(X) = 0$, and for which the first derivatives $k_{\lambda;\nu}(X)$ take all possible values. One can easily prove the following facts (see, e.g., ref. [2]):

- Any space that is isotropic about every point is also homogeneous.
- A homogeneous space that is isotropic about some point is maximally symmetric. It then follows from the previous point that also any space that is isotropic about every point is maximally symmetric.
- The converse is also true. A maximally symmetric space is homogeneous and isotropic about all points.
- Maximally symmetric spaces are uniquely specified by a constant K, and by the number of positive and negative eigenvalues of the metric. Their Riemann curvature tensor is given by

$$R_{\lambda\rho\sigma\nu} = K(g_{\nu\rho} g_{\lambda\sigma} - g_{\sigma\rho} g_{\lambda\nu}) \,. \tag{2.1.9}$$

In differential geometry these spaces are called spaces of constant curvature. There are 3 cases to distinguish, namely $K = 0$, $K > 0$ and $K < 0$. In the Riemannian case this will correspond to a flat space, a sphere or a hyperbolic Gauss-Bólyai-Lobachevski space respectively, while in the Lorentzian case this leads to Minkowski, de Sitter or Anti-de Sitter space, which will be discussed extensively in the next sections.

2.2 Geometry and causal structure of Minkowski space

To get a better idea of what a space-time looks like, it is important to study its causal structure. This will also play an important role later, in the context of holography. In order to understand the causal structure of a spacetime, we also have to know what happens at 'infinity' (see [3, 4]). To study this, we can do a conformal compactification

$$ds^2 \to d\tilde{s}^2 = \Lambda^2(\vec{r}, t) ds^2 \,, \quad \Lambda \neq 0 \,, \tag{2.2.1}$$

which will not change the causal structure of the spacetime. We want to choose Λ in such a way that points at infinity in the original metric are at finite affine parameter in the new metric. The points at infinity are not part of the original spacetime, but they are contained in the new space-time, which is a conformal compactification of the original space-time.
Take now Minkowski space(-time)* in $D(> 2)$ dimensions, which is just $\mathbb{R}^D$ with metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2_{D-2} \,, \tag{2.2.2}$$

where $d\Omega^2_{D-2}$ is the volume element of a unit $(D-2)$-sphere S^{D-2}. Doing the following coordinate transformation

$$\left\{ \begin{array}{rcll} t - r & = & \tan \tilde{U} & -\pi/2 < \tilde{U} < \pi/2 \\ t + r & = & \tan \tilde{V} & -\pi/2 < \tilde{V} < \pi/2 \end{array} \right\} \begin{array}{l} \text{with } \tilde{V} \geq \tilde{U} \\ \text{since } r \geq 0 \,, \end{array} \tag{2.2.3}$$

the metric becomes

$$ds^2 = \left(2 \cos \tilde{U} \cos \tilde{V} \right)^{-2} \left[-4 d\tilde{U}\, d\tilde{V} + \sin^2 \left(\tilde{V} - \tilde{U} \right) d\Omega^2_{D-2} \right] . \tag{2.2.4}$$

The metric of the conformal compactification of the spacetime then can be chosen to be

$$d\tilde{s}^2 = -4 d\tilde{U} d\tilde{V} + \sin^2 \left(\tilde{V} - \tilde{U} \right) d\Omega^2_{D-2} \,. \tag{2.2.5}$$

*We will often just talk about space, when we really mean space-time, but it should always be clear from the context what we mean.

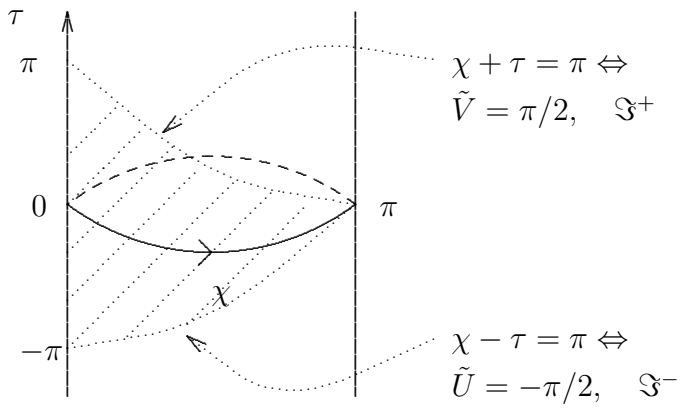


Figure 2.1: Einstein static universe and Minkowski space (from [4])

The added points at infinity are

$$\left.\begin{array}{rcl}\tilde{U} &=& -\pi/2\\ \tilde{V} &=& \pi/2\end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{c} r \to \infty \\ t \text{ finite}\end{array}\right\} \text{spatial } \infty,\, i_0\,,$$

$$\left.\begin{array}{rcl}\tilde{U} &=& \pm\pi/2\\ \tilde{V} &=& \pm\pi/2\end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{c} t \to \pm\infty \\ r \text{ finite}\end{array}\right\} \begin{array}{l}\text{past and future}\\ \text{temporal } \infty,\, i_\pm\,,\end{array}$$

$$\left.\begin{array}{rcl}\tilde{U} &=& -\pi/2\\ |\tilde{V}| &\neq& \pi/2\end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{c} r \to \infty \\ t \to -\infty \\ r + t \text{ finite}\end{array}\right\} \begin{array}{l}\text{past null } \infty\\ \Im^-\,,\end{array}$$

$$\left.\begin{array}{rcl}|\tilde{U}| &\neq& \pi/2\\ \tilde{V} &=& \pi/2\end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{c} r \to \infty \\ t \to \infty \\ r - t \text{ finite}\end{array}\right\} \begin{array}{l}\text{future null } \infty\\ \Im^+\,,\end{array}$$

Introducing the new time and space coordinates τ, χ by

$$\tau = \tilde{V} + \tilde{U}, \quad \chi = \tilde{V} - \tilde{U}\,, \tag{2.2.6}$$

we have

$$d\tilde{s}^2 = -d\tau^2 + d\chi^2 + \sin^2\chi d\Omega^2_{D-2}\,, \tag{2.2.7}$$

where χ is an angular variable which must be restricted to the interval $[0, \pi[$. If no other restriction is placed on the ranges of τ and χ, then in the case $D = 4$ this metric $d\tilde{s}^2$ is that of the Einstein static universe, of topology $\mathbb{R}$ (time) $\times$ S^3 (space).
If we represent each 2-sphere of constant χ as a point, the Einstein static universe can be drawn as half a cylinder, but compactified Minkowski space is conformal to the triangular region drawn in figure 2.1. If we roll out the cylinder we get the Carter-Penrose diagram of Minkowski space, drawn in figure 2.2.

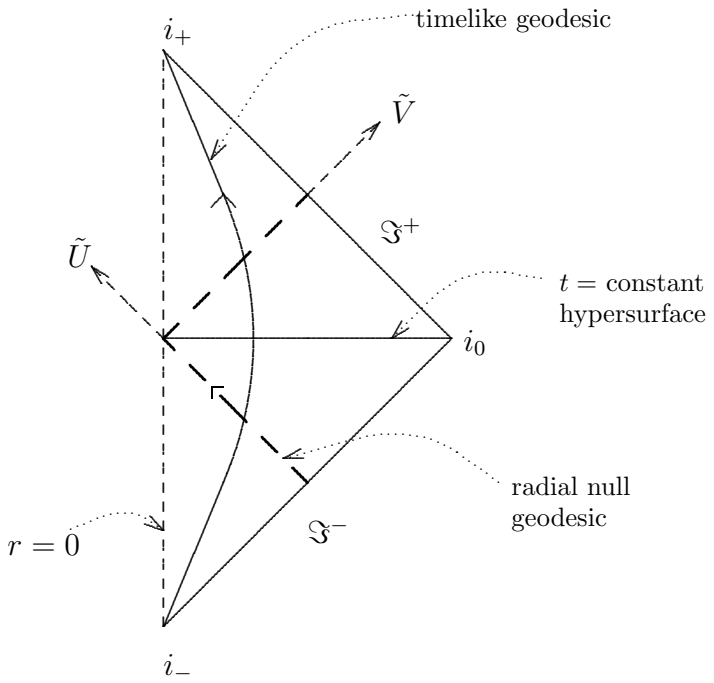


Figure 2.2: Carter-Penrose diagram of Minkowski space (from [4]). Each point represents a 2-sphere, except points on $r = 0$ and $i_0, i_\pm$. Light rays travel at 45^0 from $\Im^-$ through $r = 0$ and then out to $\Im^+$. [$\Im^\pm$ are null hypersurfaces].

2.3 Geometry and causal structure of de Sitter space

D-dimensional de Sitter space (dS_D) can be described as the hypersurface

$$\eta_{AB} X^A X^B = l^2 \,, \quad \eta_{AB} = diag(-1, 1, 1, ..., 1) \,, \quad A, B = 0, ..., D \,, \tag{2.3.1}$$

in flat $D+1$-dimensional Minkowski space with metric

$$ds^2 = \eta_{AB} dX^A dX^B \,, \tag{2.3.2}$$

where l is a length parameter called the de Sitter radius. The de Sitter metric is the induced metric. As mentioned before de Sitter space is a maximally symmetric space, which implies that it obeys the vacuum (= no matter) Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \,, \tag{2.3.3}$$

with a positive cosmological constant, given by

$$\Lambda = \frac{(D-2)(D-1)}{2l^2} \,. \tag{2.3.4}$$

Equations (2.3.1) and (2.3.2) also make it clear that the isometry group of dS_D is $O(D,1)$, which indeed has dimension $(D+1)D/2$, as required for a maximally symmetric D-dimensional space.
We will now introduce a number of coordinate systems on dS_D, which will reveal different characteristics of dS_D(see [3, 5, 6]). To avoid some subtleties when $D \leq 2$, we will always assume $D > 2$ from here on.

Global coordinates

Putting $l = 1$ from here on, we can solve equation (2.3.1) with

$$\begin{aligned} X^0 &= \sinh\tau\,, \\ \sum_{i=1}^{D}(X^i)^2 &= \cosh^2\tau\,. \end{aligned} \tag{2.3.5}$$

Substituting this into (2.3.2), the metric of dS_D becomes

$$ds^2 = -d\tau^2 + (\cosh^2\tau)d\Omega_{D-1}^2\,. \tag{2.3.6}$$

These coordinates cover the whole manifold and are therefore called global coordinates. In these coordinates dS_D looks like $(D-1)$-sphere, which starts out infinitely large at $\tau = -\infty$, then contracts to a minimal size at $\tau = 0$ and then re-expands to infinite size as $\tau \to \infty$.

Conformal coordinates

One of the strange features of dS_D is that a single observer cannot access the whole space-time. Intuitively this is because of the exponential expansion that occurs in the future: space expands so quickly, that even light cannot keep up. To make the causal structure of dS_D clear, we can again make a conformal compactification of the space, like we did for Minkowski space in the previous section. If we set

$$\cosh\tau = \frac{1}{\cos T}\,, \tag{2.3.7}$$

we get

$$ds^2 = \frac{1}{\cos^2 T}(-dT^2 + d\Omega_{D-1}^2)\,, \tag{2.3.8}$$

where $-\pi/2 < T < \pi/2$. We then can choose the following metric for the conformal compactification,

$$\begin{aligned} d\tilde{s}^2 &= (\cos^2 T)ds^2 = -dT^2 + d\Omega_{D-1}^2\,, \\ &= -dT^2 + d\chi^2 + \sin^2\chi\, d\Omega_{D-2}^2\,. \end{aligned} \tag{2.3.9}$$

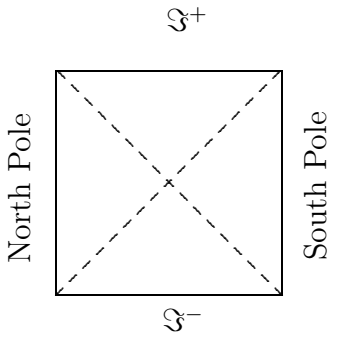


Figure 2.3: Carter-Penrose diagram of dS_D (from [5]). The North Pole and South Pole are timelike lines, while every point in the interior represents an S^{d-2}. A horizontal slice is an S^{d-1}. The dashed lines are the past and future horizons of an observer at the South Pole. The conformal time coordinate T runs from $-\pi/2$ at $\Im^-$ to $+\pi/2$ at $\Im^+$.

Just like for Minkowski space, we have conformally mapped de Sitter space to part of the Einstein static universe. The added points at infinity are

$$T = -\frac{\pi}{2} \quad \Leftrightarrow \quad \tau \to -\infty \quad \text{past temporal and null infinity, } \Im^-$$
$$T = \frac{\pi}{2} \quad \Leftrightarrow \quad \tau \to \infty \quad \text{future temporal and null infinity, } \Im^+$$

This leads to the Carter-Penrose diagram shown in figure 2.3.
In contrast to Minkowski space, where $\Im^-$ and $\Im^+$ are null hypersurfaces, in de Sitter space $\Im^-$ and $\Im^+$ are spacelike. This is related to the existence of a past and future event horizon for a geodesic (timelike) observer in de Sitter space. Take an observer, sitting at the South Pole of the sphere, at $\chi = \pi$. From the form of the metric (2.3.9), we see that the propagation of any signal is bounded by $|d\chi/dT| \leq 1$. This means that a signal (e.g a light ray) sent at $\chi < \pi/2 + T$ at time T will never be able to reach the observer at the South Pole, since spacetime 'ends' at $T = \pi/2$ and $\chi < \pi$ at this time. A geodesic observer thus has a past event horizon and will only be able to see signals that originated inside the region marked as $\mathcal{O}^-$ in figure 2.4, called the causal past. Similarly, if an observer at the South Pole ($\chi = \pi$) sends a signal, say at $T = -\pi/2$, it will never be able to reach that part of spacetime for which $\chi < \pi/2 - T$ at time T. The region $\chi \geq \pi/2 - T$ is called the causal future and is marked as $\mathcal{O}^+$ in figure 2.4 and $\chi = \pi/2 - T$ is the future event horizon of a geodesic observer sitting at the South Pole. The intersection $\mathcal{O}^+ \cap \mathcal{O}^-$ is called the (southern) causal diamond, and only this region of spacetime is fully accessible to an observer sitting at the South Pole. Finally note that for given χ and T the $(D-2)$-sphere in de Sitter space has metric $(\sin\chi/\cos T)^2 d\Omega_{D-2}$, so its area will be proportional to $(\sin\chi/\cos T)^{D-2}$. This shows that the horizons ($\chi = \pi/2 \pm T$) have a time-independent horizon area.

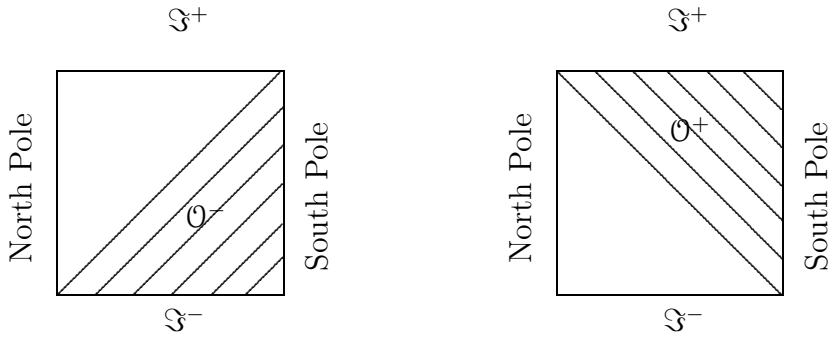


Figure 2.4: These diagrams show the regions $\mathcal{O}^-$ and $\mathcal{O}^+$ corresponding respectively to the causal past and future of an observer at the South Pole (from [5]).

Planar coordinates

Take

$$\begin{aligned} X^0 &= \sinh\bar{t} - \frac{1}{2}x_i x^i e^{-\bar{t}}\,, \\ X^i &= x^i e^{-\bar{t}}\,, \qquad i = 1,\ldots,D-1, \\ X^D &= \cosh\bar{t} - \frac{1}{2}x_i x^i e^{-\bar{t}}\,, \end{aligned} \tag{2.3.10}$$

the metric then becomes

$$ds^2 = -d\bar{t}^2 + e^{-2\bar{t}}dx_i dx^i. \tag{2.3.11}$$

These coordinates are useful because they only cover the region $\mathcal{O}^-$ and are therefore appropriate for an observer on the South Pole. The surfaces of constant $\bar{t}$ are spatial sections of dS_D which are infinite volume $(D-1)$-planes with the flat metric. In these coordinates the only manifest symmetries are translations and rotations of the x^i, the vector $\partial/\partial\bar{t}$ is not a Killing vector.

Static coordinates

If we set

$$\begin{aligned} X^0 &= \sqrt{1-r^2}\sinh t\,, \\ &\sum_{i=1}^{D-1}(X^i)^2 = r^2\,, \\ X^D &= \sqrt{1-r^2}\cosh t\,, \end{aligned}$$

the metric takes the form

$$ds^2 = -(1-r^2)dt^2 + \frac{dr^2}{1-r^2} + r^2 d\Omega^2_{D-2}\,. \tag{2.3.12}$$

The value of this coordinate system is that it has an explicit timelike Killing vector, namely $\partial/\partial t$. It covers the southern causal diamond ($0 \leq r < 1$) with South Pole at $r = 0$ and horizons at $r = 1$. From (2.3.12) we see that $\partial/\partial t$ becomes null at the horizons. Figure 2.5 shows what happens if we extend the Killing vector field $\partial/\partial t$ to the various other diamonds of dS_D. It becomes spacelike in the top and bottom diamond, and points towards the past in the Northern diamond. There is no global timelike Killing vector in de Sitter space! We can therefore use $\partial/\partial t$ only to define a sensible time evolution in the southern diamond of dS_D.

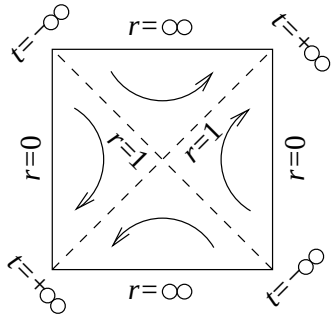


Figure 2.5: This diagram shows the direction of the flow generated by the Killing vector $\partial/\partial t$ in static coordinates. The horizons at $r = 1$ are represented by dotted lines, and the southern causal diamond is the region with $0 \leq r \leq 1$ on the right hand side. Past and future null infinity $\Im^{\pm}$ are at $r = \infty$ (from [5]).

Hyperbolic coordinates

Just like global coordinates foliate dS_D with spheres and planar coordinates foliate dS_D with planes, we can also foliate dS_D with spaces of constant negative curvature by using hyperbolic coordinates. They are defined by

$$\begin{aligned} X^0 &= \sinh\bar{\tau}\cosh\psi\,, \\ \sum_{i=1}^{D-1}(X^i)^2 &= \sinh^2\bar{\tau}\sinh^2\psi\,, \\ X^D &= \cosh\bar{\tau}\,. \end{aligned} \tag{2.3.13}$$

The metric is then given by

$$ds^2 = -d\bar{\tau}^2 + \sinh^2\bar{\tau}(d\psi^2 + \sinh^2\psi d\Omega^2_{D-2})\,. \tag{2.3.14}$$

These different ways of foliating de Sitter space with spaces of constant curvature will be important later when we discuss the relation between 4-dimensional de Sitter space and Robertson-Walker cosmologies.

2.4 Geometry and causal structure of Anti-de Sitter space

D-dimensional Anti-de Sitter space (AdS_D) can be described as the hypersurface

$$\eta_{AB}X^A X^B = -l^2\,, \quad \eta_{AB} = diag(-1,1,1,...,-1)\,, \quad A,B = 0,...,D\,, \tag{2.4.1}$$

in flat $D+1$-dimensional space with metric

$$ds^2 = \eta_{AB}dX^A dX^B\,, \tag{2.4.2}$$

where l is a length parameter called the Anti-de Sitter radius. The Anti-de Sitter metric is again the induced metric. Just like Minkowski space and de Sitter space this is a maximally symmetric Lorentzian space. It obeys the vacuum Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,, \tag{2.4.3}$$

with a negative cosmological constant, given by

$$\Lambda = -\frac{(D-2)(D-1)}{2l^2}\,, \tag{2.4.4}$$

and its isometry group is $SO(2,D-1)$. As in the previous sections, we will study the space by introducing a few coordinate systems that should help to reveal some of the characteristics of AdS_D (based on [3, 7]) and we always assume $D > 2$.

Global coordinates

Putting $l = 1$ from here on, we can solve equation (2.4.1) with

$$\begin{aligned} X^0 &= \cosh\rho\cos\tau\,,\\ \sum_{i=1}^{D-1}(X^i)^2 &= \sinh^2\rho\,,\\ X^D &= \cosh\rho\sin\tau\,. \end{aligned} \tag{2.4.5}$$

Equation (2.4.2) then leads to the following expression for the AdS_D metric,

$$ds^2 = -\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega^2_{D-2}\,. \tag{2.4.6}$$

If we take $\rho \geq 0$ and $0 \leq \tau < 2\pi$, these coordinates cover the whole manifold once. However, since the metric behaves as $ds^2 \simeq -d\tau^2 + d\rho^2 + \rho^2 d\Omega^2_{D-2}$ near $\rho = 0$, AdS_D has the topology of $S^1 \times \mathbb{R}^{D-1}$, with the S^1 representing closed timelike curves in the τ direction. Luckily the space is not simply connected, so that we can get rid of these closed timelike curves by simply unwrapping the S^1 (taking $-\infty < \tau < \infty$ without identifications) to get the universal covering space. It is usually this universal covering space that is meant when people talk about AdS_D and from now on when we refer to AdS_D we will also mean the universal covering space. Remark that AdS_D has a global timelike Killing vector, namely $\partial/\partial\tau$, in contrast to dS_D. Since the metric (2.4.6) has the typical form of a static spacetime, we could also have called these coordinates static.

Conformal coordinates

To study the causal structure of AdS_D we set

$$\sinh\rho = \tan\chi\,, \tag{2.4.7}$$

with $0 \leq \chi < \pi/2$. The metric (2.4.6) then takes the form

$$ds^2 = \frac{1}{\cos^2\chi}(-d\tau^2 + d\chi^2 + \sin^2\chi d\Omega_{D-2}^2)\,. \tag{2.4.8}$$

We then can do a conformal compactification by multiplying the metric with $\cos^2\chi$,

$$d\tilde{s}^2 = \cos^2\chi ds^2 = -d\tau^2 + d\chi^2 + \sin^2\chi d\Omega_{D-2}^2\,. \tag{2.4.9}$$

This shows that AdS_D can be conformally mapped into one half of (the higher dimensional generalization of) the Einstein static universe (see (2.2.7)), since χ only runs from 0 to $\pi/2$ (instead of π). The added points at infinity are

$$\chi = \frac{\pi}{2} \Leftrightarrow \rho \to \infty \quad \text{spatial and null infinity, } \Im\,.$$

Remark that here $\Im$ is a timelike surface, in contrast to dS_D where $\Im^-$ en $\Im^+$ are spacelike. There is also a past and future temporal infinity at $\tau \to \pm\infty$. Since any conformal transformation that takes temporal infinity to a finite affine parameter also scales the space sections by an infinite amount, temporal infinity is represented by two disjoint points i_- en i_+. Finally one can show that all the timelike geodesics from any point p in space, reconverge to an image point q, and then diverge from this image point to refocus to another image point, and so on [3]. This is summarized in the Carter-Penrose diagram shown in figure 2.6.

Poincaré coordinates

This coordinate system is often used in the AdS/CFT correspondence. It is defined by

$$\begin{aligned} X^0 &= \frac{1}{2u}(1 + u^2(1 + \vec{x}^2 - t^2))\,, \\ X^i &= ux^i\,, \quad i = 1, ..., D-2\,, \\ X^{D-1} &= \frac{1}{2u}(1 - u^2(1 - \vec{x}^2 + t^2))\,, \\ X^D &= ut\,. \end{aligned} \tag{2.4.10}$$

These coordinates do not cover the whole of AdS_D, but only a triangular region. The AdS_D metric in Poincaré coordinates is given by

$$ds^2 = \frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2))\,. \tag{2.4.11}$$

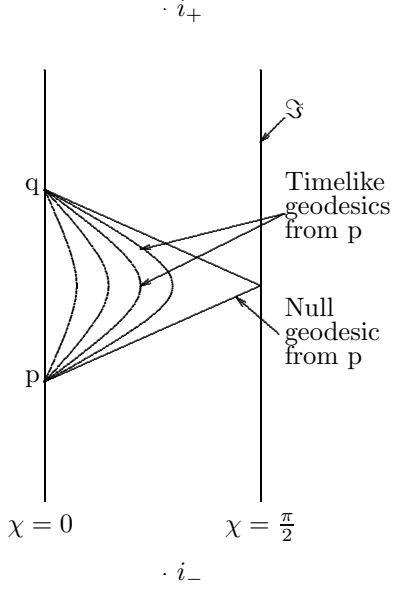


Figure 2.6: Carter-Penrose diagram of AdS_D (modified version of figure 20 in [3]).

This form makes the subgroups $ISO(1, D-2)$ and $SO(1,1)$ of the $SO(2, D-2)$ isometry group manifest, where $ISO(1, D-2)$ is the group of Poincaré transformations on $(t, \vec{x})$ and $SO(1,1)$ is

$$(t, \vec{x}, u) \to (ct, c\vec{x}, c^{-1}u), \quad c > 0\,. \tag{2.4.12}$$

In the AdS/CFT correspondence, this transformation is identified with the dilatation symmetry in the conformal symmetry group of $\mathbb{R}^{(1,D-2)}$[7].
Finally remark that the timelike Killing vector $\partial/\partial t$ becomes null at $u = 0$ (Killing horizon), unlike the timelike Killing vector $\partial/\partial\tau$ in global coordinates, which is everywhere nonzero.

Hyperbolic coordinates

Analogous to the case of dS_D, the hyperbolic coordinates are defined by

$$\begin{aligned} X^0 &= \sin\bar{\tau}\cosh\psi\,, \\ \sum_{i=1}^{D-1}(X^i)^2 &= \sin^2\bar{\tau}\sinh^2\psi\,, \\ X^D &= \cos\bar{\tau}\,, \end{aligned} \tag{2.4.13}$$

and the metric is given by

$$ds^2 = -d\bar{\tau}^2 + \sin^2\bar{\tau}(d\psi^2 + \sinh^2\psi d\Omega_{D-2}^2)\,. \tag{2.4.14}$$

The merit of this coordinate system is that now the metric has the typical form of the Robertson-Walker metrics used in cosmology. Again, only part of AdS_D is covered by this coordinate system.

2.5 Cosmology

In this section we will review some of the basic facts of standard cosmology. Our presentation is mostly based on [2], [8] and [9]. For the section on the concordance model, we also used [10], while for the section on inflation we found [11] and [12] very useful. It should be clear that this is not meant to be a complete review in any way. We just want to highlight those facts that give some motivation for the work in later chapters and that should help to put those chapters in a broader perspective.

2.5.1 The Robertson-Walker metric

One of the main assumptions in cosmology is that our universe is spatially homogeneous (='looks the same at every point') and isotropic (='looks the same in every direction') at very large scales . Direct evidence that the universe is isotropic around us comes from the smoothness of the temperature of the cosmic microwave background. The spatial homogeneity of the universe is more difficult to test, but some evidence comes from number counts of galaxies[8]. However, we can also assume what is usually called the Copernican principle, namely that we do not live in a special place in the universe. From the fact that the universe appears isotropic around us, we can then conclude that the universe should be isotropic around every point. We then know from section 2.1 that this implies that (the spatial part of) our universe is also homogeneous and even maximally symmetric. The most general space-time metric consistent with these assumptions is the Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] . \tag{2.5.1}$$

Here k determines the curvature of the spatial sections (= slices at constant cosmic time t). $k = +1$ corresponds to locally spherical spatial sections, $k = 0$ to locally flat and $k = -1$ to locally hyperbolic. By making a simple change of coordinates in the radial direction we can also write the Robertson-Walker metric as

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + f^2(\chi)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] , \tag{2.5.2}$$

where

$$f(\chi) = \begin{cases} \sin(\chi), & 0 \le \chi < \pi & \text{if } k = +1 \\ \chi, & 0 \le \chi < \infty & \text{if } k = 0 \\ \sinh(\chi), & 0 \le \chi < \infty & \text{if } k = -1 \end{cases} . \tag{2.5.3}$$

The scale factor $a(t)$ in the metric has an interesting consequence. If a photon with wavelength λ_e is emitted at time t_e, then today, at time t_0 it is observed with wavelength

λ_0, obeying

$$\frac{\lambda_0}{\lambda_e} = \frac{a_0}{a_e} \equiv 1 + z \,, \tag{2.5.4}$$

where $a_e \equiv a(t_e)$ and $a_0 \equiv a(t_0)$. If $z > 0$, photons get redshifted and if $z < 0$, photons get blueshifted. Observations show clearly that they are redshifted more and more with distance, implying that our universe is expanding, $\dot{a} \equiv da/dt > 0$. We can derive the famous Hubble law by looking at nearby sources, $z \ll 1$. This corresponds to small $(t_0 - t_e)$, implying we can expand $a(t_e)$ as

$$a(t_e) = a_0 - \dot{a}(t_0)(t_0 - t_e) \,. \tag{2.5.5}$$

To leading order in z, the difference between the present time and the emission time is equal to the distance to the source r (in units were the speed of light is equal to 1). Defining the Hubble parameter,

$$H(t) = \frac{\dot{a}(t)}{a(t)} \,, \tag{2.5.6}$$

and denoting its present value by H_0, we can write (2.5.5) as

$$a(t_e) = a_0(1 - H_0 r) \,. \tag{2.5.7}$$

To leading nontrivial order in z this leads to

$$1 + z = \frac{1}{1 - H_0 r} \approx 1 + H_0 r \,. \tag{2.5.8}$$

This finally gives the Hubble law

$$z = H_0 r \,. \tag{2.5.9}$$

This linear relation between z and r for small z is confirmed to high accuracy by modern observations. For larger redshift this linear relation between the redshift z and the distance r is no longer valid. Studying the relation between z and r for larger z is very useful as the relation can tell us about the expansion rate of the universe. The present data for large z come from type Ia supernova observations and they have led to a surprising conclusion. The universe seems to undergo a phase of accelerated expansion today(!) (and at relatively small z, i.e at late times), while at higher redshift the expansion was decelerating (as one would expect). We will come back to these remarkable claims later.

2.5.2 The Friedmann equations

Not counting the curvature parameter k, there is only one free parameter left in the Robertson-Walker metric, namely the scale factor $a(t)$. This factor will be determined by the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.5.10}$$

and the matter in the universe which can be specified by choosing a specific energy-momentum tensor $T_{\mu\nu}$. For simplicity and because it is consistent with the assumptions of a homogeneous and isotropic universe, one takes for $T_{\mu\nu}$ the perfect fluid form

$$T^{\mu}_{\nu} = diag(-\rho, p, p, p) \tag{2.5.11}$$

where ρ is the energy density and p the pressure of the fluid (in the rest frame of the fluid). Putting the Robertson-Walker metric (2.5.2) in the Einstein equations (2.5.10) with matter of the form (2.5.11) leads to two equations. The first is known as the Friedmann equation and is given by

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{k}{a^2}\,, \tag{2.5.12}$$

where the i-index denotes all different types of energy/matter in the universe, some of which will be discussed shortly. The second equation is usually called the acceleration equation and has the form

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i (\rho_i + 3p_i)\ . \tag{2.5.13}$$

It is common to use the Friedmann equation to define, at any given time, a critical energy density

$$\rho_c \equiv \frac{3H^2}{8\pi G}\,, \tag{2.5.14}$$

for which the spatial sections must be precisely flat ($k = 0$). Defining density parameters by

$$\Omega_i \equiv \frac{\rho_i}{\rho_c}\,, \qquad \Omega_{\rm total} \equiv \frac{\rho}{\rho_c}\,, \qquad \rho = \sum_i \rho_i\,, \tag{2.5.15}$$

we can then relate the total energy density in the universe to its local geometry via

$$\begin{aligned} \Omega_{\rm total} > 1 \quad &\Leftrightarrow \quad k = +1\,, \\ \Omega_{\rm total} = 1 \quad &\Leftrightarrow \quad k = 0\,, \\ \Omega_{\rm total} < 1 \quad &\Leftrightarrow \quad k = -1\,. \end{aligned} \tag{2.5.16}$$

For the Friedmann equation and the acceleration equation to be mutually consistent we also need the energy conservation relation

$$\dot{\rho} + 3H(\rho + p) = 0\,, \tag{2.5.17}$$

which is nothing but the vanishing of the covariant divergence of the energy-momentum tensor,

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{2.5.18}$$

To solve equations (2.5.12), (2.5.13) and (2.5.17), which we will collectively refer to as 'the Friedmann equations', we have to specify the matter content a little more. We will restrict ourselves to perfect fluids that obey an equation of state

$$p = w\rho\,. \tag{2.5.19}$$

with w a constant. Fortunately a lot of useful matter sources do obey this equation. For instance, pressureless matter or dust (e.g massive nonrelativistic particles, stars, galaxies,...) corresponds to $w = 0$ and radiation (e.g photons or ultrarelativistic particles) has $w = 1/3$. We can also introduce a cosmological constant Λ in the Einstein equations by including an energy-momentum tensor of the form

$$T_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu} \,. \tag{2.5.20}$$

This corresponds to a perfect fluid with $p = -\rho = -\Lambda/(8\pi G)$ and thus $w = -1$. Using (2.5.17) we find for matter obeying the equation of state (2.5.19) that

$$\rho = \frac{const}{a^{3(1+w)}} \,, \tag{2.5.21}$$

For $w = 0$, ρ scales like $1/a^3$, just as the inverse of the physical volume, as one would intuitively expect. For radiation, ρ goes like $1/a^4$. The extra factor $1/a$ can be understood as due to the cosmological redshift. Finally observe that the energy density ρ of a cosmological constant is constant, and will thus dominate over matter and radiation at large times in an expanding universe. Matter with $w < -1$ is usually thought of as unphysical, we have certainly seen nothing like it in the laboratory. It has an energy density that even grows in an expanding universe, and usually leads to instabilities of the theory. Nonetheless, observations have not yet excluded the possibility that a form of matter like this does exist in the universe, and there are some papers were this possibility is investigated. However, we will restrict ourselves to $w \geq -1$ from here on.

Both dust and radiation, and all conventional matter sources we know, have positive energy density ρ and nonnegative pressure p. If we assume that our universe only contains conventional matter sources, we see immediately from the acceleration equation (2.5.13) that this implies an expanding universe would be decelerating. This is what one would intuitively expect, the matter in the universe slows the expansion down. One can show that depending on the curvature k, the universe will either be expanding forever with $\dot{a}$ approaching some limit value which is nonzero for $k = -1$ and zero for $k = 0$, or it will start to contract at some point if $k = +1$. However, as mentioned in the previous section, recent observations strongly suggest that the universe is accelerating today. This means there must be some unconventional stuff around in the universe and the simplest explanation, which is also in complete agreement with several other observations, is that our universe has a (small positive) cosmological constant.

At late times a cosmological constant will dominate over conventional matter sources in an expanding universe, so it is useful to study how the universe would look like if there was just a cosmological constant and no matter at all. Suppose first that $\Lambda < 0$. Ω is then negative, and the Friedmann equation (2.5.12) then shows that $k = -1$. The solution of the Friedmann equations is then given by

$$a(t) = \sqrt{\frac{-3}{\Lambda}} \sin\left(\sqrt{\frac{-\Lambda}{3}} t\right) \,. \tag{2.5.22}$$

This is of course nothing but AdS_4 in hyperbolic coordinates (see section 2.4). In case $\Lambda > 0$, we have three possibilities. If $k = -1$, the Friedmann equations are solved by

$$a(t) = \sqrt{\frac{3}{\Lambda}} \sinh\left(\sqrt{\frac{\Lambda}{3}} t\right) , \tag{2.5.23}$$

which is dS_4 in hyperbolic coordinates (see section 2.3). In case the universe is flat ($k = 0$) and vacuum dominated, the solution has the form

$$a(t) \propto \exp\left(\pm\sqrt{\frac{\Lambda}{3}} t\right) . \tag{2.5.24}$$

This is also dS_4, but this time in planar coordinates. Finally, we can also have $k = +1$, leading to

$$a(t) = \sqrt{\frac{3}{\Lambda}} \cosh\left(\sqrt{\frac{\Lambda}{3}} t\right) , \tag{2.5.25}$$

which is again dS_4, now in global coordinates.

2.5.3 The concordance model: Λ-CDM

In this section we will present the current standard cosmological model of our universe, also known as the concordance model or the Λ-CDM model*. We will give a concise summary of the current estimates for the Hubble parameter H_0, the local geometry k and the matter/energy content of the present universe.
The Hubble constant is usually presented in terms of a dimensionless quantity h, defined as

$$H_0 = 100\, h \frac{\text{km}}{\text{sec Mpc}} . \tag{2.5.26}$$

One of the most reliable results on the Hubble constants comes from the Hubble Space Telescope Key project[13]. They find

$$h = 0.71 \pm 0.06 , \tag{2.5.27}$$

which is in agreement with several other observations [14].
A lot of information about the universe is gained by the WMAP satellite, which measures the cosmic microwave background (CMB). The CMB is radiation, which has an excellent blackbody spectrum of temperature

$$T_0 = 2.725 \pm 0.001 K . \tag{2.5.28}$$

It is (presumably) a relic of the early universe, when the density and energy densities were high enough so that matter and radiation were in thermal equilibrium in each point of

*Here CDM stands for cold dark matter and Λ for the presence of a cosmological constant (or something similar called dark energy) in the model.

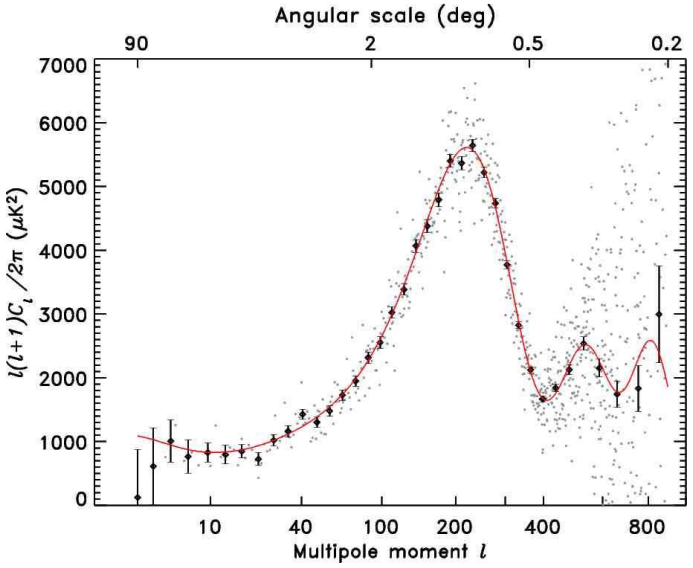


Figure 2.7: The CMB power spectrum from the CMB satellite[15]. The solid line represents the best-fit cosmological model, the gray dots are the unbinned data [16].

space (to a very good approximation). Since blackbody radiation is emitted by bodies in thermal equilibrium, and the universe is not in thermal equilibrium today, the blackbody spectrum of the CMB is convincing evidence for the Big Bang theory. The CMB also shows that the early universe was approximately isotropic and homogeneous. There are however small relative angular anisotropies of the CMB temperature, $\Delta T/T_0$, of order 10^{-4} to 10^{-5} and studying these gives a wealth of information on the universe. They are usually analyzed by decomposing the temperature, as function of the direction, in spherical harmonics Y_{lm} via

$$\frac{\Delta T}{T_0} = \sum_{l,m} a_{lm} Y_{lm}(\theta, \phi)\,, \tag{2.5.29}$$

where θ and ϕ are spherical polar angles on the sky. Defining*

$$C_l = < |a_{lm}|^2 >\,, \tag{2.5.30}$$

and plotting l against $l(l+1)C_l$ gives a plot of what is usually called the CMB power spectrum. The famous results from WMAP are shown in figure 2.7. The position of the peaks in the CMB power spectrum are sensitive to the spatial curvature k and from the position of the first peak one can conclude that the universe is spatially flat to a high precision. Assuming a reasonable value for h, e.g $h = 0.7$, the estimate for the total density of the universe is [10]

$$\Omega_{\text{total}} = 1.02 \pm 0.02\,. \tag{2.5.31}$$

*For fluctuations in an isotropic universe, the right hand side of eq. (2.5.30) is independent of m.

From the relative height of the first and the second peak one can estimate the density of baryons (=strongly interacting particles like protons and neutrons) in the universe, leading to [8]

$$\Omega_B = 0.047 \pm 0.006\,, \tag{2.5.32}$$

where $k = 0$ was assumed to tighten the bounds. This value is in good agreement with the theory of Big Bang nucleosynthesis and observations of primordial abundances of light elements (like helium, deuterium , lithium,...), which is quite remarkable, given that this is a completely different way of constraining Ω_B. Note that most of the baryons in our universe are dark, since direct measurements of the mass density of stars give an estimate [9]

$$\Omega_{\text{stars}} \sim 0.005\,, \tag{2.5.33}$$

which is an order of magnitude smaller than Ω_B. From electric neutrality we can also conclude that the total mass density of electrons will be negligible, since the number density of electrons should be about the same as that of baryons (and there mass is much smaller). A rough estimate of the photon density is given by [9]

$$\Omega_\gamma \approx 6.10^{-4}\,. \tag{2.5.34}$$

The only stable particles we did not talk about yet are neutrino's, and they turn out to be quite interesting. There are 3 different types (flavors) and observations of neutrino oscillations suggest they have a small but nonzero mass. Current particle physics experiments show that the mass of every type of neutrino must be smaller than 2.6 eV [9]. On the other hand the absence of any observable effect of neutrinos on the formation of structure leads to a bound

$$\Omega_{\nu,\text{total}} < 0.01\,, \tag{2.5.35}$$

which in terms of the neutrino masses corresponds to

$$\sum m_{\nu_\alpha} < 0.42\,\text{eV}\,. \tag{2.5.36}$$

Cosmology thus gives a much tighter bound. Since particle physics experiments also tell us that the mass of at least one neutrino must be larger than 0.02 eV, it might even be possible to measure the neutrino masses by cosmological observations in the (near) future! The experimental results summarized above lead to another remarkable fact. Since the known forms of matter lead to a total density of approximately 0.05 and we know from WMAP that $\Omega_{\text{total}} \approx 1$, we have to conclude that most of the matter/energy in the universe is in a form we do not know (yet). As we already mentioned several times, observations from high-redshift type Ia supernova strongly suggest that our universe is accelerating today (see e.g [17]), requiring a form of energy with negative pressure ($w < -1/3$), usually called dark energy. The simplest solution is a positive cosmological constant, and this fits well with the current data, but more complicated forms of dark energy are possible, which however we will not discuss. Evidence for strange (=unknown) forms of

matter comes from the determination of the matter density Ω_M based on the dynamics of galaxies and clusters. They lead to

$$0.1 \leq \Omega_M \leq 0.4\,. \tag{2.5.37}$$

From this we can conclude that most of the matter is not made of the particles we know today. This missing matter is called dark matter. One of the few things we know about dark matter is that it must be cold, meaning that it is nonrelativistic today and must have been that way for a very long time. The most natural candidate is a weakly interacting massive stable (or approximately stable) particle (WIMP). A good candidate might be the lightest supersymmetric particle in supersymmetric extensions of the Standard Model. Crude estimates show that a mass of order 1 TeV might be about the right magnitude, so perhaps we can expect to see some new (supersymmetric) particles in the near future, when the LHC is running.
All these different observations then converge remarkably well to the following extravagant picture of our universe:

$$\begin{aligned} \Omega_B &\approx 0.05\,, \\ \Omega_{DM} &\approx 0.25\,, \\ \Omega_\Lambda &\approx 0.7\,. \end{aligned} \tag{2.5.38}$$

Most of the matter/energy density in the universe is in the form of (cold) dark matter (CDM) and dark energy (Λ) instead of in the form of particles we are familiar with in the laboratory. Although perhaps hard to believe, this is what current observations strongly suggest. Remark that to derive these conclusions it was assumed that Einstein's equations are valid at cosmological distances. One could try to change general relativity at cosmological distances, but the problem is that the theory should reduce to general relativity at distances from a fraction of a millimeter to at least tens of megaparsecs, and it turns out to be very difficult to do that without making the theory highly contrived or sick in some way. Up to date there has been no convincing modification of general relativity that explains the observations. The sole model that explains all the data reasonably well is the Λ-CDM model, which we have presented in this section.

2.5.4 Inflation

Although we will not be concerned with inflation specifically in the rest of this thesis, we will give a small summary of the basic ideas involved, since inflation is today an integrable part of cosmology and since it will show the importance of de Sitter space once again.

The Hot Big Bang theory and its problems

In the Hot Big Bang theory the universe starts expanding from an initial singularity. Right after its birth, the universe was in a very hot and super-dense phase. It was radiation-dominated and matter and radiation were in thermal equilibrium. Eventually,

after roughly 400,000 years, the expanding universe cooled down enough so that matter and radiation decoupled and the universe, which had been very opaque until this point, became transparent for light. By studying the CMB background, which is excellent evidence for the Hot Big Bang scenario, we can look back to this relatively early period in the universe.
Despite its many successes, the Hot Big Bang theory also has several problems, which have to do with initial conditions for the universe. Although we cannot use the classical Einstein equations when curvature and energy densities or of order of Planck values or higher*, it is reasonable to assume that they are valid soon after the Planck epoch. The properties of the universe soon after this epoch is what is usually meant by initial conditions in the Hot Big Bang theory.

- *Homogeneity and isotropy problem*
  Although we do not know the correct quantum theory of gravity and matter, it seems natural to assume that the initial conditions are more or less random: different parts of the universe have different properties after the Planck epoch, spatial curvature and energy density would be close to the Planck values, etc. However, with such random initial conditions the universe would not be so homogeneous and isotropic as it is today. In a decelerating expanding universe† there are parts in the universe that can be observed today, but that were not in causal contact at any previous point in time. Yet they look incredibly similar. For instance the cosmic microwave background on opposite sides of the sky has the same temperature to a high degree, while no causal signal could have influenced them. This is usually called the *horizon problem.*

- *Flatness problem*
  We know from observations that the universe today is almost (or even completely) spatially flat. Looking at the Friedmann equation (2.5.12), we see that the curvature contribution scales like $1/a^2$, while we learn from (2.5.21) that the contributions from matter and radiation scale like $1/a^3$ en $1/a^4$ respectively. This means that the curvature term was even less important in the early universe and thus was extremely small at the beginning. Why is this so? Put it another way, comparing the radius of spatial curvature a with the radius of spacetime curvature H^{-1}, one can ask why initial conditions are such that a is so big.

- *Structure formation problem*
  We have seen in the previous section that the early universe was not exactly homogeneous: there were small density perturbations. These are good, since these

*The Planck energy density is M_P^4 and the Planck mass is given by $M_P = \sqrt{\frac{\hbar c}{G}} \sim 10^{19}$ GeV, where G is Newton's universal gravitational constant.

†We know from observations that the universe was decelerating until relatively recently when the small cosmological constant(or some other form of dark energy) became important enough to start accelerating the universe.

inhomogeneities later gave rise to structures in our universe like galaxies, galactic clusters, etc. The problem of the Hot Big Bang theory is that the origin of these primordial perturbations cannot be explained, they have to be built in as initial conditions.

There are many other problems like the entropy problem, the monopole problem, etc., a discussion of these can be found for instance in chapter 1 of [18].

Inflation as the solution

Inflation is a dynamical mechanism that naturally solves the problems of the Hot Big Bang theory. The basic idea is that after the Planck epoch, but before the Hot Big Bang, the universe was in a vacuum-like state and underwent a period of rapid accelerated expansion. This will quickly expand a small region of space to a huge size, thereby flattening the universe out, diluting any initial inhomogeneities and greatly enlarging the horizon size, so that distant points on the CMB were in causal contact in the early universe. The universe we observe is then just a small part of an inflated universe! This obviously explains why our universe looks so flat, homogeneous and isotropic. What is more remarkable is that inflation also provides a mechanism to generate the primordial density perturbations. They originate from vacuum fluctuations of quantum fields that get enhanced during the inflationary expansion of the universe.
Although the basic idea is simple, there are many specific models of inflation that implement these ideas in different ways. The most popular models introduce a new scalar field, called the inflaton, which drives inflation and automatically provides an exit from the inflationary phase. We will present the most simple of these models, which goes by the name chaotic inflation.
Consider a universe with just one real scalar field ϕ with potential $V(\phi)$. The energy-momentum tensor for ϕ is

$$T_{\mu\nu} = (\nabla_\mu\phi)(\nabla_\nu\phi) - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}(\nabla_\alpha\phi)(\nabla_\beta\phi) + V(\phi)\right] . \tag{2.5.39}$$

If we make the mild assumption that there is a sufficiently large patch in the universe, somewhat larger than the Planck size, were $\phi(x,t)$ is reasonably homogeneous (= the gradient term in the energy density is smaller than the potential term), then we can use the Robertson-Walker metric and take ϕ to be independent of the spatial coordinates [9]. Such a homogeneous real scalar field behaves as a perfect fluid with

$$\begin{aligned} \rho &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ p &= \frac{1}{2}\dot{\phi}^2 - V(\phi) . \end{aligned} \tag{2.5.40}$$

The equation of motion of the scalar field is given by

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{\partial V}{\partial\phi} = 0 . \tag{2.5.41}$$

Note that with respect to the usual equation of motion for a scalar field in Minkowski space there is an extra friction term due to the expansion of the universe. The Friedmann equation has the form

$$H^2 + \frac{k}{a^2} = \frac{8\pi}{3M_P^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) . \tag{2.5.42}$$

Looking at equations (2.5.40) and (2.5.41), we see that the scalar field will stay almost constant and behave effectively as a cosmological constant when the following conditions are fulfilled,

$$\begin{aligned} H\dot{\phi} &\gg \ddot{\phi} , \\ V(\phi) &\gg \dot{\phi}^2 . \end{aligned} \tag{2.5.43}$$

These are called the slow-roll conditions. We see from equation (2.5.42) that if $V(\phi)$ is large enough, H will be large and therefore $V(\phi)$ will change slowly according to equation (2.5.41), so the universe will automatically expand exponentially once it is in the slow-roll regime. Because the scale factor $a(t)$ will grow very rapidly, we will then quickly be in the regime were also $H^2 \gg k/a^2$. Under these conditions the scalar field equation and the Friedmann equation become

$$\begin{aligned} 3H\dot{\phi} &= -\frac{\partial V}{\partial \phi} , \\ H^2 &= \frac{8\pi}{3M_P^2} V(\phi) . \end{aligned} \tag{2.5.44}$$

Using equations (2.5.44), the slow-roll conditions (2.5.43) can be rewritten as follows,

$$V'' \ll \frac{V}{M_P^2} , \tag{2.5.45}$$

$$V' \ll \frac{V}{M_P} , \tag{2.5.46}$$

where the prime denotes a derivative with respect to ϕ. For power-law potentials we have

$$V' \sim \frac{V}{\phi} , \qquad V'' \sim \frac{V}{\phi^2}, \tag{2.5.47}$$

so the slow-roll conditions are satisfied if ϕ is large enough, $\phi \gg M_P$. Although the scalar field has to take Planck scale values, we can usually still do without Planck scale physics. The potentials always come with a dimensionless coupling constant λ, e.g. $V(\phi) = \lambda\phi^4$, and the energy densities will be below the Planck scale if $\lambda \ll 1$. There is direct observational evidence that at least at the end of inflation, the energy scale is well below the Planck scale[9],

$$H < 10^{-4} M_P \quad \text{at the end of inflation} . \tag{2.5.48}$$

This bound comes from the (absence of an) effect of gravitational waves generated at inflation on the CMB anisotropy.

The idea of chaotic inflation is that our universe initially consisted of many domains with chaotically distributed scalar field ϕ. In those domains where ϕ is large enough,$\phi_b \gg M_P$, inflation will start and these domains will grow vary rapidly, effectively behaving like de Sitter space, and will typically be much greater than the size of the observable part of the universe when inflation has ended (in that domain). During inflation the field will slowly roll down the potential, until it reaches the value $\phi_e \sim M_P$. At that point the slow-roll regime has ended, and the field will quickly roll down to $\phi \approx 0$, and then oscillate around its minimum $\phi = 0$. These field oscillations generally lead to particle creation from vacuum, damping the oscillations and filling the universe with particles. This process is called reheating and ends when the oscillations stop and the created particles are in thermal equilibrium. At that point the universe can be described by the Hot Big Bang theory.

To make sure the problems of the Hot Big Bang theory are solved, we have to check that inflation lasts long enough, so that space can become big enough. This is usually done by calculating the number of e-folds, which is defined as

$$N_{\mathrm{e-folds}} \equiv \ln\left(\frac{a(t_{\mathrm{end}})}{a(t_{\mathrm{begin}})}\right) = \int_{t_{\mathrm{begin}}}^{t_{\mathrm{end}}} H dt\,. \tag{2.5.49}$$

Using equations (2.5.44) this can be written as (omitting factors of order 1)

$$N_{\mathrm{e-folds}} \sim \int_{\phi_e}^{\phi_b} \frac{V}{M_P^2 V'} d\phi\,. \tag{2.5.50}$$

For power-law potentials, and $\phi_b \gg \phi_e$ we then get the following estimate

$$N_{\mathrm{e-folds}} \sim \int_{\phi_e}^{\phi_b} \frac{\phi d\phi}{M_P^2} \sim \frac{\phi_b^2}{M_P^2}\,. \tag{2.5.51}$$

Since at the beginning of inflation, $\phi_b \gg M_P$, we see that the number of e-folds is naturally very large, as required.

Inflation thus leads to an elegant solution of the problems of the Hot Big Bang theory. Remember, however, that what we have described here is just one of the many models of inflation. These models lead to different detailed predictions for the density perturbations and gravitational waves. Hopefully it will be possible to discriminate between these models in the not so far future.

Chapter 3

Holography

According to the holographic principle [25], a quantum theory of gravity in D (noncompact) dimensions has less degrees of freedom than one would naively expect and can be described by a quantum theory in $D-1$ dimensions. One of the most promising candidates for a quantum theory of gravity is string theory and it turns out that the holographic principle is realized there in several interesting ways. In this chapter, we will mainly focus on the AdS/CFT correspondence and its de Sitter counterpart, highlighting those facts that are important for the next chapter as a motivation or as technical result. Some prior knowledge of string theory is assumed. The reader can get some idea of what string theory is about by reading the introduction of this work. For more in-depth knowledge we will have to refer the reader to some of the many excellent books and reviews on the subject, see e.g. [26, 27].

3.1 The AdS/CFT correspondence

3.1.1 The Maldacena conjecture for $\mathcal{N} = 4$ SYM

Consider the closed type IIB string theory in flat 10D Minkowski space with N coinciding D3-branes. One of the key factors in the derivation of the Maldacena conjecture [19] is that these D3 branes can be described in 2 different ways, either as a dynamical hyperplane where open strings end on, or as solitonic solutions of the 10D IIB supergravity (=SUGRA) theory, which is the low-energy limit of IIB string theory.
Let us start with the 'stringy' picture, in which the D3-branes are viewed as (3+1)-dimensional planes where open strings end on. These open strings describe excitations of the D-branes, while the closed strings describe excitations of (bulk) space-time. If we consider the system at energies sufficiently lower than the string scale $1/\sqrt{\alpha'}$, then only massless string states get excited and we can describe the system accurately with an effective action containing only these massless states and their interactions. The complete

effective action will have the following form

$$\begin{aligned} S &= S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}}\,, \qquad \text{with} \\ S_{\text{bulk}} &= \text{Type IIB SUGRA with higher derivative corrections}\,, \\ S_{\text{brane}} &= \mathcal{N}=4\,U(N) \text{ SYM with higher derivative corrections}\,, \\ S_{\text{int}} &= \text{interactions between bulk and brane modes}\,. \end{aligned} \tag{3.1.1}$$

S_{bulk} can be expanded as a free quadratic part plus interactions, which are proportional to positive powers of κ, the gravitational coupling constant [7]. Similarly, the interaction Lagrangian S_{int} will be proportional to κ. The higher derivative corrections are proportional to positive powers of α'. In the low-energy limit $\alpha' \to 0$, we have $\kappa \sim g_s \alpha'^2 \to 0$, with g_s the string coupling constant, so we will get two decoupled systems, free gravity in the bulk and $\mathcal{N}=4$ Super Yang Mills (=SYM) on the brane(s).
Now consider the D-branes from the other point of view. They can be described as solutions of the Type IIB supergravity equations of motion as follows: the metric is given by

$$ds^2 = H^{-1/2}(r)(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H^{1/2}(r)(dr^2 + d\Omega_5^2)\,, \tag{3.1.2}$$

and the dilaton and RR 4-form potential are equal to

$$e^{\phi} = const.\,, \qquad C_{(4)0123} = \frac{1}{H(r)} - 1\,, \tag{3.1.3}$$

with

$$H(r) = 1 + \frac{R^4}{r^4}\,, \qquad R^4 \equiv 4\pi g_s \alpha'^2 N\,. \tag{3.1.4}$$

We want to look at excitations in this background and consider again the low-energy limit. Near $r=0$, where the N D3-branes sit, the D3-metric develops a long throat and becomes $AdS_5 \times S^5$ in the near-horizon limit ($r \to 0$), while for large r the D3-metric rapidly approaches flat space. From the point of view of an observer at infinity there are then two kinds of low-energy excitations, namely low-energy excitations in the bulk or arbitrary excitations close enough to $r=0$ (since the energy gets redshifted by the factor $g_{tt}^{-1/2} = H^{-1/4}(r) \sim r/R$). The low-energy excitations in the bulk do not notice the throat because their wave-length is much bigger than the size of the throat, while the excitations close to $r=0$ cannot climb the gravitational potential and escape from the throat. Thus we again get two decoupled systems, free supergravity in the bulk and supergravity on $AdS_5 \times S^5$ near the horizon. The radii of AdS_5 and S^5 are equal and given by

$$R^2_{AdS_5} = R^2_{S^5} = \alpha' \sqrt{4\pi N g_s}\,. \tag{3.1.5}$$

The classical solution in (3.1.2-3.1.4), is only reliable when these radii are big with respect to the string scale, otherwise the supergravity approximation is invalid. It can also be shown that the Yang-Mills coupling constant is related to the string coupling according to $g^2_{YM} = 4\pi g_s$. We thus find that the supergravity approximation is valid if

$$\frac{R^4}{(\alpha')^2} \sim g_s N \sim g^2_{YM} N \gg 1\,, \tag{3.1.6}$$

and if

$$g_s \ll 1\,, \tag{3.1.7}$$

since the supergravity approximation can not be trusted if string perturbation theory is not valid. Comparing this 'supergravity picture' with the previous 'stringy picture' finally led Maldacena to the following conclusion*:

Classical Type IIB SUGRA on $AdS_5 \times S^5$ is equivalent to
$(3+1)D$ $\mathcal{N}=4$ $U(N)$ *SYM in the limit* $N \to \infty$, $\lambda \equiv N g_{YM}^2 \to \infty$.

This is the so called weak form of the Maldacena conjecture. We will discuss a stronger form of this conjecture shortly, but even its weak form is already very remarkable. It relates a quantum gauge theory with a classical supergravity theory and a nongravitational theory in 4 (noncompact) dimensions to a gravity theory in 5 (noncompact) dimensions, the hallmark of holography!
Maldacena took the correspondence even further and conjectured that this duality is not just valid in the above large N, large 't Hooft coupling λ limit, but for all values of the parameters. However since supergravity is not valid in the whole parameter space and it's just an approximation to superstring theory it is natural to conjecture:

Type IIB superstring theory on $AdS_5 \times S^5$ is equivalent to
$(3+1)D$ $\mathcal{N}=4$ $U(N)$ *SYM, with* $g_{YM}^2 = 4\pi g_s$ *and* $g_{YM}^2 N = \frac{R^4}{\alpha'^2}$.

In the 't Hooft limit, $N \to \infty$ and λ fixed, string loop corrections are negligible ($g_s \ll 1$) and we can use tree level string theory. If we also take λ large, classical supergravity is a good approximation, while for $\lambda \ll 1$, Yang-Mills perturbation theory is valid.

3.1.2 AdS/CFT: a concrete proposal

We have seen in the previous section that there are reasons to believe that string theory on $AdS_5 \times S^5$ is somehow dual to $\mathcal{N}=4$ SYM. If this conjecture is correct, we should be able to compute correlators in the (gauge) field theory in terms of the fields of supergravity or string theory. A practical recipe was given in [20, 21]. Before we can discuss their proposal, we have to take a closer look to AdS space and especially its boundary, which will play a crucial role.
Using slightly different variables than in the previous chapter we can describe AdS_{D+1} (not its universal cover here) as the hypersurface

$$uv - \sum_{i,j}^{D} \eta_{ij}\, x^i x^j = 1\,, \tag{3.1.8}$$

*There is a subtlety here. Instead of $U(N)$ the gauge group might also be $SU(N)$. It turns out that we were a bit imprecise when we said that there are 2 kinds of low-energy excitations. There are also some zero modes which live in the region connecting the throat and the bulk and depending on whether we include these in the AdS theory or not, we get a theory dual to $SU(N)$ or $U(N)$ SYM [7].

in $(D+2)$-dimensional Minkowski space with metric

$$ds^2 = -dudv + \sum_{i,j}^{D} \eta_{ij}\, dx^i dx^j \,, \tag{3.1.9}$$

with $\eta = diag(-,+,+,...+)$. The boundary of AdS_{D+1} can then be described by coordinates $u, v, x^1, ..., x^D$ obeying

$$uv - \sum_{i,j}^{D} \eta_{ij}\, x^i x^j = 0 \,, \tag{3.1.10}$$

and subject to an overall scaling equivalence ($u \to su, v \to sv, x^i \to sx^i$, with $s > 0$). Generically $v \neq 0$ and we can use the scaling equivalence to set $v = \pm 1$ and then solve for u. This leaves the coordinates x^i, covering Minkowski space. The quadric however also contains 'point at infinity', at $v = 0$. The boundary is thus equal to a compactification of Minkowski space and is topologically $S^1 \times S^{D-1}$. In fact it turns out to be equal to the conformal compactification of Minkowski space studied in section 2.2, but with $\Im^-$ identified with $\Im^+$ and the three points i^0, $i^\pm$ identified as one point so that we get a compact manifold without boundary [28]. This space obviously has closed timelike curves just like AdS itself. We can solve this as usual by taking the universal cover of AdS. The boundary of the universal cover is then given by the universal cover of the boundary of AdS and will not have closed timelike curves anymore. What is crucial in this story is that the isometry group of AdS_{D+1}, namely $SO(2, D)$ naturally acts on the boundary as the group of conformal symmetries. It is also common knowledge that $(3+1)D$ $\mathcal{N} = 4$ SYM is a conformal theory. The Maldacena conjecture can therefore be seen as a duality between a string theory on $AdS_5 \times S^5$ and a conformal field theory (=CFT) defined on the boundary and is therefore truly holographic in the sense of [25]. This immediately suggests a generalization:

A string/gravity theory on $AdS \times X$ with X compact,
is equivalent to a CFT on the boundary of AdS .

For each conformal field $\mathcal{O}$ of the CFT we can compute the generating functional for correlators involving $\mathcal{O}$ in the standard way,

$$Z(\phi_0) = \left\langle e^{\int d^D x \phi_0(x) \mathcal{O}(x)} \right\rangle_{CFT} \,, \tag{3.1.11}$$

where ϕ_0 is an arbitrary source for $\mathcal{O}$. We now want to calculate the same object using the bulk (string) theory. In [20, 21] it was proposed that we should identify ϕ_0 with the boundary value of a bulk field ϕ and that the generating functional $Z(\phi_0)$ is then obtained by functionally integrating over ϕ in the bulk theory, with the restriction that the boundary value of ϕ be ϕ_0,

$$Z(\phi_0) = \int_{\phi \to \phi_0} D\phi e^{-S(\phi)} \,. \tag{3.1.12}$$

In general S will be the string field theory action. However, in the special case where the supergravity limit can be trusted, $N \to \infty$ and $\lambda \gg 1$, the above formula simplifies and we get

$$Z(\phi_0) = e^{-S_{SUGRA}(\phi_{cl})}\,, \tag{3.1.13}$$

where S_{SUGRA} is the classical supergravity action and ϕ_{cl} is a solution of the supergravity equations of motion with boundary value ϕ_0.
The above expressions are of course purely formal and a lot of technical issues show up that one has to solve to make everything well-defined, see e.g [21, 22]. It would take us too far to go into these problems, it is enough for us to know that they can be solved (at least in simple cases) and that meaningful calculations can be done. One can for instance take a look at some of the many references given in the review article [7].
The above proposal implies that for every field ϕ there is a corresponding operator $\mathcal{O}$ in the CFT. It turns out that the mass of the of the field ϕ is related to the scaling dimension Δ of the operator. The relation is different for particles of different spin. In the case of scalar particles in AdS_{D+1} we have

$$\Delta_\pm = \frac{1}{2}(D \pm \sqrt{D^2 + 4m^2l^2})\,, \tag{3.1.14}$$

where l is the AdS radius. Unitarity of the CFT leads to a restriction $\Delta > (D-2)/2$. This usually leads to the choice $\Delta = \Delta_+$, although sometimes also $\Delta = \Delta_-$ is possible. We will come back to this in the next section. For particles with spin different from zero, formulas can be found in [7, 21, 23].
As a final remark, we want to mention that there exist even more general (and speculative) formulations of the AdS/CFT correspondence, where the bulk theory is defined on spaces that are (only) asymptotically AdS with boundaries that can have multiple disconnected components, see e.g [21, 24].

3.1.3 Intermezzo: stability of AdS

AdS spaces are natural solutions of (gauged) supergravity theories, and with the discovery of the AdS/CFT correspondence they acquired a newfound importance. In a supergravity theory, gravity is coupled to a specific amount of fields of spin lower than 2 (for more details, see the next chapter). The most symmetric solutions of this theory will be those backgrounds in which the spin 1/2, spin 1 and spin 3/2 fields vanish and it will be these solutions, and particularly the AdS solutions among them, we are most interested in. We can restrict ourselves therefore to the gravitational and scalar part of the supergravity action, which is of the following form*

$$S = \int d^Dx\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}a_{ij}(\phi)g^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j - V(\phi)\right]\,. \tag{3.1.15}$$

*For convenience, we will use units $\kappa^2 = 1$ from here on. Throughout the thesis, the reader can assume $\kappa^2 = 1$, unless explicitly stated otherwise.

The action leads to the following equations of motion,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu} \,, \tag{3.1.16}$$

$$T_{\mu\nu} = a_{ij} \partial_\mu \phi^i \partial_\nu \phi^j - g_{\mu\nu} \left[\frac{1}{2} a_{ij} \partial^\sigma \phi^i \partial_\sigma \phi^j - V(\phi) \right] \,, \tag{3.1.17}$$

$$\frac{1}{\sqrt{|g|}} \partial_\mu (a_{ij} \sqrt{|g|} g^{\mu\nu} \partial_\nu \phi^j) + \frac{\partial V}{\partial \phi^i} = 0 \,. \tag{3.1.18}$$

If we want a solution with constant scalar fields, $\phi^i(x) = \phi_0$, equation (3.1.18) shows us that we have to look for critical points of the potential V. The Einstein equation then becomes

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu} V(\phi_0) = 0 \,, \tag{3.1.19}$$

and we see that the potential in the critical point plays the role of the cosmological constant. For $V(\phi_0) < 0$, the maximally symmetric solution is AdS space. The question we want to ask ourselves is whether the AdS solution is stable under fluctuations of the metric and scalar fields. This question can be answered by finding a conserved energy functional and checking whether the energy is positive for all suitable fluctuations. In the AdS case the energy functional can be defined using a global time-like Killing vector [29]. Splitting the metric and scalar fields into background plus perturbations, $g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x)$ and $\phi^i(x) = \phi_0^i + \lambda^i(x)$, the energy functional is given by the following integral over a hypersurface at fixed time,

$$E = \int d^{D-1}x \sqrt{-\bar{g}} \left[t^{0\mu} + T^{0\mu} - \bar{g}^{0\mu} V(\phi_0) \right] k_\mu \,, \tag{3.1.20}$$

where $k^\mu(x)$ is the time-like Killing vector, $T^{\mu\nu}$ is the stress tensor of (3.1.17) and $t^{\mu\nu}$ is the gravitational pseudotensor, which one obtains by expanding $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ and only retaining the terms of second order and higher in $h_{\mu\nu}$. Indices are raised/lowered with the metric $\bar{g}$.

Let us first look at small fluctuations, so that we only have to retain terms that are at most quadratic in $h_{\mu\nu}$ and λ^i. The metric and scalar fluctuations are related by constraint equations, but for small fluctuations they can be treated separately [30]. Positivity of E for metric fluctuations was shown in [29], we will focus here on the scalar perturbations. The nature of these small scalar fluctuations is determined by the mass matrix

$$V^i_j = a^{ik} \frac{\partial^2 V(\phi)}{\partial \phi^k \partial \phi^j} \bigg|_{\phi = \phi_0} \,. \tag{3.1.21}$$

We can diagonalize this matrix and denote its eigenvectors by $h^i(x)$ and corresponding eigenvalues by $-l^{-2}\alpha^i$, with $l^2 = -(D-2)(D-1)/(2V(\phi_0))$, the AdS radius. These eigenvectors will obey the massive wave equation in AdS space,

$$\Box h + m^2 h = 0 \,, \tag{3.1.22}$$

with

$$m^2 = -l^{-2}\alpha\,, \tag{3.1.23}$$

and $\Box = \frac{1}{\sqrt{|g|}}\,\partial_\mu \sqrt{|g|} g^{\mu\nu}\partial_\nu$, the Laplacian in AdS. Taking for instance global coordinates on AdS_D (see section 2.4), the two independent solutions to (3.1.22) behave at spatial infinity ($\rho \to \infty$) as $e^{-\lambda\rho/l}$, with λ obeying

$$\lambda^2 - (D-1)\lambda + \alpha = 0\,. \tag{3.1.24}$$

The two solutions to (3.1.24) are

$$\lambda_\pm = \frac{D-1}{2} \pm \sqrt{\frac{(D-1)^2}{4} - \alpha}\,. \tag{3.1.25}$$

It was shown in [30, 31] that for the Killing energy E to be conserved and positive we need λ to be real, leading to the condition

$$\alpha \leq \frac{(D-1)^2}{4}\,. \tag{3.1.26}$$

Thus, unlike in Minkowski space, tachyonic ($m^2 < 0$) fluctuations do not lead to instabilities in AdS as long as m^2 is not too negative. This is an important observation, since a lot of potentials in gauged supergravity have local maxima or saddle points with $V(\phi_0) < 0$ and the above shows that these AdS vacua can be (and in a lot of cases are) perfectly stable. For the Killing energy to be well defined, the scalar fluctuations have to fall off fast enough and one therefore has to impose boundary conditions on the fluctuations. When $m^2 > 1 - (D-1)^2/(4l^2)$ there is a unique boundary condition (only retaining the λ_+ modes), while in the case $1 - (D-1)^2/(4l^2) > m^2 > -(D-1)^2/(4l^2)$ both λ_+ and λ_- modes fall off fast enough and there are two possible boundary conditions [30, 31]. This is in complete agreement with our expectations from the AdS/CFT correspondence in the previous section. In this context we have $\Delta_\pm = \lambda_\pm$ and stability of AdS in (super)gravity corresponds to unitarity of the dual CFT.

So far we have only investigated the stability under small fluctuations. In the case of Minkowski space, stability under (not necessarily small) perturbations was proven by Witten and Nester in [32] by using a method inspired by (but not restricted to) supersymmetry. Subsequently, this method was generalized and applied to AdS vacua in [33]. One learned that supersymmetric vacua are always (nonperturbatively) stable, but that the converse is not true. Nonsupersymmetric solutions of (gauged) supergravity can also be stable [34]. The method can also be used to determine restrictions on the potential V that arise when one requires the existence of at least one stable, not necessarily supersymmetric, AdS vacuum [34, 35]. For instance, in the case of one scalar field (with the standard kinetic term), the most general potential for which one can prove that a stable AdS vacuum exists with the above methods, has the form

$$V = 2(D-2)\left[(D-2)(w')^2 - (D-1)w^2\right]\,, \tag{3.1.27}$$

where $w(\phi)$ is an arbitrary function that has at least one critical point and the prime indicates differentiation with respect to ϕ. This potential has stable AdS vacuum at the critical points of w, but it might also have other critical points at $w'' = (D-1)w/(D-2)$, which may be stable or unstable. In the case of multiple scalars, a potential of the form (3.1.27) is still sufficient to guarantee stability, but it might not be the most general form [35]. Remark also that for the critical point ϕ_0 at $w' = 0$, it follows from (3.1.27) that

$$\frac{V''(\phi_0)}{4(D-2)^2} = [w''(\phi_0)]^2 - \frac{(D-1)}{2(D-2)} w(\phi_0) w''(\phi_0)\,. \tag{3.1.28}$$

This is a quadratic equation for w'' and has a real solution if and only if

$$\frac{V''(\phi_0)}{V(\phi_0)} \leq \frac{D-1}{2(D-2)}\,. \tag{3.1.29}$$

This is just the condition for stability under small fluctuations,

$$\alpha \leq \frac{(D-1)^2}{4}\,, \tag{3.1.30}$$

written in a slightly different way. We recover in this manner the condition for perturbative stability from a condition for nonperturbative stability.

3.1.4 AdS/CFT and gauged supergravity

We have seen in the previous sections that there are good reasons to believe that $\mathcal{N} = 4$ SYM is dual to string theory on $AdS_5 \times S^5$. However at the moment this is just a conjecture, which needs to be tested, or even better, proven. Unfortunately, a direct comparison of correlation functions is in general not possible with our current knowledge. For instance to test the strong form of the Maldacena conjecture we would have to quantize string theory on $AdS_5 \times S^5$ and at the present time nobody knows how to do this. Most checks are therefore done in the limit were classical supergravity is valid. However, in this limit the 't Hooft coupling λ is large, making the Yang-Mills theory strongly coupled. Since our main tool in quantum field theory is perturbation theory, calculating SYM correlation functions becomes very difficult in this limit. Fortunately some of the properties of both theories do not depend on the coupling constants and they can be (and have been) compared, and they agree perfectly, supporting the AdS/CFT correspondence. For instance, the global symmetries of the theories match, certain correlators that do not get quantum corrections can be compared, ..., etc., see [7, 23] and references therein.
Assuming the conjecture is correct, we can learn more about the large N, large 't Hooft coupling limit of $\mathcal{N} = 4$ $SU(N)$ SYM theory by studying Type IIB SUGRA on $AdS_5 \times S^5$. The compactification of Type IIB SUGRA on S^5 gives rise to 5D $\mathcal{N} = 8$ $SO(6)$ gauged supergravity plus terms involving an infinite number of more massive modes, called Kaluza-Klein modes. It is believed, but not rigorously proven, that it is consistent to put these Kaluza-Klein modes to zero, truncating the theory to just 5D $\mathcal{N} = 8$ $SO(6)$ gauged

SUGRA. Such a truncation is called a Kaluza-Klein truncation and it is called consistent if all the solutions of the lower-dimensional truncated theory can be lifted to the full higher-dimensional theory. There is convincing evidence that the Kaluza-Klein truncation of 11D SUGRA on S^7 and S^4 to respectively 4D and 7D $\mathcal{N} = 8$ gauged SUGRA is consistent [36]. We will assume here, like almost everybody else in the supergravity/string community, that also the truncation of IIB SUGRA to 5D $\mathcal{N} = 8$ gauged SUGRA is consistent. See [37] and references therein for some partial results on this issue.
Taking this assumption of consistent truncation together with the AdS/CFT correspondence leads to some interesting implications [38]:

- The AdS/CFT correspondence implies that the $\mathcal{N} = 8$ gauged supergravity multiplet corresponds to the $\mathcal{N} = 4$ Yang-Mills energy-momentum tensor supermultiplet. Consistent truncation then implies that the operator product of fields in the energy-momentum supermultiplet must be closed at large N. This is indeed the case, as demonstrated in [39]. Moreover at finite N the operator algebra does not close, suggesting that gauged $\mathcal{N} = 8$ SUGRA is not a good quantum theory and needs to be completed (to a string theory).

- In gauged supergravity theories the scalar fields typically get a potential $V(\phi)$ proportional to the gauge coupling constant g^2. One of the critical points of the $SO(6)$ gauged 5D $\mathcal{N} = 8$ SUGRA potential will correspond to the $AdS_5 \times S^5$ solution of the Type IIB theory, but there are also other AdS critical points. The corresponding 10D solutions will have the form of a warped product:

$$d\hat{s}^2 = \Delta^{-2/3} ds^2_{AdS_5} + ds^2_M \,. \tag{3.1.31}$$

  ds^2_M is a deformed metric on S^5 and Δ is a function which depends on the deformed S^5 coordinates, called the warp factor. When the AdS critical points are stable, they usually correspond to nontrivial conformal fixed points of the deformed SYM theory. The supergravity scalars whose vacuum expectation values lead to the new critical point tell one by which (relevant) operators the SYM has to be deformed. The renormalization group (RG) flow in the deformed theory can be described in the dual supergravity theory by a domain wall solution interpolating between the AdS vacua. To construct these solutions one takes the metric in the 5D gauged theory to be:

$$ds^2 = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu + dr^2 \,. \tag{3.1.32}$$

  The scalars are assumed to depend only on the radial coordinate r. As $r \to \infty$ one requires that the scalars approach the $\mathcal{N} = 8$ supersymmetric AdS vacuum (corresponding to $AdS_5 \times S^5$ in 10D), while for $r \to -\infty$ one requires the scalars to approach one of the other critical points. One also has to require $A(r) \sim const.|r|$ as $r \to \pm\infty$. To interpret these domain wall solutions correctly as RG flows, one usually has to lift these solutions back to 10D. Finding these solutions directly in the IIB theory is quite difficult however. The 5D gauged theory is therefore very useful to generate solutions. The 5D $\mathcal{N} = 8$ gauged theory is, however, still quite

complicated and in order to find solutions one usually truncates the 5D gauged $\mathcal{N} = 8$ theory in a consistent way to a 5D gauged $\mathcal{N} = 4$ or a 5D gauged $\mathcal{N} = 2$ theory.
We end this section with two remarks. First, all we have said only applies to ($\mathcal{N} = 4$) (S)YM in the large N limit. In particular the critical points of the supergravity potential, might only represent some nontrivial phase of large N YM theory. Asking whether there is also a nontrivial phase at finite N corresponds to asking whether the supergravity solution is a good (approximation to a) string theory vacuum. The general belief, based on quite some evidence, is that supersymmetric solutions have a good chance to correspond to (approximations of) string vacua. A lot less is know about nonsupersymmetric vacua in string theory. There is thus more danger that nonsupersymmetric supergravity solutions just probe large N pathologies. However, since nonsupersymmetric vacua are currently so badly understood, we will approach them in this thesis with an open mind. (They are there and they might be there for a reason.)
As a final note, remark that the above can all be repeated in different dimensions, since the more general form of the AdS/CFT correspondence works for AdS_D for any D. This has led to a renewed interest in gauged supergravities in diverse dimensions, with the main object of interest being the scalar potential.

3.2 A dS/CFT correspondence ?

The AdS/CFT correspondence can be used to give a nonperturbative definition of string theory or quantum gravity in asymptotically AdS spaces. Since we expect a quantum theory of gravity to be holographic, it is natural to try to mimic the AdS/CFT correspondence for asymptotically flat and asymptotically dS spaces. In this section we will discuss the case of dS space, we will deal with flat space in the next section.

3.2.1 dS/CFT correspondence: a few proposals

Remember (see chapter 2 if you do not) that dS space has two spacelike conformal boundaries, denoted $\Im^{\pm}$. Because the boundaries are spacelike, we expect the dual field theory to be Euclidean. Just like in AdS space, the isometry group of dS_D, namely $O(D, 1)$, works on the boundary as the Euclidean conformal group in $D-1$ dimensions. We therefore guess that the dual field theory is a Euclidean CFT.
What is not so clear, is whether this dual theory should be defined on one of the boundaries or on both. In [40], Strominger proposed that quantum gravity on dS_D is holographically dual to a Euclidean conformal theory on a single S^{D-1}. On the other hand, Balasubramanian et al. proposed in [41] that the dual CFT should involve two disjoint but possibly entangled factors.
There is also a proposal from Banks [42] that the Hilbert space of quantum gravity in dS spaces is finite-dimensional, which if true would seem to contradict a description in terms

of a dual field theory. There are however some possible ways out, see [6, 43]. Besides the above, there are a wide variety of other suggestions for a possible dS/CFT correspondence. It is impossible to mention them all, but for some of the — in my opinion — more interesting ones, see [44] and the review articles [45].
Unfortunately, there is no string theory 'derivation' of the dS/CFT correspondence, like in the AdS case. This makes it difficult to decide which of these proposals, if any, are correct. One of the things we have learned from the AdS/CFT correspondence is that a lot of information can be extracted from the gauged supergravities and their AdS solutions. It is therefore natural to study also the dS solutions of gauged supergravity and we will do just that in chapter 4.

3.2.2 Stability of dS space

Just like in the AdS case, we can search for dS solutions in supergravity and ask ourselves when they will be stable. We will again only look at backgrounds where all spin 1/2, spin 1 and spin 3/2 fields vanish. The analysis in section 3.1.3 for small fluctuations can then be repeated, with a few minor but important changes.
As in the AdS case, we consider the Killing energy E, defined as*

$$E = \int d^{D-1}x \sqrt{-\bar{g}} \left[t^{0\mu} + T^{0\mu} - \bar{g}^{0\mu} V(\phi_0) \right] k_\mu \,, \tag{3.2.1}$$

where $k^\mu(x)$ is the time-like Killing vector and the integral is over a hypersurface at a fixed time. The difference is that there is no global timelike Killing vector in dS space. Every observer has an event horizon and a timelike Killing vector inside the horizon becomes spacelike outside the horizon (see section 2.3). Abbott and Deser showed in [29] that for positive mass ($m^2 \geq 0$) fluctuations inside the horizon, the energy is positive and conserved. For fluctuations outside the event horizon, the energy gets negative contributions and stability could not be proved. Also for negative mass ($m^2 < 0$) fluctuations positivity of the energy could not be proved, they will most likely lead to instabilities. All this is for small fluctuations. Unlike for the AdS case there is no nonperturbative proof of stability for dS space known. This might be related to the fact that dS space cannot be supersymmetric.
We can summarize the above as follows:

dS space is at least stable under small fluctuations which have
$m^2 \geq 0$ *and are inside the event horizon of the observer* .

Assuming (quantum) gravity on dS_D is dual to a CFT, we can again try to associate the scalar fields to operators in the dual CFT. The relation between the mass of the scalar field and the dimension of the operator can be calculated analogously as in the AdS/CFT

*There are also other proposals for a definition of energy in dS space, see [46]. We will stick here to the definition of Abbott and Deser [29].

correspondence, and is given by

$$\Delta_{\pm} = \frac{1}{2}(D-1 \pm \sqrt{(D-1)^2 - 4m^2l^2})\,. \tag{3.2.2}$$

This formula turns out to be just an analytic continuation in the cosmological constant ($l^2 \to -l^2$) of the corresponding formula in the AdS/CFT correspondence. However the dS/CFT can not be seen as just an analytic continuation of the AdS/CFT correspondence. While some classical facts of dS space follow from analytic continuation of the (Euclidean) AdS results, others do not. Moreover, the quantum mechanical physics of de Sitter space can also not be obtained from analytic continuation. For instance, the Green function for a scalar field in a de Sitter vacuum state is not the analytic continuation of the Euclidean AdS Green function [47].
Looking at equation (3.2.2) we see that for sufficiently positive m^2 the conformal dimensions become imaginary. This suggests that the dual CFT is not unitary. Strominger proposed in [40] that this might mean that consistent theories of de Sitter quantum gravity have no stable scalars with $m^2l^2 > (D-1)^2/4$. On the other hand, since the CFT is Euclidean, there is no obvious physical reason why the CFT should be unitary. In any case, there does not seem to be a connection between classical stability and unitarity as in the AdS/CFT correspondence.

3.3 Holography in flat space

Finding a holographic dual of quantum gravity in asymptotically flat spaces has proved to be as difficult as for asymptotically dS spaces. There are again several proposals[24]. We will list some (and probably most) of these, without claiming to be complete.

- After the discovery in the mid 90's that the 5 superstring theories and 11D supergravity were all related, there was (and still is) a widespread belief that there is a (probably 11D) quantum theory called M-theory that underlies all the string theories. Until now there is no background independent definition of this theory. However for M theory in space-times with 6 or more noncompact asymptotically flat directions and 16 or 32 supercharges, there is good evidence that it can be described by a theory called Matrix theory[48]. Matrix theory is a quantum mechanical theory of $N \times N$ matrices, which can be seen as either a regularized version of the quantum supermembrane theory or as a theory describing N D0-branes of Type IIA string theory. It is a nonperturbative, background dependent definition of M-theory and it is also holographic, but in a seemingly very different way than in the AdS/CFT correspondence [49].

- There have been efforts to take a decompactification limit to flat space of the AdS/CFT correspondence (see e.g. [50]), but no precise prescription for flat space holography has emerged.

- Other people have tried to imitate the AdS/CFT correspondence, by looking at the conformal boundary of Minkowski space and in particular to past and future null infinity $\Im^{\pm}$. The asymptotic symmetry group at null infinity is the so called Bondi-Metzer-Sachs (BMS) group (see e.g. [28]) and is infinite-dimensional. One might therefore guess that the dual theory 'lives' at null infinity and carries representations of the BMS group. Constructing such a theory proved difficult however, see e.g. [51].

- R. Bousso has given a general formulation of (what he considers to be) the holographic principle and has constructed so called 'holographic screens' in various spacetimes, including AdS, Minkowski and dS space [52]. According to Bousso, the properties of these holographic screens might give clues about the nature of a manifestly holographic theory. However, so far this has not led to an explicit dual description.

- Another approach divides Minkowski space in AdS and dS slices [53]. The idea is to apply AdS/CFT and dS/CFT in each slice separately and then combine the results. However, besides the drawback that also the dS/CFT correspondence is not well understood, there is the problem that the AdS/dS slicing of Minkowski does not survive small deformations.

Chapter 4

de Sitter space in supergravity

Both the holographic principle and the recent observations in cosmology, which we reviewed in the previous two chapters, motivated us to study dS space in supergravity. In this chapter, we will give an overview of de Sitter vacuum solutions and their properties in supergravity. We will restrict/focus ourselves to 4D and 5D supergravity and their possible 10D (or 11D) origin. This chapter will be partly a presentation of my own work [100, 117] and partly a review of older related results.

4.1 de Sitter vacua in 4D supergravity

4.1.1 Gauged supergravity in 4D: general features

Supergravity theories in 4D can have different amounts of supersymmetry, usually indicated by the capital letter $\mathcal{N}$, where $\mathcal{N} = 1, \ldots, 8$. All these theories have a similar form. The generic 4D $\mathcal{N}$-extended supergravity action (omitting four-fermion terms) is given by [54]*:

$$\begin{aligned}
(\sqrt{-g})^{-1}\mathcal{L} &= \frac{1}{2}R[g] + \frac{i}{4}\left(\bar{\mathcal{N}}_{\Lambda\Sigma}F^{-\Lambda}_{\mu\nu}F^{-\Sigma\mu\nu} - \mathcal{N}_{\Lambda\Sigma}F^{+\Lambda}_{\mu\nu}F^{+\Sigma\mu\nu}\right) - \hat{P}^{IA}_{\mu}\hat{P}^{\mu}_{IA} \\
&+ \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\left(\bar{\psi}^{A}_{\mu}\gamma_{\sigma}D_{\nu}\psi_{A\lambda} - \bar{\psi}_{A\mu}\gamma_{\sigma}D_{\nu}\psi^{A}_{\lambda}\right) - \frac{1}{2}\left(\bar{\lambda}^{I}\gamma^{\mu}\nabla_{\mu}\lambda_{I} + \bar{\lambda}_{I}\gamma^{\mu}\nabla_{\mu}\lambda^{I}\right) \\
&+ \hat{P}^{IA}_{\mu}\bar{\lambda}_{I}\gamma^{\nu}\gamma^{\mu}\psi_{A\nu} + \hat{P}_{IA\mu}\bar{\lambda}^{I}\gamma^{\nu}\gamma^{\mu}\psi^{A}_{\nu} \\
&- \frac{F^{\Lambda}_{\mu\nu}}{2}\mathcal{N}_{\Lambda\Sigma}\left(L^{\Sigma}{}_{AB}\overline{\psi}^{\mu A}\psi^{\nu B} + iL^{\Sigma}{}_{IA}\overline{\psi}^{\mu A}\gamma^{\nu}\lambda^{I} - L^{\Sigma}{}_{IJ}\overline{\lambda}^{I}\gamma^{\mu\nu}\lambda^{J} + \text{h.c.}\right) \\
&- 2S_{AB}\bar{\psi}^{A}_{\mu}\gamma^{\mu\nu}\psi^{B}_{\nu} - 2\overline{S}^{AB}\bar{\psi}_{\mu A}\gamma^{\mu\nu}\psi_{\nu B} \\
&- iN_{I}{}^{A}\bar{\lambda}^{I}\gamma^{\mu}\psi_{\mu A} - iN^{I}{}_{A}\bar{\lambda}_{I}\gamma^{\mu}\psi^{A}_{\mu} + \mathcal{M}^{IJ}\bar{\lambda}_{I}\lambda_{J} + \mathcal{M}_{IJ}\bar{\lambda}^{I}\lambda^{J} - V(q)\,, \qquad (4.1.1)
\end{aligned}$$

where q^u are the scalar fields, λ_I are the spin $\frac{1}{2}$ fermions, $F^{\pm\Lambda}_{\mu\nu}$ are the (anti-)selfdual field strengths of the gauge fields A^{Λ}_{μ} and $\psi_{A\mu}$ are the spin $\frac{3}{2}$ gravitino fields. The indices

*$F^{\pm}_{\mu\nu}$ here is $-2\mathcal{F}^{\pm}_{\mu\nu}$ in [54].

$A, B, \ldots$, denote the fundamental representation of the R-symmetry group, which equals $SU(\mathcal{N}) \times U(1)$ for $\mathcal{N} = 1, \ldots 6$ and $SU(8)$ for $\mathcal{N} = 7, 8$, with lower(upper) position indicating their left (right) chirality. The index I enumerates the spin $\frac{1}{2}$ fields and is a shorthand notation. If the spin $\frac{1}{2}$ fermions belong to vector multiplets, they also transform in the fundamental of the R-symmetry group and we have to set $I \to IA$, while if they belong to the gravitational multiplet, they transform in the $\mathcal{N}(\mathcal{N}-1)(\mathcal{N}-2)/3!$ antisymmetric representation and we set $I \to [ABC]$. In other cases (e.g. hypermultiplets in $\mathcal{N} = 2$) they do not transform under the R-symmetry.
The matrices $\mathcal{N}_{\Lambda\Sigma}$, S_{AB}, $N^I{}_A$, $L^\Sigma{}_{AB}$, $L^\Sigma{}_{IA}$, $L^\Sigma{}_{IJ}$, $\mathcal{M}_{IJ}$ and their hermitian conjugates $\bar{\mathcal{N}}_{\Lambda\Sigma}$, $\bar{S}^{AB}$, $N_I{}^A$, $\mathcal{M}^{IJ}$ appearing in the Lagrangian all depend on the scalar fields q^u. The matrices S_{AB}, $N^I{}_A$, $\mathcal{M}_{IJ}$ are of order g in the gauge coupling constant and the scalar potential $V(q)$ is of order g^2. D_μ is the Lorentz covariant derivative (containing the spin connection), while ∇_μ usually also contains the composite connections of the scalar manifold and extra terms of order g. Finally, $\hat{P}_\mu^{IA}$ are the gauged vielbein 1-forms of the scalar manifold defined by

$$\hat{P}_{IA\mu} = P_{IAu}(\partial_\mu q^\mu + g A_\mu^\Lambda k_\Lambda^u)\,, \tag{4.1.2}$$

where P_{IAu} is the ordinary vielbein of the scalar manifold and k_Λ^u is a Killing vector associated to an isometry of the scalar manifold. Because kinetic terms have to be positive, the scalar manifolds of supergravity are always Riemannian, i.e the metric of the scalars has to be positive definite, and likewise for the metrics of the vectors and fermions in the theory. When $\mathcal{N} > 2$ the scalar manifold is always a coset G/H with G a noncompact group and H its maximally compact subgroup. In this case the matrices $L^\Sigma{}_{AB}$, $L^\Sigma{}_{IA}$, $L^\Sigma{}_{IJ}$ are coset representatives. When $\mathcal{N} = 2$, the scalar manifold is a product of a special Kähler manifold and a quaternionic-Kähler manifold and $L^\Sigma{}_{AB}$, $L^\Sigma{}_{IA}$, $L^\Sigma{}_{IJ}$ are special geometry objects. Finally, in the case $\mathcal{N} = 1$ the scalar manifold is Hodge-Kähler, and the L-matrices are related to the kinetic matrix of the vectors.
In this chapter, we will look for (de Sitter) vacuum solutions of 4D and 5D supergravity. In general relativity, a vacuum solution is a maximally symmetric solution and it is common practice to refer also to the maximally symmetric solutions of supergravity as vacua. However, also the solutions that preserve the maximum amount of supersymmetry, the maximally supersymmetric solutions, are quite often called supergravity vacua, see e.g. [55]. A maximally supersymmetric solution is not necessarily maximally symmetric, but can be. Conversely, a maximally symmetric solution can be maximally supersymmetric but does not have to be. When we talk about vacuum solutions of 4D and 5D SUGRA we will mean a maximally symmetric solution. To preserve D-dimensional Lorentz invariance (+ translations), these vacuum solutions are necessarily configurations with constant scalar fields, a dS, AdS or Minkowski metric and all other fields put to zero. In order for such a configuration to solve the e.o.m. derived from (4.1.1) the constant values of the scalars also must be extrema of the scalar potential. We will thus be mostly interested in the scalar potential $V(q)$, which in $\mathcal{N} > 1$ SUGRAS is completely determined by supersymmetry and the group one chooses to gauge.

Let us now take a look at the supersymmetry transformations given by [54]:

$$\begin{aligned}
\delta\psi_{A\mu} &= \nabla_\mu\varepsilon_A + iT^{(-)}_{AB\mu\nu}\gamma^\nu\varepsilon^B - iS_{AB}\gamma_\mu\varepsilon^B + \cdots\,, && (4.1.3)\\
\delta\lambda_I &= -\hat{P}_{uIA}\gamma^\mu\varepsilon^A\nabla_\mu q^u + T^{(-)A}_{I\mu\nu}\gamma^{\mu\nu}\varepsilon_A + N_I{}^A\varepsilon_A + \cdots && (4.1.4)\\
\delta e^a_\mu &= -\overline{\psi}_A\gamma_\mu\varepsilon^A - \overline{\psi}^A\gamma_\mu\varepsilon_A\,, && (4.1.5)\\
\delta A^\Lambda_\mu &= 2f^{\Lambda[AB]}\overline{\psi}_{A\mu}\varepsilon_B + f^\Lambda{}_I{}^A\bar{\lambda}^I\gamma_\mu\varepsilon_A\,, && (4.1.6)\\
&+ 2f^\Lambda{}_{[AB]}\overline{\psi}^A_\mu\varepsilon^B + f^{\Lambda I}{}_A\bar{\lambda}_I\gamma_\mu\varepsilon^A\,,\\
\delta q^u P_{uIA} &= \bar{\lambda}_I\varepsilon_A\,, && (4.1.7)\\
\delta q^u P^{IA}_u &= \bar{\lambda}^I\varepsilon^A\,, && (4.1.8)
\end{aligned}$$

where the dots denote trilinear fermion terms. For $\mathcal{N} > 1$, $f^{\Lambda[AB]}$ and $f^\Lambda{}_I{}^A$ are related to the symplectic sections on the scalar manifold of the vector multiplets, while for $\mathcal{N} = 1$ we have $f^{\Lambda[AB]} \equiv 0$ and $f^\Lambda{}_I{}^A = \delta^\Lambda_I$. $T^{(-)}_{AB\mu\nu} = i\bar{f}^{-1}{}_{AB\Lambda}F^{(-)\Lambda}_{\mu\nu}/2$ and $T^{(-)A}_{I\mu\nu} = i\bar{f}^{-1}{}_I{}^A{}_\Lambda F^{(-)\Lambda}_{\mu\nu}/2$ are the dressed field strengths of the graviphotons and of the vector multiplets.
Requiring invariance (up to a total derivative) of the action (4.1.1) under the above supersymmetry transformations leads to some interesting relations*:

$$\begin{aligned}
\delta^A_B\,V(q) &= -12\bar{S}^{AC}\,S_{CB} + N_I{}^A\,N^I{}_B\,, && (4.1.9)\\
\frac{\partial V}{\partial q^u}P_{IAu} &= 4iN_I{}^B\,S_{BA} + 2\mathcal{M}_{IJ}\,N^J{}_A\,. && (4.1.10)
\end{aligned}$$

From the supersymmetry transformations, we see that a vacuum solution q_0 preserves $\mathcal{N}_0$ supersymmetries if there exists $\mathcal{N}_0$ $SU(\mathcal{N})$ vectors ρ^A_l $(l = 1, ..., \mathcal{N}_0)$, obeying

$$\begin{aligned}
S_{AB}\rho^B_l &\sim \rho_{lA}\,, && (4.1.11)\\
N^I{}_A(q_0)\rho^A_l &= 0\,. && (4.1.12)
\end{aligned}$$

Contracting equation (4.1.9) with ρ^B we get

$$\rho^A\,V(q) = -12\bar{S}^{AC}\,S_{CB}\rho^B\,, \qquad (4.1.13)$$

and multiplying with $N^I{}_A$ leads to

$$N^I{}_A(S^*S)^A{}_B\rho^B = 0\,. \qquad (4.1.14)$$

Because S^{AB} can be chosen to be real† the above formulas imply

$$N^{IA}S_{AB}\rho^B = 0\,. \qquad (4.1.15)$$

*We mention here only those involving the scalar potential, other relations can be found in [54].

†To show this (and to derive the supergravity theories in the first place) one needs to know some properties of spinors in 4D. We will not go into this, since it would lead us too far astray. The interested reader can find a good introduction to supersymmetry and spinors in arbitrary dimensions in [56].

Equations (4.1.12) and (4.1.15) together with (4.1.10) then lead to

$$\left.\frac{\partial V}{\partial q^u}\right|_{q=q_0} = 0\,. \tag{4.1.16}$$

We thus find that a supersymmetric vacuum solution automatically extremizes the potential [57]. Moreover, from (4.1.13) we see that a supersymmetric vacuum is necessarily an AdS or Minkowski vacuum. This is not a special property of 4D supergravity, dS vacua always break supersymmetry completely in conventional supergravity. There are supergravity theories where dS is a supersymmetric solution, but these are all pathological, containing negative kinetic terms for some of the physical fields[58].
In the following sections we will look at the potential of some of the gauged $\mathcal{N}$-extended supergravities in more detail. Most work has been done in theories where $\mathcal{N}$ is a power of 2, since their higher-dimensional and string theory origin is usually more clear, and we will also limit ourselves to these cases. Moreover we will also not discuss the $\mathcal{N} = 1$ supergravity, since in this case the potential is not fully determined by the gauging, and an almost arbitrary potential can be constructed.

4.1.2 $\mathcal{N} = 8$ gauged supergravity

In $\mathcal{N} = 8$ supergravity the scalar manifold is completely determined by supersymmetry and in 4D the 70 real scalars parametrize the coset space $E_{7(-7)}/SU(8)$. The equations of motion have a global $E_{7(-7)}$ symmetry but the action is only invariant under a smaller group $G \subset E_{7(-7)}$. In fact there are different actions of ungauged $\mathcal{N} = 8$ supergravity*, each having a different global invariance group G. The most well known case is $G = SL(8,\mathbb{R})$, described in [59]. The vector fields then transform in the 28 antisymmetric representation of $SL(8,\mathbb{R})$ and we can gauge a subgroup $K \subset SL(8,\mathbb{R})$. For the gauging to preserve supersymmetry, a number of so called T-identities has to be satisfied. This leads to restrictions on the possible gauge groups K. It was proved in [60] that the $CSO(p,q,r)$ gaugings with $p+q+r = 8$ constructed in [61] are the only admissible ones. Calculating the potential corresponding to these gaugings is a highly nontrivial task. To have an explicit expression of the potential in terms of the 70 real scalars, one needs an explicit parametrization of the $E_{7(-7)}/SU(8)$ coset space, making the calculations very hard. In [62] the problem was reduced to dealing with $SU(8)/SO(8)$ instead of $E_{7(-7)}/SU(8)$, at the cost of making the expressions for the potential more complicated. However, calculating an explicit expression for the potential was still not possible and in

*Different actions can lead to the same set of equations of motion and Bianchi identities, but have a different global invariance group if only a subset of the symmetries of the equations of motion and Bianchi identities can be realized at the Lagrangian level. Since gauging a theory involves making a global symmetry local by using (electric) gauge fields, ungauged supergravity actions with different global invariance groups lead to different gauged supergravity theories. Recently a new formulation of 4D supergravity was discovered, in which electric and magnetic gauge fields are on equal footing [72, 73], allowing for non-Abelian gauge symmetries involving both electric and magnetic gauge fields. This formulation eliminates the need to start from various different ungauged supergravity actions, a single action can describe all the possible gaugings. However, in the thesis we will stick to the old formalism.

fact even with today's computers it is probably not feasible yet*. Moreover, even if one had an explicit form for the potential, finding its critical points would be another difficult problem. The standard way one deals with these problems, following [64], is to choose a compact subgroup of the gauge group and restrict attention to the singlets under this group, setting all other scalars to zero. Because the potential is invariant under the gauge group by construction, the extrema of the truncated potential will also be extrema of the full potential, since terms involving nonsinglets will be at least of quadratic order. In this way several vacua of the $\mathcal{N}=8$ gauged theories were found, see e.g. [65, 68, 69]. We will review here the known dS vacua.

The groups $CSO(p,q,r)$ have a corresponding Lie algebra given by

$$[L_{ab}, L_{cd}] = L_{ad}\eta_{bc} - L_{ac}\eta_{bd} - L_{bd}\eta_{ac} + L_{bc}\eta_{ad}\,, \tag{4.1.17}$$

where

$$\eta_{ab} = \begin{pmatrix} \mathbf{1}_{p\times p} & 0 & 0 \\ 0 & -\mathbf{1}_{q\times q} & 0 \\ 0 & 0 & 0_{r\times r} \end{pmatrix}\,, \tag{4.1.18}$$

$a, b = 1, \cdots, 8$ and $L_{ab} = -L_{ba}$. We have $CSO(p,q,0) = SO(p,q)$ and $CSO(p,q,1) = ISO(p,q)$, while $CSO(p,0,q)$ is usually abbreviated to $CSO(p,q)$. The potentials of the $SO(p,q)$ and the $CSO(p,q)$ ($p+q=8$ and $p,q \geq 0$) gaugings where found explicitly in [66] for the case in which there is only one nonzero scalar ϕ, the only scalar invariant under $SO(p)\times SO(8-p)$. The potentials can be written in terms of a superpotential $w_{p,q,\xi}$ [67] as†

$$w_{p,q,\xi} = \frac{g}{8\sqrt{2}}\left(pe^{\sqrt{\frac{q}{2p}}\phi} + q\xi e^{-\sqrt{\frac{p}{2q}}\phi}\right)\,, \tag{4.1.19}$$

where

$$V_{p,q,\xi} = 4(2(\partial_\phi w_{p,q,\xi}(\phi))^2 - 3w_{p,q,\xi}(\phi)^2)\,, \tag{4.1.20}$$

as in equation (3.1.27). The parameter ξ takes values $-1, 0+1$, with $\xi = 1$ corresponding to $SO(8)$ gauging, $\xi = -1$ to $SO(p,q)$ gauging and $\xi = 0$ to $CSO(p,q)$ gauging. With this form of the potential the scalar ϕ has a canonical kinetic term. To make clear which normalizations were used we also give the relevant part of the $\mathcal{N}=8$ gauged supergravity, which here is

$$S_{p,q,\xi} = \int d^4x\sqrt{-g}\left(\frac{R}{2} - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V_{p,q,\xi}(\phi)\right)\,. \tag{4.1.21}$$

Three cases lead to dS vacua, $(p=q=4, \xi=-1)$, $(p=5, q=3, \xi=-1)$ and $(p=3, q=5, \xi=-1)$. In all cases the dS critical point is a maximum of the potential and $m^2 \equiv V''(\phi_c) = -2V(\phi_c)$.

The above approach was generalized to $CSO(p,q,r)$ gaugings in [68]. Looking at $SO(p)\times$

*See [63] for some recent interesting computer-assisted calculations.

†There is a factor 2 mismatch between equations (2.12) and (2.13) in [67]. We have chosen the superpotential in such a way that the corresponding potential corresponds to the expressions in equation (2.13).

$SO(q) \times SO(r)$ invariant directions, the (truncated) potentials now depend on 2 scalar fields and can again be given in terms of a superpotential as follows:

$$w^{p,q,r}_{\xi,\zeta}(s,t) = \frac{g}{8\sqrt{2}} \left(p e^{\sqrt{\frac{4q}{p(p+q)}}s - \sqrt{\frac{r}{2(p+q)}}t} + q\xi e^{-\sqrt{\frac{4p}{q(p+q)}}s - \sqrt{\frac{r}{2(p+q)}}t} + r\xi\zeta e^{\sqrt{\frac{p+q}{2r}}t} \right) , \quad (4.1.22)$$

and

$$V^{p,q,r}_{\xi,\zeta}(s,t) = 4\left(2(\partial_s w^{p,q,r}_{\xi,\zeta}(s,t))^2 + 2(\partial_t w^{p,q,r}_{\xi,\zeta}(s,t))^2 - 3(w^{p,q,r}_{\xi,\zeta}(s,t))^2\right) . \quad (4.1.23)$$

Just like before, the scalar fields s and t have canonical kinetic terms and the curvature term comes with a factor 1/2. The parameters ξ and ζ take discrete values $0, \pm 1$, since the potentials can only depend on one continuous parameter as shown in [60]. The gauge group is one of the CSO groups with η_{ab} in (4.1.17) given by

$$\eta_{ab} = \begin{pmatrix} \mathbf{1}_{p\times p} & 0 & 0 \\ 0 & \xi\mathbf{1}_{q\times q} & 0 \\ 0 & 0 & \xi\zeta\mathbf{1}_{r\times r} \end{pmatrix} . \quad (4.1.24)$$

So $\xi = 1$ and $\zeta = -1$ corresponds to $SO(p+q, r)$ gauging, $\xi = 1$ and $\zeta = 0$ to $CSO(p+q, r)$ gauging, and so on. The dS critical points of the potentials in (4.1.22),(4.1.23) were studied in [12, 70]. They observed that all these dS vacua have the following peculiar properties:

- All the dS extrema are saddle points, the mass matrix has a positive and a negative eigenvalue, and the dS vacua are therefore unstable. The tachyonic mass is always given by
$$m^2_{\text{tach}} = -2V|_0 = -\frac{6}{l^2} , \quad (4.1.25)$$
where l^2 is the radius of dS space and as usual we have used units with $\kappa^2 = 1$.

- All the scalar masses in the dS vacua are quantized in terms of the cosmological constant,
$$m^2 = \frac{k}{l^2} , \quad (4.1.26)$$
with k some integers of the order 1, independent of the parameters of the theory.

The above properties make the discussed potentials unfit for slow-roll inflation, since the slow-roll condition (2.5.45) is not satisfied [12]. However, they might play a role in explaining the current acceleration of the universe.
At the present time, no explanation is known for the above properties nor whether they are general properties of dS vacua in $\mathcal{N} = 8$ supergravity. We remark also that only the masses of the singlet scalars were calculated. It would be interesting to also calculate the masses of the truncated scalars, to see whether the above properties still hold.
One might wonder whether it is the form of the (super)potential that makes the dS vacua unstable, i.e., can a superpotential of the form

$$w = c_1 e^{-a_1\phi} + c_2 e^{-a_2\phi} \quad (4.1.27)$$

have stable dS vacua if the arbitrary constants a_1, a_2, c_1, c_2 are chosen appropriately? It turns out the answer is yes. Without losing generality, we can choose $c_1 = 1$. A few elementary calculations then show that the potential V corresponding to the superpotential in (4.1.27) has a dS minimum in

$$\phi = \ln\left[\frac{3-2a_1^2}{(-3+2a_2^2)c_2}\right], \tag{4.1.28}$$

if $c_2 < 0$ and $a_1 a_2 < -3/2$.
Besides the $CSO(p,q,r)$ gaugings we discussed here, other gauge groups are possible if we start from an ungauged action with a different global symmetry group G, see [71]. Their potentials have not yet been investigated as thoroughly as those of the $CSO(p,q,r)$ groups and no dS vacua are known for these gaugings.

4.1.3 $\mathcal{N} = 2$ gauged supergravity

$\mathcal{N} = 2$ supergravity is not as rigid as maximal supergravity. The pure $\mathcal{N} = 2$ supergravity, containing a vielbein, one vector called the graviphoton and two spin 3/2 fermions called gravitini, can be coupled to an arbitrary number of vector multiplets and hypermultiplets. The $\mathcal{N} = 2$ vector multiplets consist of one vector, two spin 1/2 fermions and a complex scalar, and the hypermultiplets contain two spin 1/2 fermions and four real scalars. The scalars of the vector multiplets are the coordinates of a special Kähler manifold, while those of the hypermultiplets parametrise a quaternionic-Kähler manifold. The total scalar manifold is just the direct product of a special Kähler manifold and a quaternionic-Kähler manifold.

Special Kähler geometry

Before we can define a special Kähler manifold we have to recall a few definitions and facts of complex geometry which can be found for instance in [74].

Definition 1 *A 2n-dimensional manifold M that admits a globally defined (1,1) tensor J with local expression $J_m{}^n dx^m \otimes \partial_n$ having the property*

$$J_m{}^n J_n{}^k = -\delta_m{}^k, \tag{4.1.29}$$

is called an almost complex manifold and J is called an almost complex structure.

Using the almost complex structure J we can introduce the Nijenhuis tensor N, which is defined as

$$N_{mn}{}^p = \frac{1}{6} J_m{}^k \partial_{[k} J_{n]}{}^p - (m \leftrightarrow n). \tag{4.1.30}$$

Definition 2 *An almost complex structure with vanishing Nijenhuis tensor is called a complex structure and a manifold admitting a complex structure is called a complex manifold.*

Given a metric h_{mn} on an (almost) complex manifold we can construct a new metric g_{mn} as follows,

$$g_{mn} = \frac{1}{2}(h_{mn} + J_m{}^k J_n{}^l h_{kl})\,. \tag{4.1.31}$$

The metric g is positive definite if h is and it satisfies

$$g_{mn} = J_m{}^k J_n{}^l g_{kl}\,. \tag{4.1.32}$$

Definition 3 *A metric obeying (4.1.32) is called an (almost) hermitian metric and an (almost) complex manifold endowed with such a metric is called an (almost) hermitian manifold.*

Defining

$$J_{mn} \equiv J_m{}^k g_{kn}\,, \tag{4.1.33}$$

equation (4.1.32) is equivalent to

$$J_{mn} = -J_{nm}\,. \tag{4.1.34}$$

Thus, on an (almost) hermitian manifold, the (almost) complex structure defines a natural two-form, which is called the fundamental two-form.
On a complex manifold there exists a holomorphic atlas, i.e. a collection of charts (U_j, z_j) with the coordinates z_j taking values in $\mathbb{C}^n$ and with the transition functions, which relate coordinates in overlapping coordinate patches U_j and U_k, being holomorphic. We will denote the complex coordinates with Greek indices, e.g. $\{z^\alpha,\, \bar{z}^{\bar{\alpha}}\}$ with $\alpha = 1, \ldots n$, and the real coordinates with Latin indices, e.g $\{x^m\}$ with $m = 1, \ldots, 2n$. One can show that on a complex manifold there exists a holomorphic atlas such that

$$J_\alpha{}^\beta = i\delta_\alpha^\beta\,, \quad J_{\bar{\alpha}}{}^{\bar{\beta}} = -i\delta_{\bar{\alpha}}{}^{\bar{\beta}}\,, \quad J_\alpha{}^{\bar{\beta}} = J_{\bar{\alpha}}{}^\beta = 0\,. \tag{4.1.35}$$

A hermitian metric then takes the form

$$ds^2 = g_{\alpha\bar{\beta}} dz^\alpha dz^{\bar{\beta}}\,, \tag{4.1.36}$$

or put another way, $g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$, i.e components which are pure in its indices are zero, only mixed components are nonzero.

Theorem 1 *On a hermitian manifold there is a unique connection called the hermitian connection with the properties:*

1. *The covariant derivative of the metric vanishes.*
2. *The covariant derivative of the complex structure vanishes.*
3. *The torsion $\Gamma_{[mn]}{}^r$ is pure in its indices.*

One can show that the hermitian connection is pure in its indices and given by

$$\Gamma_{\alpha\beta}{}^{\gamma} = g^{\gamma\bar{\delta}}\partial_{\alpha}g_{\beta\bar{\delta}}\,. \tag{4.1.37}$$

From this it follows that the only nonzero components of the curvature tensor are of the form

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}}\,, \quad R_{\bar{\beta}\alpha\gamma\bar{\delta}}\,, \quad R_{\alpha\bar{\beta}\bar{\delta}\gamma}\,, \quad R_{\bar{\beta}\alpha\bar{\delta}\gamma}\,. \tag{4.1.38}$$

Their explicit form can be calculated from

$$R_{\alpha\bar{\beta}\bar{\gamma}}{}^{\bar{\delta}} = -R_{\bar{\beta}\alpha\bar{\gamma}}{}^{\bar{\delta}} = \partial_{\alpha}\Gamma_{\bar{\beta}\bar{\gamma}}{}^{\bar{\delta}}\,. \tag{4.1.39}$$

Definition 4 *A hermitian manifold is called Kähler if the fundamental two-form is closed, i.e.* $\partial_{[\alpha}J_{\beta\gamma]} = 0$.

The fact that the fundamental two-form is closed implies that a hermitian metric on a Kähler manifold, called a Kähler metric, in each coordinate patch can be written in terms of a real function $K = \bar{K} = K(z,\bar{z})$, known as the Kähler potential:

$$g_{\alpha\bar{\beta}} = \partial_{\alpha}\partial_{\bar{\beta}}K\,. \tag{4.1.40}$$

While the metric is defined globally, the Kähler potential is usually only defined locally. Two Kähler potentials on some nontrivial overlap of two patches are related by a Kähler transformation

$$K'(z,\bar{z}) = K(z,\bar{z}) + Ref(z)\,, \tag{4.1.41}$$

which leaves the metric invariant.
The hermitian connection of a Kähler manifold is symmetric in its lower indices, i.e. it is torsionless, and coincides with the Levi-Civita connection. The curvature tensor has some extra symmetries, it is symmetric in its holomorphic and anti-holomorphic indices,

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\gamma\bar{\beta}\alpha\bar{\delta}} = R_{\alpha\bar{\delta}\gamma\bar{\beta}}\,. \tag{4.1.42}$$

The Ricci tensor

$$R_{mn} = R^{k}{}_{mkn}\,, \tag{4.1.43}$$

has only mixed components

$$R_{\alpha\bar{\beta}} = g^{\gamma\bar{\delta}}R_{\bar{\delta}\alpha\gamma\bar{\beta}} = -g^{\gamma\bar{\delta}}R_{\alpha\bar{\beta}\gamma\bar{\delta}}\,. \tag{4.1.44}$$

Definition 5 *Consider a line bundle* $\mathcal{L}$ *over a Kähler manifold M. The first Chern form is given by*

$$\begin{aligned} c_1 &= \frac{i}{2\pi}\,\bar{\partial}\,(h^{-1}\partial h) = \frac{i}{2\pi}\,\bar{\partial}\,\partial\,\log\,h \\ &= \frac{i}{2\pi}\,\bar{\partial}\,\partial\log\,||\xi(z)||^2\,, \end{aligned} \tag{4.1.45}$$

where the real function $h(z,\bar{z})$ *is some hermitian fibre metric on* $\mathcal{L}$ *and* $||\xi(z)||^2 = h(z,\bar{z})\xi(z)\bar{\xi}(\bar{z})$ *is the norm of a holomorphic section of* $\mathcal{L}$.

Since $d = \partial + \bar{\partial}$ and $\partial^2 = \bar{\partial}^2 = 0$ on a complex manifold, it's easy to see that the first Chern two-form is closed, $dc_1 = 0$.

Definition 6 *Consider a set of closed p-forms ω_p and call two p-forms ω_p and ω'_p equivalent if*

$$\omega_p = \omega'_p + d\alpha_{p-1} \tag{4.1.46}$$

for some (p-1)-form α_{p-1}. An equivalence class of closed forms which only differ by exact forms $d\alpha$ is called a de Rham cohomology class.

We can thus associate a cohomology class to the first Chern form called the first Chern class and one can show that this class is independent of the fiber metric so that we can call it the first Chern class of the bundle $\mathcal{L}$ and we will denote this class with $c_1(\mathcal{L})$. Moreover the first Chern class is an integer cohomology class, meaning that if we integrate the first Chern two-form over any 2-cycle in the Kähler manifold with integer coefficients*, we obtain an integer. The fundamental 2-form J of a Kähler manifold is often called the Kähler form and since it is closed we can again associate a (de Rham) cohomology class with it. We will define the Kähler form κ to be $J/2\pi$ and denote its cohomology class with $[\kappa]$.

Definition 7 *A Hodge-Kähler manifold is a Kähler manifold for which there exists a complex line bundle $\mathcal{L}$ with the property:*

$$c_1(\mathcal{L}) \;=\; \frac{1}{2}[\kappa]\,. \tag{4.1.47}$$

Hodge-Kähler manifolds are the scalar manifolds of $\mathcal{N} = 1$ supergravity. The above definition is slightly nonstandard and implies that the Kähler form of a Hodge-Kähler manifold has even integer cohomology. In mathematics one usually defines a Hodge-Kähler manifold by requiring $c_1(\mathcal{L}) = [\kappa]$ instead, only demanding $[\kappa]$ to have integer cohomology. However, due to the presence of fermions, the Kähler forms have to have even integer cohomology in supergravity (see e.g. [75]), hence the above definition. An example of a Kähler-Hodge manifold is the n-dimensional complex projective space $\mathbb{C}P^n$ with the Fubini-Study metric as the Kähler metric.

Definition 8 *A special Kähler manifold is an n-dimensional Hodge-Kähler manifold $\mathcal{M}$ with the following properties [76]:*

- *There exists a holomorphic $Sp(2(n+1), \mathbb{R})$ vector bundle $\mathcal{H}$ over $\mathcal{M}$, and a holomorphic section $v(z)$ of $\mathcal{L} \otimes \mathcal{H}$ such that the Kähler potential is given by*

$$K = -\log[i\langle v, \bar{v}\rangle]\,, \tag{4.1.48}$$

where

$$\langle v, w\rangle \equiv v^T \Omega w\,, \quad \text{with} \quad \Omega = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\,, \tag{4.1.49}$$

denotes the symplectic inner product.

*A 2-cycle with integer coefficients is a formal sum of the form $\sum_i a_i N_i$ where a_i are integers and N_i are smooth 2-dimensional oriented submanifolds without boundary.

- *The section v obeys*

$$\langle \mathcal{D}_\alpha v, \mathcal{D}_\beta v \rangle = 0\,, \tag{4.1.50}$$

with $\mathcal{D}_\alpha v \equiv \partial_\alpha v + \partial_\alpha K$ the Kähler covariant derivative of v.

Using the notation

$$v = \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix}\,, \tag{4.1.51}$$

we can define

$$\mathcal{N}_{\Lambda\Sigma} = \begin{pmatrix} \mathcal{D}_{\bar\alpha}\bar F_\Lambda & F_\Lambda \end{pmatrix} \begin{pmatrix} \mathcal{D}_{\bar\alpha}\bar X^\Sigma & X^\Sigma \end{pmatrix}^{-1}\,. \tag{4.1.52}$$

This matrix completely determines the vector kinetic term, which is given by

$$\frac{1}{4}(Im\,\mathcal{N})_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\mu\nu\Sigma} - \frac{i}{8}(Re\,\mathcal{N})_{\Lambda\Sigma}\epsilon^{\mu\nu\rho\sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}\,. \tag{4.1.53}$$

One can show that the matrix $\begin{pmatrix} \mathcal{D}_{\bar\alpha}\bar X^\Lambda & X^\Lambda \end{pmatrix}$ is always invertible, so that $\mathcal{N}_{\Lambda\Sigma}$ in (4.1.52) is well defined. The matrix $\begin{pmatrix} \mathcal{D}_\alpha X^\Lambda & X^\Lambda \end{pmatrix}$ on the other hand is only invertible if a so called prepotential exists. While in general F_Λ can be independent of X^Λ, if a prepotential exists, then $F_\Lambda(X) = \partial F/\partial X^\Lambda$, with $F(X)$ a homogeneous function of second degree in the variables X^Λ, called the prepotential. One can also show, see [76] for all this, that there is always a symplectic transformation

$$\begin{aligned} v &\rightarrow \tilde v = \begin{pmatrix} A & B \\ C & D \end{pmatrix} v\,, \\ \mathcal{N} &\rightarrow (C + D\mathcal{N})(A + B\mathcal{N})^{-1}\,, \end{aligned} \tag{4.1.54}$$

with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n+2, \mathbb{R})\,, \tag{4.1.55}$$

so that $\tilde v$ can be written in terms of a prepotential. Since a symplectic transformation leaves the Kähler metric invariant, this means we can always describe the scalar manifold in terms of a prepotential. However, the kinetic term for the vector fields (4.1.53) is not always covariant under (4.1.54). Just like in $\mathcal{N} = 8$ supergravity, only the equations of motion are covariant under these symplectic transformations (also called duality transformations*), the Lagrangian is only covariant (up to a total derivative) under a subgroup ($B = 0$ in (4.1.54)). Symplectically rotated supergravity theories are therefore not necessarily equivalent. They might for instance allow for different gauge groups like in the $\mathcal{N} = 8$ case. Moreover, it was shown in [77] that partial symmetry breaking from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ can only occur in theories without a prepotential. The duality transformations also allow for the introduction of de Roo-Wagemans angles, which we will explain in more

*Duality transformations that do not change the prepotential up to a real quadratic polynomial in the X^Λ (i.e., $\tilde F(\tilde X) = F(\tilde X) + \frac{1}{2}C_{\Lambda\Sigma}\tilde X^\Lambda \tilde X^\Sigma$, with $C_{\Lambda\Sigma}$ components of a real symmetric matrix C) are called duality invariances or duality symmetries. They give rise to isometries of the special Kähler scalar manifold.

detail later on.
Just like the metric is the derivative of a more fundamental object (the Kähler potential), the holomorphic Killing vectors of a Kähler manifold can be written in terms of Killing prepotentials or moment maps P_Λ [78]:

$$\delta z^\alpha = \epsilon^\Lambda k^\alpha_\Lambda(z)\,, \quad k^\alpha_\Lambda(z) = ig^{\alpha\bar\beta}\partial_{\bar\beta}P_\Lambda\,, \quad P^*_\Lambda = P_\Lambda\,. \tag{4.1.56}$$

The symmetries of the scalar manifold can only be gauged if they are symmetries of the full supergravity theory. Since $\mathcal{N}_{\Lambda\Sigma}$ depends on the scalars, a transformation $z \to z'$ will induce a transformation $\mathcal{N} \to \mathcal{N}'$. This requires that the subgroup of the isometry group that is gauged, $G \subset Iso(\mathcal{M})$, can be embedded in the symplectic group $Sp(2n+2,\mathbb{R})$. Moreover, since the gauge fields are by definition in the adjoint of the gauge group G, we actually need [79]

$$\begin{aligned} t_\Lambda \to T_\Lambda &= \begin{pmatrix} t_\Lambda & 0 \\ C_\Lambda & -t^T_\Lambda \end{pmatrix} \in Sp(2n+2,\mathbb{R})\,, \\ (t_\Lambda)^\Delta{}_\Sigma &= f_{\Lambda\Sigma}{}^\Delta\,. \end{aligned} \tag{4.1.57}$$

with t_Λ the generators of the gauge group, $f_{\Lambda\Sigma}{}^\Delta$ the structure constants of the gauge algebra:

$$[t_\Lambda,\, t_\Sigma] = f_{\Lambda\Sigma}{}^\Delta t_\Delta\,, \tag{4.1.58}$$

and C_Λ symmetric matrices with components $C_{\Lambda,\Sigma\Delta}$ obeying

$$f_{\Gamma\Pi}{}^\Lambda C_{\Lambda,\Sigma\Delta} = 2f_{\Delta[\Gamma}{}^\Lambda C_{\Pi],\Sigma\Lambda} + 2f_{\Sigma[\Gamma}{}^\Lambda C_{\Pi],\Delta\Lambda}\,, \tag{4.1.59}$$

$$C_{\Lambda,\Sigma\Delta}X^\Lambda X^\Sigma X^\Delta = 0\,. \tag{4.1.60}$$

The Killing prepotentials can be calculated from the following expression [78, 79]:

$$P_\Lambda = \exp(K)\langle\bar v|T_\Lambda v\rangle\,. \tag{4.1.61}$$

In the rest of the thesis, we will restrict to semisimple gaugings, in which case $C_\Lambda = 0$.
All the symmetric special Kähler manifolds were classified in [80]. They found the following possibilities:

$$\frac{SU(1,n)}{SU(n)\times U(1)}\,, \tag{4.1.62}$$

$$\frac{SU(1,1)}{U(1)} \times \frac{SO(n-1,2)}{SO(n-1)\times SO(2)}\,, \tag{4.1.63}$$

$$\frac{Sp(6,\mathbb{R})}{U(3)}\,, \quad \frac{U(3,3)}{U(3)\times U(3)}\,,$$

$$\frac{SO^*(12)}{U(6)}\,, \quad \frac{E_{7,-26}}{E_6\times SO(2)}\,, \tag{4.1.64}$$

with $n = 1, 2, \ldots$ All these spaces are of the form $\mathcal{G}/\mathcal{H}$ with $\mathcal{G}$ a noncompact semisimple group and $\mathcal{H}$ the maximal compact subgroup of $\mathcal{G}$. The special Kähler manifolds in (4.1.62) can be described in terms of a quadratic prepotential

$$F = -iX^\Lambda \eta_{\Lambda\Sigma} X^\Sigma \,, \tag{4.1.65}$$

with η a constant real symmetric matrix of signature $(+-\cdots-)$. All the other symmetric spaces can be described with a cubic prepotential

$$F = \frac{d_{\alpha\beta\gamma} X^\alpha X^\beta X^\gamma}{X^0} \,. \tag{4.1.66}$$

Special Kähler manifolds with a cubic prepotential are called very special Kähler, because they are related by dimensional reduction to very special real spaces, which are the scalar manifolds of 5D $\mathcal{N} = 2$ supergravity coupled to vector multiplets (see section 4.2.2 for more info). When the cubic prepotential can be written as

$$F(X) = \frac{X^1(\eta_{ab} X^a X^b)}{X^0} \,, \quad a, b = 2, \ldots n \,, \tag{4.1.67}$$

it describes the special Kähler space in (4.1.63), which is the only special Kähler space that is also a product manifold, as proved in [81].
For all the known 2n-real-dimensional special Kähler manifolds $\mathcal{SM}$ we have $Iso(\mathcal{SM}) \subset Sp(2n+2, \mathbb{R})$. It was proved in [82] that this is always true for the very special Kähler manifolds. Whether there are any special Kähler manifolds with isometries that do not give rise to duality symmetries, i.e., which are only symmetries of the scalar sector and not of the full supergravity equations, is not known.
For completeness, we end this section with some formulas that will come in handy later on. Defining

$$\mathcal{V} = \begin{pmatrix} L^\Lambda \\ M_\Sigma \end{pmatrix} \equiv e^{K/2} v \,, \quad U_\alpha = \begin{pmatrix} f_\alpha^\Lambda \\ h_{\Sigma|\alpha} \end{pmatrix} \equiv \mathcal{D}_\alpha \mathcal{V} \,, \tag{4.1.68}$$

$$C_{\alpha\beta\gamma} \equiv i\langle \mathcal{D}_\alpha U_\beta, U_\gamma \rangle = i\langle \mathcal{V}, \partial_\alpha \partial_\beta \partial_\gamma \mathcal{V} \rangle \,, \tag{4.1.69}$$

one can derive

$$\mathcal{D}_\alpha \mathcal{D}_{\bar\beta} L^\Lambda = g_{\alpha\bar\beta} L^\Lambda \,, \qquad \mathcal{D}_\alpha \mathcal{D}_\beta L^\Lambda = C_{\alpha\beta\gamma} \mathcal{D}^\gamma \bar L^\Lambda \,, \tag{4.1.70}$$

while the curvature tensor has the following characteristic form,

$$R_{\bar\alpha\beta\bar\gamma\delta} = 2 g_{\bar\gamma(\beta} g_{\delta)\bar\alpha} - \bar C_{\bar\alpha\bar\gamma\bar\epsilon} C_{\beta\delta\epsilon} g^{\bar\epsilon\epsilon} \,. \tag{4.1.71}$$

Quaternionic-Kähler geometry

Definition 9 *A quaternionic-Kähler manifold is a 4m-dimensional real manifold with the following properties [78]:*

- *There exist three complex structures J^x that satisfy the quaternionic algebra*

$$J^x J^y = -\delta^{xy} \mathbb{1} + \epsilon^{xyz} J^z \,, \tag{4.1.72}$$

and a metric h

$$ds^2 = h_{uv}(q) dq^u dq^v \,; \quad u, v = 1, \ldots, 4m \,, \tag{4.1.73}$$

which is hermitian with respect to these complex structures.

- *There exists a principal SU(2)-bundle $\mathfrak{SU}$ such that*

  1.

$$\nabla K^x \equiv dK^x + \epsilon^{xyz} \omega^y \wedge K^z = 0 \,, \tag{4.1.74}$$

  where K^x is a triplet of 2-forms, called the HyperKähler form, and defined by

$$K^x = K^x_{uv} dq^u dq^v \,, \quad K^x_{uv} = h_{uw} (J^x)^w_v \,, \tag{4.1.75}$$

  and ω^x is a connection on the bundle $\mathfrak{SU}$,

  2. *the curvature of the SU bundle, defined by*

$$\Omega^x \equiv d\omega^x + \frac{1}{2} \epsilon^{xyz} \omega^y \wedge \omega^z \,, \tag{4.1.76}$$

  is proportional to the HyperKähler 2-form,

$$\Omega^x = \lambda K^x \,, \tag{4.1.77}$$

  with λ a nonvanishing real number.

It can be shown that quaternionic-Kähler manifolds are Einstein ($R_{uv} \sim h_{uv}$), see e.g. appendix B in [83], with scalar curvature $R = 8\lambda m(m+2)$, where we have adjusted for differences in notation, λ in [78] is $\nu/2$ in [83]. In supergravity, λ determines the normalization of the kinetic term for the hyperscalars. We have*

$$\mathcal{L} = \sqrt{-g} \left[\frac{R}{2} - g_{\alpha\bar{\beta}}(z, \bar{z}) \partial_\mu z^\alpha \partial^\mu z^{\bar{\beta}} + \lambda h_{uv}(q) \partial_\mu q^u \partial^\mu q^v + \ldots \right] . \tag{4.1.78}$$

Because the kinetic terms have to be positive, supergravity requires λ to be negative. The most common choices are $\lambda = -1$ and $\lambda = -1/2$. In this section we mostly use the notation of [78] and therefore put $\lambda = -1$.
The isometries of the hyperscalar quaternionic-Kähler manifold $\mathcal{HM}$ are global symmetries of the supergravity Lagrangian and we can therefore gauge a subgroup $H \subset Iso(\mathcal{HM})$. If H is non-Abelian, H also has to be a subgroup of the isometry group of the vector scalar manifold, since one needs vector fields transforming in the adjoint of H. In order

*The reader should not confuse the space-time metric $g_{\mu\nu}$ with the special Kähler metric $g_{\alpha\bar{\beta}}$.

to gauge symmetries of the quaternionic-Kähler manifold we need triholomorphic Killing vectors [78]

$$\vec{k}_\Lambda = k^u_\Lambda \frac{\partial}{\partial q^u}\,, \tag{4.1.79}$$

satisfying the Lie algebra of H. A Killing vector is triholomorphic if it leaves the quaternionic structure invariant up to SU(2) rotations in the SU(2) bundle $\mathcal{SU}$, i.e

$$\mathcal{L}_\Lambda K^x = \epsilon^{xyz} K^y W^z_\Lambda\,, \quad \mathcal{L}_\Lambda \omega^x = \nabla W^x_\Lambda \equiv dW^x_\Lambda + \epsilon^{xyz}\omega^y P^z_\Lambda\,, \tag{4.1.80}$$

where $\mathcal{L}$ is the Lie-derivative and W^x_Λ is an SU(2) compensator associated to the Killing vector k^u_Λ. The triholomorphicity of the Killing vectors is necessary for the existence of moment maps P^x_Λ, obeying

$$2k^u_\Lambda \Omega^x_{uv} = \nabla_v P^x_\Lambda \equiv \partial_v P^x_\Lambda + \epsilon^{xyz}\omega^y_v P^z_\Lambda\,. \tag{4.1.81}$$

They also have to satisfy the so called equivariance condition

$$2k^u_\Lambda k^v_\Sigma \Omega^x_{uv} - \epsilon^{xyz} P^y_\Lambda P^z_\Sigma = f_{\Lambda\Sigma}{}^\Delta P^x_\Delta\,, \tag{4.1.82}$$

where the structure constants are defined by

$$k^u_\Lambda \partial_u k^v_\Sigma - k^u_\Sigma \partial_u k^v_\Lambda = -f_{\Lambda\Sigma}{}^\Delta k^v_\Delta\,. \tag{4.1.83}$$

Triholomorphicity of the Killing vectors is not really a restriction, since it can be shown that all isometries of a quaternionic-Kähler manifold are triholomorphic [84]. Moment maps always exist for isometries of quaternionic-Kähler manifolds.
By dimensionally reducing a 4D $\mathcal{N} = 2$ supergravity coupled to vector multiplets to 3D, one obtains a map, called the c-map, relating a special Kähler manifold to a quaternionic-Kähler manifold (see [85] for a review). Not all quaternionic-Kähler manifolds can be obtained this way. The quaternionic-Kähler manifolds that can be related to a (very) special Kähler manifold by the c-map are called (very) special quaternionic-Kähler. For instance, the symmetric special Kähler manifolds in equations (4.1.62)-(4.1.64) have corresponding symmetric quaternionic-Kähler manifolds given by

$$\frac{SU(2,n+1)}{SU(n+1)\times SU(2)\times U(1)}\,, \quad n = 1,2,\dots\,, \tag{4.1.84}$$

$$\frac{G_{2(+2)}}{SU(2)\times SU(2)}\,, \quad \frac{SO(n+1,4)}{SO(n+1)\times SO(4)}\,, \quad n = 2,\dots\,, \tag{4.1.85}$$

$$\frac{F_{4(+4)}}{USp(6)\times USp(2)}\,, \quad \frac{E_{6(+2)}}{SU(6)\times SU(2)}\,, \quad \frac{E_{7(-8)}}{SO(12)\times SU(2)}\,, \quad \frac{E_{8(-24)}}{E_7\times SU(2)}\,. \tag{4.1.86}$$

The only other known symmetric quaternionic-Kähler spaces are

- $\frac{U(1,2)}{U(1)\times SU(2)}$, which can be obtained by dimensionally reducing pure 4D $\mathcal{N}=2$ supergravity to 3D,
- $\frac{USp(2n+2,2)}{USp(2n+2)\times USp(2)}$, which is not special,
- the Wolf spaces [86], which are compact versions of the spaces we already mentioned. They have positive scalar curvature and are therefore not allowed in supergravity.

The scalar potential

The scalar potential can be written as the sum of three distinct contributions[78, 87]:

$$\begin{aligned} V &= V_1 + V_2 + V_3\,, \\ V_1 &= g_{\alpha\bar\beta} k^\alpha_\Lambda \bar k^{\bar\beta}_\Sigma\, \bar L^\Lambda L^\Sigma = i\left(M_\Delta f_{\Lambda\Gamma}{}^\Delta \bar L^\Lambda\right)\left(L^\Sigma f_{\Sigma\Pi}{}^\Gamma \bar L^\Pi\right) + \text{h.c.}\,, \\ V_2 &= 4 h_{uv} k^u_\Lambda k^v_\Sigma\, \bar L^\Lambda L^\Sigma\,, \\ V_3 &= \left(U^{\Lambda\Sigma} - 3\bar L^\Lambda L^\Sigma\right)\, P^x_\Lambda P^x_\Sigma\,. \end{aligned} \tag{4.1.87}$$

The contributions V_1 and V_2 are manifestly positive, only V_3 can be negative. The term V_1 is only nonzero for non-Abelian gauge groups, since supersymmetry demands the vector scalars to transform in the adjoint like the vector fields. Abelian factors therefore have vanishing Killing vectors k^α_Λ. The last expression for V_1 can be derived by using the equivariance relation

$$\frac{i}{2} g_{\alpha\bar\beta}(k^\alpha_\Lambda \bar k^{\bar\beta}_\Sigma - k^\alpha_\Sigma \bar k^{\bar\beta}_\Lambda) = \frac{1}{2} f_{\Lambda\Sigma}{}^\Gamma P_\Gamma\,, \tag{4.1.88}$$

with $f_{\Lambda\Sigma}{}^\Gamma$ the structure constants of a semisimple Lie-algebra, and

$$k^\alpha_\Lambda L^\Lambda = \bar k^{\bar\alpha}_\Lambda \bar L^\Lambda = 0\,. \tag{4.1.89}$$

The term V_2 is only nonzero when there are hypermultiplets charged under the gauge group, otherwise we have $k^u_\Lambda = 0$. The Killing vectors k^u_Λ can be nonzero also for Abelian groups. The term V_3 describes interactions between the vector and hyper scalars. It can be nonzero even in the absence of hypers due to the possibility of including Fayet-Iliopoulos terms when the gauge group contains a $U(1)$ or $SO(3)$ factor.

Fayet-Iliopoulos terms

In the absence of hypermultiplets, equation (4.1.81) is meaningless and equation (4.1.82) reduces to

$$\epsilon^{xyz} P^y_\Lambda P^z_\Sigma = -f_{\Lambda\Sigma}{}^\Delta P^x_\Delta\,. \tag{4.1.90}$$

This equation can be solved if at least one of the factors G_i in the gauge group G,

$$\begin{aligned} G &= G_0 \times G_1 \times \ldots G_r\,, \\ \dim(G_k) &= d_k\,, \quad k = 0, \ldots r\,, \end{aligned} \tag{4.1.91}$$

is a $U(1)$ or $SO(3)$ factor. For instance for each Abelian U(1) generator $t_{\Lambda_\odot}$ in the gauge algebra, we can solve (4.1.90) by setting

$$P^x_\Sigma = \begin{cases} g\,\delta^x_3 & \text{for} \quad \Sigma = \Sigma_\odot\,, \\ 0 & \text{for} \quad \Sigma = \text{otherwise}, \end{cases} \tag{4.1.92}$$

where g is the coupling constant. Instead we can introduce a Fayet-Iliopoulos (FI) term for one of the $SO(3)$ factors by setting

$$P^x_\Sigma = \begin{cases} g\,\delta^x_y & \text{for} \quad \Sigma = y\,, \\ 0 & \text{for} \quad \Sigma = \text{otherwise}, \end{cases} \tag{4.1.93}$$

where for notational simplicity we have assumed the SO(3) factor is in the 1-2-3 directions, i.e. $f_{xy}{}^z = g\,\epsilon_{xyz}$ for $x, y, z = 1, 2, 3$. Generalizing to other cases is trivial.

de Roo-Wagemans angles

Before gauging the theory, we can perform a symplectic rotation on the holomorphic section of the special Kähler manifold, which will map the supergravity theory to an equivalent theory at the ungauged level. However, this will usually lead to inequivalent gauged models. The de Roo-Wagemans angles parametrize a rotation of the relative embeddings of the groups G_k inside $Sp(2n+2, \mathbb{R})$ and can be introduced by performing a symplectic rotation:

$$v \to v_R \equiv R(\vec{\theta})\, v\,, \tag{4.1.94}$$

with the following properties:

$$R(\vec{\theta}) \in Sp(2n+2, \mathbb{R})/Iso(\mathcal{SM})\,, \quad R^{-1}(\vec{\theta})\, G\, R(\vec{\theta}) = G\,. \tag{4.1.95}$$

A similar rotation was first introduced in $\mathcal{N} = 4$ supergravity in [89, 90, 91].

dS extrema: some simple examples

Minimal coupling with $U(1)$ FI term

Let us consider the simplest case of one vector multiplet minimally coupled to supergravity. In minimal coupling the special Kähler manifold is determined by the quadratic prepotential in (4.1.65). For the case of one vector multiplet we can therefore take the symplectic section to be

$$v = \begin{pmatrix} 1 \\ z \\ -2i \\ 2iz \end{pmatrix}\,, \quad \mathcal{V} \equiv e^{K/2} v\,. \tag{4.1.96}$$

$$K = -\log\left[4(1 - z\bar{z})\right]\,, \quad |z|^2 < 1\,. \tag{4.1.97}$$

The only contributions to the scalar potential come from possible FI terms. Taking $P_0^x = g_0\delta_3^x$ and $P_1^x = g_1\delta_3^x$, we have

$$V = g_0^2\left(-\frac{1}{4} - \frac{1}{2(1-|z|^2)}\right) + g_1^2\left(\frac{1}{4} - \frac{|z|^2}{2(1-|z|^2)}\right) - g_0 g_1\left(\frac{z+\bar{z}}{2(1-|z|^2)}\right). \quad (4.1.98)$$

As we have seen in previous sections, the isometry group $U(1,1)$ of the scalar manifold can be lifted to a symmetry of the full supergravity e.o.m. by embedding the isometry group in the symplectic group and accompanying a transformation of the scalars with a symplectic transformation of the vector fields. However, the only semisimple* (sub)group that can be implemented on the vector potentials and will thus leave the Lagrangian invariant is the real subgroup SO(1,1). In general, the gauging will break this global invariance of the ungauged theory. However, these transformations can still be useful. The SO(1,1) transformation

$$X^\Lambda \to X'^\Lambda = O^\Lambda{}_\Sigma X^\Sigma\,, \quad (4.1.99)$$

will map a gauged theory with coupling constants g_0 and g_1 to an identical theory but with gauge coupling constants g_0' and g_1', where

$$g'_\Lambda = \eta_{\Lambda\Sigma}O^\Sigma{}_\Gamma\eta^{\Gamma\Delta}g_\Delta\,. \quad (4.1.100)$$

Two Lagrangians with coupling constants related by (4.1.100) are therefore equivalent [92]. There are three inequivalent cases, characterized by the SO(1,1) invariant:

$$g_0^2 - g_1^2 > 0\,, \quad (4.1.101)$$
$$g_0^2 - g_1^2 = 0\,, \quad (4.1.102)$$
$$g_0^2 - g_1^2 < 0\,. \quad (4.1.103)$$

We may parametrize these cases as

$$1)\quad g_0 \neq 0\,,\ g_1 = 0\,, \quad (4.1.104)$$
$$2)\quad g_0 = g_1 \neq 0\,, \quad (4.1.105)$$
$$3)\quad g_0 = 0\,,\ g_1 \neq 0\,. \quad (4.1.106)$$

The potential in (4.1.98) becomes

$$1)\quad V = -g_0^2\left(\frac{1}{4} + \frac{1}{2(1-|z|^2)}\right), \quad (4.1.107)$$
$$2)\quad V = -g_0^2\frac{|1+z|^2}{2(1-|z|^2)}\,, \quad (4.1.108)$$
$$3)\quad V = g_1^2\left(\frac{1}{4} - \frac{|z|^2}{2(1-|z|^2)}\right). \quad (4.1.109)$$

*As mentioned before, we restrict to semisimple gaugings in this thesis. The only possibility is then to gauge $SO(1,1)$.

We find the following extrema

$$\begin{aligned} &1) \qquad z=0\,, \quad \Lambda = V|_0 = -\frac{3}{4}g_0^2\,, \\ &\qquad \frac{3m^2}{|\Lambda|} = 3\,\frac{\partial_z\partial^z V}{|V|}\bigg|_0 = -2\,, \qquad (4.1.110) \\ &2) \qquad \text{no extremum for } |z|<1 \qquad (4.1.111) \\ &3) \qquad z=0\,, \quad \Lambda = V|_0 = +\frac{1}{4}g_1^2\,, \\ &\qquad \frac{3m^2}{\Lambda} = 3\,\frac{\partial_z\partial^z V}{V}\bigg|_0 = -6\,. \qquad (4.1.112) \end{aligned}$$

Case 3) leads to an unstable de Sitter vacuum with characteristic tachyonic mass $m^2 = -2\Lambda$, as in (4.1.25).

$U(1)$ FI terms cannot lead to stable dS vacua: a theorem

We have seen in the previous paragraph that in minimal coupling a $U(1)$ FI term gives rise to an unstable dS vacuum. This turns out to be a special case of a general theorem: if the only contribution to the potential comes from a $U(1)$ FI term, then this potential has no stable dS extrema [87, 93]. We repeat here the proof given in [87]. Taking

$$P_\Lambda^x = g_\Lambda \delta^{x3}\,, \qquad (4.1.113)$$

the potential can be written as

$$V = g^{\alpha\bar\alpha}\mathcal{D}_\alpha W\,\mathcal{D}_{\bar\alpha}\bar W - 3|W|^2\,, \qquad W \equiv g_\Lambda L^\Lambda\,. \qquad (4.1.114)$$

Using the special geometry relations (4.1.70) we get the following condition for extrema of the potential:

$$\partial_\alpha V = C_{\alpha\beta\gamma}\mathcal{D}^\beta\bar W\,\mathcal{D}^\gamma\bar W - 2\bar W\mathcal{D}_\alpha W = 0\,. \qquad (4.1.115)$$

The second derivatives of the potential are

$$\begin{aligned} \mathcal{D}_\alpha\partial_\beta V &= (\mathcal{D}_\alpha C_{\beta\gamma\delta})\,\mathcal{D}^\gamma\bar W\,\mathcal{D}^\delta\bar W\,, \\ \partial^\beta\partial_\alpha V &= 2C_{\alpha\gamma\delta}\bar C^{\beta\gamma\epsilon}\mathcal{D}_\epsilon W\,\mathcal{D}^\delta\bar W - 2\mathcal{D}^\beta\bar W\,\mathcal{D}_\alpha W - 2\delta_\alpha^\beta|W|^2\,. \qquad (4.1.116) \end{aligned}$$

Using (4.1.115) and its complex conjugate, we find

$$\partial^\beta\partial_\alpha V\,\mathcal{D}_\beta W|_0 = -2V\,\mathcal{D}_\alpha W|_0\,, \qquad (4.1.117)$$

at an extremum of the potential. From (4.1.114) and (4.1.115), we see that $\mathcal{D}_\alpha W|_0 = 0$ leads to AdS vacua. A dS vacuum therefore always has a negative eigenvalue $-2V$ in the holomorphic-antiholomorphic derivative and is thus always unstable. Potentials with stable dS vacua therefore necessarily involve contributions coming from non-Abelian gaugings. We will study two simple examples of non-Abelian gaugings in the next paragraphs.

Minimal coupling with $SU(2)$ FI term

To gauge $SU(2)$ in minimal coupling we need at least 3 vector multiplets, and for simplicity we will take exactly 3 vector multiplets. We parametrize the symplectic section as

$$v = \begin{pmatrix} 1 \\ z^\alpha \\ -2i \\ 2iz^\alpha \end{pmatrix} , \quad \mathcal{V} \equiv e^{K/2} v , \quad \alpha = 1,2,3 . \tag{4.1.118}$$

$$K = -\log\left[4(1-|z|^2)\right] , \quad |z|^2 = \sum_{\alpha=1}^{3} z^\alpha \bar{z}^\alpha < 1 . \tag{4.1.119}$$

The potential is a sum of two terms,

$$V = V_1 + V_3 . \tag{4.1.120}$$

For V_1 we find

$$V_1 = \frac{1}{8(1-|z|^2)^2} \left(z^\alpha f_{\mu\nu}{}^\alpha \bar{z}^\mu f_{\beta\gamma}{}^\nu z^\beta \bar{z}^\gamma + \text{h.c.} \right) . \tag{4.1.121}$$

Taking $f_{\alpha\beta}{}^\gamma = g\,\epsilon_{\alpha\beta\gamma}$ with $\epsilon_{\alpha\beta\gamma}$ completely antisymmetric and $\epsilon_{123} = 1$, we can rewrite (4.1.121) as

$$V_1 = \frac{g^2}{4(1-|z|^2)^2} \left[\left(\sum_\alpha z^\alpha \bar{z}^\alpha \right)^2 - \left(\sum_\alpha z^\alpha z^\alpha \right) \left(\sum_\alpha \bar{z}^\alpha \bar{z}^\alpha \right) \right] . \tag{4.1.122}$$

The contribution due to the $SU(2)$ FI term is given by

$$V_3 = g^2 \left(\frac{3}{4} - \frac{\sum_\alpha z^\alpha \bar{z}^\alpha}{2(1-|z|^2)} \right) . \tag{4.1.123}$$

The potential V has two dS extrema with the following properties [92]:

$$\begin{aligned} 1) \qquad & z^\alpha = 0 , \quad \Lambda = V|_0 = \frac{3}{4} g^2 , \\ & SU(2) \text{ is unbroken} , \\ & 6 \text{ scalars with } \frac{3m^2}{\Lambda} = -2 , \end{aligned} \tag{4.1.124}$$

$$
\begin{aligned}
2) \qquad & |z|^2 = \frac{1}{2}\,, \sum_\alpha z^\alpha z^\alpha = 0\,, \quad \Lambda = V|_0 = \frac{1}{2}g^2\,, \\
& SU(2) \text{ is broken}\,, \\
& 1 \text{ scalar with } \frac{3m^2}{\Lambda} = 12\,, \\
& 2 \text{ scalars with } \frac{3m^2}{\Lambda} = -6\,, \\
& 3 \text{ massless scalars, giving masses to the } SU(2) \text{ gauge fields} \\
& \text{by the Brout-Englert-Higgs (BEH)-effect}\,. \qquad (4.1.125)
\end{aligned}
$$

The dS vacua are again unstable with all masses quantized in units of the cosmological constant.

Minimal coupling with $SO(2,1)$ gauging

To gauge $SO(2,1)$ in minimal coupling we only need 2 vector multiplets, because we can also use the gauge field in the gravitational multiplet. However, to allow for the possibility of including a U(1) FI term we will again study supergravity minimally coupled to 3 vector multiplets and use the same symplectic section as in the previous paragraph. If we gauge the $SO(2,1)$ symmetry between X^0, X^1 and X^2, we find

$$
\begin{aligned}
V_1 &= \frac{g^2}{4(1-|z|^2)^2}\left[\left(1-z_1^2-z_2^2\right)\left(1-\bar{z}_1^2-\bar{z}_2^2\right)-\left(1-|z_1|^2-|z_2|^2\right)^2\right]\,, \\
&= \frac{g^2}{4(1-|z|^2)^2}\left[-(z_1-\bar{z}_1)^2-(z_2-\bar{z}_2)^2+(z_1\bar{z}_2-z_2\bar{z}_1)^2\right]\,. \qquad (4.1.126)
\end{aligned}
$$

This potential has a family of Minkowski vacua at $z_1 = \bar{z}_1$, $z_2 = \bar{z}_2$, $z_3 = 0$. We can easily lift these vacua to dS vacua by adding also a $U(1)$ FI term

$$
P^x_\Lambda = g'\delta^{x3}\delta_{\Lambda 3}\,. \qquad (4.1.127)
$$

We find

$$
V_3 = g'^2\left(\frac{1}{4}-\frac{|z_3|^2}{2(1-|z|^2)}\right)\,. \qquad (4.1.128)
$$

The potential $V = V_1 + V_3$ has extrema with the following properties:

$$
\begin{aligned}
& z_1 = \bar{z}_1\,, z_2 = \bar{z}_2\,, z_3 = 0\,, \quad \Lambda = V|_0 = \frac{1}{4}g'^2\,, \\
& SO(2,1) \text{ broken to } SO(2)\,, \\
& 2 \text{ scalars with } \frac{3m^2}{\Lambda} = -6\,, \\
& 2 \text{ scalars with } \frac{3m^2}{\Lambda} = 12\frac{g^2}{g'^2}\,, \\
& 2 \text{ massless scalars, giving masses to the gauge fields associated} \\
& \text{with the noncompact generators of } SO(2,1) \text{ by the BEH-effect}\,. \qquad (4.1.129)
\end{aligned}
$$

The dS extrema are again unstable, but in this case not all the masses are quantized, some of them depend on g^2/g'^2, where g and g' are the coupling constants of the $SO(2,1)$ and $U(1)$ gauge group respectively.

Stable dS solutions

Given all the examples of unstable dS extrema we have discussed so far, one might wonder whether it is possible at all to have stable dS vacua in $\mathcal{N}=2$ supergravity. It turns out it is indeed possible, but it is not easy. The only known examples to date were constructed in [87], see also [88] for a short summary of the results.
All their models are based on the symmetric special Kähler manifold

$$\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2)\times SO(n)}\,. \tag{4.1.130}$$

While the above special Kähler manifolds can be described with a cubic prepotential, a symplectic section is used for which no prepotential exists,

$$v = \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix}\,, \quad X^\Lambda = \begin{pmatrix} \frac{1}{2}(1+y^2) \\ \frac{i}{2}(1-y^2) \\ y_a \end{pmatrix}\,, \quad F_\Lambda = S\eta_{\Lambda\Sigma}X^\Sigma\,, \tag{4.1.131}$$

$$\eta_{\Lambda\Sigma} = (+,+,-,\ldots,-)\,, \quad y^2 = \sum_{a=0}^{n-1}(y_a)^2\,, \quad \Lambda = 0,\ldots n+2\,, \tag{4.1.132}$$

The coordinates $\{z^\alpha\} = \{S, y_a\}\,(a=0,\ldots n-1)$ are called Calabi-Vesentini coordinates (see the appendix of [78] for more details). In this symplectic gauge, only the $SO(2,n)$ subgroup of the isometry group leaves the Lagrangian invariant. The $SU(1,1)$ transformations mix electric and magnetic fields and can not be gauged. In all models of [87] the $SO(2,1)$ symmetry between X^0, X^1, X^2 was gauged, together with a $U(1)$ or $SU(2)$ group, either with a FI term or with hypermultiplets. Another essential ingredient was a de Roo-Wagemans rotation between the $SO(2,1)$ and the $U(1)$ or $SU(2)$ gauge group.
To be a bit more concrete, we will look at the simplest model in more detail. The model consists of 3 vector multiplets ($n=2$ in (4.1.130)) with $SO(2,1)\times U(1)$ gauge group and a $U(1)$ FI term

$$P^x_\Lambda = e_1\,\delta^{x3}\delta_{\Lambda 3}\,. \tag{4.1.133}$$

The de Roo-Wagemans angles are introduced before gauging by rotating the symplectic section

$$v \to v_R = R(\theta)v\,, \tag{4.1.134}$$

with

$$R(\theta) = \begin{pmatrix} \mathbb{1}_3 & 0 & 0 & 0 \\ 0 & \cos(\theta) & 0 & \sin(\theta) \\ 0 & 0 & \mathbb{1}_3 & 0 \\ 0 & -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \in \frac{\mathrm{Sp}(8,\mathbb{R})}{\mathrm{SU}(1,1)\otimes\mathrm{SO}(2,2)}\,. \tag{4.1.135}$$

This leads to

$$V_3 = -e_1^2 \frac{1}{\mathrm{Im}\, S} |\cos(\theta) - S \sin(\theta)|^2 \,. \tag{4.1.136}$$

Positivity of the kinetic term for the vector fields requires $\mathrm{Im}\,\mathcal{N} < 0$ and therefore $\mathrm{Im}\, S < 0$, so we have $V_3 > 0$ and thus $V > 0$. The total potential becomes

$$V = V_1 + V_3 = -\frac{1}{2\mathrm{Im}\, S} \left({e_1}^2 |\cos\theta - S\, \sin\theta|^2 + {e_0}^2 \frac{P_2^+(y)}{P_2^-(y)} \right), \tag{4.1.137}$$

where e_0 and e_1 are the $SO(2,1)$ and $U(1)$ coupling constants respectively, and $P_2^\pm(y)$ are polynomial functions of holomorphic degree 2,

$$P_2^\pm(y) = 1 - 2\, y_0\, \overline{y}_0 \pm 2\, y_1\, \overline{y}_1 + y^2 \bar{y}^2 \,. \tag{4.1.138}$$

This potential has a 2-parameter family of dS extrema with the following properties:

$$\begin{aligned}
&y_0 = \text{arbitrary}\,, \quad y_1 = 0\,, \quad S = \frac{\cos(\theta)}{\sin(\theta)} - i \left| \frac{e_0}{e_1 \sin(\theta)} \right| \,, \\
&\Lambda = V|_0 = |e_0 e_1 \sin(\theta)| > 0\,, \quad SO(2,1) \text{ broken to } SO(2)\,, \\
&2 \text{ scalars with } \frac{3m^2}{\Lambda} = 3\,, \quad 2 \text{ scalars with } \frac{3m^2}{\Lambda} = 6\,, \\
&2 \text{ massless scalars, giving masses to the gauge fields associated} \\
&\text{with the noncompact generators of } SO(2,1) \text{ by the BEH-effect}\,.
\end{aligned} \tag{4.1.139}$$

The masses are again quantized in units of the cosmological constant, but this time they're all positive. Similar properties hold for the other models discussed in [87]. Whether all the ingredients (noncompact gauge groups, de Roo-Wagemans angles, FI terms or charged hypermultiplets) used in [87] are also necessary for the existence of stable dS vacua in $\mathcal{N} = 2$ supergravity is not known.

4.1.4 $\mathcal{N} = 4$ gauged supergravity

Pure $\mathcal{N} = 4$ supergravity can be coupled to an arbitrary number of vector multiplets, each containing 6 real scalars, 1 vector field and 4 spin 1/2 fermions. The most general matter coupled $\mathcal{N} = 4$ supergravity was constructed in [90]. The bosonic part of the Lagrangian density of the ungauged theory is [91, 94]

$$(\sqrt{-g})^{-1} \mathcal{L} = \frac{R}{4} + \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{vector}}\,, \tag{4.1.140}$$

$$
\begin{aligned}
\mathcal{L}_{\rm scalar} &= -\frac{1}{2}\left(\partial_\mu\phi^\alpha\partial^\mu\phi_\alpha + \phi^\alpha\partial_\mu\phi_\alpha\phi^\beta\partial^\mu\phi_\beta\right) \\
&\quad +\frac{1}{2}\left(\eta_{RS}+\eta_{RT}\eta_{SU}Z_c{}^TZ_c{}^U\right)\partial_\mu Z_a{}^R\partial^\mu Z_a{}^S\,, \qquad (4.1.141)\\
\mathcal{L}_{\rm vector} &= -\frac{1}{4}\eta_{RS}F_{\mu\nu}^{+R}F^{\mu\nu\,+S}\frac{1}{\Phi_{(R)}}(\phi^1_{(R)}-\phi^2_{(R)}) + \\
&\quad +\frac{1}{2}\eta_{RS}\eta_{TU}\frac{1}{\Phi_{(R)}}F^{\mu\nu\,+R}Z_a{}^S(Q^{-1})_{ab}Z_b{}^T\frac{1}{\Phi_{(T)}}F_{\mu\nu}^{+U}+\text{h.c.}\,, \qquad (4.1.142)\\
&\quad \text{with}\quad Q_{ab}=\eta_{RS}\frac{\Phi^*_{(R)}}{\Phi_{(R)}}Z_a{}^RZ_b{}^S\,. \qquad (4.1.143)
\end{aligned}
$$

The indices $\alpha,\beta,\ldots$ take on values 1 and 2, the indices $a,b,\ldots$ the values $1,\ldots,6$ and the indices $R,S\ldots$ the values $1,\ldots,6+n$, with n the number of vector multiplets. The metric η_{RS} can be chosen as $\text{diag}(-1,-1,-1,-1,-1,-1,+1,\ldots,+1)$, with n positive entries. The scalars ϕ_α ($\phi^1=(\phi_1)^*,\phi^2=-(\phi_2)^*$) and $Z_a{}^R$ satisfy the constraints

$$
\phi^\alpha\phi_\alpha = 1\,, \qquad (4.1.144)
$$

$$
\eta_{RS}Z_a{}^RZ_b{}^S = -\delta_{ab}\,. \qquad (4.1.145)
$$

The physical scalars therefore parametrize the cosets $SU(1,1)/U(1)$ and $SO(6,n)/SO(6)\times SO(n)$. For each vector field we can introduce a de Roo-Wagemans angle α_R,

$$
\phi^1_{(R)}=e^{i\alpha_R}\phi^1\,,\quad \phi^2_{(R)}=e^{-i\alpha_R}\phi^2\,,\quad \Phi_{(R)}=e^{i\alpha_R}\phi^1+e^{-i\alpha_R}\phi^2\,. \qquad (4.1.146)
$$

Here, ϕ^1 and ϕ^2 are complex fields and the de Roo-Wagemans angles α_R are real constants. These angles appear in the kinetic terms of the vectors $\mathcal{L}_{\rm vector}$ and in the scalar potential $V(\phi,Z)$. The gauge group has to be a subgroup of $SO(6,n)$. For a semisimple gauge group the α_R have to be the same for all R belonging to the same factor of the gauge group. Gauging induces a scalar potential of the form

$$
V(\phi,Z)=\left(\frac{1}{4}Z^{RU}Z^{SV}(\eta^{TW}+\frac{2}{3}Z^{TW})-\frac{i}{36}Z^{RSTUVW}\right)\Phi^*_{(R)}f_{RST}\Phi_{(U)}f_{UVW}\,, \qquad (4.1.147)
$$

where $Z^{RS}=Z_a{}^RZ_a{}^S$ and $Z^{RSTUVW}=\epsilon^{abcdef}Z_a{}^RZ_b{}^SZ_c{}^TZ_d{}^UZ_e{}^VZ_f{}^W$. The structure constants $f_{RST}=f_{RS}{}^T\eta_{VT}$ are totally antisymmetric.

$SU(2)\times SU(2)$ gauged $\mathcal{N}=4$ supergravity without matter

Let us study the scalar potential in a simple case, gauged $\mathcal{N}=4$ supergravity without vector multiplets. The largest semisimple group we can gauge in this case is $SU(2)\times SU(2)$. We can take the structure constants to be $f_{AB}{}^C=g_1\epsilon_{ABC456}$ for $A,B,C,=1,2,3$, $f_{AB}{}^C=g_2\epsilon_{123ABC}$ for $A,B,C,=4,5,6$, and zero otherwise. Calling the de Roo-Wagemans angles associated to the two $SU(2)$ groups α_1 and α_2 and solving the scalar constraint (4.1.145) by $Z_a{}^R=\delta_a{}^R$, the scalar potential becomes [90, 91]

$$
\begin{aligned}
V &= -\frac{1}{2}\left(g_1^2|\Phi_1|^2+g_2^2|\Phi_2|^2\right)-ig_1g_2\left(\Phi_1^*\Phi_2-\Phi_2^*\Phi_1\right)\,, \\
&= -\frac{1}{2}|g_1\Phi_1+ig_2\Phi_2|^2-g_1g_2\sin(\alpha_1-\alpha_2)\,, \qquad (4.1.148)
\end{aligned}
$$

with $\Phi_j = e^{i\alpha_j}\phi_1 + e^{-i\alpha_j}\phi_2$. Solving the constraint (4.1.144) as in [90] by,

$$\phi_1 = \frac{e^{i\beta}}{\sqrt{1-|Z|^2}}\,, \quad \phi_2 = \frac{Ze^{i(\alpha_1+\alpha_2)}}{\sqrt{1-|Z|^2}}\,, \quad |Z|<1\,, \tag{4.1.149}$$

where $(\alpha \equiv \alpha_1 - \alpha_2)$

$$e^{i\beta} = \frac{e^{i\alpha}g_1^2 + e^{-i\alpha}g_2^2}{|e^{i\alpha}g_1^2 + e^{-i\alpha}g_2^2|}\,, \tag{4.1.150}$$

the scalar potential becomes

$$V = -\frac{1}{2}\frac{1}{1-|Z|^2}\left[(g_1^2+g_2^2)(1+|Z|^2) - 2|e^{i\alpha}g_1^2 + e^{-i\alpha}g_2^2|X\right] - 2g_1g_2\sin\alpha\,, \tag{4.1.151}$$

with $Z = X + iY$. For $\sin\alpha = 0$ the potential has no critical point in the positivity domain $|Z|<1$. For $\sin\alpha \neq 0$ the potential has a critical point at

$$\begin{aligned} X_0 &= Z_0 = \frac{g_1^2+g_2^2-2|g_1g_2\sin\alpha|}{|e^{i\alpha}g_1^2+e^{-i\alpha}g_2^2|}\,, \\ Y_0 &= 0. \end{aligned} \tag{4.1.152}$$

The potential at the critical point is $V|_0 = -|g_1g_2\sin\alpha| - 2g_1g_2\sin\alpha$. For $2g_1g_2\sin\alpha < 0$ we have a dS critical point with the following properties*

$$\begin{aligned} &\Lambda = \kappa^2 V|_0 = 2|g_1g_2\sin\alpha|\,, \quad \text{with } \kappa^2 = 2\,, \\ &2 \text{ real scalars with } \frac{3m^2}{\Lambda} = -6\,. \end{aligned} \tag{4.1.153}$$

To compare with the dS extrema we discussed in the previous sections, we have adjusted for the different normalization used in [90, 91]. In general we have $(\sqrt{-g})^{-1}\mathcal{L} = R/(2\kappa^2) + \ldots$, so (4.1.140) indicates $\kappa^2 = 2$ was used, instead of the more common $\kappa^2 = 1$, which we used in previous sections.

At first sight it looks like the cosmological constant Λ depends on two arbitrary coupling constants and an arbitrary angle and can therefore be adjusted at will. However, this is an illusion. If we look at kinetic terms of the vector fields at the critical point, we find [90, 91]

$$\begin{aligned} \mathcal{L}_{\text{vector}}|_0 &= -\frac{1}{4}\sum_{R=1}^{3}(F_{\mu\nu}^{+R})^2\left(\left|\frac{g_1}{g_2\sin\alpha}\right| + i\cot\alpha\right) + \text{h.c.} \\ &\quad -\frac{1}{4}\sum_{R=4}^{6}(F_{\mu\nu}^{+R})^2\left(\left|\frac{g_2}{g_1\sin\alpha}\right| - i\cot\alpha\right) + \text{h.c.}\,. \end{aligned} \tag{4.1.154}$$

*The scalar masses were also calculated in [12], but one of their masses is wrong due to a mistake in the scalar kinetic term.

The terms proportional to $\cot\alpha$ add a total derivative to the action and can therefore be disregarded. The kinetic terms can be made canonical by rescaling the vector fields: ${A_\mu}^R \to {A'_\mu}^R/\Lambda^R$. The coupling constants will also get rescaled and we find

$$\begin{aligned} g'_1 &= \text{sign}\, g_1 \sqrt{|g_1 g_2 \sin\alpha|}\,, \\ g'_2 &= \text{sign}\, g_2 \sqrt{|g_1 g_2 \sin\alpha|}\,, \end{aligned} \tag{4.1.155}$$

suggesting that the only physical parameter is $g' = g_1 g_2 \sin\alpha$. This is also in agreement with [95, 96]. The $SU(2)\times SU(2)$ gauged $\mathcal{N}=4$ supergravity theories constructed in [95] have potentials exactly of the form (4.1.98) and it was shown in [96] that there always exists field redefinitions mapping them to one of three inequivalent theories with potentials of the form (4.1.107-4.1.109)*. It can easily be checked that for specific values of g_1, g_2 and α, the potential in (4.1.151) reduces to (4.1.98) and it seems very likely that the full $SU(2)\times SU(2)$ gauged theories we have discussed here are equivalent to those constructed in [95].

$SU(2)\times SU(2)$ gauged $\mathcal{N}=4$ supergravity with matter

We can easily extend the model in the previous section by adding one vector multiplet and retaining the same gauge group. The scalar constraint (4.1.145) can be solved by [90, 91]

$${Z_a}^7 = u_a\,, \tag{4.1.156}$$

$${Z_a}^A = {\delta_a}^A + \frac{u_a u_A}{1+\sqrt{1+u^2}}\,, \quad a, A = 1\ldots 6\,, \tag{4.1.157}$$

where $u^2 = \sum_{a=1}^6 u_a^2$. The scalar potential becomes

$$\begin{aligned} V &= -\frac{1}{2}\left(g_1^2|\Phi_1|^2 + g_2^2|\Phi_2|^2\right) - i g_1 g_2 \left(\Phi_1^*\Phi_2 - \Phi_2^*\Phi_1\right)\,, \\ &= -\frac{1}{2}\left(g_1^2|\Phi_1|^2 + g_2^2|\Phi_2|^2\right) - 2 g_1 g_2 \sin(\alpha_1-\alpha_2)\sqrt{1+u^2}\,. \end{aligned} \tag{4.1.158}$$

This potential does not depend on the de Roo-Wagemans angle of the added vector multiplet and differs from the previous potential only by the factor $\sqrt{1+u^2}$ in the last term. It has the same stationary points, with the additional scalar fields u_a equal to zero. A straightforward calculation shows that the metric of the u_a scalars is just the identity when $u_a = 0$. The dS critical points of the potential (4.1.158) therefore have the following properties:

$$\begin{aligned} &\Lambda = \kappa^2 V|_0 = 2|g_1 g_2 \sin\alpha|\,, \quad \text{with } \kappa^2 = 2\,, \\ &2 \text{ real scalars with } \frac{3m^2}{\Lambda} = -6\,, \\ &6 \text{ real scalars with} \frac{3m^2}{\Lambda} = 3\,. \end{aligned} \tag{4.1.159}$$

*The similarities with the case of $\mathcal{N}=2$ minimal coupling with FI term are not a coincidence, since it can be shown that the $\mathcal{N}=2$ theory can be obtained by truncating the $\mathcal{N}=4$ theories of [95] (see [92]).

Searching for stable dS vacua

The examples of dS vacua discussed so far again had tachyons and quantized masses and a natural questions arises: are this general properties of dS vacua in $\mathcal{N}=4$ gauged supergravity? An extensive search for stable dS vacua was done in [97]. They started with $\mathcal{N}=4$ gauged supergravity coupled to 6 vector multiplets. Splitting the indices $R,S,\ldots$ of η_{RS} in $A,B,\cdots=1,\ldots,6,(\eta_{AB}=-\delta_{AB})$ and $I,J,\cdots=7,\ldots 12,(\eta_{IJ}=+\delta_{IJ})$, the scalar constraint (4.1.145) can be rewritten as

$$XX^T - YY^T = \mathbb{1}_6\,, \tag{4.1.160}$$

with $X_a{}^A = Z_a{}^A,\ Y_a{}^{I-6} = Z_a{}^I$, which can be solved by

$$X = \frac{1}{2}\left(G + G^{-1} + BG^{-1} - G^{-1}B - BG^{-1}B\right)\,, \tag{4.1.161}$$

$$Y = \frac{1}{2}\left(G - G^{-1} - BG^{-1} - G^{-1}B - BG^{-1}B\right)\,, \tag{4.1.162}$$

with G a symmetric and B an antisymmetric 6×6 matrix. They then looked at all possible semisimple gaugings and investigated the resulting potentials at the origin of the $SO(6,6)/SO(6)\times SO(6)$ scalar manifold, given by $G=\mathbb{1}$, $B=0$. No stable dS vacua were found. Amongst that the gauge groups that lead to dS extrema they found two different classes. In one class the $SU(1,1)/U(1)$ scalars had positive m^2, but some of the matter scalars were tachyonic, while in the other class all the matter scalars had positive m^2, but the two $SU(1,1)/U(1)$ scalars were tachyons. They also found that is possible to have parameter-dependent masses, i.e. not all the masses are necessary quantized. Up to the day of this writing, no stable dS vacua are known in 4D $\mathcal{N}\geq 3$ gauged supergravity.

4.2 de Sitter vacua in 5D supergravity

Independently of the amount of supersymmetry, the bosonic part of an ungauged 5D supergravity theory can be written as

$$\begin{aligned}\mathcal{L}^{(ungauged)}_{(D=5)} &= \sqrt{-g}\left(\frac{R}{2} - \frac{1}{4}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}\,F^{\Sigma|\mu\nu} - \frac{1}{2}\,g_{ij}\,\partial_\mu\phi^i\,\partial^\mu\,\phi^j\right)\\ &\quad + \frac{1}{8}d_{\Lambda\Sigma\Gamma}\,\epsilon^{\mu\nu\rho\sigma\tau}\,F^{\Lambda}_{\mu\nu}\,F^{\Sigma}_{\rho\sigma}\,A^{\Gamma}_{\tau}\end{aligned} \tag{4.2.1}$$

where g_{ij} is the positive definite metric of the scalar manifold, $\mathcal{N}_{\Lambda\Sigma}(\phi)$ is a positive definite symmetric function of the scalars, and $d_{\Lambda\Sigma\Gamma}$ is a constant symmetric tensor.
With respect to 4D supergravity, there are some important differences:

- Due to the properties of spinors in 5D, only theories with 8, 16, 24, or 32 supercharges exist, usually denoted as $\mathcal{N}=2,4,6,8$ respectively. The R-symmetry group is $USp(\mathcal{N})$.

- There are no duality symmetries, instead a Chern-Simons term ($F \wedge F \wedge A$) is present.
- In order to gauge isometries of the scalar manifold in 4D, we had to realize them as symplectic duality symmetries on the vector fields. In 5D, the isometries of the scalar manifold have to be realized linearly on the vector fields and they have to leave the symmetric tensor $d_{\Lambda\Sigma\Gamma}$ invariant, in order to leave the full supergravity Lagrangian invariant.
- In the ungauged 5D theory, vectors A_μ and antisymmetric 2-tensors $B_{\mu\nu}$ are equivalent. They can be dualised into each other by*

$$\partial_{[\mu} A_{\nu]} = \epsilon_{\mu\nu}{}^{\lambda\rho\sigma} \partial_\lambda B_{\rho\sigma} \,. \tag{4.2.2}$$

  However, since only the 1-forms can be used as non-Abelian gauge vectors, they will be distinct in the gauged theory. In order to preserve supersymmetry when gauging, all vector fields transforming nontrivially under the gauge group and not in the adjoint representation have to be dualised to antisymmetric tensors†.

As in 4D, supersymmetry implies that the scalar potential equals 'the square of the fermion supersymmetry transformations'. The relations (4.1.9) and (4.1.10) are not special for 4D, but can be generalized to arbitrary dimensions.

4.2.1 $\mathcal{N} = 8$ gauged supergravity

The ungauged $\mathcal{N} = 8$ $D = 5$ supergravity theory has an action invariant under a global $E_{6(6)}$ symmetry, with the 27 vector fields transforming as the 27 of E_6 and the 42 scalar fields parametrising the coset space $E_{6(6)}/USp(8)$. Gauging the theory leads to a potential, which is however difficult to calculate explicitly. Just like in $D = 4$, one usually reduces the complexity by choosing a compact subgroup of the gauge group and restricting to the singlets under the chosen group.
Only one dS vacuum is known, it arises in the $SO(3,3)$ gauged supergravity, first constructed in [99]. To construct the theory, 12 vector fields are dualized to antisymmetric 2-tensors. The remaining 15 vector fields transform as the $(15,1)$ of the global symmetry group $SL(6,\mathbb{R}) \times SL(2,\mathbb{R})$, and can be used to gauge the subgroups $SO(p,q)$ of $SL(6,\mathbb{R})$, with $p+q=6$ and $p,q \geq 0$. Gauging $SO(3,3)$ and restricting to a particular‡ $SO(3) \times SO(3)$ invariant scalar λ, leads to the following scalar potential [67]

$$V(\lambda) = 6[3(\partial_\lambda w(\lambda))^2 - 4w(\lambda)^2] \,, \tag{4.2.3}$$

*This is an on-shell duality, i.e., equations of motion have to be imposed. The duality relates the Bianchi identity of the vector field A_μ to the equation of motion of the antisymmetric 2-tensor $B_{\mu\nu}$ and it relates the Bianchi identity of the antisymmetric 2-tensor $B_{\mu\nu}$ to the equation of motion of the vector field A_μ.

†Recently a new formulation of 5D supergravity was discovered, in which vectors and tensors are on equal footing, allowing for a uniform description of all possible gaugings [73, 98]. However, in this thesis we will continue to use the older formulation.

‡See page 40 of [99] for more detailed information.

with the superpotential $w(\lambda)$ given by

$$w(\lambda) = \frac{g}{8}(e^{\sqrt{\frac{2}{3}}\lambda} - e^{-\sqrt{\frac{2}{3}}\lambda}) \,. \tag{4.2.4}$$

The part of the supergravity action relevant for our purposes is

$$S = \int d^5x\sqrt{-g}\left(\frac{R}{2} - \frac{1}{2}\partial^\mu\lambda\partial_\mu\lambda - V(\lambda)\right) \,. \tag{4.2.5}$$

The scalar potential $V(\lambda)$ has a dS critical point with the following properties:

$$\begin{aligned}
&\lambda = 0\,, \quad \Lambda = V|_0 = \frac{3}{4}g^2\,, \\
&SO(3,3) \text{ broken to } SO(3)\times SO(3)\,, \\
&1 \text{ scalar with } m^2l^2 \equiv \frac{6m^2}{\Lambda} = -8\,, \\
&\text{Masses of other scalars have not been calculated}\,.
\end{aligned} \tag{4.2.6}$$

The dS vacuum is unstable, and the mass of the tachyon is again quantized.

4.2.2 $\mathcal{N} = 2$ gauged supergravity

Pure $\mathcal{N} = 2$ supergravity, containing a vielbein, 2 spin 3/2 fermions (gravitini), and a vector field (the graviphoton), can be coupled to an arbitrary number of vector, tensor and hypermultiplets. Vector multiplets consist of a vector field, two spin 1/2 fermions and a real scalar, tensor multiplets contain an antisymmetric tensor, two spin 1/2 fermions and a real scalar, and hypermultiplets have two spin 1/2 fermions and 4 real scalars. Just like in 4D, the hypermultiplet scalars parametrize a quaternionic-Kähler manifold. The combined scalars of the vector and tensor multiplets parametrize a 'very special real' manifold, which we will define now.

Very special real geometry

Very special real manifolds are completely determined by a symmetric 3-tensor C_{IJK}*. Let M be the following $n+1$ dimensional subspace of $\mathbb{R}^{n+1}$

$$M = \{h^I \in \mathbb{R}^{n+1} | N(h) = C_{IJK}h^Ih^Jh^K > 0\}\,, \tag{4.2.7}$$

with metric

$$a_{IJ} = -\frac{1}{3}\partial_I\partial_J \ln N(h)\,. \tag{4.2.8}$$

Then the very special real manifold $\mathcal{M}_n$ can be defined as the hypersurface $N(h) = 1$ with metric the induced metric from the embedding space M,

$$g_{xy} = \frac{3}{2}\overset{\circ}{a}_{IJ}h^I_{,x}h^J_{,y} = -3C_{IJK}h^Ih^J_{,x}h^K_{,y}\,, \tag{4.2.9}$$

*Most of this section is taken from the appendix in [100], which was based on the appendix in [102] and the classic paper on $5D$ $\mathcal{N} = 2$ supergravity [101].

with $h^I(\varphi)$ obeying $C_{IJK}h^I(\varphi)h^J(\varphi)h^K(\varphi)=1$, the comma denoting an ordinary derivative with respect to φ^x and*

$$\mathring{a}_{IJ}\equiv a_{IJ}|_{N=1}=-2C_{IJK}h^K+3h_Ih_J\,,\quad h_I\equiv C_{IJK}h^Jh^K=\mathring{a}_{IJ}h^J\,. \tag{4.2.10}$$

Defining

$$h^I_x\equiv-\sqrt{\frac{3}{2}}h^I_{,x}(\varphi)\,, \tag{4.2.11}$$

we have $h_Ih^I_x=0$, leading to

$$h_{Ix}\equiv\mathring{a}_{IJ}h^J_x=\sqrt{\frac{3}{2}}h_{I,x}(\varphi)\,,\quad h^Ih_{Ix}=0\,. \tag{4.2.12}$$

Using the above relations we can also write $\mathring{a}_{IJ}$ as

$$\mathring{a}_{IJ}=h_Ih_J+h^x_Ih_{Jx}\,, \tag{4.2.13}$$

and we have

$$\begin{aligned} h_{Ix;y} &= \sqrt{\frac{2}{3}}\left(h_Ig_{xy}+T_{xyz}h^z_I\right)\,,\\ h^I{}_{x;y} &= -\sqrt{\frac{2}{3}}\left(h^Ig_{xy}+T_{xyz}h^{Iz}\right)\,, \end{aligned} \tag{4.2.14}$$

where the semicolon is a covariant derivative using the Christoffel connection calculated from the metric g_{xy}, with

$$T_{xyz}\equiv C_{IJK}h^I_xh^J_yh^K_z\,. \tag{4.2.15}$$

The previous relations can be used to derive some useful identities. Comparing (4.2.13) and (4.2.10), we obtain

$$h^x_Ih_{Jx}=-2C_{IJK}h^K+2h_Ih_J\,, \tag{4.2.16}$$

and taking the covariant derivative with respect to φ^y of (4.2.16) leads to

$$T_{xyz}h^x_Ih^z_J=C_{IJL}h^L_y+h_{(I}h_{J)y}\,. \tag{4.2.17}$$

Finally, after a straightforward calculation, we get

$$T_{xyz;u} = \sqrt{\frac{3}{2}}[g_{(xy}g_{z)u}-2T_{(xy}{}^wT_{z)uw}]\,, \tag{4.2.18}$$

$$C^{IJK}{}_{,x} = 2T_{uvw;x}h^{Iu}h^{Jv}h^{Kw}\,, \tag{4.2.19}$$

where we defined

$$C^{IJK}\equiv\mathring{a}^{II'}\mathring{a}^{JJ'}\mathring{a}^{KK'}C_{I'J'K'}\,, \tag{4.2.20}$$

*It should be understood that the h^I obey $N(h)=1$ from here on.

with $\mathring{a}^{IJ}$ being the inverse of $\mathring{a}_{IJ}$. The formula (4.2.19) was found originally in [103], but with an erroneous factor 3 instead of 2.
The domain of the variables should be limited to $h^I(\varphi) \neq 0$ and the metrics $\mathring{a}_{IJ}$ and g_{xy} should be positive definite. The latter two conditions are equivalent.
The homogeneous very special real manifolds were classified in [104]. The symmetric very special real manifolds are a subclass of these, and where already found in [101] and [105]. They can be divided into two subclasses, depending on whether they are associated with Jordan algebras or not:

1. When $\mathcal{M}$ is associated with a Jordan algebra, there are two subclasses:

   - The 'generic' or 'reducible' Jordan class:

   $$\mathcal{M} = SO(1,1) \times \frac{SO(n-1,1)}{SO(n-1)}, \qquad n \geq 1. \tag{4.2.21}$$

   - The 'irreducible' or 'magical' Jordan class:

   $$\begin{aligned} \mathcal{M} &= SL(3,\mathbb{R})/SO(3), && (n=5) \\ \mathcal{M} &= SL(3,\mathbb{C})/SU(3), && (n=8) \\ \mathcal{M} &= SU^*(6)/USp(6), && (n=14) \\ \mathcal{M} &= E_{6(-26)}/F_4. && (n=26) \end{aligned}$$

2. There is one class which is not associated with a Jordan algebra, and which is therefore referred to as the 'symmetric non-Jordan family':

   $$\mathcal{M} = \frac{SO(1,n)}{SO(n)}, \qquad n > 1. \tag{4.2.22}$$

Homogeneous manifolds are manifolds for which its isometry group works transitively (on the manifold). The group G of linear transformations of the h^I that leave C_{IJK} invariant is a subgroup of the isometry group $Iso(\mathcal{M})$, but it is not the whole isometry group in general. For instance, the symmetric non-Jordan family (4.2.22) has isometries that are not in G [82]. Strictly speaking, only those homogeneous spaces for which G works transitively are classified in [104]. To the best of our knowledge there is no proof that there are no other homogeneous (or even symmetric) very special real spaces (for which $Iso(\mathcal{M})$ works transitively, but G does not).

The scalar potential

Gauged $\mathcal{N} = 2$ supergravity coupled to n vector multiplets, $2m$ tensor multiplets* and r hypermultiplets was constructed in [102, 106, 107]. We will use the following notation. The n vector fields of the vector multiplets are combined with the graviphoton into a

*The existence of an action functional requires an even number of tensor multiplets [83].

single $(n+1)$-plet of vector fields A^I_μ labeled by an index I (or J or K) running from 0 to n, while the tensor fields $B^M_{\mu\nu}$ are labeled by an index M (or N or P), running over $2m$ values. The combined scalars of the vector and tensor multiplets are denoted by φ^x, with x (or y or z) running over $n+2m$ values, and the very special real coordinates $h^{\tilde{I}}$, are labeled by an index $\tilde{I} = (I, M)$ (or $\tilde{J}$ or $\tilde{K}$), which runs over the $n+2m+1$ values of I and M.
The scalar potential can be schematically written as

$$V = V^{(T)} + V^{(H)}\,, \tag{4.2.23}$$

with $V^{(T)}$ only nonzero if tensors are present and $V^{(H)}$ the contribution coming from the hypermultiplet sector or from FI terms. Unlike in 4D, non-Abelian vectors do not contribute a term to the potential [103, 108]. The potential $V^{(T)}$ can be written as [109]*

$$V^{(T)} = \frac{3\sqrt{6}}{16} h^I \Lambda_I^{MN} h_M h_N, \tag{4.2.24}$$

with Λ^M_{IN} the transformation matrices of the tensor fields under the gauge group and

$$\Lambda_I^{MN} \equiv \Lambda^M_{IP}\Omega^{PN} = \frac{2}{\sqrt{6}}\Omega^{MR} C_{IRP}\Omega^{PN}, \tag{4.2.25}$$

where Ω^{MN} is the inverse of Ω_{MN}, which is a constant invariant antisymmetric tensor of the gauge group

$$\Omega_{MN} = -\Omega_{NM}, \qquad \Omega_{MN}\Omega^{NP} = \delta^P_M. \tag{4.2.26}$$

For the potential $V^{(H)}$ we have the following general expression†

$$V^{(H)} = -4\vec{P}\cdot\vec{P} + 2\vec{P}^x\cdot\vec{P}_x + 2\mathcal{N}_{iA}\mathcal{N}^{iA}\,, \tag{4.2.27}$$

$$\vec{P} = h^I\vec{P}_I\,, \quad \vec{P}_x = h^I_x\vec{P}_I\,, \quad \mathcal{N}^{iA} = \frac{\sqrt{6}}{4} h^I K^X_I f^{iA}_X. \tag{4.2.28}$$

Here f^{iA}_X are the quaternionic vielbeins, $f^{iA}_X f_{YiA} = g_{XY}$ with g_{XY} the metric of the quaternionic-Kähler hypermultiplet scalar manifold, K^X_I are the Killing vectors and $\vec{P}_I$ the corresponding prepotentials. In the absence of hypermultiplets, $V^{(H)}$ is not necessarily zero, since there is the possibility of $U(1)$ or $SU(2)$ FI terms, just like in 4D. Because adding FI terms can be interpreted as gauging (part of) the $SU(2)_R$ R-symmetry, we will denote the potential due to FI terms as $V^{(R)}$. We thus have

$$V^{(R)} = -4\vec{P}\cdot\vec{P} + 2\vec{P}^x\cdot\vec{P}_x\,. \tag{4.2.29}$$

When we gauge the full R-symmetry group $SU(2)_R$ we have

$$\vec{P}_I = \vec{e}_I\xi\,, \tag{4.2.30}$$

*We assume $C_{MNP} = C_{IJM} = 0$. More general tensor couplings with $C_{IJM} \neq 0$ are possible, see [102], but we will not consider these here.

†Here $\vec{P}_I$ are 3-vectors with components P^r_I, $r = 1, \ldots, 3$, and $\vec{P}_I.\vec{P}_J = \sum_{r=1}^3 P^r_I P^r_J$ is the scalar product.

where ξ is an arbitrary constant, and $\vec{e}_I$ are constants that are nonzero only for I in the range of the $SU(2)$ factor and satisfy

$$\vec{e}_I \times \vec{e}_J = f_{IJ}{}^K \vec{e}_K \,, \tag{4.2.31}$$

with $f_{IJ}{}^K$ the $SU(2)$ structure constants. From now on, we will use indices $A, B, \ldots$ for the $SU(2)$ factor. We can then take $\vec{e}_A \cdot \vec{e}_B = \delta_{AB}$ such that (using (4.2.16))

$$V^{(R)}_{SU(2)} = -4\xi^2 C^{AB\tilde{I}} h_{\tilde{I}} \delta_{AB} \,. \tag{4.2.32}$$

In the case of $U(1)_R$ gauging we have

$$\vec{P}_I = \xi_I \vec{e}, \tag{4.2.33}$$

where $\vec{e}$ is an arbitrary unit 3-vector and ξ_I are constants that define the linear combination of the vector fields A^I_μ that is used as the $U(1)_R$ gauge field

$$A_\mu[U(1)_R] = \xi_I A^I_\mu. \tag{4.2.34}$$

They have to be constrained by

$$\xi_I f^I_{JK} = 0, \tag{4.2.35}$$

with f^I_{JK} being the structure constants of the gauge group that is gauged simultaneously. Using the very special geometry identities of section 4.2.2, the $U(1)$ FI potential can be written as

$$V^{(R)}_{U(1)} = -4C^{IJ\tilde{K}} \xi_I \xi_J h_{\tilde{K}}. \tag{4.2.36}$$

Finally, we remark that when $\mathcal{M}$ is associated with a Jordan algebra [101], one has (componentwise)

$$C^{\tilde{I}\tilde{J}\tilde{K}} = C_{\tilde{I}\tilde{J}\tilde{K}} = \text{const.} \tag{4.2.37}$$

This simplifies the calculation of $V^{(R)}$ considerably.

U(1) FI potential has no stable de Sitter vacua: a theorem

Analogous to the 4D case discussed in section 4.1.3, we proved in [100] that the $U(1)$ FI potential in equation (4.2.36) has no de Sitter minima. We will repeat the proof here.
In our conventions, a critical point φ_c of the potential with $V^{(R)}(\varphi_c) > 0$ corresponds to a de Sitter vacuum. A critical point, by definition, obeys the following relation

$$\frac{\partial V^{(R)}}{\partial \varphi^x}(\varphi_c) = -4C^{IJK} \xi_I \xi_J h_{K,x}|_{\varphi_c} = 0 \,. \tag{4.2.38}$$

For the mass matrix we find

$$g^{yz} \frac{\partial V^{(R)}}{\partial \varphi^x \partial \varphi^z}\bigg|_{\varphi_c} = \left(\frac{2}{3} V^{(R)} \delta^y_x - 8\sqrt{\frac{2}{3}} \xi_I \xi_J h^{Iu} h^{Jv} T_{uvx;z} g^{zy} \right)\bigg|_{\varphi_c} , \tag{4.2.39}$$

where we have used (4.2.15), (4.2.19) and (4.2.38). We will now show that $\xi_I h^I_y(\varphi_c)$ is an eigenvector of this matrix. Equation (4.2.18) leads to

$$T_{uvx;y}h^{Iu}h^{Jv}h^{Ky}\xi_I\xi_J\xi_K = \sqrt{\frac{3}{2}}[h^{Iu}h^J_u\xi_I\xi_J\xi_K h^K_x - 2\xi_I\xi_J h^{Iu}h^{Jv}T_{uv}{}^w T_{wyx}\xi_K h^{Ky}]\,. \quad (4.2.40)$$

Since, using (4.2.17) and (4.2.38), we have

$$\xi_I\xi_J h^{Iu}h^{Jv}T_{uv}{}^w|_{\varphi_c} = \xi_I h^I \xi_J h^{Jw}|_{\varphi_c}\,, \quad (4.2.41)$$

we finally obtain

$$\begin{aligned} T_{uvx;y}h^{Iu}h^{Jv}h^{Ky}\xi_I\xi_J\xi_K|_{\varphi_c} &= \sqrt{\frac{3}{2}}[h^{Iu}h^J_u\xi_I\xi_J - 2(\xi_I h^I)^2]\xi_K h^K_x|_{\varphi_c}\,, \\ &= \sqrt{\frac{3}{8}}V^{(R)}\xi_K h^K_x|_{\varphi_c}\,. \end{aligned} \quad (4.2.42)$$

We thus find that $\xi_I h^I_y(\varphi_c)$ is indeed an eigenvector of the mass matrix with eigenvalue $-10V^{(R)}(\varphi_c)/3$. Looking at equations (4.2.29) and (4.2.33) we see that if a critical point with $V^{(R)} > 0$ exists, then also $\xi_I h^I_y(\varphi_c) \neq 0$. This proves that in case of a de Sitter extremum, the mass matrix has always at least one negative eigenvalue.
To illustrate our proof, let us give an example. The Jordan symmetric spaces only lead to Anti-de Sitter or Minkowski extrema (see [103]), but we will show now that there are also a lot of models with de Sitter extrema.
Equations (4.2.36), (4.2.38) and (4.2.13) lead to

$$C^{IJK}\xi_J\xi_K|_{\varphi_c} = -\frac{1}{4}V^{(R)}(\varphi_c)h^I(\varphi_c)\,. \quad (4.2.43)$$

This is a necessary and sufficient condition for φ_c to be a critical point. Furthermore, for φ_c in the domain where a_{IJ} is positive definite, one can always perform a linear transformation on the h^I such that $h^I(\varphi_c) = (1, 0, \ldots, 0)$ and $a_{IJ}(\varphi_c) = \delta_{IJ}$. After this transformation the polynomial $N(h)$ will take the following form

$$N(h) = (h^0)^3 - \frac{3}{2}h^0 h^i h^j \delta_{ij} + C_{ijk}h^i h^j h^k\,, \qquad i = 1, \ldots, \tilde{n}, \quad (4.2.44)$$

which is called the canonical parametrization of $N(h)$. Equation (4.2.43) then becomes

$$C_{0JK}\xi_J\xi_K = -\frac{1}{4}V^{(R)}(\varphi_c)\,, \quad (4.2.45)$$

$$C_{ijk}\xi_j\xi_k - \xi_0\xi_i = 0\,, \quad (4.2.46)$$

with summation over repeated indices. Equation (4.2.36) however leads to

$$V^{(R)}(\varphi_c) = -4C_{0JK}\xi_J\xi_K = -4\xi_0^2 + 2\sum_i \xi_i^2\,, \quad (4.2.47)$$

such that (4.2.45) is automatically fulfilled. So, given a theory (a tensor C_{ijk}) , we look for a vector ξ_I that solves equation (4.2.46) and for which

$$-4\xi_0^2 + 2\sum_i \xi_i^2 > 0\,. \tag{4.2.48}$$

As we know, this is not possible for general C_{ijk}. However, one can construct a lot of examples. Take for example $C_{ijk} = 0$. Equation (4.2.46) in this case leads to $\xi_i = 0$, corresponding to anti-de Sitter vacua or $\xi_0 = 0$, corresponding to de Sitter vacua. To study the mass matrix, we look at the particular example $n = 1$. We then have the following polynomial

$$N(h) = (h^0)^3 - \frac{3}{2}h^0(h^1)^2\,. \tag{4.2.49}$$

The constraint $N = 1$ can be solved by

$$h^0 = \frac{\varphi}{2} + \frac{\sqrt{\varphi+\varphi^4}}{2\varphi} \tag{4.2.50}$$

$$h^1 = \sqrt{\frac{2}{3}}\left(-\frac{3}{2}\varphi + \frac{\sqrt{\varphi+\varphi^4}}{2\varphi}\right)\,. \tag{4.2.51}$$

The metric on the scalar manifold is (using (4.2.9))

$$g_{xy} = \frac{3+12\varphi^3}{4(\varphi^2+\varphi^5)}\,. \tag{4.2.52}$$

We restrict to the region $\varphi > 0$, which contains the point $h^I(\varphi_c = 1/2) = (1, 0)$ and where the metric is positive definite.
Taking $\xi_0 = 0$, equation (4.2.46) is fulfilled for arbitrary values of ξ_1 and equation (4.2.47) becomes

$$V^{(R)}(\varphi_c) = 2\xi_1^2 > 0\,. \tag{4.2.53}$$

Using the metric (4.2.52) and with $\xi_0 = 0$ we get for the potential

$$V^{(R)}(\varphi) = \frac{-1-8\varphi^3-40\varphi^6+12\varphi\sqrt{\varphi+\varphi^4}+24\varphi^4\sqrt{\varphi+\varphi^4}}{2(\varphi+4\varphi^4)}\xi_1^2\,, \tag{4.2.54}$$

which indeed fulfils (4.2.53). Furthermore,

$$V^{(R)}_{,x}(\varphi_c) = 0\,, \tag{4.2.55}$$

and

$$g^{yz}V^{(R)}_{,x,z}|_{\varphi_c} = -\frac{20}{3}\xi_1^2\,, \tag{4.2.56}$$

which is indeed $-10V^{(R)}(\varphi_c)/3$ as stated in the theorem above.

de Sitter vacua from the $SU(2)$ FI potential

For the known symmetric spaces, the $SU(2)$ FI potential in equation (4.2.32) does not have any critical points (see [110]). There are, however, also a lot of models with unstable de Sitter vacua, as we will show here [100]. Proving or disproving that the $SU(2)$ FI potential has no de Sitter minima, seems to be more difficult than in the $U(1)$ case, and is still an open problem. We start from a polynomial in the canonical parametrization. In order to gauge $SU(2)_R$, the polynomial should have an $SU(2)_G$ symmetry [110]. Without charged tensors this further restricts the coefficients C_{ijk} by [111]

$$C_{ABC} = 0\,, \qquad C_{AB\alpha} = c_\alpha \delta_{AB} \qquad C_{A\alpha\beta} = 0\,, \tag{4.2.57}$$

with c_α some arbitrary constants. We have split the indices $i = 1, \ldots, n$ as $i = (A, \alpha)$ with $A, B, \ldots \in \{1, 2, 3\}$ corresponding to the $SU(2)$ factor of the gauge group. The $C_{\alpha\beta\gamma}$ are still unconstrained.
Using expression (4.2.32) for the $SU(2)$ FI potential, the equation analogous to (4.2.43) is

$$C^{ABI}\delta_{AB}|_{\varphi_c} = -V^{(R)}(\varphi_c) h^I(\varphi_c)\,. \tag{4.2.58}$$

We now assume that $h^I(\varphi_c) = (1, 0, \ldots, 0)$. Equation (4.2.58) then leads to the conditions

$$V^{(R)}(\varphi_c) = \frac{3}{2}\,, \tag{4.2.59}$$

$$c_\alpha = 0\,, \quad \forall\, \alpha\,, \tag{4.2.60}$$

and therefore $C_{ABi} = 0$, $\forall\, i$. The first condition is again automatically fulfilled and tells us that all these critical points are de Sitter.
We now investigate the stability of these de Sitter vacua. Calculating the second derivative of the potential, we get

$$V^{(R)}_{,x;y} = -C^{ABI}{}_{;y}\delta_{AB} h_{I,x} - C^{ABI}\delta_{AB} h_{I,x;y}\,. \tag{4.2.61}$$

Using $h^I(\varphi_c) = (1, 0, \ldots, 0)$, $a_{IJ}(\varphi_c) = \delta_{IJ}$ and (4.2.14) we get for the second term in the critical point

$$-C^{ABI}\delta_{AB} h_{I,x;y}|_{\varphi_c} = \frac{2}{3} V^{(R)}(\varphi_c) g_{xy}(\varphi_c) = g_{xy}(\varphi_c)\,. \tag{4.2.62}$$

For the first term we also use (4.2.19), (4.2.18), (4.2.15) and (4.2.57), which gives after some calculation

$$-C^{ABI}{}_{;y}\delta_{AB} h_{I,x}|_{\varphi_c} = -2 g_{xy}(\varphi_c) - \frac{4}{3} h^A_x h^B_y \delta_{AB}\,, \tag{4.2.63}$$

and therefore

$$g^{zy} V^{(R)}_{,x;y}|_{\varphi_c} = -\delta^z_x - \frac{4}{3} h^A_x h^{Bz} \delta_{AB}\,. \tag{4.2.64}$$

This matrix has the following eigenvectors

$$\begin{cases} h^A_x(\varphi_c) & \text{with eigenvalue } -\frac{7}{3}\,, \\ h^\alpha_x(\varphi_c) & \text{with eigenvalue } -1\,. \end{cases} \tag{4.2.65}$$

The de Sitter vacua are therefore always maxima of the potential.

Stable de Sitter vacua in 5D $\mathcal{N}=2$ gauged supergravity: a few examples

The previous section made clear that $U(1)_R$ gauging alone cannot give rise to stable de Sitter vacua. We show here that adding tensor multiplets can change this. The gauging we study in this section was first performed in [109] for $n=3$. They found de Sitter extrema, but did not check their stability. In [100] we generalized the construction to arbitrary n and showed that the obtained de Sitter vacua are all stable.
We consider $\mathcal{N}=2$ supergravity coupled to n Abelian vector multiplets and with scalar manifold $\mathcal{M}=SO(1,1)\times\frac{SO(n-1,1)}{SO(n-1)}, n\geq 1$. The polynomial can then be written in the following form

$$N(h)=3\frac{\sqrt{3}}{2}\,h^0[(h^1)^2-(h^2)^2-\ldots-(h^n)^2]\,. \tag{4.2.66}$$

With $x=1,\ldots,n$, introducing

$$\eta_{xy}=\eta^{xy}=\mathrm{Diag}(1,-1,\ldots,-1)\,, \tag{4.2.67}$$

we write the $C_{\tilde{I}\tilde{J}\tilde{K}}$ symbols as

$$C_{0\underline{xy}}=\frac{\sqrt{3}}{2}\eta_{xy}\,. \tag{4.2.68}$$

(we underline x-type indices that are in fact of type $\tilde{I}$, but take values in the x-range due to our choice of coordinates). The constraint $N=1$ can be solved by

$$h^0=\frac{1}{\sqrt{3}\|\varphi\|^2}\,,\qquad h^{\underline{x}}=\sqrt{\frac{2}{3}}\,\varphi^x\,,$$

with

$$\|\varphi\|^2=\varphi^x\eta_{xy}\varphi^y\,. \tag{4.2.69}$$

The hypersurface $N=1$ decomposes into three disconnected components:
(i) $\|\varphi\|^2>0$ and $\varphi^1>0$
(ii) $\|\varphi\|^2<0$
(iii) $\|\varphi\|^2>0$ and $\varphi^1<0$.
In the following, we will consider the "positive timelike" region (i) only, since in region (ii), g_{xy} and $\mathring{a}_{\tilde{I}\tilde{J}}$ are not positive definite , and region (iii) is isomorphic to region (i).
We now proceed by gauging the above theory. The isometry group of the scalar manifold is $SO(n-1,1)\times SO(1,1)$. We gauge the noncompact subgroup $SO(1,1)\subset SO(n-1,1)$ together with $U(1)_R\subset SU(2)_R$. The $SO(1,1)$ subgroup rotates h^1 and h^2 into each other and therefore acts nontrivially on the vector fields A^1_μ and A^2_μ. In order for the resulting theory to be supersymmetric, these vectors have to be dualized to antisymmetric tensor fields. We can thus decompose the index $\tilde{I}$ in the following way

$$\tilde{I}=(I,M)\,, \tag{4.2.70}$$

with $I,J,K,\ldots=0,3,4,\ldots,n$ and $M,N,P,\ldots=1,2$.

Furthermore, we need a vector transforming in the adjoint of $SO(1,1)$ (which means it should be inert) to act as its gauge field. Looking at $\Lambda^M_{IN} \sim C_{IRN}\Omega^{MR}$, we see that only A^0_μ couples to the tensor fields (only $C_{0RP} \neq 0$) and thus acts as the gauge field of $SO(1,1)$. The remaining vectors are called 'spectator fields' with respect to the $SO(1,1)$ gauging.
Finally, for the $U(1)_R$ gauge field we take a linear combination $A_\mu[U(1)_R] = \xi_I A^I_\mu$ of the vectors. We now have the ingredients to calculate the potentials (4.2.24) and (4.2.36) (taking $\Omega^{12} = -\Omega^{21} = -1$):

$$\Lambda^M_{0N} = \frac{2}{\sqrt{6}}\Omega^{MR}C_{0RN} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.2.71}$$

$$V^{(T)} = \frac{3\sqrt{6}}{16}h^I\Lambda^{MN}_I h_M h_N = \frac{1}{8}\frac{(\varphi^1)^2 - (\varphi^2)^2}{\|\varphi\|^6}, \tag{4.2.72}$$

$$V^{(R)} = -4\sqrt{2}\xi_0\xi_i\varphi^i\|\varphi\|^{-2} + 2|\xi|^2\|\varphi\|^2, \qquad |\xi|^2 \equiv \xi_i\xi_i, \tag{4.2.73}$$

where we defined a new index i as $I = (0,i)$. Then

$$V = V^{(T)} + V^{(R)} = \frac{1}{8}\frac{(\varphi^1)^2 - (\varphi^2)^2}{\|\varphi\|^6} - 4\sqrt{2}\xi_0\varphi^i\xi_i\|\varphi\|^{-2} + 2\|\varphi\|^2|\xi|^2. \tag{4.2.74}$$

Demanding $V_{,x} = 0$ gives the following conditions on the critical points

$$\frac{\varphi^i}{\|\varphi\|^4} = 16\sqrt{2}\xi_0\xi_i, \tag{4.2.75}$$

$$\frac{1}{\|\varphi\|^6} = -\frac{1}{2}\left(16\sqrt{2}\xi_0|\xi|\right)^2 + 8|\xi|^2, \tag{4.2.76}$$

with the constraints

$$\begin{aligned} &|\xi|^2 \neq 0, \\ &32\xi_0^2 < 1. \end{aligned} \tag{4.2.77}$$

From (4.2.76) we see that $\|\varphi\|^2$ is completely determined by the ξ_I. From (4.2.75) we see that also the φ^i are completely fixed by the ξ_I. This means that only φ^1 and φ^2 can still vary, as long as the combination $(\varphi^1)^2 - (\varphi^2)^2$ remains fixed. We thus have a one parameter family of critical points, which is due to the unbroken $SO(1,1)$. There is also an unbroken $SO(n-3)$, but the vacuum is at the symmetric point.
The value of the potential in the critical points becomes

$$V(\varphi_c) = 3\|\varphi\|^2|\xi|^2\left(1 - 32\xi_0^2\right), \tag{4.2.78}$$

which is clearly positive because of (4.2.77) and therefore corresponds to de Sitter vacua. We now show that all these vacua are stable. We use the index $m = 1,2$ for the scalars related to $M = 1,2$. Calculating the second derivatives of the potential in the critical

points gives the following Hessian

$$\begin{aligned} V_{,m,n}(\varphi_c) &= (\eta\varphi)_m(\eta\varphi)_n\left[3\|\varphi\|^{-8}+4\varphi^k\varphi^k\|\varphi\|^{-10}\right], \\ V_{,m,i}(\varphi_c) &= -4(\eta\varphi)_m\varphi^i\left[\|\varphi\|^{-8}+\varphi^k\varphi^k\|\varphi\|^{-10}\right], \\ V_{,i,j}(\varphi_c) &= \varphi^i\varphi^j\left[5\|\varphi\|^{-8}+4\varphi^k\varphi^k\|\varphi\|^{-10}\right]+\frac{1}{4}\delta_{ij}\|\varphi\|^{-6}, \end{aligned} \tag{4.2.79}$$

where $(\eta\varphi)_x \equiv \eta_{xy}\varphi^y$.
The $SO(1,1)$ invariance implies a zero eigenvector $\varphi^n(\sigma_1)_n{}^m$. Using this $SO(1,1)$ and the $SO(n-2)$ of the φ^i, we may further use for any critical point $\varphi_c = (\varphi^1, 0, \varphi^3, 0, \ldots, 0)$ with $|\varphi^3| < |\varphi^1|$. Then the zero mode is φ^2, and the sector $\varphi^4, \ldots, \varphi^n$ decouples as a unit matrix times $\frac{1}{4}\|\varphi\|^{-6}$. The relevant part of the Hessian therefore is

$$\partial\partial V = |\|\varphi\||^{-10}\begin{pmatrix} (\varphi^1)^2\left[3(\varphi^1)^2+(\varphi^3)^2\right] & -4(\varphi^1)^3\varphi^3 \\ -4(\varphi^1)^3\varphi^3 & \frac{1}{4}\left[(\varphi^1)^4+18(\varphi^1)^2(\varphi^3)^2-3(\varphi^3)^4\right] \end{pmatrix}, \tag{4.2.80}$$

where we defined $\partial\partial V \equiv \partial_x\partial_y V(\varphi_c)|_{x,y=1,3}$. The determinant and trace are

$$\det\partial\partial V = \frac{3}{4}(\varphi^1)^2\|\varphi\|^{-14}, \qquad \mathrm{Tr}\,\partial\partial V = \frac{1}{4}\|\varphi\|^{-10}\left[13(\varphi^1)^4+22(\varphi^1)^2(\varphi^3)^2-3(\varphi^3)^4\right], \tag{4.2.81}$$

which shows that the eigenvalues are positive.
The masses of the scalars are given by the eigenvalues of $g^{xy}V_{x,y}(\varphi_c)$. Using Mathematica, the scalar mass spectrum turns out to be

$$m^2l^2 \equiv \frac{6m^2}{\Lambda} = (0, 16\frac{(\varphi_1)^2-(\varphi_2)^2}{\|\varphi\|^2}, 4, 4, ..., 4)\,. \tag{4.2.82}$$

All the scalar masses but one are quantized in units of the cosmological constant. One of the scalar masses depends on the parameters ξ_I that determine the $U(1)_R$ gauge field. Like in [87, 88], the massless scalar is a Goldstone boson. It will get 'eaten' by the SO(1,1) gauge field, making the gauge field massive (Brout-Englert-Higgs effect). There will thus be only positive mass scalars left in the effective theory.
Instead of gauging $U(1)_R$ we could also have gauged the full $SU(2)_R$ R-symmetry as long as there are a sufficient number of gauge fields available ($n \geq 5$). Without loss of generality, we can choose A^3_μ, A^4_μ, A^5_μ as the $SU(2)$ gauge fields. We then find

$$V^{(R)} = \frac{3}{2}\|\varphi\|^2\,. \tag{4.2.83}$$

Looking at equation (4.2.73), we observe that we get the same potential if we do a $U(1)_R$ gauging with $\xi_0 = 0$ and $|\xi|^2 = 3/4$. $SU(2)_R$ gauging with tensors charged under $SO(1,1)$ therefore will also lead to stable de Sitter vacua. The scalar masses are in this case given by $\frac{6m^2}{\Lambda} = (0, 16, 4, 4, ..., 4)$. Observe that now all scalar masses are quantized.
We can also try to imitate the effect of R-symmetry gauging by adding charged hypermultiplets. Let us look at the simplest example, the universal hypermultiplet. The

hyperscalar geometry is the quaternionic-Kähler space $\frac{SU(2,1)}{SU(2)\times U(1)}$. The metric of the universal hypermultiplet, together with the Killing vectors and corresponding prepotentials were given in [112]. The four hyperscalars q^X are labeled as $\{\mathcal{V}, \sigma, \theta, \tau\}$ and the metric is given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}\mathcal{V}^2}{2\mathcal{V}^2} + \frac{1}{2\mathcal{V}^2}\left(\mathrm{d}\sigma + 2\theta\,\mathrm{d}\tau - 2\tau\,\mathrm{d}\theta\right)^2 + \frac{2}{\mathcal{V}}\left(\mathrm{d}\tau^2 + \mathrm{d}\theta^2\right) . \tag{4.2.84}$$

The determinant for this metric is $1/\mathcal{V}^6$ and since the metric is positive definite in $\theta = \tau = 0$ if $\mathcal{V} > 0$, the metric will be positive-definite and well-behaved everywhere as long as $\mathcal{V} > 0$. This parametrization of the universal hypermultiplet is the one that comes out naturally from the Calabi-Yau compactifications of M-theory, where $\mathcal{V}$ acquires the meaning of the volume of the Calabi-Yau manifold. The metric (4.2.84) has an SU(2,1) isometry group generated by the following eight Killing vectors k^X_α

$$\begin{aligned}
\vec{k}_1 &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{k}_2 = \begin{pmatrix} 0 \\ 2\theta \\ 0 \\ 1 \end{pmatrix}, \quad \vec{k}_3 = \begin{pmatrix} 0 \\ -2\tau \\ 1 \\ 0 \end{pmatrix}, \quad \vec{k}_4 = \begin{pmatrix} 0 \\ 0 \\ -\tau \\ \theta \end{pmatrix}, \\
\vec{k}_5 &= \begin{pmatrix} \mathcal{V} \\ \sigma \\ \theta/2 \\ \tau/2 \end{pmatrix}, \quad \vec{k}_6 = \begin{pmatrix} 2\mathcal{V}\sigma \\ \sigma^2 - \left(\mathcal{V} + \theta^2 + \tau^2\right)^2 \\ \sigma\theta - \tau\left(\mathcal{V} + \theta^2 + \tau^2\right) \\ \sigma\tau + \theta\left(\mathcal{V} + \theta^2 + \tau^2\right) \end{pmatrix}, \\
\vec{k}_7 &= \begin{pmatrix} -2\mathcal{V}\theta \\ -\sigma\theta + \mathcal{V}\tau + \tau\left(\theta^2 + \tau^2\right) \\ \frac{1}{2}\left(\mathcal{V} - \theta^2 + 3\tau^2\right) \\ -2\theta\tau - \sigma/2 \end{pmatrix}, \quad \vec{k}_8 = \begin{pmatrix} -2\mathcal{V}\tau \\ -\sigma\tau - \mathcal{V}\theta - \theta\left(\theta^2 + \tau^2\right) \\ -2\theta\tau + \sigma/2 \\ \frac{1}{2}\left(\mathcal{V} + 3\theta^2 - \tau^2\right) \end{pmatrix} .
\end{aligned} \tag{4.2.85}$$

The corresponding prepotentials P^r_α are given by

$$\begin{aligned}
\vec{P}_1 &= \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{4\mathcal{V}} \end{pmatrix}, \quad \vec{P}_2 = \begin{pmatrix} -\frac{1}{\sqrt{\mathcal{V}}} \\ 0 \\ -\frac{\theta}{\mathcal{V}} \end{pmatrix}, \quad \vec{P}_3 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{\mathcal{V}}} \\ \frac{\tau}{\mathcal{V}} \end{pmatrix}, \quad \vec{P}_4 = \begin{pmatrix} -\frac{\theta}{\sqrt{\mathcal{V}}} \\ -\frac{\tau}{\sqrt{\mathcal{V}}} \\ \frac{1}{2} - \frac{\theta^2+\tau^2}{2\mathcal{V}} \end{pmatrix}, \\
\vec{P}_5 &= \begin{pmatrix} -\frac{\tau}{2\sqrt{\mathcal{V}}} \\ \frac{\theta}{2\sqrt{\mathcal{V}}} \\ -\frac{\sigma}{4\mathcal{V}} \end{pmatrix}, \quad \vec{P}_6 = \begin{pmatrix} -\frac{1}{\sqrt{\mathcal{V}}}\left[\sigma\tau + \theta\left(-\mathcal{V} + \theta^2 + \tau^2\right)\right] \\ \frac{1}{\sqrt{\mathcal{V}}}\left[\sigma\theta - \tau\left(-\mathcal{V} + \theta^2 + \tau^2\right)\right] \\ -\frac{\mathcal{V}}{4} - \frac{1}{4\mathcal{V}}\left[\sigma^2 + \left(\theta^2 + \tau^2\right)^2\right] + \frac{3}{2}\left(\theta^2 + \tau^2\right) \end{pmatrix}, \\
\vec{P}_7 &= \begin{pmatrix} \frac{4\theta\tau+\sigma}{2\sqrt{\mathcal{V}}} \\ \frac{3\tau^2-\theta^2}{2\sqrt{\mathcal{V}}} - \frac{\sqrt{\mathcal{V}}}{2} \\ -\frac{3}{2}\tau + \frac{1}{2\mathcal{V}}\left[\sigma\theta + \tau\left(\theta^2 + \tau^2\right)\right] \end{pmatrix}, \quad \vec{P}_8 = \begin{pmatrix} -\frac{3\theta^2-\tau^2}{2\sqrt{\mathcal{V}}} + \frac{\sqrt{\mathcal{V}}}{2} \\ \frac{\sigma-4\theta\tau}{2\sqrt{\mathcal{V}}} \\ \frac{3}{2}\theta + \frac{1}{2\mathcal{V}}\left[\sigma\tau - \theta\left(\theta^2 + \tau^2\right)\right] \end{pmatrix} .
\end{aligned} \tag{4.2.86}$$

The Killing vectors K^X_I are now given by $\xi^\alpha_I k^X_\alpha$, where the components of the embedding matrix ξ^α_I are constants that determine which isometries are gauged and which vector fields are used to gauge them. The corresponding prepotentials $\vec{P}_I$ then become $\xi^\alpha_I \vec{P}_\alpha$.

We now choose to gauge the $U(1)$ (hypermultiplet) isometry generated by $2\vec{k}_4 - \vec{k}_1 - \vec{k}_6$, so we take

$$\vec{K}_I = \xi_I(2\vec{k}_4 - \vec{k}_1 - \vec{k}_6)\,, \quad \vec{P}_I = \xi_I(2\vec{P}_4 - \vec{P}_1 - \vec{P}_6) = \xi_I\vec{Q}\,, \tag{4.2.87}$$

where we have defined $\vec{Q} \equiv 2\vec{P}_4 - \vec{P}_1 - \vec{P}_6$. For the scalar manifold of the vector multiplets we choose $\mathcal{M} = SO(\tilde{n}-1,1) \times SO(1,1)/SO(\tilde{n}-1), \tilde{n} \geq 1$, and again gauge a noncompact $SO(1,1)$ isometry of this manifold by dualizing the two charged vector fields to tensor fields. For simplicity, we do not charge the hypers under this symmetry. Our gauge group is thus $SO(1,1) \times U(1)$, where two tensors are charged under the $SO(1,1)$ and the hypers are charged under the $U(1)$. We then find that

$$\frac{\partial V}{\partial q^X}(\varphi, q_c) = 0\,, \tag{4.2.88}$$

$$V(\varphi, q_c) = \frac{9}{4}V^{(R)}(\varphi) + V^{(T)}(\varphi)\,, \tag{4.2.89}$$

where $q_c = \{\mathcal{V} = 1, \sigma = 0, \theta = 0, \tau = 0\}$ and with $V^{(R)}$ and $V^{(T)}$ given in equations (4.2.73) and (4.2.72) respectively.
To verify this, first notice that $K_I^X|_{q_c} = 0$ and therefore

$$\mathcal{N}_{iA}|_{q_c} = 0\,, \qquad \left.\frac{\partial(\mathcal{N}_{iA}\mathcal{N}^{iA})}{\partial q^X}\right|_{q_c} = 0\,. \tag{4.2.90}$$

We also have $\frac{\partial V^{(T)}}{\partial q^X} = 0$ since $V^{(T)}$ only depends on the scalars of the vector multiplets. The remaining term $-4\vec{P}\cdot\vec{P} + 2\vec{P}^x \cdot \vec{P}_x$ in equation (4.2.27) can be written as

$$-4\vec{P}\cdot\vec{P}(\varphi, q) + 2\vec{P}^x \cdot \vec{P}_x(\varphi, q) = -4C^{IJK}\xi_I\xi_J h_K(\varphi)\vec{Q}\cdot\vec{Q}(q)\,, \tag{4.2.91}$$

which shows that the φ (vector multiplet) and q (hypermultiplet) dependence of this part factorizes. Since $\vec{Q}|_{q_c} = (0, 0, 3/2)$, to verify equation (4.2.88) we only need to check that $\frac{\partial Q^3}{\partial q^X}|_{q_c} = 0$, with Q^3 the third component of the vector $\vec{Q}$. Because Q^3 is quadratic in θ, σ and τ we have

$$\frac{\partial Q^3}{\partial \theta} = \frac{\partial Q^3}{\partial \sigma} = \frac{\partial Q^3}{\partial \tau} = 0 \quad \text{if} \quad \theta = \sigma = \tau = 0\,. \tag{4.2.92}$$

Finally

$$Q^3|_{\theta=\sigma=\tau=0} = 1 + 1/4\mathcal{V} + \mathcal{V}/4\,, \tag{4.2.93}$$

and it is easy to see that $\mathcal{V} = 1$ is an extremum. This proves equations (4.2.88) en (4.2.89). We thus find that in the point q_c, up to a factor $9/4$, which can be absorbed in the ξ_I, the potential reduces to the same potential for the vector multiplet scalars as in the case of $U(1)_R$ symmetry gauging. We now have to calculate the mass matrix in the critical point. Since (4.2.88) is true for any value of the vector multiplet scalars, we have

$$\left.\frac{\partial^2 V}{\partial q^X \partial \varphi^x}\right|_{q_c} = 0\,, \tag{4.2.94}$$

so we can calculate the masses of the hyper-scalars and the vector-scalars separately. Since we already calculated the mass matrix for the vector multiplet scalars, we just have to calculate the mass matrix for the hypers. After a straightforward calculation we find the matrix to be diagonal, with all entries always positive. There are only 2 different diagonal elements and they can both be written as a sum of manifestly positive terms. Because the expressions are quite messy, we will not give them in their generality here. Instead we will look at the particular case $\xi_0 = 0$. We then have $\varphi_i = 0$ in the critical point and the expressions simplify significantly. Concretely, we find

$$6\frac{\partial_X \partial^Y V}{V}\bigg|_0 = \begin{pmatrix} 16/3 & 0 & 0 & 0 \\ 0 & 16/3 & 0 & 0 \\ 0 & 0 & 8/3 & 0 \\ 0 & 0 & 0 & 8/3 \end{pmatrix}, \tag{4.2.95}$$

where derivation with respect to q^X is denoted by ∂_X and indices are raised with the (inverse) quaternionic-Kähler metric g^{XY}. This shows that potentials with stable de Sitter vacua also exist when hypermultiplets are included.

More examples?

The above examples are the first and still only examples of stable de Sitter vacua in $D > 4$ supergravity theories. In $5D$ $\mathcal{N} = 2$ supergravity it should be fairly easy to find new examples by just changing the hypermultiplet sector. However, finding stable dS vacua in models with a different vector multiplet sector seems to be more challenging. In the case where the very special real manifold $\mathcal{M}$ is one of the following symmetric spaces: $SL(3,\mathbb{R})/SO(3)$, $SL(3,\mathbb{C})/SU(3)$ or $SO(1,3)/SO(3)$, and there are no hypermultiplets, all possible potentials were analyzed in [100, 103, 110]. The results can be summarized as follows

- The $U(1)$ FI potential $V^{(R)}_{U(1)}$ only has Minkowski or AdS vacua if $\mathcal{M}$ is a Jordan symmetric space [103] and we proved that it has at most unstable dS vacua in general [110].
- When $\mathcal{M}$ is one of the magical Jordan symmetric spaces, $SU(2)_R$ gauging is impossible without also introducing charged tensors, while if $\mathcal{M}$ is a non-Jordan symmetric space, the $SU(2)$ FI potential $V^{(R)}_{SU(2)}$ has no critical points [110].
- $\mathcal{M} = SL(3,\mathbb{R})/SO(3)$. We can gauge the $SO(2,1)$ symmetry by dualizing some of the vector fields to tensor fields. The potential $V^{(T)}$ only has Minkowski vacua. We cannot gauge the full R-symmetry in this model, only $U(1)_R$. The potential $V^{(R)}_{U(1)}$ turns out to be zero. There are therefore no dS vacua [100].
- $\mathcal{M} = SL(3,\mathbb{C})/SU(3)$. We can gauge either $SU(2)$ or $SO(2,1)$ by dualizing some of the vector fields to tensor fields. The potential $V^{(T)}$ only has Minkowski vacua in both cases. In case we gauge $SU(2)$, we can also gauge $SU(2)_R$ or $U(1)_R$. The

potential $V = V^{(R)}_{SU(2)} + V^{(T)}$ has no critical points [110], while the potential $V = V^{(R)}_{U(1)} + V^{(T)}$ has Minkowski and AdS vacua, but no dS vacua [100]. In case we gauge $SO(2,1)$, we can only gauge $U(1)_R$. The total potential $V = V^{(R)}_{U(1)} + V^{(T)}$ only has Minkowski vacua [100].

- $\mathcal{M} = SO(1,3)/SO(3)$. The only way to introduce charged tensors is to gauge an $SO(2)$ symmetry. The potential $V^{(T)}$ only has Minkowski vacua. We can also gauge $U(1)_R$ or $SU(2)_R$ simultaneously. The total potential $V = V^{(R)}_{SU(2)} + V^{(T)}$ has no critical points [110], while the potential $V = V^{(R)}_{U(1)} + V^{(T)}$ has Minkowski and AdS vacua, and unstable dS vacua [100].

All known examples of stable dS vacua involve noncompact gaugings and R-symmetry gauging (or charged hypers). It is not known whether this is a necessary condition for the existence of stable dS vacua. The above examples in any case show that $U(1)_R$ gauging and tensors charged under a noncompact gauge group are not sufficient to guarantee stable dS vacua. The geometry of the vector-scalar manifold also seems to play a crucial role.

4.2.3 $\mathcal{N} = 4$ gauged supergravity

Pure $\mathcal{N} = 4$ supergravity containing a graviton $e_\mu{}^m$, four spin 3/2 gravitini ψ^i_μ, six vector fields (A^{ij}_μ, a_μ), four spin 1/2 fermions χ^i and one real scalar σ, can be coupled to an arbitrary number of vector multiplets, each consisting of a vector field A_μ, four spin 1/2 fields λ^i and 5 real scalar fields ϕ^{ij} [113]. Here, μ, m are 5D spacetime and Lorentz indices respectively and the indices $i, j = 1, \ldots, 4$ correspond to the fundamental representation of the R-symmetry group $USp(4)$. The vector field a_μ is $USp(4)$ inert, while the vector fields A^{ij}_μ and the scalar fields ϕ^{ij} transform as the **5** of $USp(4)$, i.e.,

$$A^{ij}_\mu = -A^{ji}_\mu\,, \qquad A^{ij}_\mu \Omega_{ij} = 0\,, \tag{4.2.96}$$

with Ω_{ij} being the symplectic metric of $USp(4)$. We will collectively denote the vector fields of the supergravity multiplet and the n vector multiplets as $A^{\tilde{I}}_\mu$, with $\tilde{I} = 1, \ldots, 5+n$, and the scalar fields of the vector multiplets as ϕ^x, with $x = 1, \ldots, 5n$.
As was first shown in [114], the $5n + 1$ scalar fields parametrize the manifold

$$\mathcal{M} = SO(1,1) \times \frac{SO(5,n)}{SO(5) \times SO(n)}\,, \tag{4.2.97}$$

with the scalar σ of the supergravity multiplet parametrizing the $SO(1,1)$ part. The manifold $\mathcal{M}' = SO(5,n)/[SO(5) \times SO(n)]$ spanned by the scalars ϕ^x can be described by coset representatives $L_{\tilde{I}}{}^A$, where $\tilde{I}$ is an $SO(5,n)$ index and $A = (m,a)$ is an $SO(5) \times SO(n)$ index. Denoting the inverse of $L_{\tilde{I}}{}^A$ by $L^{\tilde{I}}{}_A$ (i.e $L_{\tilde{I}}{}^A L^{\tilde{I}}{}_B = \delta^A{}_B$), the metric g_{xy} of $\mathcal{M}'$ can be written as

$$g_{xy} = \frac{1}{4} f^{ija}_x f^a_{yij}\,, \qquad f^{ija}_x = -2 L^{\tilde{I}a} \partial_x L_{\tilde{I}}{}^{ij}\,. \tag{4.2.98}$$

Here, we used the local isomorphism between $USp(4)$ and $SO(5)$ to change $SO(5)$ indices into $USp(4)$ indices. Concretely we have

$$L_{\tilde{I}i}{}^{j} = \frac{1}{2} L_{\tilde{I}}{}^{m} (\Gamma_m)_i{}^{j} \,, \tag{4.2.99}$$

with

$$(\Gamma_m)_i{}^{j} (\Gamma_n)_j{}^{k} + (\Gamma_n)_i{}^{j} (\Gamma_m)_j{}^{k} = 2\delta_{mn} \delta_i{}^{k} \,, \tag{4.2.100}$$

where $i, j = 1, \ldots, 4$ are $USp(4)$ indices and $m, n = 1, \ldots, 5$ are $SO(5)$ indices. The indices a, b and m, n are raised and lowered with δ^{ab} and δ^{mn} respectively, while for the $USp(4)$ indices we use the following convention

$$T^i = \Omega^{ij} T_j \,, \qquad T_i = T^j \Omega_{ji} \,. \tag{4.2.101}$$

The bosonic part of the Lagrangian of the ungauged theory is given by [113, 114]

$$\begin{aligned}
(\sqrt{-g})^{-1} \mathcal{L} &= \frac{R}{2} + \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{vector}} \,, && (4.2.102) \\
\mathcal{L}_{\text{scalar}} &= -\frac{1}{2} (\partial_\mu \sigma)^2 - \frac{1}{2} g_{xy} \partial_\mu \phi^x \partial^\mu \phi^y \,, && (4.2.103) \\
\mathcal{L}_{\text{vector}} &= -\frac{1}{4} \Sigma^{-4} G_{\mu\nu} G^{\mu\nu} - \frac{1}{4} \Sigma^2 a_{\tilde{I}\tilde{J}} F^{\tilde{I}}_{\mu\nu} F^{\mu\nu\tilde{J}} \\
&\quad - \frac{\sqrt{2}}{8} (\sqrt{-g})^{-1} C_{\tilde{I}\tilde{J}} \epsilon^{\mu\nu\rho\sigma\lambda} F^{\tilde{I}}_{\mu\nu} F^{\tilde{J}}_{\rho\sigma} a_\lambda \,, && (4.2.104)
\end{aligned}$$

with

$$F^{\tilde{I}}_{\mu\nu} = \partial_\mu A^{\tilde{I}}_\nu - \partial_\nu A^{\tilde{I}}_\mu \,, \qquad G_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \,, \tag{4.2.105}$$

the Abelian field strengths of the vector fields and

$$\Sigma = e^{\frac{1}{\sqrt{3}}\sigma} \,. \tag{4.2.106}$$

We also have

$$a_{\tilde{I}\tilde{J}} = L_{\tilde{I}}{}^{ij} L_{\tilde{J}ij} + L_{\tilde{I}}{}^{a} L_{\tilde{J}}{}^{a} \,, \qquad C_{\tilde{I}\tilde{J}} = L_{\tilde{I}}{}^{ij} L_{\tilde{J}ij} - L_{\tilde{I}}{}^{a} L_{\tilde{J}}{}^{a} \,, \tag{4.2.107}$$

and

$$L_{\tilde{I}}{}^{A} = a_{\tilde{I}\tilde{J}} L^{\tilde{J}A} \,. \tag{4.2.108}$$

The symmetric tensor $C_{\tilde{I}\tilde{J}}$ is the constant $SO(5,n)$ metric.
For gauge groups that are a direct product of an Abelian and a semisimple group, the gauged 5D $\mathcal{N} = 4$ supergravity theory was constructed in [113]. More recently another formulation of 5D $\mathcal{N} = 4$ gauged supergravity was proposed in [73], which also allows nonsemisimple gauge groups. Here, we will restrict to gauge groups that are a direct product of an Abelian and a semisimple group and summarize the results of [113].

The Lagrangian (4.2.102) and its $\mathcal{N}=4$ supersymmetric completion is invariant under the following global $SO(1,1)\times SO(5,n)$ transformations

$$\begin{array}{lcllcl}
\delta_{SO(5,n)}A^{\tilde{I}}_{\mu} &=& \alpha^r M^{\tilde{I}}_{(r)\tilde{J}} A^{\tilde{J}}_{\mu}\,, & \delta_{SO(1,1)}A^{\tilde{I}}_{\mu} &=& -\lambda A^{\tilde{I}}_{\mu}\,,\\
\delta_{SO(5,n)}L^{\tilde{I}}_{A} &=& \alpha^r M^{\tilde{I}}_{(r)\tilde{J}} L^{\tilde{J}}_{A}\,, & \delta_{SO(1,1)}L^{\tilde{I}}_{A} &=& 0\,,\\
\delta_{SO(5,n)}\Sigma &=& 0\,, & \delta_{SO(1,1)}\Sigma &=& \lambda\Sigma\,,\\
\delta_{SO(5,n)}a_{\mu} &=& 0\,, & \delta_{SO(1,1)}a_{\mu} &=& 2\lambda a_{\mu}\,,
\end{array} \tag{4.2.109}$$

where $M^{\tilde{I}}_{(r)\tilde{J}}\in\mathfrak{so}(5,n)$, and α^r $(r=1,\ldots,\dim(SO(5,n)))$ and λ are some infinitesimal parameters. Gauging involves making part of this symmetry local using the vector fields in the theory. Since all the vector fields transform nontrivially under the $SO(1,1)$ factor, only a subgroup $G\subset SO(5,n)$ can be gauged. The vector fields $A^{\tilde{I}}_{\mu}$ transform in the $(\mathbf{5}+\mathbf{n})$ of $SO(5,n)$. In general, we have the following decomposition

$$(\mathbf{5}+\mathbf{n})_{SO(5,n)}\rightarrow \text{singlets}(G)\oplus\text{adjoint}(G)\oplus\text{nonsinglets}(G)\,. \tag{4.2.110}$$

We will split the vector fields as

$$A^{\tilde{I}}_{\mu}=(A^{I}_{\mu},A^{M}_{\mu})\,, \tag{4.2.111}$$

where the indices $I,J,\ldots$ collectively label the singlet and adjoint vector fields and $M,N,\ldots$ label the nonsinglets. In order for the gauged theory to remain supersymmetric, the nonsinglet vector fields A^{M}_{μ} have to be dualized into selfdual tensor fields $B^{M}_{\mu\nu}$. This conversion involves replacing all the field strengths $F^{M}_{\mu\nu}$ by $B^{M}_{\mu\nu}$ and adding a kinetic term $B\wedge dB$ to the Lagrangian. To K-covariantize this term we need a term $B\wedge B\wedge A$ or $B\wedge B\wedge a$, which should originate from the dualization of a $F\wedge F\wedge A$ or $F\wedge F\wedge a$ term in the ungauged Lagrangian. However, Lagrangian (4.2.102) only contains a $F\wedge F\wedge a$ term. This has a couple of interesting consequences:

- The gauge group G is necessary of the form $G=SO(1,1)\times G_{\rm S}$ or $G=SO(2)\times G_{\rm S}$ with G_S a semisimple group and a_μ the $SO(1,1)$ or $SO(2)$ gauge field.
- The tensor fields can only be charged with respect to the $SO(1,1)$ or $SO(2)$ factor of G, unlike $\mathcal{N}=8$ or $\mathcal{N}=2$ theories, where the tensor fields can also transform nontrivially under non-Abelian gauge groups.
- Gauging of a semisimple gauge group cannot lead to the introduction of tensor fields, i.e we can only gauge a semisimple group G_S if

$$(\mathbf{5}+\mathbf{n})_{SO(5,n)}\rightarrow \text{singlets}(G_S)\oplus\text{adjoint}(G_S)\,. \tag{4.2.112}$$

Gauging a group $G=G_A\times G_S$ with G_A a one-dimensional Abelian group and G_S semisimple leads to a scalar potential of the form

$$\begin{aligned}
V &= g_A^2 V^{(A)}+g_A g_S V^{(AS)}+g_S^2 V^{(S)}\\
&= \frac{1}{2}\Big[\, g_A^2\, V^{a}_{ij}V^{aij}-36\, g_A g_S\, U_{ij}S^{ij}+g_S^2\left(T^{a}_{ij}T^{aij}-9\,S_{ij}S^{ij}\right)\Big]\,,
\end{aligned} \tag{4.2.113}$$

with

$$\begin{array}{rclcrcl} V^a_{ij} &=& \frac{1}{2\sqrt{2}}\,\Sigma^2\, f^a_{xij}\, K^x \ , & & T^a_{ij} &=& \frac{1}{2}\,\Sigma^{-1}\, L^I{}_i{}^k\, f^a_{xkj}\, K^x_I \,, \\ U_{ij} &=& \frac{\sqrt{2}}{3}\,\Sigma^2\, P^{(A)}_{ij} \,, & & S_{ij} &=& -\frac{4}{9}\,\Sigma^{-1}\, L^I{}_i{}^k\, P^{(S)}_{Ikj} \,. \end{array} \tag{4.2.114}$$

Here K^x and K^x_I are the Killing vectors corresponding to the G_A and G_S transformations respectively, and $P^{(A)}_{ij}$, $P^{(S)}_{Ikj}$ are the corresponding Killing prepotentials. The $G_A \times G_S$ transformations act on the scalars ϕ^x as isometries

$$\delta\phi^x = \alpha K^x + \alpha^I K^x_I \,, \tag{4.2.115}$$

where α and α^I are the infinitesimal G_A and G_S transformation parameters respectively, and

$$K^x = \frac{1}{2}\, f^{xija}\, \Lambda^M{}_N\, L_{Mij} L^{Na} \ , \quad K^x_I = \frac{1}{2}\, f^{xija}\, L^J{}_{ij} f_{IJ}{}^K L_K{}^a \,. \tag{4.2.116}$$

Here, $\Lambda^M{}_N$ is the G_A transformation matrix of the tensor fields and $f_{IJ}{}^K$ are the G_S structure constants. The corresponding prepotentials are given by

$$P^{(A)}_{ij} = -\frac{1}{2}\,\Lambda^M{}_N\, L^N{}_{ik} L_M{}^k{}_j, \qquad P^{(S)}_{Iij} = \frac{1}{2}\, f_{IJ}{}^K L^J{}_i{}^k\, L_{Kkj} \,. \tag{4.2.117}$$

We have

$$C_{IM} = 0 \,, \qquad f_{KL}{}^I\, C_{IJ} + f_{KJ}{}^I\, C_{IL} = 0 \,, \tag{4.2.118}$$

and

$$\Lambda^M{}_N = \frac{1}{\sqrt{2}} \Omega^{MP} C_{PN} \,, \tag{4.2.119}$$

with Ω_{MN} a constant and antisymmetric metric with inverse Ω^{MN}:

$$\Omega_{MN}\Omega^{NP} = \delta^P_M, \qquad \Omega_{MN} = -\Omega_{NM} \,. \tag{4.2.120}$$

$\mathcal{N} = 4$ gauged supergravity with 1 vector multiplet

When $g_A = 0$ or $g_S = 0$ the potential (4.2.113) only has an extremum at $\Sigma > 0$ if the potential vanishes at the critical point. dS vacua are therefore only possible when both an Abelian gauge group and a semisimple gauge group are present. With only 1 vector multiplet present, only 3 cases have to be considered, namely $G = SO(2) \times SO(3)$, $G = SO(2) \times SO(2,1)$ and $G = SO(1,1) \times SO(3)$. To calculate the potentials, we will use the so called symmetric gauge for the coset representative L, i.e. (see e.g. [115])

$$L = \exp \begin{pmatrix} 0_{5\times 5} & A \\ A^T & 0_{n\times n} \end{pmatrix} = \begin{pmatrix} \sqrt{1 + XX^T} & X \\ X^T & \sqrt{1 + X^T X} \end{pmatrix} \,, \tag{4.2.121}$$

where A and X are $5 \times n$ matrices . In the case of one vector multiplet, we have $n = 1$ and we use the notation $X^T = (x_1, x_2, x_3, x_4, x_5)$.

1) $G = SO(2) \times SO(3)$

We take $I = 1, 2, 3, 6$ and $M = 4, 5$, with

$$\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad f_{IJ}{}^{K} = \epsilon_{IJK} \quad \text{if} \quad I, J, K = 1, 2, 3\,, \tag{4.2.122}$$

where ϵ_{IJK} is the completely antisymmetric tensor, $\epsilon_{123} = 1$, and $f_{IJ}{}^{K} = 0$ if I, J, or K equals 6. This leads to the following scalar potential

$$V = -\frac{1}{2}\Sigma^{-2} g_S^2 + \frac{1}{4}\Sigma^4 g_A^2 (x_4^2 + x_5^2) - \sqrt{2}\Sigma g_A g_S \sqrt{1 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2}\,. \tag{4.2.123}$$

This potential has only AdS extrema.

2) $G = SO(2) \times SO(2, 1)$

We again take $I = 1, 2, 3, 6$ and $M = 4, 5$, with now

$$\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad f_{IJ}{}^{K} = -\epsilon_{IJL} C^{LK} \quad \text{if} \quad I, J, K = 1, 2, 6\,. \tag{4.2.124}$$

Here, $C = \text{diag}(1, 1, 1, 1, 1, -1)$, $\epsilon_{126} = 1$, and $f_{IJ}{}^{K} = 0$ if I, J, or K equals 3. This leads to the following scalar potential

$$\begin{aligned} V &= -\sqrt{2}\Sigma g_A g_S x_3 + \frac{1}{4}\Sigma^4 g_A^2 (x_4^2 + x_5^2) + \frac{1}{2}\Sigma^{-2} g_S^2 [1 + x_1^4 + x_2^4 + \\ &\quad x_2^2 (2 + x_1^2 + x_3^2 + x_4^2 + x_5^2) + x_1^2 (2 + x_2^2 + x_3^2 + x_4^2 + x_5^2)]\,. \end{aligned} \tag{4.2.125}$$

This potential has no extremum.

3) $G = SO(1, 1) \times SO(3)$

We take $I = 1, 2, 3, 4$ and $M = 5, 6$ with

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad f_{IJ}{}^{K} = \epsilon_{IJK} \quad \text{if} \quad I, J, K = 1, 2, 3\,, \tag{4.2.126}$$

where $\epsilon_{123} = 1$, and $f_{IJ}{}^{K} = 0$ if I, J, or K equals 4. This leads to the following scalar potential

$$V = -\frac{1}{2}\Sigma^{-2} g_S^2 + \sqrt{2}\Sigma g_A g_S x_4 + \frac{1}{4}\Sigma^4 g_A^2 (1 + x_1^2 + x_2^2 + x_3^2 + x_4^2)\,. \tag{4.2.127}$$

This potential also has no extremum.

We thus find that more than one vector multiplet is needed to find dS vacua in 5D $\mathcal{N} = 4$ gauged supergravity, at least if we restrict to gauge groups G of the form $G = G_A \times G_S$ as we do in this thesis.

$\mathcal{N} = 4$ gauged supergravity with more than 1 vector multiplet

With more than 1 vector multiplet, straightforwardly calculating the potential as we have done in the previous section, quickly leads to unmanageable expressions. Only when the matrix X in (4.2.121) has at most one nonzero element per row, is calculating the potential almost as easy as in the 1 vector multiplet case. In some cases, the potential has enough symmetry so that X can be reduced to such a form. For instance, consider the following gauging of $SO(3)$ and a diagonal $SO(2)$ with 2 vector multiplets: take $I = 1, 2, 3$ and $M = 4, 5, 6, 7$ with

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad f_{IJ}{}^{K} = \epsilon_{IJK}\,, \tag{4.2.128}$$

where $\epsilon_{123} = 1$. The corresponding potential V has a $SU(2) \times U(1) \times SL(2, \mathbb{R})$ symmetry [116], and can therefore be calculated by taking for example

$$X^T = \begin{pmatrix} x_1 & 0 & 0 & x_4 & 0 \\ 0 & y_2 & 0 & 0 & 0 \end{pmatrix}. \tag{4.2.129}$$

This leads to the following expression,

$$\begin{aligned} V &= -\sqrt{2}\Sigma g_A g_S \sqrt{1 + x_1^2 + x_2^2}\sqrt{1 + y_2^2} + \frac{1}{2}\Sigma^{-2} g_S^2 (x_1^2 y_2^2 - 1) + \\ &\quad \frac{1}{4}\Sigma^4 g_A^2 [y_2^2 + 2x_4^2(1 + y_2^2) + x_1^2(1 + 2y_2^2)]\,. \end{aligned} \tag{4.2.130}$$

This potential has several AdS extrema, as was shown in [116]. The coordinates ρ, r_x, r_y and θ used in [116] are related to ours by the following coordinate transformations:

$$\begin{aligned} x_1 &= \cos\theta \sinh(2r_x)\,, & x_4 &= \sin\theta \sinh(2r_x)\,, \\ y_2 &= \sinh(2r_y)\,, & \Sigma &= \rho^2\,. \end{aligned} \tag{4.2.131}$$

To compare with [116], we also have to put $g_A = g_S/\sqrt{2} = g/2$. The relative size of g_A and g_S is immaterial, since we can always change it by rescaling $\Sigma \to \Sigma' = \alpha\Sigma$. Notice that the kinetic term of Σ is invariant under this rescaling. Only the relative sign of g_A and g_S is thus important. For simplicity, we will therefore use $g_A = \pm g_S/\sqrt{2}$ and $g_S = g/\sqrt{2}$ from here on.
Unfortunately, in most cases there is not enough symmetry to reduce X to the desired form. In the rest of this section, we will therefore investigate the potential only in the 'origin' $X = 0$. With 2 vector multiplets, we have to consider 3 more gaugings, namely $G = SO(2) \times SO(2, 1)$, $G = SO(1, 1) \times SO(2, 1)$ and $G = SO(1, 1) \times SO(3)$.

1) $G = SO(2) \times SO(2, 1)$

We take $I = 1, 2, 3, 6, 7$ and $M = 4, 5$, with now

$$\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad f_{IJ}{}^{K} = -\epsilon_{IJL} C^{LK} \quad \text{if} \quad I, J, K = 1, 2, 6\,. \tag{4.2.132}$$

Here, $C = \mathrm{diag}(1,1,1,1,1,-1,-1)$, $\epsilon_{126} = 1$, and $f_{IJ}{}^K = 0$ if I, J, or K equals 3 or 7. We find that $X = 0$ extremizes $V^{(A)}$ and $V^{(S)}$, but not $V^{(AS)}$. Thus, the total potential V has no extremum in the origin $X = 0$.

2) $G = SO(1,1) \times SO(2,1)$

We take $I = 1,2,3,4,7$ and $M = 5,6$, with now

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad f_{IJ}{}^K = -\epsilon_{IJL} C^{LK} \quad \text{if} \quad I,J,K = 3,4,7\,. \tag{4.2.133}$$

Here, $C = \mathrm{diag}(1,1,1,1,1,-1,-1)$, $\epsilon_{347} = 1$, and $f_{IJ}{}^K = 0$ if I, J, or K equals 1 or 2. We find that $X = 0$ extremizes $V^{(A)}$, $V^{(S)}$ and $V^{(AS)}$, and therefore also the total potential V for any value of Σ. Since $V^{(AS)}$ vanishes at $X = 0$, the relative sign of g_A and g_S is irrelevant. We have

$$V|_{X=0} = \frac{1}{16} g^2 \Sigma^{-2} (4 + \Sigma^6)\,. \tag{4.2.134}$$

This potential has a dS extremum at $\Sigma = 2^{1/6}$ with the following properties:

$$\begin{aligned}
&\Lambda = V|_0 = \frac{3}{8} 2^{-1/3} g^2 > 0\,, \quad \text{1 scalar with } \frac{6m_\Sigma^2}{\Lambda} = 6 g^{\Sigma\Sigma} \frac{\partial_\Sigma \partial_\Sigma V}{V}|_0 = 16\,,\\
&\text{1 scalar with } \frac{6m^2}{\Lambda} = 4\,, \quad \text{2 scalars with } \frac{6m^2}{\Lambda} = 12\,,\\
&\text{2 scalars with } \frac{6m^2}{\Lambda} = 2(1 - \sqrt{33})\,, \quad \text{2 scalars with } \frac{6m^2}{\Lambda} = 2(1 + \sqrt{33})\,,\\
&\text{3 massless scalars, giving masses to the gauge fields associated}\\
&\text{with the noncompact generators of } G \text{ by the BEH-effect}\,.
\end{aligned} \tag{4.2.135}$$

Since $1 - \sqrt{33} < 0$, the dS vacuum is unstable. Notice also that some of the masses are quantized, but not all of them.

3) $G = SO(1,1) \times SO(3)$

We take $I = 1,2,3$ and $M = 4,5,6,7$ with

$$\Lambda = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \quad \lambda = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \quad \text{and} \quad f_{IJ}{}^K = \epsilon_{IJK} \quad \text{if} \quad I,J,K = 1,2,3\,, \tag{4.2.136}$$

where $\epsilon_{123} = 1$ and α is a free parameter. We find that $X = 0$ extremizes $V^{(A)}$ and $V^{(S)}$, but not $V^{(AS)}$. Thus, the total potential V has no extremum in the origin $X = 0$.

4) $G = SO(1,1) \times SO(2,1)$ with 3 vector multiplets

When there are 3 vector multiplets we find unstable dS vacua at the origin $X = 0$ for $G = SO(1,1) \times SO(2,1) \times SO(2,1)$ and $G = SO(1,1) \times SO(2,1)$. Since the results of both cases are similar, we will only look at the second case here. We take $I = 1, 2, 3, 8$ and $M = 4, 5, 6, 7$, with

$$\Lambda = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \quad \lambda = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \quad \text{and} \quad f_{IJ}{}^{K} = -\epsilon_{IJL} C^{LK} \quad \text{if} \quad I, J, K = 2, 3, 8\,. \tag{4.2.137}$$

Here, $C = \text{diag}(1, 1, 1, 1, 1, -1, -1, -1)$, $\epsilon_{238} = 1$, and $f_{IJ}{}^{K} = 0$ if I, J, or K equals 1. With respect to the case of 2 vector multiplets, there is now an extra free parameter α. The origin $X = 0$ extremizes the potential V and we have

$$V|_{X=0} = \frac{1}{16} g^2 \Sigma^{-2} (4 + (1 + \alpha^2)\Sigma^6)\,. \tag{4.2.138}$$

This potential has a dS extremum at $\Sigma = 2^{1/6}(1 + \alpha^2)^{-1/6}$ with the following properties:

$$\begin{aligned} &\Lambda = V|_0 = \frac{3}{8} 2^{-1/3} g^2 (1 + \alpha^2)^{1/3}\,, \quad 1 \text{ scalar with } \frac{6m_\Sigma^2}{\Lambda} = 16\,, \\ &4 \text{ massless scalars, of which 3 will give masses to the gauge fields} \\ &\text{associated with the noncompact generators of } G \text{ by the BEH-effect}\,. \end{aligned} \tag{4.2.139}$$

and all other scalar masses dependent on α. For instance, for $\alpha = 0$ we find

$$\frac{6m^2}{\Lambda} = (12, 12, 8, 8, 2[1 - \sqrt{33}], 2[1 + \sqrt{33}], 0, 4, 2[1 - \sqrt{33}], 4, 2[1 + \sqrt{33}])\,, \tag{4.2.140}$$

while for $\alpha = 1$ we have

$$\frac{6m^2}{\Lambda} = (10, 10, 10, 10, -6, 10, -6, 10, -8, 0, 16)\,. \tag{4.2.141}$$

Observe that all masses are quantized when $\alpha = 1$. In both cases there is also an accidental extra massless scalar. Finally there are always negative m^2 scalars, irrespective of the value of α, so that the dS vacua are always unstable.

The only dS vacua we have found so far involve the presence of both a noncompact Abelian and a noncompact non-Abelian group. We do not know at present whether this is a general feature of 5D $\mathcal{N} = 4$ supergravity, nor did we find any stable dS vacua. We plan to continue our search for stable dS vacua by generalizing our approach to at least 5 vector multiplets and hope to discover some general features of dS vacua in 5D $\mathcal{N} = 4$ gauged supergravity. The results will be reported in a future publication [117].

4.3 de Sitter in 10D and 11D supergravity

4.3.1 The theories and their potentials

Assuming 11D Lorentz symmetry, no spin > 2 particles and at most two derivatives in the action, there is a unique 11D supergravity theory, which has 32 supersymmetries and was first constructed in [118]. The bosonic sector contains a vielbein (graviton) and a 3-form potential. There are no scalar fields and thus no scalar potential, and a cosmological constant cannot be added without breaking 11D Lorentz invariance [119]. The vacuum solution of 11D supergravity is therefore 11D Minkowski space.
In contrast to supergravity in 11D, there is a rich variety of 10D supergravities. All 10D supergravity theories have 16 ($\mathcal{N} = 1$) or 32 ($\mathcal{N} = 2$) supersymmetry generators. 10D $\mathcal{N} = 1$ supergravity is the (tree level two derivative) low-energy effective theory of the closed string sector of both the Type I superstring and the heterotic superstring. An action for this theory can be found e.g. in the classic textbooks [26]. The bosonic field content consists of a graviton, a 2-form potential and a real scalar ϕ called the dilaton. There is no potential for this scalar field, so the vacuum solution of this theory is 10D Minkowski space.
10D $\mathcal{N} = 2$ supergravity comes in 4 inequivalent forms:

- Type IIB supergravity. This theory is chiral (left-right asymmetric), the 32 supersymmetries are all of the same handedness, and is the low-energy effective theory of type IIB string theory. The bosonic sector contains a graviton, a 4-form potential with corresponding selfdual field strength, two 2-form potentials and a complex scalar field. Covariant equations of motion, can be found in [26]. Because of the selfdual 5-form field strength, the construction of a covariant action for this theory is quite involved, see e.g. [120]. There is again no potential for the scalar fields, so the vacuum solution is 10D Minkowski space.

- Type IIA supergravity. This theory is nonchiral (left-right symmetric), 16 supersymmetries are of one handedness, the other 16 supersymmetries are of the opposite handedness, and is the low-energy effective theory of type IIA string theory. An action can be found in [26]. Unlike type IIB supergravity, it can be obtained by dimensional reduction of 11D supergravity. The bosonic sector contains a graviton, a 3-form potential, a 2-form potential, a 1-form potential and a real scalar field. The scalar field has no potential, the vacuum solution is therefore again 10D Minkowski space.

- Romans' massive type IIA supergravity. This theory is a generalization of type IIA supergravity and was first constructed in [121]. It arises in string theory as the low-energy theory of type IIA in a D8-brane background. An action containing a 9-form potential was constructed for this theory in [122]. Romans' theory cannot be obtained by dimensional reduction from 11D supergravity, but it can be derived from massive 11D supergravity, which is a deformation of 11D supergravity that breaks 11D Lorentz invariance [123]. The scalar field ϕ gets a scalar potential of

the form $m^2 e^{b\phi}$, where b is some nonzero fixed constant and m is a constant called the mass parameter ($m \to 0$ leads back to type IIA). Since the scalar potential has no extremum for $m \neq 0$, the theory does not have solutions preserving 10D Lorentz invariance if $m \neq 0$.

- Massive HLW IIA supergravity. This is another generalization of type IIA supergravity, inequivalent to Romans' massive IIA theory. It was first derived in [124] by introducing a conformal spin connection in 11D and compactifying on S^1. Later the authors of [125] showed that the theory also can be obtained from 11D supergravity by a generalized Scherk-Schwarz reduction, using a global scale symmetry of the 11D supergravity equations of motion. Finally, it can also be viewed as a gauged theory, where a global scale symmetry of the IIA supergravity equations of motion is gauged, see e.g [126]. The equations of motion of massive HLW IIA theory are not derivable from a Lorentz-invariant Lagrangian[124]. Given that the global scale symmetry used to construct the theory is only a symmetry of the equations of motion (of type IIA/11D supergravity) and not of the Lagrangian (of type IIA/11D supergravity) this is not too surprising. An interesting feature of massive HLW IIA supergravity is that it admits a 10D de Sitter vacuum solution [125, 127]. The graviphoton turns out to be tachyonic in this background, so that the dS vacuum is probably unstable [128]. Before ending, let us also note that Romans' massive IIA theory and the massive HLW IIA theory are the only possible massive deformations of type IIA supergravity, as was proved recently in [129].

Besides looking for maximally symmetric de Sitter solutions, it might also be interesting to search for solutions of the form $dS_4 \times M$ or $dS_5 \times M$. We will study such solutions in the next sections.

4.3.2 $dS \times H^{p,q}$ solutions

As shown in [130], the dS vacuum solutions in 4D and 5D $\mathcal{N} = 8$ supergravity we discussed previously can be lifted to 11D and Type IIB supergravity respectively. The solutions of D-dimensional supergravity have a warped product metric of the form (using the notation of [131])

$$ds^2 = f_1^2(y) dS_d^2(x) + f_2^2(y) d\Omega_{p,q}^2(y)\,, \tag{4.3.1}$$

where $dS_d^2(x)$ is the d-dimensional de Sitter metric with coordinates x, $d\Omega_{p,q}^2(y)$ is the metric on the internal space $H^{p,q}$ with coordinates y, and $D = d + p + q - 1 = 10$ or 11. The space $H^{p,q}$ is defined as the hypersurface

$$\eta_{AB} z^A z^B = R^2\,, \tag{4.3.2}$$

in $\mathbb{R}^{p+q}$, with z^A real cartesian coordinates of $\mathbb{R}^{p+q}$, R a constant and $\eta_{AB} = \mathrm{diag}(\mathbb{1}_p, -c^2\mathbb{1}_q)$ for some constant c. The metric $d\Omega_{p,q}^2$ on $H^{p,q}$ is the positive definite metric induced from the Euclidean metric on $\mathbb{R}^{p+q}$. For $p, q > 1$ the $H^{p,q}$ spaces are noncompact, nonhomogeneous, negative curvature spaces with isometry group $SO(p) \times SO(q)$.

Finally, the warp factors are given by

$$f_1 = L^a\,, \qquad f_2 = L^b\,, \tag{4.3.3}$$

for some constants a and b, with

$$L^2 = R^{-2}\left[\sum_{i=1}^{p}(z^i)^2 + c^4\sum_{i=p+1}^{p+q}(z^i)^2\right]\,. \tag{4.3.4}$$

The $d = 4$ de Sitter solutions of SO(4,4) and SO(5,3) $\mathcal{N} = 8$ gauged supergravity arise from the solutions of $D = 11$ supergravity of the above form with respectively

$$p = 4\,, \quad q = 4\,, \quad c^2 = 1\,, \quad a = 2/3\,, \quad b = -1/3\,, \tag{4.3.5}$$

and

$$p = 5\,, \quad q = 3\,, \quad c^2 = 3\,, \quad a = 2/3\,, \quad b = -1/3\,. \tag{4.3.6}$$

The 4-form field strength of $D = 11$ supergravity is proportional to R times the volume form on dS_4.
Similarly, the $d = 5$ de Sitter solution of SO(3,3) $\mathcal{N} = 8$ supergravity arises from a solution of $D = 10$ Type IIB supergravity of the above form with

$$p = 3\,, \quad q = 3\,, \quad c^2 = 1\,, \quad a = 1/2\,, \quad b = -1/2\,, \tag{4.3.7}$$

and with the selfdual 5-form field strength proportional to R and a selfdual combination of the volume forms on dS_5 and $H^{3,3}$.
Note that since the dS critical points of 4D and 5D $\mathcal{N} = 8$ supergravity are unstable, the corresponding 10D and 11D solutions must also be unstable.

4.3.3 The Maldacena/Nunez no go theorem

The solutions of the form $dS \times H^{p,q}$, which we discussed in the previous section, have the drawback that the internal manifold $H^{p,q}$ is noncompact, making them unsuitable for phenomenological purposes. The natural question arises whether 10D or 11D supergravity also has solutions of the form $dS \times M$, where M is a compact manifold. While some noncompact manifolds can be compactified by identifying points under the action of a discrete subgroup of the isometry group, e.g. $\mathbb{R}^n$ can be compactified to a n-dimensional torus T^n, this does not work for the $H^{p,q}$ spaces since their isometry group is compact. We will therefore have to look for other solutions. The following theorem [132] (see also [133]) suggests we might be looking for quite a while...

Theorem 2 *Consider a D-dimensional gravity theory, with $D > 2$, compactified down to d dimensions, and satisfying the following conditions:*

- *The scalar potential is nonpositive, $V \leq 0$.*

- *The theory contains massless fields that have field strengths that are n-forms $F_{i_1,\ldots,i_n}$ and positive kinetic terms.*
- *The d-dimensional effective Newton's constant is finite.*
- *The gravity action does not contain higher curvature corrections.*

Then there are no nonsingular compactifications to dS_d.

Let us sketch the proof. Take the Einstein equations in D dimensions,

$$R_{MN} = T_{MN} - \frac{1}{D-2} g_{MN} T^L{}_L \,. \tag{4.3.8}$$

Writing the D-dimensional metric as

$$ds_D^2 = \Omega^2(y) \left(\tilde{g}_{\mu\nu} dx^\mu dx^\nu + \hat{g}_{mn} dx^m dx^n \right) \,, \tag{4.3.9}$$

where $\tilde{g}$ is the dS or Minkowski metric, and substituting this in (4.3.8) gives

$$R_{\mu\nu} = R_{\mu\nu}(\tilde{g}) - \tilde{g}_{\mu\nu} \left(\hat{\nabla}^2 \log\Omega + (D-2)(\hat{\nabla}\log\Omega)^2 \right) = T_{\mu\nu} - \frac{1}{D-2} \Omega^2 \tilde{g}_{\mu\nu} T^L{}_L \,. \tag{4.3.10}$$

Multiplying both sides of (4.3.10) with $\tilde{g}^{\mu\nu}$ gives

$$\hat{\nabla}^2 \log\Omega + (D-2)(\hat{\nabla}\log\Omega)^2 = \frac{1}{(D-2)\Omega^{D-2}} \hat{\nabla}^2 \Omega^{D-2} = R(\tilde{g}) + \Omega^2 \tilde{T} \,, \tag{4.3.11}$$

where we have defined

$$\tilde{T} \equiv -T^\mu{}_\mu + \frac{d}{D-2} T^L{}_L \,. \tag{4.3.12}$$

It was shown in [132] that all contributions to $\tilde{T}$, coming from the scalar potential and the n-form field strengths (fluxes), are positive, i.e. $\tilde{T} \geq 0$. Multiplying (4.3.11) by a power of Ω and using $\tilde{T} \geq 0$ we get

$$\Omega^{(D-2)} \hat{\nabla}^2 \Omega^{(D-2)} \geq 0 \,, \tag{4.3.13}$$

for dS or Minkowski compactifications. Assuming the d-dimensional Newton constant

$$\frac{1}{G_N^d} \sim \int d^{(D-d)} y \sqrt{\hat{g}}\, \Omega^{(D-2)} \tag{4.3.14}$$

is finite, we can integrate (4.3.13) by parts over the compact internal manifold and find

$$\int d^{(D-d)} y \sqrt{\hat{g}}\, (\hat{\nabla}\Omega^{(D-2)})^2 \leq 0 \,, \tag{4.3.15}$$

which is possible only if Ω is constant. In this case the right hand side of (4.3.11) has to be zero and this can only happen if the d-dimensional space is Minkowski space, and only

1-form and/or $(D-1)$-form fluxes are turned on [132], Q.E.D.
Type IIA, type IIB and 11D supergravity all obey the conditions of the above no-go theorem, however the massive deformations of Type IIA do not, since they have a positive scalar potential. Nevertheless, Maldacena and Nuñez showed in [132] that there are no nonsingular compactifications to dS or Minkowski space in Romans' massive IIA theory. The massive HLW IIA theory on the other hand has solutions of the form $dS_d \times S^{(10-d)}$, with $2 \leq d \leq 10$ [127].
Every (no go) theorem contains assumptions and can therefore be evaded. Since type IIA and type IIB supergravity are low-energy effective actions of string theories, we can ask whether string theory can lead to (effective) theories allowing compactifications to dS space. It turns out there are (at least) two ways in which string theories can evade the no-go theorem:

- String theory leads to higher derivative corrections to the Einstein equations. These corrections come with powers of the string tension α'.
- String theory contains localized sources with negative tension, e.g. orientifold planes. They give a negative contribution to $\tilde{T}$.

We will study the second possibility in more detail in the next chapter, focusing mainly on the resulting effective theories.

4.4 Summary and open problems

In this chapter, we have searched for dS vacua in 4D and 5D supergravity and briefly looked at their possible 10D and 11D embedding. The results can be summarized as follows:

- No stable dS vacua are known in 4D and 5D $\mathcal{N} \geq 3$ gauged supergravity, nor in $D > 5$ supergravity.
- Masses of scalar fluctuations in dS supergravity vacua are often quantized in units of the cosmological constant.
- The 4D and 5D $\mathcal{N} = 2$ supergravity theories with known stable dS vacua share some peculiar features. They involve noncompact gauge groups and FI terms, together with de Roo-Wagemans angles in 4D and charged antisymmetric tensors in 5D.
- The only 10D or 11D supergravity theory with a dS vacuum is the 10D HLW IIA theory. The dS vacuum is probably unstable.
- dS 'compactifications' from 10D or 11D to 4D or 5D necessary involve either non-compact internal manifolds, or the incorporation of stringy effects.

In the papers [100, 117], we concentrated on 5D $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supergravity, leading to the following highlights:

- We found the first examples of stable dS vacua in 5D $\mathcal{N} = 2$ supergravity.
- We showed that the $U(1)$ FI potential of 5D $\mathcal{N} = 2$ supergravity cannot have stable dS vacua.
- In 5D $\mathcal{N} = 4$ supergravity we only found unstable dS vacua. They all involve gauging of noncompact groups.
- On the other hand, in 5D $\mathcal{N} = 2$ supergravity we found lots of examples of compact gaugings leading to unstable dS vacua.
- As expected, many of the results were similar in 4D and 5D.

The above results lead to some obvious questions:

- Can we find stable dS vacua in 4D and 5D $\mathcal{N} \geq 3$ gauged supergravity or can we prove that for instance $\mathcal{N} = 8$ gauged supergravity has no stable dS vacua?
- Can we find stable dS vacua in $D > 5$ supergravity? It might be interesting to look at 6D supergravity with 8 supercharges. Supergravity theories in $D > 6$, necessarily have more than 8 supercharges and are therefore less likely to have stable dS vacua, since they are more restricted by supersymmetry.
- Is there a (deep) reason for the quantized values of some of the scalar masses?
- Can we find more examples of stable dS vacua in 4D or 5D $\mathcal{N} = 2$ supergravity?
- What are necessary requirements for a supergravity theory to have stable dS vacua?
- What are sufficient requirements for a supergravity theory to have stable dS vacua?

Some of these questions will probably have to be answered before we can say what role, if any, is in store for the dS vacua of supergravity.

Chapter 5

Flux compactifications and orientifolds

Orientifold planes not only make compactifications to de Sitter space possible, but also allow for compactifications with fluxes to Minkowski space. In the last five years these so called flux compactifications have become an increasingly popular subject in string theory. While the main goal of this chapter is the presentation of our own work on orientifold truncations [134], which we describe in section 5.2, to sketch the broader context, we will first very briefly review some of the salient features of flux compactifications.

5.1 Flux compactifications in string theory: some basic ideas

If string theory is truly a unified theory of nature, it has to contain the Standard Model of particle physics, which has been tested and found to be correct to great accuracy. Since low-energy supersymmetry is a possible solution to the gauge-hierarchy problem*, and supersymmetric string backgrounds are under much better calculational control than nonsupersymmetric, one usually looks for a supersymmetric string theory that contains the Standard Model as a subsector. Supersymmetric string theories naturally live in 10 dimensions and in order to make contact with the apparently 4D world we live in, 6 dimensions have to be compactified. From the mid eighties until the mid nineties, the most promising string theory for phenomenological purposes was the $E_8 \times E_8$ heterotic string theory. The authors of [135] discovered that heterotic backgrounds of the form $M_4 \times K_6$, with M_4 a maximally symmetric 4-dimensional space and K_6 a compact 6-dimensional space, only preserve supersymmetry if M_4 is Minkowski space and K_6 is a Calabi-Yau manifold, a Kähler manifold with vanishing first Chern class. This was the start of much work on Calabi-Yau compactifications. With the discovery of D-branes in 1995, and the

*The gauge-hierarchy problem is basically the question of why the mass scale of electro-weak symmetry breaking is so small compared to other fundamental scales like the GUT scale or the Planck scale.

realization that the Standard Model particles might live on such a D-brane (or intersections of D-branes), also Calabi-Yau compactifications in type II string theories became phenomenologically interesting.
One of the problems of Calabi-Yau compactifications is that a Calabi-Yau manifold often comes with a lot of (Kähler and complex structure) moduli, parametrizing the size and shape of the manifold, which are not determined by string theory. These free moduli appear as massless scalar particles in the 4D world, and since they interact at least gravitationally, they usually lead to disagreements with observations. Fortunately, introducing fluxes leads to a potential for some of the moduli, improving the situation. Moreover, by also taking into account nonperturbative corrections, one can generically fix all moduli (see [136] for a recent review).
Besides offering a solution to the moduli problem, flux compactifications also play an important role in the construction of de Sitter vacua in string theory, following the influential paper [137], commonly known as 'the KKLT paper'. In this paper, the authors Kachru, Kallosh, Linde and Trivedi outlined a procedure to construct (metastable) de Sitter vacua. The construction starts with a flux compactification of type IIB on a Calabi-Yau. To add fluxes on a compact space, the positive tension of the fluxes has to be canceled and therefore orientifold planes are added. Compactifying type II on a Calabi-Yau orientifold leads to an effective 4D $\mathcal{N} = 1$ supergravity theory. Switching on fluxes corresponds to gauging some of the isometries of the scalar manifold in the effective 4D theory and induces a potential of the form

$$V = e^K (K^{IJ} D_I W D_J \bar{W} - 3|W|^2) + \frac{1}{2} (\mathrm{Re} f)^{-1\,\alpha\beta} D_\alpha D_\beta \,, \tag{5.1.1}$$

with K the Kähler potential, K^{IJ} the inverse of the Kähler metric $K_{IJ} = \partial_I \partial_J K$, and the Kähler covariant derivatives defined as

$$D_I W = \partial_I W + W \partial_I K \,. \tag{5.1.2}$$

The functions $f_{\alpha\beta}$ are the components of the metric in the kinetic term of the gauge fields, and D_α are the prepotentials of the gauged symmetries. The first term in the potential V is referred to as the 'F-term' and the last term is called the 'D-term'. The KKLT method consists of first looking for supersymmetric AdS vacua in which all moduli are stabilized, and then lift the AdS vacua to dS vacua. In type IIB flux compactifications the D-term is zero and supersymmetric vacua are solutions of the equation $D_I W = 0$ [136]. Generically, in order to stabilize all moduli, nonperturbative corrections to the superpotential W, coming from D-brane instantons or gaugino condensation on D-branes, have to be taken into account. Once a supersymmetric AdS vacuum is found in which all moduli are stabilized, the final step is to lift the AdS vacuum to a dS vacuum. In the original KKLT paper, this was done by adding anti-D3 branes, which breaks supersymmetry explicitly and gives a positive contribution to the potential that changes the AdS into a dS vacuum. Another way to get a dS vacuum is to add fluxes of gauge fields within D7-branes [138]. This leads to a D-term in the potential and has the advantage that supersymmetry is not explicitly broken, but only spontaneously in the dS vacuum. A recent paper suggests we

might also look for supersymmetric Minkowski vacua and try to lift these [139]. Finally, we might of course also try to look directly for stable nonsupersymmetric dS extrema of the potential V induced by the fluxes and nonperturbative effects, but this is usually more difficult.
Explicit examples illustrating the ideas on moduli-fixing discussed in the KKLT paper can be found in [140] and attempts to construct dS vacua from flux compactifications in [141]. Some other ways to construct dS vacua were suggested in [142].

5.2 Calabi-Yau orientifolds and special geometry

Compactifying type II string theory on a 6-dimensional Calabi-Yau manifold leads in the low-energy limit to 4D $\mathcal{N} = 2$ supergravity. A Calabi-Yau manifold with $h^{(1,1)}$ Kähler structure moduli and $2h^{(1,2)}$ complex structure moduli gives rise to $h^{(1,1)}$ vector multiplets and $h^{(1,2)} + 1$ hypermultiplets in the IIA case, and to $h^{(1,2)}$ vector multiplets and $h^{(1,1)} + 1$ hypermultiplets in the IIB case. A well known conjecture states that type IIA string theory compactified on a Calabi-Yau threefold Y is equivalent to type IIB string theory compactified on a mirror Calabi-Yau threefold $\tilde{Y}$, with $h^{(1,1)}(Y) = h^{(1,2)}(\tilde{Y})$ and $h^{(1,2)}(Y) = h^{(1,1)}(\tilde{Y})$ (see [143] for a review). As we reviewed in section 4.1.3, the scalar manifold of the vector multiplets is a special Kähler manifold, while the scalars of the hypermultiplets parametrize a quaternionic-Kähler manifold. In the case of Calabi-Yau compactifications, the quaternionic-Kähler manifold is a special quaternionic-Kähler manifold, the geometry is completely determined by a holomorphic prepotential $F(X)$, which is homogeneous of degree two [144]. In order to display the mirror map at the supergravity level, one has to take the large volume limit in the IIB theory and the large complex structure limit in the IIA theory [145]. In this limit the prepotentials on the two sides can be identified and are given by $F = id_{ABC}X^A X^B X^C/X^0$, with d_{ABC} a completely symmetric tensor, i.e. the quaternionic-Kähler manifolds are very special.
Introducing orientifold planes results in truncating the 4D low-energy effective theory, thereby breaking supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 1$. Due to the remaining $\mathcal{N} = 1$ supersymmetry, the reduced scalar manifold has to be Hodge-Kähler. In the hypermultiplet sector, this leads to some interesting mathematics. A theorem in [146, 147] states that any Kähler submanifold of a quaternionic-Kähler manifold can have a real dimension of at most half the real dimension of the quaternionic-Kähler manifold. The orientifold truncation puts to zero half of the hyperscalars and therefore gives rise to a Kähler submanifold with a dimension that exactly saturates the upper bound. Moreover, it was found in [148] that truncations that arise in type IIB Calabi-Yau compactifications with O3 planes lead to Kähler submanifolds (of very special quaternionic-Kähler manifolds) that factorize into a Kähler manifold times $SU(1,1)/U(1)$. Since the very special quaternionic-Kähler spaces are related by the c-map to a special Kähler space, one obtains in this way a map between a special Kähler space and a 'dual' Kähler space. These dual Kähler spaces were further investigated in [149], where it was found that the symmetric special Kähler spaces

and their corresponding duals are equal as coset spaces and have the same curvature tensor. Motivated by their work we studied more general truncations, corresponding to the presence of both O3 and O7 planes in type IIB, in [134]. For the symmetric special quaternionic-Kähler spaces we studied the truncated isometry algebra and gave a complete overview of the Kähler spaces one gets after truncation. For the very special spaces we also explicitly verified the Kähler structure of the truncated spaces and studied the dual Kähler spaces by calculating their curvature tensor. We will present this work in the following sections.

5.2.1 Type IIB on Calabi-Yau manifolds and orientifolds

Type IIB supergravity consists of a real scalar $\hat{\phi}$ called the dilaton, a space-time metric $\hat{g}$ and a two-form $\hat{B}_2$ (all coming from the NS-NS sector in string theory), together with the axion $\hat{C}_0$, a two-form $\hat{C}_2$ and a four-form $\hat{C}_4$ with selfdual field strength (all coming from the R-R sector in string theory)*. To compactify the theory, we make a product ansatz $M \times K$ for the 10D space-time, and we expand all the fields in terms of a complete set of eigenfunctions on the compact internal space K. This leads to an infinite tower of generally massive fields in the lower-dimensional theory, but if the internal manifold K is sufficiently small, the massive fields will be very massive, and we can restrict to the massless sector. For instance, consider a massless scalar $\hat{\phi}$ in flat $(D+1)$-dimensional space. It satisfies

$$\partial^M \partial_M \hat{\phi} = (\partial^\mu \partial_\mu + \partial^z \partial_z)\hat{\phi} = 0\,, \tag{5.2.1}$$

where x^μ are the coordinates of a D-dimensional space, z is a coordinate of a 1-dimensional space, and $X^M = (x^\mu, z)$ are coordinates of the flat $(D+1)$-dimensional space. We can then compactify the z-direction (on a circle S^1) and expand the scalar field in terms of eigenfunctions of $\partial^z \partial_z$[150]:

$$\hat{\phi}(x,z) = \sum_n \phi_n(x) e^{i\,n\,z/L}\,. \tag{5.2.2}$$

The lower-dimensional fields $\phi_n(x)$ then obey the wave equation for a scalar field of mass $|n|/L$:

$$\partial^\mu \partial_\mu \phi_n - \frac{n^2}{L^2}\phi_n = 0\,. \tag{5.2.3}$$

If the radius L is small enough, the nonzero modes will have huge masses and can be neglected.

The zero-modes of form fields (antisymmetric tensor fields) are in 1-1 correspondence with harmonic forms† on K and their multiplicities are given by the dimensions of the cohomology groups of K [151]. The numbers $h^{(p,q)}$ of inequivalent harmonic (p,q)-forms

*All 10D fields will have a hat in this section, while lower-dimensional fields arising from a compactification will be denoted without hats.

†A form ω is harmonic if $\Delta\omega = 0$ with Δ a Laplacian on K. For Kähler metrics the various Laplacians are identical, $\Delta_d \equiv dd^\dagger + d^\dagger d = 2\Delta_\partial \equiv 2\partial\partial^\dagger + 2\partial^\dagger\partial = 2\Delta_{\bar\partial} \equiv 2\bar\partial\bar\partial^\dagger + 2\bar\partial^\dagger\bar\partial$.

on a Kähler manifold K are usually summarized in a Hodge diamond, and for a 6D Calabi-Yau manifold (also called a Calabi-Yau 3-fold) we have

$$\begin{array}{ccccccc} & & & h^{(0,0)} & & & \\ & & h^{(1,0)} & & h^{(0,1)} & & \\ & h^{(2,0)} & & h^{(1,1)} & & h^{(0,2)} & \\ h^{(3,0)} & & h^{(2,1)} & & h^{(1,2)} & & h^{(0,3)} \\ & h^{(3,1)} & & h^{(2,2)} & & h^{(1,3)} & \\ & & h^{(3,2)} & & h^{(2,3)} & & \\ & & & h^{(3,3)} & & & \end{array} \quad = \quad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{(1,1)} & & 0 & \\ 1 & & h^{(2,1)} & & h^{(2,1)} & & 1 \\ & 0 & & h^{(1,1)} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \quad . \tag{5.2.4}$$

A Calabi-Yau 3-fold has a unique harmonic $(3,0)$-form Ω, and at least one harmonic $(1,1)$-form, the Kähler form $J = ig^0_{i\bar{j}} dy^i \wedge dy^{\bar{j}}$, so $h^{(1,1)} \geq 1$. More info on Calabi-Yau manifolds can be found for instance in [152].
Compactifying type IIB supergravity on a Calabi-Yau 3-fold (and truncating to massless fields) leads to a 4D $\mathcal{N} = 2$ supergravity theory. The 4D massless fields arising from the spacetime metric $\hat{g}$ are a 4D space-time metric g and 4D massless scalar fields coming from deformations δg of the Calabi-Yau metric g^0 that respect the Ricci-flatness condition. These deformations are in 1-1 correspondence with harmonic $(1,1)$- and $(2,1)$-forms on the Calabi-Yau 3-fold[151, 152, 153]. The deformations of the Kähler form J giver rise to $h^{(1,1)}$ real scalars v^A, while the complex structure deformations lead to $h^{(2,1)}$ complex scalars z^K. The 10D space-time metric $\hat{g}_{MN}$ thus decomposes as follows[145, 154]:

$$\begin{aligned} \hat{g}_{\mu\nu}(x,y) &= g_{\mu\nu}(x)\,, & \hat{g}_{\mu i}(x,y) = \hat{g}_{\mu\bar{i}}(x,y) = 0\,, \\ \hat{g}_{i\bar{j}}(x,y) &= -i\,u^A(x)(\omega_A)_{i\bar{j}}(y)\,, & \hat{g}_{ij}(x,y) = i\,z^K(x)\left(\frac{(\bar{\chi}_K)_{i\bar{k}\bar{l}}\Omega^{\bar{k}\bar{l}}{}_j}{|\Omega|^2}\right)(y)\,, \end{aligned} \tag{5.2.5}$$

with ω_A and $\bar{\chi}_K$ a basis of harmonic $(1,1)$-forms and $(2,1)$-forms respectively, and $|\Omega|^2 \equiv \frac{1}{3!}\Omega_{ijk}\bar{\Omega}^{ijk}$. The reduction of the 10D space-time metric does not give rise to 4D vector fields, because there are no harmonic 1-forms on a Calabi-Yau 3-fold.
The type IIB gauge potentials can be expanded as follows[145, 154]:

$$\begin{aligned} \hat{\phi}(x,y) &= \phi(x)\,, & \hat{B}_2(x,y) &= B_2(x) + b^A(x)\,\omega_A(y)\,, \\ \hat{C}_0(x,y) &= C_0(x)\,, & \hat{C}_2(x,y) &= C_2(x) + c^A(x)\,\omega_A(y)\,, \\ \hat{C}_4(x,y) &= D_2^A(x)\wedge\omega_A(y) + V_1^{\hat{K}}(x)\wedge\alpha_{\hat{K}}(y) - U_{1\hat{K}}(x)\wedge\beta^{\hat{K}}(y) + \rho_A(x)\,\tilde{\omega}^A(y)\,. \end{aligned} \tag{5.2.6}$$

Here $\tilde{\omega}^A$ are a basis of harmonic $(2,2)$-forms dual to ω^A, and $(\alpha_{\hat{K}}, \beta^{\hat{L}})$ form a basis of harmonic 3-forms, with $\hat{K} = 0, \ldots, h^{(1,2)}$. We have

$$\begin{aligned} \int_K \omega_A \wedge \tilde{\omega}^B &= \delta_A^B\,, & \int_K \alpha_{\hat{K}} \wedge \beta^{\hat{L}} &= \delta_{\hat{K}}^{\hat{L}}\,, \\ \int_K \alpha_{\hat{K}} \wedge \alpha_{\hat{L}} &= 0\,, & \int_K \beta^{\hat{K}} \wedge \beta^{\hat{L}} &= 0\,. \end{aligned} \tag{5.2.7}$$

The gauge potential $\hat{C}_4$ has a selfdual 5-form field strength, and therefore half of the degrees of freedom of $\hat{C}_4$ should be eliminated. As in [145], we choose to keep D_2^A and $V_1^{\hat{K}}$ and eliminate ρ_A and $U_{1\hat{K}}$.
The fields $(g_{\mu\nu}, V_1^0)$ are the bosonic components of a gravitational multiplet, (V_1^K, z^K) form the bosonic parts of $h^{(2,1)}$ vector multiplets, and (D_2^A, u^A, b^A, c^A) are the bosonic fields of $h^{(1,1)}$ tensor multiplets. Finally, the fields (B_2, C_2, ϕ, C_0) are the bosonic components of one double-tensor multiplet. In 4D an antisymmetric tensor is dual to a scalar. The double-tensor and tensor multiplets can therefore be dualized to hypermultiplets. We will call the scalars dual to B_2, C_2 and $E_2^A \equiv D_2^A + \frac{1}{3}(B_2 c^A - C_2 b^A)$ (see [145] for technical details*), respectively h, $\tilde{h}$ and g_A. A summary of the 4D massless spectrum is given in table 5.1.

gravity multiplet	1	$(g_{\mu\nu}, V_1^0)$
vector multiplets	$h^{(2,1)}$	(V_1^K, z^K)
hypermultiplets	$h^{(1,1)}$	(u^A, b^A, c^A, g_A)
	1	$(\phi, C_0, h, \tilde{h})$

Table 5.1: $\mathcal{N} = 2$ multiplets for type IIB supergravity compactified on a Calabi-Yau manifold (after dualizing all 4D 2-forms to scalars).

Finally, in order to get the standard 4D $\mathcal{N} = 2$ action, some small field redefinitions are performed[145]: a 4-dimensional dilation ϕ_4 and 'rotated' moduli fields v^a are introduced by

$$e^{-2\phi_4} = \mathcal{V}(v) e^{-2\phi}\,, \qquad v^A = u^A e^{\phi/2}\,, \tag{5.2.8}$$

with $\mathcal{V}(v)$ the volume of the Calabi-Yau 3-fold (measured in the string frame†),

$$\mathcal{V}(v) = \frac{1}{6} d_{ABC} v^A v^B v^C\,, \tag{5.2.9}$$

and d_{ABC} the intersection numbers of the Calabi-Yau 3-fold,

$$d_{ABC} = \int_K \omega^A \wedge \omega^B \wedge \omega^C\,. \tag{5.2.10}$$

As we mentioned before, in the following sections, we will focus on the hypermultiplet sector and study how the quaternionic-Kähler geometry of the hypermultiplets gets reduced

*To compare our notation with that of [145], see table 5.2 in section 5.2.2.

†The volume of K is $\mathcal{V} = \frac{1}{6}\int_K J \wedge J \wedge J$. In the string frame, the 10D Einstein-Hilbert term takes the form $\int \frac{1}{2} e^{-2\hat{\phi}} \sqrt{-\hat{g}} \hat{R}$, while in the Einstein frame it is $\int \frac{1}{2}\sqrt{-\hat{g}}\hat{R}$. Therefore J_S, obtained from the internal part of the 10D string frame metric, is related to J_E in the Einstein frame by $J_S = e^{\phi/2} J_E$[155]. (The metric $\hat{g}$ in equation (5.2.5) is the Einstein frame metric, and $\hat{g}_{i\bar{j}} = -iJ_{i\bar{j}}$.)

to a Kähler-Hodge geometry when orientifolds are presented. Before we come to that, we have to know when orientifold planes can be introduced. A sufficient condition for the existence of IIB Calabi-Yau orientifolds is the presence of an isometric and holomorphic involution σ on K [156, 157, 154]. By definition, a holomorphic isometry leaves both the metric and the complex structure, and therefore also the Kähler form J invariant:

$$\sigma^* J = J \,, \tag{5.2.11}$$

with σ^* the pullback of σ. We also have

$$\sigma^* H^{(3,0)} = H^{(3,0)} \,, \qquad (\sigma^*)^2 = \mathrm{id} \,, \tag{5.2.12}$$

with $H^{(3,0)}$ the (Dolbeault) cohomology class of (3,0)-forms. Thus, we have to distinguish two different cases [154, 155]:

$$(1) \quad O3/O7: \quad \sigma^*\Omega = -\Omega \,, \qquad\qquad (2) \quad O5/O9: \quad \sigma^*\Omega = +\Omega \,. \tag{5.2.13}$$

The orientifold planes always fill the full uncompactified 4D space-time and are even-dimensional (space directions + time) in IIB, because σ is holomorphic. The dimensionality of the fixed point set of σ in K determines the actual dimensionality of the orientifold planes.
For more detailed information on (Calabi-Yau) orientifolds in string theory, we refer the reader to [158, 155] and references therein.

5.2.2 The O3/O7 orientifold truncation

In this section we will describe truncations of the hypermultiplet sector of 4D $\mathcal{N} = 2$ supergravity that occur when O3 and/or O7 planes are present in Calabi-Yau compactifications of IIB string theory. Because of mirror symmetry we can also study these truncations in the dual IIA theory. Compactifying IIA/IIB supergravity on a (mirror) Calabi-Yau leads to a 4D $\mathcal{N} = 2$ supergravity in IIA/IIB coordinates respectively. Explicit formulas relating the IIA and IIB coordinates of the hypermultiplet sector were derived in [145]. Using these formulas we can translate the truncation of the IIB fields into a corresponding truncation of the IIA fields. To conform with the notation of [148, 159], to which we shall refer extensively, we will use IIA coordinates.
We parametrize the $(4n + 4)$-dimensional very special quaternionic-Kähler manifolds by

- the special coordinates $z^A = \frac{1}{2}(x^A + iy^A), \quad A = 1, ..., n,$
- the extra coordinates $D, \sigma, A^I, B_I, \quad I = (0, A) = 0, ..., n.$

These IIA coordinates are related to the IIB coordinates of the previous section by [145]:

$$\begin{aligned}
x^A &= \frac{2}{3}b^A\,,\\
y^A &= v^A\,,\\
A^0 &= \sqrt{2}C_0,\\
A^A &= \frac{2\sqrt{2}}{3}(C_0 b^A - c^A)\,,\\
B_A &= -\frac{\sqrt{2}}{4}g_A + \frac{\sqrt{2}}{18}C_0 d_{ABC}b^B b^C - \frac{\sqrt{2}}{18}d_{ABC}b^B c^C\,,\\
B_0 &= \frac{\sqrt{2}}{4}\tilde{h} - \frac{\sqrt{2}}{81}C_0 d_{ABC}b^A b^B b^C + \frac{\sqrt{2}}{81}d_{ABC}b^A b^B c^C\,,\\
\sigma &= h + \frac{1}{2}C_0\tilde{h} + \frac{1}{3}c^A g_A - \frac{1}{3}C_0 b^A g_A - \frac{2}{81}d_{ABC}b^A c^B c^C + \frac{2}{81}C_0 d_{ABC}b^A b^B c^C\,,\\
D &= \phi_4\,.
\end{aligned} \tag{5.2.14}$$

To compare our notation with those of the papers [145, 148, 159], see table 5.2.

Here (IIB)	[145]	Here (IIA)	[145]	[148]	[159]
u^A	$\hat{v}^a$	x^A	x^a	x^a	y^A
v^A	v^a	y^A	y^a	y^a	$-x^A$
b^A	$\hat{b}^{1a}$	A^0	ρ^0	ρ^0	A^0
c^A	$\hat{b}^{2a}$	A^A	ρ^a	ρ^a	A^A
g_A	g_a	B_0	$\tilde{\rho}_0$	$\tilde{\rho}_0$	B_0
D_2^A	$\hat{D}^a_{\mu\nu}$	B_A	$\tilde{\rho}_a$	$\tilde{\rho}_a$	B_A
E_2^A	$D^a_{\mu\nu}$	D	ϕ_A	D	D
ϕ	ϕ	σ	$\tilde{\phi}$	$\tilde{\Phi}$	σ
ϕ_4	ϕ_4				
C_0	l				
h	h_1				
$\tilde{h}$	h_2				

Table 5.2: Comparison of notations.

The $\mathcal{N} = 2$ Lagrangian for $n + 1$ hypermultiplets with very special quaternionic-Kähler

target space (in IIA coordinates) is (see [145]):

$$
\begin{aligned}
(\sqrt{-g})^{-1}\mathfrak{L} \;=\; & \frac{R}{2} - (\partial_\mu D)^2 - \frac{1}{4}G_{AB}\partial_\mu y^A \partial^\mu y^B - \frac{1}{4}G_{AB}\partial_\mu x^A \partial^\mu x^B \\
& -\frac{1}{8}e^{2D}\mathcal{V}(\partial_\mu A^0)^2 - \frac{1}{2}e^{2D}\mathcal{V}G_{AB}\big(x^B\partial_\mu A^0 - \partial_\mu A^B\big)\big(x^A\partial^\mu A^0 - \partial^\mu A^A\big) \\
& -2e^{2D}\mathcal{V}^{-1}G^{-1AB}\Big(\partial_\mu B_A + \frac{1}{8}(\kappa xx)_A\partial_\mu A^0 - \frac{1}{4}(\kappa x)_{AC}\partial_\mu A^C\Big)\times \\
& \qquad\qquad \Big(\partial^\mu B_B + \frac{1}{8}(\kappa xx)_B\partial^\mu A^0 - \frac{1}{4}(\kappa x)_{BD}\partial^\mu A^D\Big) \\
& -2e^{2D}\mathcal{V}^{-1}\Big(\partial_\mu B_0 + x^A\partial_\mu B_A + \frac{1}{24}(\kappa xxx)\partial_\mu A^0 - \frac{1}{8}(\kappa xx)_A\partial_\mu A^A\Big)^2 \\
& -\frac{1}{4}e^{4D}\big(\partial_\mu\sigma + A^C\partial_\mu B_C - B^C\partial_\mu A_C\big)^2 ,
\end{aligned}
\tag{5.2.15}
$$

with

$$
G_{AB} = -6\left(\frac{(\kappa y)_{AB}}{(\kappa yyy)} - \frac{3}{2}\frac{(\kappa yy)_A(\kappa yy)_B}{(dyyy)^2}\right) , \tag{5.2.16}
$$

and

$$
\mathcal{V} = \frac{1}{6}(\kappa yyy) = \frac{1}{6}d_{ABC}y^A y^B y^C , \quad (\kappa xx)_A = d_{ABC}x^B x^C , \quad (\kappa x)_{AB} = d_{ABC}x^C . \tag{5.2.17}
$$

Splitting the index A into $A = (a, \alpha)$, with $a = 1, .., n_-$ and $\alpha = 1, .., n_+$ ($n_+ + n_- = n$), the Lagrangian (5.2.15) is invariant under the following discrete symmetries:

$$
\begin{aligned}
& y_\pm \to \pm y_\pm , \qquad D \to D , \\
& x_\pm \to \mp x_\pm , \qquad \sigma \to -\sigma ,
\end{aligned}
\tag{5.2.18}
$$

and

1. O3/O7

$$
\begin{aligned}
& A_\pm \to \mp A_\pm , \qquad A^0 \to A^0 , \\
& B_\pm \to \pm B_\pm , \qquad B_0 \to -B_0 ,
\end{aligned}
\tag{5.2.19}
$$

2. O5/O9

$$
\begin{aligned}
& A_\pm \to \pm A_\pm , \qquad A^0 \to -A^0 , \\
& B_\pm \to \mp B_\pm , \qquad B_0 \to B_0 ,
\end{aligned}
\tag{5.2.20}
$$

if

$$
d_{++-} = d_{---} = 0 . \tag{5.2.21}
$$

Keeping only the fields invariant under the discrete symmetry, i.e. setting the noninvariant fields to zero, truncates the theory in a consistent manner to an $\mathcal{N} = 1$ theory*. Case 1)

*See [146] for a general analysis of supersymmetry reduction in supergravity.

(equation (5.2.19)) leads to a truncation that occurs when O3/O7 orientifold planes are present, while case 2) (equation (5.2.20)) leads to a truncation that occurs when O5/O9 orientifold planes are present(see [154] for details).
In the rest of this thesis, we will focus on the O3/O7 orientifold truncations*. The case $n_- = 0$ (only O3-planes present) was studied in detail in [148, 149]. We will extend their results to the case $n_- \neq 0$. Looking at equation (5.2.19), we see that the O3/O7 orientifold truncations are given by

$$\begin{aligned} &x^\alpha = y^a = \sigma = 0\,, \\ &A^\alpha = B_a = B_0 = 0\,. \end{aligned} \tag{5.2.22}$$

The remaining nonzero fields are: $\{D, x^a, y^\alpha, A^0, A^a, B_\alpha\}$.
For a consistent truncation, one must also demand (see equation (5.2.21))

$$d_{\alpha\beta a} = d_{abc} = 0\,. \tag{5.2.23}$$

Performing the truncation (5.2.22) on the Lagrangian (5.2.15) gives rise to the following Lagrangian with $2n+2$ real variables (we omit the trivially contracted index μ from here on):

$$\begin{aligned} (\sqrt{-g})^{-1}\mathfrak{L} \;=\; & \frac{R}{2} - (\partial D)^2 - \frac{1}{4}G_{\alpha\beta}\partial y^\alpha \partial y^\beta - \frac{1}{4}G_{ab}\partial x^a \partial x^b - \frac{1}{8}e^{2D}\mathcal{V}(\partial A^0)^2 \\ & -\frac{1}{8}e^{2D}\mathcal{V}G_{ab}(x^a\partial A^0 - \partial A^a)(x^b\partial A^0 - \partial A^b) \\ & -2e^{2D}\mathcal{V}^{-1}(G^{-1})^{\alpha\beta}\big[\partial B_\alpha + \frac{1}{8}(\kappa xx)_\alpha \partial A^0 \\ & -\frac{1}{4}(\kappa x)_{\alpha c}\partial A^c\big]\big[\partial B_\beta + \frac{1}{8}(\kappa xx)_\beta \partial A^0 - \frac{1}{4}(\kappa x)_{\beta d}\partial A^d\big]\,, \end{aligned} \tag{5.2.24}$$

with truncated elements of the special Kähler metric G_{AB} given by

$$\begin{aligned} G_{\alpha\beta} &= -6\left(\frac{(\kappa y)_{\alpha\beta}}{(\kappa yyy)} - \frac{3}{2}\frac{(\kappa yy)_\alpha(\kappa yy)_\beta}{(dyyy)^2}\right)\,, \\ G_{\alpha a} &= 0\,, \qquad G_{ab} = -6\frac{(\kappa y)_{ab}}{(\kappa yyy)}\,. \end{aligned} \tag{5.2.25}$$

and we have introduced the following notation†:

$$\begin{array}{lll} (\kappa y)_{\alpha\beta} = d_{\alpha\beta\gamma}y^\gamma\,, & (\kappa y)_{ab} = d_{ab\gamma}y^\gamma\,, & (\kappa x)_{\alpha a} = d_{ab\alpha}x^b\,, \\ (\kappa yy)_\alpha = d_{\alpha\beta\gamma}y^\beta y^\gamma\,, & (\kappa xy)_a = d_{ab\gamma}x^b y^\gamma\,, & (\kappa xx)_\alpha = d_{ab\alpha}x^a x^b\,, \\ (\kappa yyy) = d_{\alpha\beta\gamma}y^\alpha y^\beta y^\gamma = \kappa\,, & (\kappa xxy) = d_{ab\gamma}x^a x^b y^\gamma\,, & \mathcal{V}(y) = \frac{1}{6}(\kappa yyy)\,. \end{array}$$

Note that the truncated special Kähler metric in (5.2.25) has no mixed components. From this, and the restriction (5.2.23) on the d-symbols, one can deduce a restriction on the

*Study of the O5/O9 truncations probably will lead to similar results as in the O3/O7 case. A few results for the O5/O9 truncations can be found in [154].

†It will always be clear which variables have a greek index, and which have a latin index, so that there should not be any confusion between $(\kappa xx)_\alpha$ and $(\kappa yy)_\alpha$.

matrix C^{ABC} that appears in the curvature tensor $R_A{}^C{}_B{}^D$ (see equation (4.1.71)). It is defined by

$$C^{ABC} = -27 G^{AD} G^{BE} G^{CF} d_{DEF} (\kappa y y y)^{-2} \,, \tag{5.2.26}$$

and one sees easily that

$$C^{\alpha\beta a} = C^{abc} = 0 \,. \tag{5.2.27}$$

5.2.3 The Kähler potential

Supersymmetry requires the target space of $\mathcal{N} = 1$ σ-model Lagrangian in (5.2.24) to be Kähler. The aim of this section is to find the right coordinate transformations that allow us to write this Lagrangian in complex coordinates and to find the Kähler potential. We start with the same change of variables as in [148]

$$e^{2\Phi} := \mathcal{V}(y) e^{2D}, \qquad \chi^\alpha := y^\alpha e^{-\frac{\Phi}{2}} \,, \tag{5.2.28}$$

followed by

$$p_\alpha := \frac{1}{2} (\kappa \chi \chi)_\alpha = \frac{1}{2} d_{\alpha\beta\gamma} \chi^\beta \chi^\gamma \,, \tag{5.2.29}$$

so that

$$\partial \chi^\beta = ((\kappa\chi)^{-1})^{\beta\alpha} \partial p_\alpha \,. \tag{5.2.30}$$

Now we define

$$\lambda^a := x^a e^{-\Phi} \,, \tag{5.2.31}$$

and we also perform a rescaling $A^a \to A^a \sqrt{2}$, $A^0 \to A^0 \sqrt{2}$ and $B_\alpha \to B_\alpha / 2\sqrt{2}$. This gives us the following Lagrangian in the new variables:

$$\begin{aligned}
(\sqrt{-g})^{-1} \mathfrak{L} \;=\; & \frac{R}{2} - \frac{1}{4} e^{\Phi} G_{ab}(\chi) [\partial \lambda^a \partial \lambda^b + \partial A^a \partial A^b] \\
& + \frac{1}{2} e^{\Phi} G_{ab} \lambda^b [-\partial\Phi \partial \lambda^a + e^{\Phi} \partial A^a \partial A^0] \\
& - \frac{1}{4} (G_{ab} + 1) \lambda^a \lambda^b [(\partial\Phi)^2 + e^{2\Phi} (\partial A^0)^2] \\
& - \frac{1}{4} \mathcal{V}(\lambda)^{-2} (G^{-1})^{\alpha\beta} [\partial B_\alpha \partial B_\beta + \partial p_\alpha \partial p_\beta + e^{2\Phi} (\kappa\lambda\lambda)_\beta \partial A^0 \partial B_\alpha \\
& - 2 e^{\Phi} (\kappa\lambda)_{\beta c} \partial B_\alpha \partial A^c + \frac{1}{4} e^{4\Phi} (\kappa\lambda\lambda)_\alpha (\kappa\lambda)_\beta (\partial A^0)^2 \\
& e^{3\Phi} (\kappa\lambda\lambda)_\alpha (\kappa\lambda)_{\beta c} \partial A^0 \partial A^c + e^{2\Phi} (\kappa\lambda)_{\alpha c} (\kappa\lambda)_{\beta d} \partial A^c \partial A^d] \,.
\end{aligned} \tag{5.2.32}$$

Finally, we perform the following transformation:

$$p_\alpha = -t_\alpha + \frac{1}{2} e^{\Phi} (\kappa\lambda\lambda)_\alpha = -t_\alpha + \frac{1}{2} e^{\Phi} d_{\alpha ab} \lambda^a \lambda^b \,. \tag{5.2.33}$$

After some calculation, we find that the Lagrangian can now be written in the following complex coordinates:

$$\begin{aligned}
\eta &= \exp(-\Phi) + i A^o = \eta_1 + i \eta_2 \,, \\
\epsilon_\alpha &= t_\alpha + i B_\alpha \,, \\
\rho^a &= \lambda^a + i A^a \,.
\end{aligned} \tag{5.2.34}$$

This gives us

$$
\begin{aligned}
(\sqrt{-g})^{-1}\mathcal{L} &= \frac{R}{2} - \frac{1}{4}\big[\frac{G_{ab}}{\eta_1} + \frac{g^{\alpha\beta}}{{\eta_1}^2}(\kappa\lambda)_{\alpha a}(\kappa\lambda)_{\beta b}\big][\partial\rho^a\partial\bar{\rho}^b] - \frac{1}{4}g^{\alpha\beta}[\partial\epsilon_\alpha\partial\bar{\epsilon}_\beta] \\
&\quad -\frac{1}{4{\eta_1}^2}\big[1 + \frac{1}{4{\eta_1}^2}g^{\alpha\beta}(\kappa\lambda\lambda)_\alpha(\kappa\lambda\lambda)_\beta + \frac{1}{\eta}G_{ab}\lambda^a\lambda^b\big](\partial\eta)^2 \\
&\quad +\frac{1}{4{\eta_1}^2}\big[G_{ab}\lambda^b + \frac{1}{2\eta_1}g^{\alpha\beta}(\kappa\lambda\lambda)_\alpha(\kappa\lambda)_{\beta a}\big][\partial\eta\partial\bar{\rho}^a + \partial\bar{\eta}\partial\rho^a] \\
&\quad +\frac{1}{4\eta_1}g^{\alpha\beta}(\kappa\lambda)_{\beta a}[\partial\epsilon_\alpha\partial\bar{\rho}^a + \partial\bar{\epsilon}_\alpha\partial\rho^a] \\
&\quad -\frac{1}{8{\eta_1}^2}g^{\alpha\beta}(\kappa\lambda\lambda)_\beta[\partial\epsilon_\alpha\partial\bar{\eta} + \partial\eta\partial\bar{\epsilon}_\alpha]\,,
\end{aligned}
\tag{5.2.35}
$$

with $g^{\alpha\beta} := (G^{-1})^{\alpha\beta}/\mathcal{V}^2$.
The scalar metric can be derived from the following Kähler-potential:

$$
K = -2\log\mathcal{V}\big(\chi^\alpha(p(t_\beta,\lambda^a,\eta_1))\big) - \log(\eta_1)\,,
\tag{5.2.36}
$$

since we can write the Lagrangian as

$$
\begin{aligned}
(\sqrt{-g})^{-1}\mathcal{L} &= \frac{R}{2} - \frac{1}{4}\big[\bar{g}_{ab}(\partial\rho^a\partial\bar{\rho}^b) + \bar{g}^{\alpha\beta}(\partial\epsilon_\alpha\partial\bar{\epsilon}_\beta) + \bar{g}_{00}(\partial\eta)^2 \\
&\quad +\bar{g}_{a0}(\partial\eta\partial\bar{\rho}^a + \partial\bar{\eta}\partial\rho^a) + \bar{g}^\alpha{}_0(\partial\epsilon_\alpha\partial\bar{\eta} + \partial\eta\partial\bar{\epsilon}_\alpha) \\
&\quad +\bar{g}^\alpha{}_a(\partial\epsilon_\alpha\partial\bar{\rho}^a + \partial\bar{\epsilon}_\alpha\partial\rho^a)\big]\,,
\end{aligned}
\tag{5.2.37}
$$

where

$$
\begin{aligned}
\bar{g}^{\alpha\beta} &= \frac{\partial^2 K}{\partial t_\alpha\partial t_\beta}\,, \qquad \bar{g}_{00} = \frac{\partial^2 K}{\partial\eta_1\partial\eta_1}\,, \qquad \bar{g}_{ab} = \frac{\partial^2 K}{\partial\lambda^a\partial\lambda^b}\,, \\
\bar{g}_{a0} &= \frac{\partial^2 K}{\partial\lambda^a\partial\eta_1}\,, \qquad \bar{g}^\alpha{}_0 = \frac{\partial^2 K}{\partial t_\alpha\partial\eta_1}\,, \qquad \bar{g}^\alpha{}_a = \frac{\partial^2 K}{\partial t_\alpha\partial\lambda^a}\,.
\end{aligned}
\tag{5.2.38}
$$

The Kähler potential depends only on the real parts of the complex variables. Contrary to the case $n_- = 0$ studied in [148], the $(\eta,\bar{\eta})$-part does not decouple when $n_- \neq 0$, and therefore does not describe a separate $SU(1,1)/U(1)$ σ-model. The Kähler space that is obtained, has complex dimension $n+1$ and is therefore different from the original Kähler space upon which the **c**-map is performed (which has complex dimension n).
It is now instructive to study the $\mathcal{N}=1$ potential (5.1.1) in a little more detail. The D-term always gives a positive contribution to the potential, while the F-term is the difference of two positive terms. In the following we will concentrate on the F-term and assume the D-term is equal to zero. The potential can be written then as

$$
V = e^K\left[(K^{IJ}\partial_I K\partial_J\bar{K} - 3)|W|^2 + K^{IJ}\partial_I W\partial_J\bar{W}\right]\,.
\tag{5.2.39}
$$

Because the Kähler metric is positive definite, the second term in (5.2.39) will always be positive. To study the first term, we need the inverse Kähler metric. Defining $(\eta^\Lambda) \equiv$

$(\eta, \epsilon_\alpha, \rho^a)$, we have for the metric $\bar{g}_{\Lambda\bar{\Sigma}}$

$$\begin{aligned}
\bar{g}^{\alpha\beta} &= g^{\alpha\beta}\,, \\
\bar{g}_{00} &= \frac{1}{\eta_1^2}(1 + \frac{1}{4\eta_1^2} g^{\alpha\beta}(\kappa\lambda\lambda)_\alpha(\kappa\lambda\lambda)_\beta + \frac{1}{\eta_1} G_{ab}\lambda^a\lambda^b)\,, \\
\bar{g}_{ab} &= \frac{g^{\alpha\beta}}{\eta_1^2}(\kappa\lambda)_{\alpha a}(\kappa\lambda)_{\beta b} + \frac{1}{\eta_1} G_{ab}\,, \\
\bar{g}_{a0} &= -(\frac{1}{\eta_1^2} G_{ab}\lambda^b + \frac{1}{2\eta_1^3} g^{\alpha\beta}(\kappa\lambda\lambda)_\beta(\kappa\lambda)_{\alpha a})\,, \\
\bar{g}^{\alpha}{}_0 &= \frac{1}{2\eta_1^2} g^{\alpha\beta}(\kappa\lambda\lambda)_\beta\,, \\
\bar{g}^{\alpha}{}_a &= -\frac{g^{\alpha\beta}}{\eta_1}(\kappa\lambda)_{\beta a}\,.
\end{aligned} \tag{5.2.40}$$

A straightforward calculation gives the following expressions for the inverse metric $(\bar{g}^{-1})^{\Lambda\bar{\Sigma}}$

$$\begin{aligned}
(\bar{g}^{-1})_{\alpha\beta} &= (g^{-1})_{\alpha\beta} + \frac{1}{4\eta_1^2}(\kappa\lambda\lambda)_\alpha(\kappa\lambda\lambda)_\beta + \frac{(G^{-1})^{ab}}{\eta_1}(\kappa\lambda)_{\alpha a}(\kappa\lambda)_{\beta b}\,, \\
(\bar{g}^{-1})^{00} &= \eta_1^2\,, \\
(\bar{g}^{-1})^{ab} &= \lambda^a\lambda^b + \eta_1(G^{-1})^{ab}\,, \\
(\bar{g}^{-1})^{a0} &= \eta_1\lambda^a\,, \\
(\bar{g}^{-1})^{0}{}_\alpha &= \frac{1}{2}(\kappa\lambda\lambda)_\alpha\,, \\
(\bar{g}^{-1})^{a}{}_\alpha &= \frac{\lambda^a}{2\eta_1}(\kappa\lambda\lambda)_\alpha + (G^{-1})^{ac}(\kappa\lambda)_{\alpha c}\,.
\end{aligned} \tag{5.2.41}$$

We then find

$$\frac{\partial K}{\partial \Re\eta^\Lambda}(\bar{g}^{-1})^{\Lambda\Sigma}\frac{\partial K}{\partial \Re\eta^\Sigma} = 4\,. \tag{5.2.42}$$

The Kähler potentials thus give rise to positive potentials*. Later on, we will show that some of the Kähler spaces described in this section do factorize, namely when one performs the truncation on the homogeneous very special quaternionic-Kähler spaces. In this case, a $SU(1,1)/U(1)$ part splits off and the remaining part gives rise to flat potentials, potentials that are identically zero, like in [148]. Some older examples of flat and positive potentials in 4D supergravity can be found in [93, 160].

5.2.4 Surviving isometries of special quaternionic-Kähler manifolds

We analyze here which isometries of the special quaternionic-Kähler manifold survive the truncation. As is discussed extensively in [159], the isometries of a special quaternionic-Kähler manifold consist of the duality transformations inherited from the special Kähler

*The results in this section were also found in [157, 154], using a different coordinate system.

space under the **c**-map, which are characterized by parameters ω^i, and in addition, $2n+4$ extra isometries characterized by $\epsilon^0, \epsilon^+, \alpha^I, \beta_I$. There are also hidden symmetries, which can exist when certain conditions are satisfied by the quaternionic-Kähler manifold, characterized by $\hat{\alpha}^I, \hat{\beta}_I, \epsilon^-$. It was proven in [159, 161] that all hidden symmetries are realized if and only if the quaternionic-Kähler manifold is symmetric. The infinitesimal isometries are generated by

$$\delta = \epsilon^0 \underline{\epsilon}_0 + \epsilon^+ \underline{\epsilon}_+ + \epsilon^- \underline{\epsilon}_- + \alpha^I \underline{\alpha}_I + \beta_I \underline{\beta}^I + \hat{\alpha}^I \underline{\hat{\alpha}}_I + \hat{\beta}_I \underline{\hat{\beta}}^I + \omega^i \underline{\omega}_i \,, \tag{5.2.43}$$

and act on the coordinates of the quaternionic-Kähler manifold as (note that we have redefined $D = \frac{1}{2}\log\phi$) [159]

$$\begin{aligned}
\delta\phi &= \phi\left(-\epsilon^0 + 2\sigma\epsilon^- + \hat{\alpha}^I B_I - \hat{\beta}_I A^I\right) , \\
\delta\sigma &= \epsilon^+ + \frac{1}{2}\left(\alpha^I B_I - \beta_I A^I\right) + (\sigma^2 - \phi^2)\epsilon^- + \sigma\left(\frac{1}{2}\hat{\alpha}^I B_I - \frac{1}{2}\hat{\beta}_I A^I - \epsilon^0\right) + \mathcal{D}h \,, \\
\delta A^I &= \alpha^I + \sigma\hat{\alpha}^I + B^I{}_J(\omega)\, A^J - D^{IJ}(\omega)\, B_J \\
&\quad + \left(\epsilon^-\sigma + \frac{1}{2}\hat{\alpha}^J B_J - \frac{1}{2}\hat{\beta}_J A^J - \frac{1}{2}\epsilon^0\right) A^I - \partial^I \mathcal{D}\left(h + \frac{1}{2}\phi\mathcal{Z}_2\right) , \\
\delta B_I &= \beta_I + \sigma\hat{\beta}_I + C_{IJ}(\omega)\, A^J - B^J{}_I(\omega)\, B_J \\
&\quad + \left(\epsilon^-\sigma + \frac{1}{2}\hat{\alpha}^J B_J - \frac{1}{2}\hat{\beta}_J A^J - \frac{1}{2}\epsilon^0\right) B_I + \partial_I \mathcal{D}\left(h + \frac{1}{2}\phi\mathcal{Z}_2\right) , \\
\delta X^I &= B^I{}_J(\omega)\, X^J + \tfrac{1}{2} i D^{IJ}(\omega)\, F_J \\
&\quad + \mathcal{D}\left(-\frac{1}{2} i\bar{\mathcal{B}}^I\left(X^J\mathcal{B}_J\right) + \frac{1}{16} i (N^{-1})^{IJ}\mathcal{B}^K \bar{F}_{JKL}\mathcal{B}^L\left(XN\bar{X}\right)\right) ,
\end{aligned} \tag{5.2.44}$$

where

$$\mathcal{D}h = \left(\epsilon^- + \hat{\alpha}^I\frac{\partial}{\partial A^I} + \hat{\beta}_I\frac{\partial}{\partial B_I}\right) h \,, \tag{5.2.45}$$

the function $\mathcal{Z}$ is given by the expression:

$$\mathcal{Z}_2 \equiv \mathcal{B}_I\bar{\mathcal{B}}^I - 2\frac{(X^I\mathcal{B}_I)(\bar{X}^J\bar{\mathcal{B}}_J)}{XN\bar{X}} \,, \tag{5.2.46}$$

and the function h is the real function defined by

$$\begin{aligned}
h(X,\bar{X},A,B) &= -\frac{1}{16}\left\{(\mathcal{B}_I\bar{\mathcal{B}}^I)^2 - \frac{1}{6}\left[(F_{IJK}\bar{\mathcal{B}}^I\bar{\mathcal{B}}^J\bar{\mathcal{B}}^K)(X^L\mathcal{B}_L) + \text{h.c.}\right]\right. \\
&\quad \left. + \frac{1}{16}(XN\bar{X})\bar{\mathcal{B}}^I\bar{\mathcal{B}}^J F_{IJK} N^{-1\ KL}\bar{F}_{LMN}\mathcal{B}^M\mathcal{B}^N\right\} ,
\end{aligned} \tag{5.2.47}$$

with

$$\begin{aligned}
\mathcal{B}_I &= B_I + \tfrac{1}{2} i F_{IJ} A^J \equiv N_{IJ}\mathcal{B}^J \,, \\
N_{IJ} &= \frac{1}{4}(F_{IJ} + \bar{F}_{IJ}) \,,
\end{aligned} \tag{5.2.48}$$

and $F(X)$ a holomorphic function which is homogeneous of second degree in the variables X^I. The subscripts I, J, ... on F denote differentiation with respect to X^I, X^J, etc. In these expressions, the fields X^I are not independent complex fields. The independent fields are introduced through the ratios $z^A = X^A/X^0 = \frac{1}{2}(x^A + iy^A)$ (with $z^0 := 1$)*.
In the 'very special' case, $F(X) = id_{ABC}X^AX^BX^C/X^0$, and the matrix elements N_{IJ} are given by

$$N_{00} = \frac{1}{2}i(\kappa zzz) + \text{h.c.}\,, \quad N_{0A} = -\frac{3}{4}i(\kappa zz)_A + \text{h.c.}\,, \quad N_{AB} = -\frac{3}{2}(\kappa y)_{AB}\,. \quad (5.2.49)$$

For later use, we give also the truncated elements:

$$\begin{aligned} N_{\alpha\beta} &= -\frac{3}{2}(\kappa y)_{\alpha\beta}\,,\ N_{ab} = -\frac{3}{2}(\kappa y)_{ab}\,, \\ N_{a\alpha} &= N_{\alpha 0} = 0\,, \\ N_{a0} &= \frac{3}{4}(\kappa xy)_a\,, \\ N_{00} &= \frac{1}{8}(\kappa yyy) - \frac{3}{8}(\kappa xxy)\,. \end{aligned} \quad (5.2.50)$$

The matrices B, C and D characterize the infinitesimal duality symmetries,

$$\tilde{X}^I = X^I + B^I{}_J X^J + \frac{1}{2}D^{IJ}F_J\,, \quad (5.2.51)$$

$$\tilde{F}_I = F_I - B^J{}_I F_J + 2iC_{IJ}X^J\,. \quad (5.2.52)$$

Demanding $\tilde{F}(\tilde{X}) = F(\tilde{X})$ leads to

$$iC_{IJ}X^IX^J - B^I_J F_I X^J - \frac{1}{4}iD^{IJ}F_IF_J = 0\,. \quad (5.2.53)$$

In the 'very special' case, equation (5.2.53) can be solved by

$$B^I{}_J = \begin{pmatrix} \beta & a_B \\ b^A & \tilde{B}^A_B + \frac{1}{3}\beta\,\delta^A_B \end{pmatrix}, \quad C_{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & 3d_{ABC}\,b^C \end{pmatrix}, \quad D^{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{4}{9}C^{ABC}a_C \end{pmatrix}. \quad (5.2.54)$$

The parameters b^A and β correspond to isometries inherited from the special Kähler manifold under the **c**-map and are always present, while the parameters a_B correspond to hidden symmetries. The elements $\tilde{B}^A_B$ correspond to the symmetries of d_{ABC}, i.e. $\tilde{B}^D_{(A}d_{BC)D} = 0$ (see [159] for more details).
We also list the nonzero commutators between the generators of the isometries. First there are those not involving the duality transformations:

*For convenience we use here the notation of [159]. Hence, F in chapter 4 corresponds to $-iF/4$ in this chapter. The reader should also not confuse N_{IJ} in this chapter with $\mathcal{N}_{IJ}$ in the previous chapter. They are different objects.

$$
\begin{array}{ll}
[\underline{\epsilon}_0, \underline{\epsilon}_\pm] = \pm \underline{\epsilon}_\pm \,, & [\underline{\epsilon}_-, \underline{\epsilon}_+] = 2\underline{\epsilon}_0 \,, \\
[\underline{\epsilon}_0, \underline{\alpha}_I] = \tfrac{1}{2}\underline{\alpha}_I \,, & [\underline{\epsilon}_0, \underline{\hat{\alpha}}_I] = -\tfrac{1}{2}\underline{\hat{\alpha}}_I \,, \\
[\underline{\epsilon}_0, \underline{\beta}^I] = \tfrac{1}{2}\underline{\beta}^I \,, & [\underline{\epsilon}_0, \underline{\hat{\beta}}^I] = -\tfrac{1}{2}\underline{\hat{\beta}}^I \,, \\
[\underline{\epsilon}_-, \underline{\alpha}_I] = -\underline{\hat{\alpha}}_I \,, & [\underline{\epsilon}_+, \underline{\hat{\alpha}}_I] = \underline{\alpha}_I \,, \\
[\underline{\epsilon}_-, \underline{\beta}^I] = -\underline{\hat{\beta}}^I \,, & [\underline{\epsilon}_+, \underline{\hat{\beta}}^I] = \underline{\beta}^I \,, \\
[\underline{\alpha}_I, \underline{\beta}^J] = -\delta_I^J \underline{\epsilon}_+ \,, & [\underline{\hat{\alpha}}_I, \underline{\hat{\beta}}^J] = -\delta_I^J \underline{\epsilon}_- \,.
\end{array}
\tag{5.2.55}
$$

The commutators involving the duality transformations are:

$$
\begin{aligned}
[\alpha^I \, \underline{\alpha}_I + \beta_I \, \underline{\beta}^I, \omega^i \, \underline{\omega}_i] &= \alpha'^I \, \underline{\alpha}_I + \beta'_I \, \underline{\beta}^I \,, \\
[\hat{\alpha}^I \, \underline{\hat{\alpha}}_I + \hat{\beta}_I \, \underline{\hat{\beta}}^I, \omega^i \, \underline{\omega}_i] &= \hat{\alpha}'^I \, \underline{\hat{\alpha}}_I + \hat{\beta}'_I \, \underline{\hat{\beta}}^I \,,
\end{aligned}
\tag{5.2.56}
$$

where

$$
\begin{pmatrix} \alpha'^I \\ \beta'_J \end{pmatrix} = \begin{pmatrix} B^I{}_K(\omega) & -D^{IL}(\omega) \\ C_{JK}(\omega) & -B^L{}_J(\omega) \end{pmatrix} \begin{pmatrix} \alpha^K \\ \beta_L \end{pmatrix} \,,
\tag{5.2.57}
$$

and likewise for $\hat{\alpha}'$ and $\hat{\beta}'$. Finally

$$
[\alpha^I \, \underline{\alpha}_I + \beta_I \, \underline{\beta}^I, \hat{\alpha}^I \, \underline{\hat{\alpha}}_I + \hat{\beta}_I \, \underline{\hat{\beta}}^I] = (\alpha^I \hat{\beta}_I - \hat{\alpha}^I \beta_I) \, \underline{\epsilon}_0 + \omega^i(\alpha, \beta, \hat{\alpha}, \hat{\beta}) \, \underline{\omega}_i \,.
\tag{5.2.58}
$$

It can be shown that duality transformations corresponding to the parameters $\omega(\alpha, \beta, \hat{\alpha}, \hat{\beta})$ correspond to the matrices [159]

$$
\begin{aligned}
B^I{}_J(\alpha, \beta, \hat{\alpha}, \hat{\beta}) &= -\frac{1}{2}(\hat{\alpha}^I \beta_J + \alpha^I \hat{\beta}_J) - \partial^I \partial_J h''(\alpha, \beta, \hat{\alpha}, \hat{\beta}) \,, \\
C_{IJ}(\alpha, \beta, \hat{\alpha}, \hat{\beta}) &= -\hat{\beta}_{(I} \beta_{J)} + \partial_I \partial_J h''(\alpha, \beta, \hat{\alpha}, \hat{\beta}) \,, \\
D^{IJ}(\alpha, \beta, \hat{\alpha}, \hat{\beta}) &= -\hat{\alpha}^{(I} \alpha^{J)} + \partial^I \partial^J h''(\alpha, \beta, \hat{\alpha}, \hat{\beta}) \,,
\end{aligned}
\tag{5.2.59}
$$

where

$$
\partial^I := \frac{\partial}{\partial B_I} \;,\; \partial_I := \frac{\partial}{\partial A^I} \,,
$$

and

$$
h''(\alpha, \beta, \hat{\alpha}, \hat{\beta}) \equiv (\alpha \cdot \partial + \beta \cdot \partial)(\hat{\alpha} \cdot \partial + \hat{\beta} \cdot \partial) \, h(X, \bar{X}, A, B) \,.
\tag{5.2.60}
$$

From analyzing the infinitesimal transformations, we can now find the remaining isometries after the truncation (5.2.22). We demand that fields that were put to zero remain zero after performing an infinitesimal transformation. One easily finds that the isometries corresponding to ϵ^0, α^a and β_α survive the truncation. The truncation also restricts the possible duality transformations. For our convenience, we start by rewriting the variation of the fields X^A as function of the x^A and y^A. For the 'very special' case we obtain

$$
\begin{aligned}
\frac{1}{2}\delta x^A &= B^A{}_0 + \frac{1}{2}(B^A{}_B - B^0{}_0 \delta^A{}_B) x^B - \frac{1}{4}(\frac{3}{2} D^{AB} d_{BCD} + B^0{}_C \delta^A{}_D (x^C x^D - y^C y^D)) \,, \\
\frac{1}{2}\delta y^A &= \frac{1}{2}(B^A{}_B - B^0{}_0 \delta^A{}_B) y^B - \frac{1}{4}(\frac{3}{2} D^{AB} d_{BCD} + B^0{}_C \delta^A{}_D (x^C y^D + y^C x^D)) \,,
\end{aligned}
\tag{5.2.61}
$$

disregarding for now the possible existence of any hidden symmetries. From this, and the variations of A^I and B_I we infer the following conditions on the $Sp(2n+2,\mathbb{R})$ matrix of the duality transformations:

$$\begin{aligned} B^{\alpha}{}_{0} &= b^{\alpha} = 0\,, \\ B^{\alpha}{}_{a} &= B^{a}{}_{\alpha} = 0\,, \\ D^{\alpha\beta} &= D^{ab} = 0\,. \end{aligned} \tag{5.2.62}$$

From the last equations, using (5.2.23), (5.2.27), and (5.2.54) we can further deduce that

$$\begin{aligned} a_{\alpha} &= 0\,, \\ C_{\alpha\beta} &= C_{ab} = 0\,. \end{aligned} \tag{5.2.63}$$

Finally, there are the possible hidden symmetries $\{\hat{\alpha}, \hat{\beta}, \epsilon_-\}$. For these, the analysis is more complicated. We start by demanding that $\mathcal{D}h = 0$ after truncation, a condition that is necessary for the variation $\delta\sigma$ to vanish. From this condition, a short calculation based on results in [159] for the expression $\mathcal{D}h$ reveals that the hidden isometries $\hat{\alpha}^a, \hat{\beta}^{\alpha}$ and ϵ^- are certainly broken. For the remaining isometries $\hat{\alpha}^{\beta}, \hat{\beta}^a$, we must check that they satisfy the following conditions (after truncation)

$$\begin{aligned} \partial^{\alpha}\mathcal{D}h &= \partial_a\mathcal{D}h = 0\,, \\ \partial^{\alpha}\mathcal{Z}_2 &= \partial_a\mathcal{Z}_2 = 0\,. \end{aligned} \tag{5.2.64}$$

After some calculation, one finds that this is indeed the case. Finally, one can check that also the last term in the variation of X^I vanishes when calculating the variations δx^{α} and δy^a for the remaining isometries. Crucial here is the fact that the matrix elements $N_{\alpha a}$ and $N_{\alpha 0}$ vanish identically upon truncation (see (5.2.50)). We conclude that the remaining isometries are characterized by parameters $\{\alpha^0, \alpha^a, \hat{\alpha}^{\beta}, \beta_{\alpha}, \hat{\beta}_0, \hat{\beta}_a, \epsilon_0\}$ and by duality transformations satisfying (5.2.62) and (5.2.63). As an extra check one can verify, using (5.2.59), that the duality transformations corresponding to the hidden symmetries also satisfy (5.2.62) and (5.2.63). Note that of course only hidden symmetries that were there in the first place, will remain after the truncation.

The same analysis can be done for the minimal coupling case, $F(X) = X^I X^J \eta_{IJ}$, which gives rise to a class of symmetric special quaternionic-Kähler spaces*. In this case, the duality transformations are not of the form shown in equation (5.2.54), but obey the following equations: $B_{IJ} = -B_{JI}$, where $B_{IJ} \equiv \eta_{IK} B^K{}_J$, and $C_{IJ} = \frac{1}{4}\eta_{IK} D^{KL} \eta_{LJ}$. This means the group of duality transformations is $U(n,1)$, with the generators $B^I{}_J$ forming a $SO(n,1)$ subgroup. Using the transformations (5.2.44), which are valid for all special quaternionic-Kähler spaces, we find the following result after the truncation:

- broken isometries:
  $\{B^{\alpha}{}_0, B^0{}_{\alpha}, B^a{}_{\alpha}, B^{\alpha}{}_a, D^{\alpha\beta}, D^{ab}, C_{\alpha\beta}, C_{ab}, \epsilon_-, \epsilon_+, \hat{\alpha}^0, \hat{\alpha}^a, \alpha^{\beta}, \hat{\beta}_{\alpha}, \beta_0, \beta_a\}$,
- remaining isometries:
  $\{\alpha^0, \alpha^a, \hat{\alpha}^{\beta}, \beta_{\alpha}, \hat{\beta}_0, \hat{\beta}_a\, \epsilon_0, B^{\alpha}{}_{\beta}, B^a{}_b, D^{\alpha a}, D^{a\alpha}, D^{\alpha 0}, D^{0\alpha}, C_{\alpha a}, C_{a\alpha}, C_{\alpha 0}, C_{0\alpha}\}$.

*η_{IJ} is a constant diagonal matrix with eigenvalues $(1, -1, -1, ..., -1)$.

In the next section, we will take a closer look at the isometry algebra that arises when truncating homogeneous special quaternionic-Kähler manifolds. For these manifolds, it is known which hidden symmetries are realized.

5.2.5 Truncation of homogeneous very special quaternionic-Kähler manifolds

The complete classification of homogeneous very special quaternionic-Kähler manifolds (HVSQ manifolds from now on) was performed by de Wit and Van Proeyen in [104], and the symmetry structure of the isometries was further studied in [159]. The solutions are denoted by $L(q,P)$, where q and P are integers with $q \geq -1$ and $P \geq 0$. For q equal to a multiple of 4, there exist additional solutions denoted by $L(4m,P,\dot{P})$, with $m \geq 0$ and $P,\dot{P} \geq 1$. Here $L(4m,P,\dot{P}) = L(4m,\dot{P},P)$. The different solutions are listed in Table 5.3, where the new spaces discovered in [104] are indicated by a $\star$.

	Real ($\dim_{\mathbb{R}} = n-1$)	Kähler ($\dim_{\mathbb{R}} = 2n$)	Quaternionic ($\dim_{\mathbb{R}} = 4n+4$)
$L(-1,0)$	$SO(1,1)$	$\left[\frac{SU(1,1)}{U(1)}\right]^2$	$\frac{SO(3,4)}{(SU(2))^3}$
$L(-1,P)$	$\frac{SO(P+1,1)}{SO(P+1)}$	$\star$	$\star$
$L(0,0)$	$[SO(1,1)]^2$	$\left[\frac{SU(1,1)}{U(1)}\right]^3$	$\frac{SO(4,4)}{SO(4)\otimes SO(4)}$
$L(0,P)$	$\frac{SO(P+1,1)}{SO(P+1)} \otimes SO(1,1)$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(P+2,2)}{SO(P+2)\otimes SO(2)}$	$\frac{SO(P+4,4)}{SO(P+4)\otimes SO(4)}$
$L(0,P,\dot{P})$	$Y(P,\dot{P})$	$K(P,\dot{P})$	$W(P,\dot{P})$
$L(q,P)$	$X(P,q)$	$H(P,q)$	$V(P,q)$
$L(4m,P,\dot{P})$	$\star$	$\star$	$\star$
$L(1,1)$	$\frac{S\ell(3,\mathbb{R})}{SO(3)}$	$\frac{Sp(6)}{U(3)}$	$\frac{F_4}{USp(6)\otimes SU(2)}$
$L(2,1)$	$\frac{S\ell(3,\mathbb{C})}{SU(3)}$	$\frac{SU(3,3)}{SU(3)\otimes SU(3)\otimes U(1)}$	$\frac{E_6}{SU(6)\otimes SU(2)}$
$L(4,1)$	$\frac{SU^*(6)}{Sp(3)}$	$\frac{SO^*(12)}{SU(6)\otimes U(1)}$	$\frac{E_7}{SO(12)\otimes SU(2)}$
$L(8,1)$	$\frac{E_6}{F_4}$	$\frac{E_7}{E_6\otimes U(1)}$	$\frac{E_8}{E_7\otimes SU(2)}$

Table 5.3: Homogeneous special real spaces and their corresponding Kähler and quaternionic-Kähler spaces. The integers P, $\dot{P}$, q and m can take all values ≥ 1.

The symmetric spaces are the three varieties corresponding to $L(-1,0)$, $L(0,P)$, $L(1,1)$, $L(2,1)$, $L(4,1)$, and $L(8,1)$, and the real spaces corresponding to $L(-1,P)$. For those cases the isometry group G is given. For the nonsymmetric spaces the isometry group G is not semisimple, the isotropy group H is always its maximal compact subgroup.

For HVSQ manifolds the components of d_{ABC} can be brought into the canonical form with nonzero components

$$d_{122} = 1\,, \qquad d_{1\mu\nu} = -\delta_{\mu\nu}\,, \qquad d_{2ij} = -\delta_{ij}\,, \qquad d_{\mu ij} = \gamma_{\mu ij}\,, \tag{5.2.65}$$

where the coordinates y^A are decomposed into y^1, y^2, y^μ and y^i, and the indices μ and i run over $q+1$ and $r = (P + \dot{P})\,\mathcal{D}_{q+1}$ values, respectively. Thus we have $n = 3 + q + r$. The coefficients $\gamma_{\mu ij}$ are the generators of a $(q+1)$-dimensional real Clifford algebra with positive signature, denoted by $\mathcal{C}(q+1,0)$,

$$\gamma_{\mu ik}\,\gamma_{\nu kj} + \gamma_{\nu ik}\,\gamma_{\mu kj} = 2\delta_{ij}\delta_{\mu\nu}\,, \tag{5.2.66}$$

and $\mathcal{D}_{q+1}$ denotes the dimension of an irreducible representation of this Clifford algebra. In the case $q \neq 0 \bmod 4$, the irreducible representations for a given q are unique and thus the gamma matrices are unique once we specify the number of irreducible representations, denoted by P. For $q = 0 \bmod 4$ the representations γ_μ and $-\gamma_\mu$ are not equivalent, and a reducible representation is characterized by the multiplicity of each of these representations, P and $\dot{P}$. Hence $r = (P + \dot{P})\,\mathcal{D}_{q+1}$ in general. The symmetric spaces are characterized by $\Gamma_{ijkl} = 0$, with Γ_{ijkl} defined by

$$\Gamma_{ijkl} \equiv \frac{3}{8}\left[\gamma_{\mu(ij}\,\gamma_{kl)\mu} - \delta_{(ij}\,\delta_{kl)}\right]\,. \tag{5.2.67}$$

For calculational purposes it's sometimes convenient to combine the indices 2 and μ in one index M. We can then rewrite the relations in section 5.2.5. The nonzero components of the tensor d_{ABC} are

$$d_{1MN} = -\eta_{MN}\,, \qquad d_{Mij} = \gamma_{Mij}\,. \tag{5.2.68}$$

The γ_{Mij} obey

$$\gamma_{Mik}\,\gamma_N{}^{kj} + \gamma_{Nik}\,\gamma_M{}^{kj} = 2\delta_i^j\eta_{MN}\,, \tag{5.2.69}$$

with $\eta = (-1,1,1,...,1)$, $\gamma_{\mu\,ij} = \gamma_\mu{}^{ij}$ and $\gamma_2{}^{ij} = -\gamma_{2\,ij} = \delta_{ij}$. The symmetric spaces are then characterized by $\Gamma_{ijkl} = \frac{3}{8}\left[\gamma_{M(ij}\,\gamma_{kl)N}\eta^{MN}\right] = 0$.

For the orientifold truncation, we take the index α of the previous sections to run over 1 and M (this is necessary, considering the restriction (5.2.23) on the d-symbols), while the index a is identified with the index i (this can be generalized in some cases, as we will show in section 5.2.6)*. The symmetries of the d-symbols are encoded in the matrix $\tilde{B}^A{}_B$, which appears in the duality transformations and has the following structure (see [159])

$$\tilde{B}^A{}_B = \begin{pmatrix} -2\lambda & 0 & 2\xi_j \\ 0 & \lambda\,\delta_N^M + A^M{}_N & -\zeta^k\,\gamma^M{}_{kj} \\ -\zeta^i & \gamma_N{}^{ik}\xi_k & -\frac{1}{2}\lambda\delta_j^i + \frac{1}{4}A^{PQ}\,(\gamma_{PQ})^i{}_j + S^i{}_j \end{pmatrix}\,. \tag{5.2.70}$$

*The case $(\alpha) = (1, M, i)$ was studied in [148, 149].

The matrices $A^M{}_N$ generate infinitesimal $SO(q+1,1)$-symmetries and $S^i{}_j$ generate the group $\mathcal{S}_q(P,\dot{P})$ which is the centralizer of the Clifford algebra representation [159]. Now using (5.2.62), we can see that all isometries will survive the orientifold truncation, except the ones with parameters ξ_j and ζ^j, which are broken. The full set of isometries of the HVSQ manifold consists of the above isometries coming from the symmetries of the d-symbols, plus the extra symmetries one gets from performing the **r**-map and **c**-map. For nonsymmetric HVSQ manifolds, the following hidden symmetries are realized: $\{\underline{a}^M, \underline{\hat{\alpha}}_1, \underline{\hat{\beta}}^M, \underline{\hat{\beta}}^0\}$. The analysis of the various commutation relations and the isometry algebras that arise, is performed in [159], resulting in the classification given in table 5.3. We will not repeat all the results here, and we refer to [159] for the details. Our aim is to study what happens to these manifolds upon performing the orientifold truncation. The important results are that the resulting Kähler manifolds are also homogenous manifolds that have one half of the dimension of the HVSQ manifold. Furthermore, they are always product manifolds of the form $\mathcal{M} \times SU(1,1)/U(1)$, with $\mathcal{M}$ having the same dimension as the original special Kähler space that is the target of the **c**-map.

Homogeneity

We have to show that the remaining isometries act transitively on the truncated manifold. It is obvious that this is so for the remaining shifts and scale transformation. To study the effect of the remaining duality transformations, we write down the truncated version of (5.2.61)

$$\begin{aligned} \delta x^a &= b^a + B^a{}_b x^b - \beta x^a - \frac{3}{4} D^{a\alpha}(d_{\alpha cd} x^c x^d + d_{\alpha\beta\gamma} y^\beta y^\gamma) - \frac{1}{2} a_c x^c x^a \,, \\ \delta y^\alpha &= B^\alpha{}_\beta y^\beta - \beta y^\alpha - \frac{3}{2} D^{\alpha b} d_{bc\gamma} x^c y^\gamma - \frac{1}{2} a_c x^c y^\alpha \,. \end{aligned} \tag{5.2.71}$$

Looking at the form of the matrix $\tilde{B}^A_B$ in (5.2.70), it is now clear that one can connect any two points in the truncated manifold by performing shifts b^a (for the coordinates x^a), rotations $A^M{}_N$ and scale transformations β (for the coordinates y^α). Thus, the truncated manifold is homogeneous. To find the elements of the isotropy group, we can choose a point with $A^0 = A^a = B_\alpha = x^a = y^\mu = 0$ (i.e. only y^1, y^2 different from 0). This point is left invariant by generators of $\mathcal{S}_q(P,\dot{P})$ and $SO(q+1)$, and by the generators of the hidden symmetries.

Splitting off an $SU(1,1)/U(1)$-part

We will show that there exists an $SU(1,1)$- subalgebra of the isometry algebra, of which the three generators commute with all the other generators. First, we give a list of the isometries of the HVSQ spaces that survive the truncation:
$\{\underline{\lambda}, \underline{A}^M{}_N, \underline{S}^i{}_j, \underline{\beta}, \underline{\epsilon}_0, \underline{\beta}^M, \underline{\alpha}_i, \underline{b}_i, \underline{\hat{\alpha}}_1, \underline{\hat{\beta}}^0, \underline{\alpha}_0, \underline{\beta}^1\}$.
Next, we write down the commutators between the generators $\underline{\alpha}$,$\underline{\beta}$, $\underline{a}$ and $\underline{b}$ which were

not given explicitly in [159]. Using (5.2.58), one finds the following relations

$$
\begin{array}{ll}
[\underline{\alpha}_0, \underline{\hat{\alpha}}_\beta] = 0\,, & [\underline{\hat{\beta}}^0, \underline{\alpha}_a] = \underline{b}_a\,, \\
[\underline{\alpha}_0, \underline{\hat{\beta}}^b] = -\underline{a}^b\,, & [\underline{\hat{\beta}}^0, \underline{\beta}^\alpha] = 0\,, \\
[\underline{\alpha}_0, \underline{\hat{\beta}}^0] = \underline{\epsilon}_0 - \frac{3}{2}\underline{\beta}\,, & \\
[\underline{\alpha}_a, \underline{\hat{\alpha}}_\beta] = 3 d_{\beta a b} \underline{a}^b\,, & \\
[\underline{\beta}^\alpha, \underline{\hat{\beta}}^a] = -\frac{4}{9} C^{\alpha a b} \underline{b}_b\,, & \\
[\underline{\beta}^1, \underline{\hat{\alpha}}_1] = -\underline{\epsilon}_0 + \frac{2}{3}\underline{\lambda} - \frac{1}{2}\underline{\beta}\,, & \\
[\underline{\beta}^2, \underline{\hat{\alpha}}_2] = -\underline{\epsilon}_0 - \frac{1}{3}\underline{\lambda} - \frac{1}{2}\underline{\beta}\,, & \\
[\underline{\beta}^\mu, \underline{\hat{\alpha}}_\nu] = -\underline{A}_{\mu\nu} + \delta_{\mu\nu}(-\underline{\epsilon}_0 - \frac{1}{3}\underline{\lambda} - \frac{1}{2}\underline{\beta})\,, & \\
[\underline{a}^b, \underline{\hat{\beta}}^0] = \hat{\beta}^b\,, & [\underline{b}_a, \underline{\alpha}_0] = -\underline{\alpha}_a\,, \\
[\underline{a}^b, \underline{\hat{\alpha}}_\beta] = 0\,, & [\underline{b}_a, \underline{\alpha}_b] = -3 d_{\alpha a b} \underline{\beta}^\alpha\,, \\
[\underline{a}^b, \underline{\hat{\beta}}^a] = -\frac{4}{9} C^{\beta a b} \underline{\alpha}_\beta\,, & [\underline{b}_a, \underline{\beta}^\alpha] = 0\,.
\end{array}
\tag{5.2.72}
$$

When one now considers the explicit form of the d-symbols and the tensor C^{ABC}, one finds that the set of generators $\{\underline{\hat{\alpha}}_1, \underline{\beta}^1, \underline{\epsilon} \equiv -\underline{\epsilon}_0 + \frac{2}{3}\underline{\lambda} - \frac{1}{2}\underline{\beta}\}$ generates an $SU(1,1)$-subalgebra, and furthermore commutes with all other generators. The Kähler space resulting from the truncation will thus contain a $SU(1,1)/U(1)$-part which splits off from the manifold, with the $U(1)$ subgroup generated by the compact generator $\hat{\alpha}_1$. This result will be confirmed when we analyze the metric and curvature tensor later on.

5.2.6 Truncation of symmetric special quaternionic-Kähler manifolds

SVSQ manifolds: the generic truncation

In the previous section, we showed that a truncated HVSQ manifold takes the form $\mathcal{M} \times SU(1,1)/U(1)$. It is therefore interesting to treat the manifold $\mathcal{M}$ as a 'dual' Kähler manifold in the sense of [148], and to look closer at its properties. We shall start by looking at the subclass of symmetric very special quaternionic-Kähler manifolds (SVSQ manifolds). These manifolds take the form of coset spaces G/H, where the isometry group G is a semisimple Lie-group, and the isotropy group H is its maximal compact subgroup. The SVSQ manifolds are the ones denoted by $L(-1,0)$, $L(0,P)$, $L(1,1)$, $L(2,1)$, $L(4,1)$, and $L(8,1)$ in table (5.3). For these manifolds, all hidden symmetries are realized. The full set of isometries and the commutators are given in table 5.4, which we have taken from [159], completed with the additional hidden symmetries.

generator	$\mathcal{S}_q$	$so(q+2,2)$	$\underline{\lambda}'$	$\underline{\epsilon}_0$	$\underline{\lambda}'-2\underline{\epsilon}_0$	$\underline{\lambda}'+2\underline{\epsilon}_0$
$\underline{\epsilon}_+$	0	0	0	1	-2	2
$(\underline{\alpha}_1, \underline{\beta}^M, \underline{\beta}^0)$	0	v	1	$\frac{1}{2}$	0	2
$\underline{b}_1$	0	0	2	0	2	2
$(\underline{\alpha}_i, \underline{\beta}^i)$	v	s	0	$\frac{1}{2}$	-1	1
$(\underline{\xi}^i, \underline{b}_i)$	v	$\bar{s}$	1	0	1	1
$(\hat{\underline{\alpha}}_1, \hat{\underline{\beta}}^M, \hat{\underline{\beta}}^0)$	0	v	1	$-\frac{1}{2}$	2	0
$(\underline{\alpha}_0, \underline{\alpha}_M, \underline{\beta}^1)$	0	v	-1	$\frac{1}{2}$	-2	0
$\underline{\epsilon}_-$	0	0	0	-1	2	-2
$(\hat{\underline{\alpha}}_0, \hat{\underline{\alpha}}_M, \hat{\underline{\beta}}^1)$	0	v	-1	$-\frac{1}{2}$	0	-2
$\underline{a}^1$	0	0	-2	0	-2	-2
$(\hat{\underline{\alpha}}_i, \hat{\underline{\beta}}^i)$	v	$\bar{s}$	0	$-\frac{1}{2}$	1	-1
$(\underline{\zeta}_i, \underline{a}^i)$	v	s	-1	0	-1	-1

Table 5.4: Roots of the isometries of the symmetric very special quaternionic-Kähler spaces: $so(q+2,2)$ is generated by $\{\underline{A}^M{}_N, \underline{a}^M, \underline{b}_M, \underline{\beta} - \frac{1}{3}\underline{\lambda}\}$ and $\lambda' \equiv \frac{2}{3}\lambda + \beta$.

The isometry algebra can be decomposed as follows (see [159])

$$\begin{aligned}
\mathcal{U} &= \mathcal{U}_0 + \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_{-1} + \mathcal{U}_{-2}\,, \\
\mathcal{U}_0 &= \underline{\epsilon}' \oplus so(q+3,3) \oplus \mathcal{S}_q(P,\dot{P})\,, \\
\mathcal{U}_1 &= (\underline{\xi}^i, \underline{b}_i) \oplus (\underline{\alpha}_i, \underline{\beta}^i) = (1, s, v)\,, \\
\mathcal{U}_2 &= \underline{\epsilon}_+ \oplus (\underline{\alpha}_1, \underline{\beta}^M, \underline{\beta}^0) \oplus \underline{b}_1 = (2, v, 0)\,, \\
\mathcal{U}_{-1} &= (\underline{a}^i, \underline{\zeta}_i) \oplus (\hat{\underline{\alpha}}_i, \hat{\underline{\beta}}^i) = (-1, s, v)\,, \\
\mathcal{U}_{-2} &= \underline{\epsilon}_- \oplus (\hat{\underline{\alpha}}_0, \hat{\underline{\alpha}}_M, \hat{\underline{\beta}}^1) \oplus \underline{a}^1 = (-2, v, 0)\,,
\end{aligned} \tag{5.2.73}$$

where the representations of $\mathcal{U}_1$,$\mathcal{U}_2$, $\mathcal{U}_{-1}$ and $\mathcal{U}_{-2}$ are indicated according to the three subalgebras of $\mathcal{U}_0$ and $\underline{\epsilon}' \equiv 2\underline{\epsilon}_0 + \underline{\beta} + \frac{2}{3}\underline{\lambda}$.
After the orientifold projection, the following set of isometries will survive:
$\{\underline{\lambda}, \underline{A}^M{}_N, \underline{S}^i{}_j, \underline{\beta}, \underline{\epsilon}_0, \underline{\beta}^M, \underline{\alpha}_i, \underline{b}_i, \hat{\underline{\alpha}}_1, \hat{\underline{\beta}}^0, \underline{\alpha}_0, \underline{\beta}^1, \underline{a}^i, \hat{\underline{\beta}}^i, \hat{\underline{\alpha}}_M\}$.
Counting the number of compact and noncompact generators gives us:

- $\dim G = 2n + 6 + \frac{1}{2}q(q+3) + 2r + \dim \mathcal{S}_q$ remaining isometries ,
- $\dim H = n + 1 + \frac{1}{2}q(q+1) + r + \dim \mathcal{S}_q$ remaining compact isometries ,

so that the dimension of the truncated manifolds is given by

$$\dim \mathcal{M}_{trunc} = \dim G - \dim H = n + r + q + 5 = 2n + 2. \tag{5.2.74}$$

This is consistent with the result $\mathcal{M}_{trunc} = \mathcal{K}_{dual} \times SU(1,1)/U(1)$. Also, looking at the root diagrams presented in [159] and [162] reveals the following fact: if a root corresponds to an isometry that is projected out, the opposite root also belongs to a broken isometry. In other words, opposing roots are always projected out together.
After performing the truncation on the isometry algebra (5.2.73) and throwing away the $SU(1,1)/U(1)$ part that splits off, we find the following structure for the isometry algebra of $\mathcal{K}_{dual}$,

$$\begin{aligned}
&\mathcal{W} = \mathcal{W}_0 + \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_{-1} + \mathcal{W}_{-2}\,,\\
&\mathcal{W}_0 = \underline{\epsilon}' \oplus (\underline{\epsilon}_0 - \frac{3}{2}\underline{\beta}, \underline{\alpha}_0, \underline{\hat{\beta}}_0) \oplus so(q+1,1) \oplus \mathcal{S}_q(P, \dot{P})\,,\\
&\mathcal{W}_1 = \underline{b}^i \oplus \underline{\alpha}_i = (1, s, s, v)\,,\\
&\mathcal{W}_2 = \underline{\beta}^M = (2, 0, v, 0)\,,\\
&\mathcal{W}_{-1} = \underline{a}_i \oplus \underline{\hat{\beta}}^i = (-1, s, s, v)\,,\\
&\mathcal{W}_{-2} = \underline{\hat{\alpha}}_M = (-2, 0, v, 0)\,.
\end{aligned} \tag{5.2.75}$$

From this, we see that the generators $\underline{\epsilon}'$, $\underline{\beta}^M$, $\underline{\hat{\alpha}}_M$, together with the generators $\underline{A}_{MN}$ of $so(q+1,1)$ form the algebra $so(q+2,2)$. We can then decompose the isometry algebra with respect to the grading under $\underline{\epsilon}_0 - \frac{3}{2}\underline{\beta}$

$$\begin{aligned}
&\mathcal{W} = \mathcal{W}'_0 + \mathcal{W}'_1 + \mathcal{W}'_2 + \mathcal{W}'_{-1} + \mathcal{W}'_{-2}\,,\\
&\mathcal{W}'_0 = \underline{\epsilon}_0 - \frac{3}{2}\underline{\beta} \oplus so(q+2,2) \oplus \mathcal{S}_q(P, \dot{P})\,,\\
&\mathcal{W}'_1 = \underline{a}^i \oplus \underline{\alpha}_i = (1, s, v)\,,\\
&\mathcal{W}'_2 = \underline{\alpha}_0 = (2, 0, 0)\,,\\
&\mathcal{W}'_{-1} = \underline{b}_i \oplus \underline{\hat{\beta}}^i = (-1, s, v)\,,\\
&\mathcal{W}'_{-2} = \underline{\hat{\beta}}_0 = (-2, 0, 0)\,.
\end{aligned} \tag{5.2.76}$$

This has exactly the same structure as the isometry algebra of the symmetric very special Kähler spaces (see [159]). The isotropy group is again the maximal compact subgroup of the isometry group. Another way to get this result, is by looking at all possible semisimple subgroups of the isometry groups of the symmetric very special quaternionic-Kähler manifolds and imposing the constraints that these subgroups have the right dimension and the right number of noncompact generators. One then always finds a unique possibility.
From these results, we can conclude that the dual Kähler spaces are the same cosets as the original special Kähler spaces. Note that this result does not necessarily imply that the dual space has the same curvature. We will investigate this in the next section.

A generalisation for the case $L(0,P)$

When $P>1$, we can split the index i in two parts, $(i)=(i_+,i_-)$ where i_+ runs over $P_+\mathcal{D}_{q+1}$ values, i_- runs over $P_-\mathcal{D}_{q+1}$ values, $P=P_++P_-$, and in such a way that $\gamma_{\mu i_+ i_-}=0$. We can take the α-index of the previous sections to run over $1,2,\mu$ and i_+, and identify the a-index with i_-. Therefore, from now on, we will write the i_+-index as $\tilde{M}$ and the i_--index as a. We can then start with the isometry algebra for the very special quaternionic-Kähler spaces given in (5.2.73), specialised to the case $q=\dot{P}=0$, and with $(i)=(\tilde{M},a)$. After the truncation, the $SU(1,1)$ generated by $\{\underline{\hat{\alpha}}_1,\underline{\beta}^1,\underline{\epsilon}\equiv-\underline{\epsilon}_0+\frac{2}{3}\underline{\lambda}-\frac{1}{2}\underline{\beta}\}$, does not split off anymore if $P_+\neq 0$. We find the following structure for the truncated isometry algebra,

$$\begin{aligned}
&\mathcal{W}=\mathcal{W}_0+\mathcal{W}_+ +\mathcal{W}_-\,,\\
&\mathcal{W}_0=SO(2,2)_+\oplus SO(P_+)\oplus SO(2,2)_-\oplus SO(P_-)\,,\\
&\mathcal{W}_+=(\underline{\xi}^{\tilde{M}},\underline{\hat{\alpha}}_{\tilde{M}},\underline{\beta}^{\tilde{M}},\underline{\zeta}_{\tilde{M}})=(v,v,0,0)\,,\\
&\mathcal{W}_-=(\underline{a}^b,\underline{\alpha}_b,\underline{b}_a,\underline{\hat{\beta}}^a)=(0,0,v,v)\,,
\end{aligned}\tag{5.2.77}$$

where

- $SO(2,2)_+$ is generated by $\{\underline{\hat{\alpha}}_1,\underline{\beta}^1,\underline{\epsilon},\underline{\beta}^2+\underline{\beta}^3,\underline{\alpha}^2+\underline{\alpha}^3,-\underline{\epsilon}-2\underline{A}_3^2\}$,
- $SO(2,2)_-$ by $\{\underline{\hat{\beta}}^0,\underline{\alpha}_0,\underline{\epsilon}_0-\frac{3}{2}\underline{\beta},\underline{\beta}^2-\underline{\beta}^3,\underline{\alpha}^2-\underline{\alpha}^3,-\underline{\epsilon}+2\underline{A}_3^2\}$,
- $SO(P_+)$ by $S^{\tilde{M}}_{\tilde{N}}$ and $SO(P_-)$ by S^a_b.

It is also easy to see that the generators in W_+ commute with those in W_-, since their commutators will be proportional to either $\delta^{a\tilde{M}}$, $\gamma^{\mu a\tilde{M}}$ or $S^{a\tilde{M}}$ and those are all zero. Thus, we find that the isometry group surviving the truncation is $SO(P_++2,2)\times SO(P_-+2,2)$. It is again straightforward to show that these isometries act transitively on the manifold, so that the resulting Kähler space is*

$$\frac{SO(P_++2,2)}{SO(P_++2)\times SO(2)}\times\frac{SO(P_-+2,2)}{SO(P_-+2)\times SO(2)}\,.\tag{5.2.78}$$

The other SSQ manifolds

Finally, we can try to extend our classification by analyzing the effect of the truncation on symmetric special quaternionic-Kähler manifolds (SSQ manifolds) that are not very special. There are three cases to consider: (see [162] for an overview)

- The space $U(1,2)/(U(1)\times U(2))$, which one gets by acting with the **c**-map on the empty special Kähler space.

*For $P_+=0$, which was discussed in section 5.2.6, or for $P_-=0$, which was discussed in [148, 149] this reduces to $[SU(1,1)/U(1)]^2\times[SO(P+2,2)/(SO(P+2)\times SO(2))]$.

- The space $G_2/(SU(2) \times SU(2))$, resulting from the **c**-map on the special Kähler manifold $SU(1,1)/U(1)$ that is characterized by the prepotential $F = i(X^1)^3/X^0$.

- The spaces denoted by $L(-2,P)$ in [162], resulting from the **c**-map on $U(P+1,1)/(U(P+1) \times U(1))$: Kähler spaces characterized by a quadratic prepotential $F(X) = X^I \eta_{IJ} X^J$. Because of the quadratic prepotential, the truncation conditions from [146] are trivially satisfied, so that no extra conditions need to be imposed. This means that different truncations are possible, depending on how we split the index I into indices $(0,a,\alpha)$.

The analysis of the duality symmetries is performed in [92] and [161], to which we refer for details. These are the results for the truncated manifolds*

- In the first case, the coset $SU(1,1)/U(1)$, the set of three surviving isometries being $\{\underline{\alpha}_0, \underline{\hat{\beta}}^0, \underline{\epsilon}_0\}$.

- In the second case, the coset $(SU(1,1)/U(1))^2$, with the two commuting $SU(1,1)$-subalgebras generated by $\{\underline{\alpha}_0, \underline{\hat{\beta}}^0, \underline{\epsilon}_0 - \frac{3}{2}\underline{\beta}\}$ and $\{\underline{\hat{\alpha}}_1, \underline{\beta}^1, -\underline{\epsilon}_0 - \frac{1}{2}\underline{\beta}\}$. (Although the space is usually not considered to be 'very special', the duality transformations are still of the form (5.2.54).)

- In the third case, the structure of the isometry algebra before the truncation is

$$\begin{aligned}
\mathcal{U} &= \mathcal{U}_0 + \mathcal{U}_{\frac{1}{2}} + \mathcal{U}_1 + \mathcal{U}_{-\frac{1}{2}} + \mathcal{U}_{-1}\,, \\
\mathcal{U}_0 &= \underline{\epsilon}_0 \oplus U(P+1,1)\,, \\
\mathcal{U}_{\frac{1}{2}} &= (\underline{\alpha}_I, \underline{\beta}^I) = (\frac{1}{2}, P+2) \oplus (\frac{1}{2}, \overline{P+2})\,, \\
\mathcal{U}_1 &= \epsilon^+\,, \\
\mathcal{U}_{-\frac{1}{2}} &= (\underline{\hat{\alpha}}_I, \underline{\hat{\beta}}^I) = (-\frac{1}{2}, P+2) \oplus (-\frac{1}{2}, \overline{P+2})\,, \\
\mathcal{U}_{-1} &= \epsilon^-\,,
\end{aligned} \tag{5.2.79}$$

  forming $SU(P+2,2)$. For all possible truncations, we find the same result: a total number of $\frac{1}{2}(P+4)(P+3)$ isometries is preserved, of which $2(P+2)$ are noncompact. The truncated isometry algebra has the following structure:

$$\begin{aligned}
\mathcal{U} &= \mathcal{U}_0 + \mathcal{U}_{\frac{1}{2}} + \mathcal{U}_{-\frac{1}{2}}\,, \\
\mathcal{U}_0 &= \underline{\epsilon}_0 \oplus SO(P+1,1)\,, \\
\mathcal{U}_{\frac{1}{2}} &= (\underline{\alpha}_0, \underline{\alpha}_a, \underline{\beta}^\alpha) = (\frac{1}{2}, v)\,, \\
\mathcal{U}_{-\frac{1}{2}} &= (\underline{\hat{\alpha}}_0, \underline{\hat{\alpha}}_\alpha, \underline{\hat{\beta}}^a) = (-\frac{1}{2}, v)\,,
\end{aligned} \tag{5.2.80}$$

*As before the remaining isometries act transitively, so that we again find homogeneous Kähler submanifolds of the corresponding quaternionic-Kähler spaces.

forming $SO(P+2,2)$. From this we conclude that the truncated spaces consist of the class $SO(P+2,2)/(SO(P+2)\times SO(2))$. These are symmetric Kähler spaces that are not special (see [80] for the classification). Note that no $SU(1,1)$-part splits off.

The results for the truncation of all SSQ manifolds are summarized in table 5.5.

	$\mathcal{M}_Q$ $(\dim_{\mathbb{R}} = 4n+4)$	$\mathcal{M}_{KH}$ $(\dim_{\mathbb{R}} = 2n+2)$
SG_4	$\frac{U(2,1)}{U(2)\times U(1)}$	$\frac{SU(1,1)}{U(1)}$
$L(-2,P)$	$\frac{SU(P+2,2)}{SU(P+2)\times SU(2)\times U(1)}$	$\frac{SO(P+2,2)}{SO(P+2)\times SO(2)}$
$L(-1,0)$	$\frac{SO(3,4)}{(SU(2)^3)}$	$[\frac{SU(1,1)}{U(1)}]^3$
$L(0,0)$	$\frac{SO(4,4)}{SO(4)\times SO(4)}$	$[\frac{SU(1,1)}{U(1)}]^4$
$L(0,P)$	$\frac{SO(P+4,4)}{SO(P+4)\times SO(4)}$	$\frac{SO(P_+ +2,2)}{SO(P_+ +2)\times SO(2)} \times \frac{SO(P_- +2,2)}{SO(P_- +2)\times SO(2)}$
SG_5	$\frac{G_{2(2)}}{SO(4)}$	$[\frac{SU(1,1)}{U(1)}]^2$
$L(1,1)$	$\frac{F_{4(4)}}{USp(6)\times USp(2)}$	$\frac{SU(1,1)}{U(1)} \times \frac{Sp(6)}{U(3)}$
$L(2,1)$	$\frac{E_{6(2)}}{SU(6)\times SU(2)}$	$\frac{SU(1,1)}{U(1)} \times \frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$
$L(4,1)$	$\frac{E_{7(-8)}}{SO(12)\times SU(2)}$	$\frac{SU(1,1)}{U(1)} \times \frac{SO^*(12)}{U(6)}$
$L(8,1)$	$\frac{E_{8(-24)}}{E_7\times SU(2)}$	$\frac{SU(1,1)}{U(1)} \times \frac{E_{7(-26)}}{E_6\times SO(2)}$

Table 5.5: Truncation of symmetric special quaternionic-Kähler manifolds.

5.2.7 Dual Kähler spaces

In the previous sections we studied the result of the orientifold truncation (5.2.22) on the symmetric special quaternionic-Kähler spaces. It was noticed in [148] that if one starts with the very special quaternionic-Kähler spaces, characterized by a d-symbol d_{ABC}, and takes the truncation $(A) = (\alpha)$, that is $n_- = 0$, the resulting Kähler spaces are always of the form $\mathcal{M}_{trunc} = SU(1,1)/U(1) \times \mathcal{K}_{dual}$. When we take orientifold truncations with $n_- \neq 0$, no $SU(1,1)/U(1)$ part splits off in general. However, for specific d-symbols, e.g those characterizing the HVSQ manifolds, we found a truncation with $n_- \neq 0$ where a $SU(1,1)/U(1)$ part splits off (see section 5.2.5)*. We can then, as in [148, 149], view the

*The fact that in this case again a $SU(1,1)/U(1)$ part splits off does not depend on the fact that γ_{Mij} obey (5.2.69) and so there are also many nonhomogeneous VSQ manifolds for which one can find truncations with $n_- \neq 0$ and a $SU(1,1)/U(1)$ part splitting off, namely all VSQ spaces with $d_{1MN} = -\eta_{MN}$, d_{Mij} arbitrary and all other d-symbol components equal to zero. These are precisely the VSQ manifolds that can be obtained by dimensional reduction from $(2,0)$ $D=6$ supergravity with $q+1$ tensor multiplets and r vector multiplets [163].

remaining manifold as a dual Kähler manifold. In the previous section, we found that in the symmetric case, the dual Kähler manifolds are the same cosets as the original very special Kähler spaces. In the next sections, we will verify that the dual Kähler spaces also have the same curvature tensor in the symmetric case. For the nonsymmetric case, the curvature tensors are different in general, as we will show by computing the scalar curvature for a specific case. Another important property to check is whether these dual Kähler spaces give rise to flat potentials, like in [148]. We will find in the last subsection that this is indeed the case.

The curvature tensor of the truncated VSQ manifolds

In this section we will give the components of the Christoffel connection and the curvature tensor which are necessary to calculate the scalar curvature of the truncated very special quaternionic-Kähler manifold (VSQ manifold). We also give all other components of the curvature tensor which are nonzero in the symmetric case. The connection and the Riemann tensor are defined as

$$\begin{aligned} \bar{\Gamma}^{\Lambda}{}_{\Sigma\Delta} &= (\bar{g}^{-1})^{\Lambda\bar{\Lambda}}\partial_{\Sigma}\bar{g}_{\bar{\Lambda}\Delta}\,, \\ \bar{R}_{\Sigma}{}^{\Gamma}{}_{\Delta}{}^{\Lambda} &= -(\bar{g}^{-1})^{\Gamma\bar{\Gamma}}\partial_{\bar{\Gamma}}\bar{\Gamma}^{\Lambda}{}_{\Sigma\Delta}\,. \end{aligned} \tag{5.2.81}$$

The Christoffel connection is symmetric in the last two indices and, because the metric $\bar{g}_{\Lambda\bar{\Sigma}}$ is Kähler, the curvature tensor is symmetric under exchange of the first and the third index and/or the second and the fourth index. Also note that since we have defined $(\eta^{\Lambda}) \equiv (\eta, \epsilon_{\alpha}, \rho^{a})$, a Λ up-index will stay an up-index if $\Lambda = 0$ or a, but become a down-index if $\Lambda = \alpha$.
After some long calculations, using equations (5.2.40) and (5.2.41) in section 5.2.3, we found the following results*

$$\begin{aligned} \bar{\Gamma}_{\gamma}{}^{\alpha\beta} &= (\bar{g}^{-1})_{\gamma\bar{\gamma}}\frac{\partial\bar{g}^{\bar{\gamma}\beta}}{\partial t_{\alpha}} + (\bar{g}^{-1})_{\gamma}{}^{0}\frac{\partial\bar{g}_{0}{}^{\beta}}{\partial t_{\alpha}} + (\bar{g}^{-1})_{\gamma}{}^{a}\frac{\partial\bar{g}_{a}{}^{\beta}}{\partial t_{\alpha}} \\ &= d_{\bar{\alpha}\bar{\beta}\gamma}(\kappa^{-1})^{\alpha\bar{\alpha}}(\kappa^{-1})^{\beta\bar{\beta}} + \frac{6}{\kappa}\delta^{(\alpha}_{\gamma}\lambda^{\beta)} - \frac{3}{\kappa}(\kappa^{-1})^{\alpha\beta}\kappa_{\gamma}\,, \\ \bar{\Gamma}_{\gamma}{}^{\alpha}{}_{0} &= \frac{1}{2\eta_1^2}(\kappa\lambda\lambda)_{\beta}\bar{\Gamma}_{\gamma}{}^{\alpha\beta} + \frac{1}{\eta_1^2}(\kappa\lambda)_{a\bar{\gamma}}(\kappa^{-1})^{\bar{\gamma}\alpha}(\kappa^{-1})^{ac}(\kappa\lambda)_{\gamma c} - \frac{3}{\kappa\eta_1^2}(\kappa\lambda\lambda)_{\gamma}\lambda^{\alpha}\,, \\ \bar{\Gamma}_{\gamma}{}^{\alpha}{}_{a} &= -\frac{1}{\eta_1}(\kappa\lambda\lambda)_{\beta a}\bar{\Gamma}_{\gamma}{}^{\alpha\beta} - \frac{1}{\eta_1}d_{ab\bar{\gamma}}(\kappa^{-1})^{\bar{\gamma}\alpha}(\kappa^{-1})^{bc}(\kappa\lambda)_{\gamma c} + \frac{3}{\kappa\eta_1}(\kappa\lambda)_{\gamma a}\lambda^{\alpha}\,, \\ \bar{\Gamma}^{a\alpha}{}_{b} &= -(\kappa^{-1})^{ac}d_{bc\bar{\gamma}}(\kappa^{-1})^{\bar{\gamma}\alpha} + \frac{3}{\kappa}\delta^{a}_{b}\lambda^{\alpha}\,, \qquad \bar{\Gamma}^{0\alpha}{}_{0} = 0\,, \qquad \bar{\Gamma}^{a\alpha}{}_{0} = -\bar{\Gamma}^{a\alpha}{}_{b}\frac{\lambda^{b}}{\eta_1}\,, \\ \bar{\Gamma}^{0}{}_{a0} &= \frac{1}{\eta_1}G_{ab}\lambda^{b}\,, \qquad \bar{\Gamma}^{0}{}_{00} = -\frac{1}{\eta_1^2}G_{ab}\lambda^{a}\lambda^{b} - \frac{2}{\eta_1}\,, \qquad \bar{\Gamma}^{0}{}_{ab} = -G_{ab}\,, \end{aligned}$$

*We used the following shorthand notation, $\kappa_{..} \equiv (\kappa\chi)_{..}, (\kappa^{-1})^{\cdot\cdot} \equiv ((\kappa\chi)^{-1})^{\cdot\cdot}$.

$$
\begin{aligned}
\bar{\Gamma}^a{}_{b0} &= \frac{1}{2\eta_1^2} d_{bc\bar{\gamma}} (\kappa^{-1})^{\bar{\gamma}\alpha} (\kappa\lambda\lambda)_\alpha (\kappa^{-1})^{ac} - \frac{1}{\eta_1^2} (\kappa^{-1})^{ac} (\kappa\lambda)_{\bar{\gamma}c} (\kappa\lambda)_{\bar{\beta}b} (\kappa^{-1})^{\bar{\gamma}\bar{\beta}} \\
&\quad + \frac{3}{\kappa\eta_1^2} (\kappa\lambda\lambda)_b \lambda^a + \frac{3}{2\kappa\eta_1^2} (\kappa\lambda\lambda)_\alpha \lambda^\alpha \delta_b^a - \frac{6\lambda^a}{\kappa\eta_1^2} \kappa_{bc} \lambda^c - \frac{1}{\eta_1} \delta_b^a \,, \\
\bar{\Gamma}^a{}_{bc} &= \frac{2}{\eta_1} d_{\bar{\gamma}d(c} (\kappa\lambda)_{b)\alpha} (\kappa^{-1})^{\bar{\gamma}\alpha} (\kappa^{-1})^{ad} - \frac{6}{\kappa\eta_1} \delta^a_{(b} (\kappa\lambda\lambda)_{c)} + \frac{6\lambda^a}{\kappa\eta_1} \kappa_{bc} \,, \\
\bar{\Gamma}_{\alpha ab} &= \frac{2}{\eta_1^2} (\kappa^{-1})^{ce} (\kappa^{-1})^{\beta\bar{\beta}} (\kappa\lambda)_{\alpha e} d_{\beta c(a} (\kappa\lambda)_{b)\bar{\beta}} - \frac{3}{\kappa\eta_1^2} (\kappa^{-1})^{\beta\gamma} (\kappa\lambda)_{\beta a} (\kappa\lambda)_{\gamma b} \kappa_\alpha \\
&\quad + \frac{d_{\bar{\beta}\bar{\gamma}\alpha}}{\eta_1^2} (\kappa^{-1})^{\beta\bar{\beta}} (\kappa^{-1})^{\gamma\bar{\gamma}} (\kappa\lambda)_{\beta a} (\kappa\lambda)_{\gamma b} - \frac{d_{\alpha ab}}{\eta^1} + \frac{3}{\kappa\eta_1^2} (\kappa\lambda\lambda)_\alpha \kappa_{ab} \,,
\end{aligned} \tag{5.2.82}
$$

$$
\begin{aligned}
\bar{R}^\beta{}_\alpha{}^\mu{}_\gamma &= -(\bar{g}^{-1})_{\alpha\bar{\alpha}} \frac{\partial}{\partial t_{\bar{\alpha}}} \bar{\Gamma}_\gamma{}^{\beta\mu} - (\bar{g}^{-1})_\alpha{}^a \frac{\partial}{\partial \lambda^a} \bar{\Gamma}_\gamma{}^{\beta\mu} - (\bar{g}^{-1})_\alpha{}^0 \frac{\partial}{\partial \eta_1} \bar{\Gamma}_\gamma{}^{\beta\mu} \\
&= -\delta^\beta_{(\alpha} \delta^\mu_{\gamma)} - d_{\bar{\beta}\bar{\mu}(\alpha} \kappa_{\gamma)} (\kappa^{-1})^{\beta\bar{\beta}} (\kappa^{-1})^{\mu\bar{\mu}} + (\kappa^{-1})^{\beta\mu} \kappa_{\alpha\gamma} \\
&\quad + \frac{\kappa}{3} (\kappa^{-1})^{\bar{\beta}(\beta} (\kappa^{-1})^{\mu)\bar{\mu}} (\kappa^{-1})^{\delta\bar{\delta}} d_{\bar{\beta}\delta\alpha} d_{\bar{\mu}\bar{\delta}\gamma} \,, \\
\bar{R}_c{}^a{}_b{}^d &= -(\bar{g}^{-1})^a{}_{\bar{\alpha}} \frac{\partial}{\partial t_{\bar{\alpha}}} \bar{\Gamma}^d{}_{bc} - (\bar{g}^{-1})^{ae} \frac{\partial}{\partial \lambda^e} \bar{\Gamma}^d{}_{bc} - (\bar{g}^{-1})^{a0} \frac{\partial}{\partial \eta_1} \bar{\Gamma}^d{}_{bc} \\
&= -\delta^a_{(c} \delta^d_{b)} + (\kappa^{-1})^{ad} \kappa_{cb} + \frac{\kappa}{3} (\kappa^{-1})^{e(a} (\kappa^{-1})^{d)f} (\kappa^{-1})^{\gamma\bar{\gamma}} d_{\gamma ec} d_{\bar{\gamma}fb} \,, \\
\bar{R}_0{}^0{}_0{}^0 &= -2 \,, \qquad \bar{R}^\beta{}_{\alpha 0}{}^0 = 0 \,, \qquad \bar{R}_b{}^a{}_0{}^0 = -\delta_b^a \,, \qquad R_{a\gamma b}{}^0 = -d_{\gamma ab} \,, \\
\bar{R}^\beta{}_{\alpha b}{}^a &= -\frac{1}{2} \delta^\beta_\alpha \delta^a_b + \frac{1}{2} d_{\bar{\beta}eb} \kappa_\alpha (\kappa^{-1})^{\beta\bar{\beta}} (\kappa^{-1})^{ae} - \frac{\kappa}{6} (\kappa^{-1})^{ae} (\kappa^{-1})^{cf} (\kappa^{-1})^{\bar{\gamma}\beta} d_{\alpha ce} d_{\bar{\gamma}bf} \\
&\quad - \frac{\kappa}{6} (\kappa^{-1})^{ae} (\kappa^{-1})^{\beta\gamma} (\kappa^{-1})^{\bar{\gamma}\bar{\beta}} d_{\alpha\gamma\bar{\gamma}} d_{\bar{\beta}be} \,, \\
\bar{R}^{\alpha a}{}_0{}^b &= \frac{\kappa}{6} (\kappa^{-1})^{ae} (\kappa^{-1})^{bc} (\kappa^{-1})^{\gamma\alpha} d_{ec\gamma} - \frac{1}{2} (\kappa^{-1})^{ab} \lambda^\alpha \,.
\end{aligned} \tag{5.2.83}
$$

All other components of the curvature tensor are zero in points of the manifold were $\lambda^a = 0$ for all a. For the dual symmetric spaces this implies that these components are zero in every point.

The curvature tensor of the truncated HVSQ manifolds

We restrict ourselves to the homogeneous very special quaternionic-Kähler manifolds. These are characterized by the d-symbols (5.2.65) mentioned in section 5.2.5. Inserting these d-symbols in the expressions for the metric (5.2.40)*, it is easy to see that in this case, the Kähler manifolds have an $\frac{SU(1,1)}{U(1)}$ part that splits off. Indeed, we find $\bar{g}^{1M} = \bar{g}^1{}_a = \bar{g}^1{}_0 = 0$, and therefore also, since the metric is Kähler, that $\bar{g}^{11}$ only depends on t_1 and that all other components are independent of t_1. We can now calculate the scalar curvature of the 'dual' Kähler space. Since we know from the previous sections that the

*Remember, we take $(\alpha) = (1, M)$ and $(a) = (i)$.

Kähler submanifolds we get by the truncation are again homogeneous, we can calculate the constant scalar curvature in one convenient point. Thus, we look at a point where $\lambda^\mu = 0$, $\lambda^2 \neq 0$, since the expressions then simplify considerably. We get

$$\bar{R} = -(q+2)^2 - 2(q+2)r - r^2 - 2 + \frac{1}{2}\gamma_{Mfa}\gamma^M_{be}\delta^{ae}\delta^{bf} - \frac{1}{2}\gamma_{Mfb}\gamma^M_{ae}\delta^{ae}\delta^{bf}\,. \quad (5.2.84)$$

For the homogeneous nonsymmetric space $L(0,1,1)$ we find $\bar{R} = -16$ for the dual space, while the original special Kähler space has $R = -18$, showing that generically the curvature tensors are different. For the symmetric very special quaternionic-Kähler manifolds, we also have $\gamma_{M(fb}\gamma^M_{ae)} = 0$, so for their dual Kähler spaces we finally get

$$\bar{R} = -(q+2)^2 - 2 - \frac{1}{2}qr - r^2 - 4r\,. \quad (5.2.85)$$

It is now easy to verify that we get the same value as for the original special Kähler manifolds that gave rise to the quaternionic-Kähler manifolds via the **c**-map, just like in [149]. In fact, in the case $r = 0$, since there are no γ_M-matrices, the only difference with the case studied in [148, 149] is the $SU(1,1)$ that was discarded. The truncated manifold is of the form $(SU(1,1)/U(1))_1 \times (SU(1,1)/U(1))_2 \times \mathcal{M}$, with $SU(1,1)_1$ generated by $\{\underline{\hat{\beta}}^0, \underline{\alpha}_0, \underline{\epsilon}_0 - \frac{3}{2}\underline{\beta}\}$ and $SU(1,1)_2$ generated by $\{\underline{\hat{\alpha}}_1, \underline{\beta}^1, \underline{\epsilon}\}$. Taking $\mathcal{K}_{dual} = (SU(1,1)/U(1))_2 \times \mathcal{M}$ corresponds to the case discussed in [148, 149], while taking $\mathcal{K}_{dual} = (SU(1,1)/U(1))_1 \times \mathcal{M}$ corresponds to the case discussed here. Since both $SU(1,1)/U(1)$ parts have scalar curvature -2, as is easy to verify, there is actually no difference between the two cases and so the dual Kähler spaces and the original very special Kähler spaces have, in the case of symmetric spaces, the same curvature tensor, like in [149].
To study the full curvature tensor in the general case where $r \neq 0$, we make use of an extension of the computer program used in [149]*. Let z^A denote the coordinates of the original very special Kähler space, and $y^A = \Im z^A$. The metric can then be derived from the Kähler potential $K = -\log(\mathcal{V}(y))$ and the curvature tensor is given by

$$R_A{}^C{}_B{}^D = -2\delta^C_{(A}\delta^D_{B)} + \frac{4}{3}d_{ABE}\,C^{CDE}\,, \quad (5.2.86)$$

where C^{ABC} is defined as in (5.2.26). In the symmetric case, it turns out that C^{ABC} is constant [93], so that the curvature tensor $R_A{}^C{}_B{}^D$ is constant (in this coordinate system). Similarly, we will denote the complex coordinates of the dual Kähler space as $(\eta^A) = (\eta, \epsilon_M, \rho^i)$ (see previous notation) and the curvature tensor as $\bar{R}_A{}^C{}_B{}^D$. All components which are nonzero in the symmetric case are listed in the previous section and they can be derived from the Kähler potential $K = -2\log\mathcal{V}\big(\chi^\alpha(p(t_\beta, \lambda^a, \eta_1))\big) - \log(\eta_1)$ (see section 5.2.3). Using the computer program mentioned before, we then find that the curvature tensor of the dual spaces are also constant in the symmetric case. To prove that the curvature tensors of the original special Kähler spaces and their dual spaces are the same, we need a (linear) coordinate transformation that relates the two tensors. This

*We would like to thank M. Trigiante for providing us with this program.

transformation can be found for all cases. We first transform the dual curvature tensor $\bar{R}$ to $\bar{R}'$,

$$\bar{R}'_A{}^C{}_B{}^D = U_A{}^{A'} U_B{}^{B'} R_{A'}{}^{C'}{}_{B'}{}^{D'} (U^{-1})_{C'}{}^C (U^{-1})_{D'}{}^D \,, \tag{5.2.87}$$

with $U_A{}^C (U^{-1})_C{}^B = \delta_A{}^B$ and $U_A{}^B = v^A \delta_A^B$ where $v^1 = 1$, $v^2 = -2$, $v^\mu = 2$ and $v^i = 1$, and as before $\mu = 3, ..., q+3$ runs over $q+1$ values, and $i = q+4, ..., n$ runs over r values. We then find

$$R_A{}^C{}_B{}^D = \bar{R}'_C{}^A{}_D{}^B \,. \tag{5.2.88}$$

In the cases where $r = 0$, the transformation U has no effect and we find $R_A{}^C{}_B{}^D = \bar{R}_C{}^A{}_D{}^B$, as in [149], in agreement with our previous arguments.

Flat and no-scale potentials

A straightforward calculation, using equations (5.2.36),(5.2.41) and (5.2.65), shows that

$$\frac{\partial K}{\partial \Re \eta^A} (\bar{g}^{-1})^{AB} \frac{\partial K}{\partial \Re \eta^B} = 3 \,, \tag{5.2.89}$$

where $(\eta^A) = (\eta, \epsilon_M, \rho^i)$. Looking at the potential in (5.2.39), we see that the first term becomes identically zero. Moreover, if the superpotential is independent of η^A, we get a potential that is identically zero, usually called a flat potential. In type IIB flux compactifications, the superpotential is independent of most (but not all) of the scalars in the hypermultiplet sector (see e.g. [136]), and we get so called no-scale potentials, potentials that are independent of some of the scalar fields (moduli) of the theory.
The calculations do not depend on the specific form of e.g $d_{\mu ij}$ and can therefore be generalized to nonhomogeneous Kähler manifolds. Notice that all the results in this section depend on the specific form (5.2.65) of the d-symbols and so do not apply to the case SG_5, characterized by e.g. $d_{111} = 1$. In this case the original special Kähler space is $SU(1,1)/U(1)$ with scalar curvature $-2/3$ and $\mathcal{M}_{trunc} = (SU(1,1)/U(1))^2$, with one $SU(1,1)/U(1)$ also having scalar curvature $-2/3$ and the other having scalar curvature -2. The $SU(1,1)/U(1)$ manifold with scalar curvature $-2/3$ gives rise to a flat potential, the other one does not.

5.3 Summary and open problems

In this chapter, we studied the orientifold truncation that arises when compactifying type II string theory on Calabi-Yau orientifolds with $O3/O7$-planes. This truncation provides a way of reducing supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 1$, and we have looked at the reduction of the hypermultiplet sector of $\mathcal{N} = 2$ supergravity with scalars spanning a $(4n + 4)$-dimensional very special quaternionic-Kähler manifold. The results can be summarized as follows:

- We established that the truncated $(2n + 2)$-dimensional manifold is Kähler, by explicitly determining the Kähler potential.

- We studied the truncated isometry algebra. For homogeneous very special quaternionic-Kähler spaces, in the generic case, the truncated manifold splits into $SU(1,1)/U(1)$ times a $2n$-dimensional Kähler space that can be viewed as dual to the original special Kähler space, in the sense of [148]. These spaces also give rise to flat potentials.

- For the case of symmetric spaces, the dual Kähler space consists of the same coset space as the original special Kähler space and has the same curvature tensor.

- We also studied a more general truncation for the symmetric spaces $L(0,P)$, and found that generically no $SU(1,1)/U(1)$ splits off anymore.

- Finally, we truncated symmetric special quaternionic-Kähler spaces that are not very special (this includes the case of minimal coupling): in this case the truncated manifold is still symmetric, but does not always factorize.

For the symmetric special quaternionic-Kähler spaces, we have given a complete overview of the Kähler spaces one gets after truncation in table 5.5. In principle, the same methods can be used to give a similar classification for all homogeneous nonsymmetric special quaternionic-Kähler spaces.
Special geometry played an important role both in this chapter and in the previous chapter. We will therefore conclude this thesis with some interesting open problems in special geometry:

- $\frac{SU(1,1)}{U(1)} \times \frac{SO(n-1,2)}{SO(n-1)\times SO(2)}$ is the only special Kähler manifold that is also a product manifold, as was proven in [81]. Similarly, $SO(1,1) \times \frac{SO(n-1,1)}{SO(n-1)}$ is the only known very special real product manifold. Can we prove there are no others? See [164] for some inspiration.

- In [104], the subclass of homogeneous very special real manifolds were classified for which $Inv(C_{IJK})$, the group of linear transformations that leave C_{IJK} invariant, works transitively. We have $Inv(C_{IJK}) \subset Iso(\mathcal{M})$, but not $Inv(C_{IJK}) = Iso(\mathcal{M})$ in general [82]. It is is therefore not clear whether all homogeneous (or even symmetric) very special real manifolds are known.

- Similarly, it is not clear whether all homogeneous special Kähler manifolds are known. However, all symmetric special Kähler manifolds were classified in [80].

- It was proven in [82] that for all very special Kähler manifolds the isometries give rise to duality symmetries, i.e. $Iso(\mathcal{M}) \subset Sp(2n+2,\mathbb{R})$. It is not known whether this is true for all special Kähler manifolds.

- It was proven in [165] that all homogenous quaternionic-Kähler manifolds of positive scalar curvature are necessarily symmetric. They were classified by Wolf in [86]

and are usually called Wolf spaces. They are the only known compact quaternionic-Kähler manifolds of positive scalar curvature. In fact, the LeBrun-Salomon conjecture [166] states that all compact quaternionic-Kähler manifolds of positive scalar curvature are symmetric and thus Wolf spaces, but this conjecture remains to be proven (see [167] for a short summary of the current status).

- There are on the other hand lots of (noncompact) quaternionic-Kähler manifolds of negative scalar curvature. However, the only known homogeneous quaternionic-Kähler manifolds of negative scalar curvature are the Alekseevskiĭ spaces. These are quaternionic-Kähler manifolds that admit a simply transitive, real solvable group of isometries and they were classified in [104, 168, 169]. The Alekseevskiĭ conjecture [168] states that the Alekseevskiĭ spaces are the only homogeneous quaternionic-Kähler manifolds of negative scalar curvature, but remains to be proven.

The main topic of the thesis was (the search for) de Sitter vacua in supergravity. Our main motivation came from results in cosmology and holography, which we discussed in chapters 2 and chapters 3 respectively. We then went on to look for de Sitter vacua in various supergravity theories, and investigated their properties, in chapter 4. Finally, in chapter 5, we studied orientifold truncations in 4D $\mathcal{N} = 2$ supergravity. Contact was made with the main theme of the thesis by noting that orientifolds are often a crucial ingredient in flux compactifications, which is the most popular method to construct de Sitter vacua in superstring theory. Many interesting questions remain (see sections 4.4 and 5.3). Hopefully, future research will lead to satisfactory answers.

Appendix A

Conventions

Throughout the thesis we will use mostly plus conventions for the space-time metric $g_{\mu\nu}$. The space-time curvature tensor is defined by

$$R^{\mu}{}_{\nu\rho\sigma} \equiv \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\tau}\Gamma^\tau_{\nu\sigma} - \Gamma^\mu_{\sigma\tau}\Gamma^\tau_{\nu\rho}\,, \tag{A.0.1}$$

$$\Gamma^\mu_{\nu\rho} \equiv \frac{1}{2} g^{\mu\sigma}(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho})\,. \tag{A.0.2}$$

while the Ricci tensor and the scalar curvature are given by

$$R_{\mu\nu} \equiv R^{\rho}{}_{\nu\rho\mu}\,, \qquad R \equiv g^{\mu\nu} R_{\mu\nu}\,. \tag{A.0.3}$$

The Einstein equation reads

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu}\,, \tag{A.0.4}$$

where κ^2 is the gravitational coupling constant

$$\kappa^2 = \frac{D-2}{D-3} A_{D-2} \frac{G}{c^4}\,, \tag{A.0.5}$$

with D the space-time dimension, A_{D-2} the area of a $(D-2)$-dimensional unit sphere, and G Newton's universal gravitational constant. In $D = 4$ we have the familiar $\kappa^2 = 8\pi G/c^4$. The Einstein equation can be derived from the following action

$$S = \int d^D x \left(\sqrt{-g} \frac{R}{2\kappa^2} + \mathcal{L}_m \right)\,, \tag{A.0.6}$$

where $g \equiv det(g_{\mu\nu})$ and $\mathcal{L}_m$ is the Lagrangian of the matter fields. The energy-momentum tensor in the Einstein equations is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}}\,. \tag{A.0.7}$$

A cosmological constant Λ can be introduced by taking $\mathcal{L}_m = -\sqrt{-g}\,\Lambda/\kappa^2$ or equivalently

$$T_{\mu\nu} = -\frac{\Lambda}{\kappa^2} g_{\mu\nu}\,. \tag{A.0.8}$$

With these definitions, we have $\Lambda > 0$ for de Sitter space (dS space).
Unless explicitly stated otherwise, we will use units $\hbar = c = \kappa^2 = 1$.

Appendix B

Samenvatting

B.1 Elementaire deeltjes, supersnaren en M-theorie

Mensen, auto's, bomen, zelfs de lucht die we inademen, alles wat we rondom ons waarnemen is verbazingwekkend genoeg opgebouwd uit een klein aantal elementaire deeltjes die met elkaar interageren op maar 4 verschillende manieren. Deze 4 manieren van interageren kunnen gezien worden als het resultaat van 4 fundamentele krachten die op de deeltjes werken. Elk deeltje is een bron van krachtvelden en oefent krachten uit op alle andere deeltjes, waardoor interacties tussen de deeltjes ontstaan. De krachten waar we het meest vertrouwd mee zijn in het dagelijkse leven zijn de zwaartekracht, ook gravitatie genoemd, en elektromagnetisme, maar op subatomaire schaal worden 2 andere krachten belangrijk, de zwakke en de sterke kracht. Terwijl op grote (lengte)schalen, de schaal van melkwegstelsels of zelfs de schaal van het dagelijks leven, gravitatie de dominerende kracht, voor de energieën die op dit moment haalbaar zijn in deeltjesversnellers gravitatie zo zwak ten opzichte van de 3 andere krachten dat haar invloed kan genegeerd worden. Alle huidige experimenten met deeltjesversnellers kunnen met een indrukwekkende precisie verklaard worden door een theorie die het Standaard Model genoemd wordt, een zogenaamde relativistische kwantumveldentheorie van de sterke, zwakke en de elektromagnetische kracht. In dergelijke theorieën worden de interacties tussen de materiedeeltjes, die fermionen genoemd worden, beschreven door het uitwisselen van krachtdeeltjes die bosonen genoemd worden, en bewegen de deeltjes volgens de tegenintuïtieve regels van de (relativistische) kwantummechanica.
Hoewel het Standaard Model met groot succes de resultaten van de experimenten met deeltjesversnellers heeft voorspeld, heeft het toch een aantal wiskundige en conceptuele problemen. Het hoofdprobleem van het Standaard Model is dat sommige fysische grootheden niet met de theorie kunnen worden bepaald, het zijn parameters die door een experiment moeten worden vastgelegd. Het Standaard Model bevat 29 van dergelijke parameters. Een ander probleem, een probleem van de meeste kwantumveldentheorieën, is het veelvuldig voorkomen van oneindigheden in het berekenen van fysische grootheden. Opdat er zinvolle voorspellingen zouden kunnen gemaakt worden, dienen deze oneindigheden verwijderd te worden, en hiervoor werd er een procedure bedacht, de renormalisatie.

Hoewel deze procedure heel goed werkt in kwantumveldentheorieën van de sterke, zwakke en elektromagnetische kracht, welke speciale kwantumveldentheorieën zijn die ijktheorieën genoemd worden, is de procedure toch niet helemaal bevredigend vanuit strikt wiskundig oogpunt. Bovendien blijkt renormalisatie te falen in een kwantumveldentheorie van gravitatie. Ofschoon we geen kwantumgravitatietheorie nodig hebben om de huidige experimenten te verklaren, bij hogere en hogere energieën, wanneer we kleinere en kleinere afstanden aftasten, zal kwantumgravitatie almaar belangrijker worden*. We weten daarom al dat op een bepaald moment, het Standaard Model dient vervangen te worden door een completere theorie die ook kwantumgravitatie bevat.
De meest beloftevolle kandidaat voor zo'n theorie is de supersnaartheorie. Volgens deze theorie zijn de elementaire deeltjes eigenlijk heel kleine vibrerende snaren. De supersnaartheorie heeft een aantal aantrekkelijke eigenschappen, welke de theorie heel aannemelijk maken. Bijvoorbeeld, in plaats van verscheidene elementaire deeltjes, heeft de supersnaartheorie maar 2 verschillende soorten snaren, open en gesloten snaren, en alle elementaire deeltjes zijn gewoon snaren die op verschillende wijzen trillen. Het graviton, het krachtdeeltje dat de zwaartekracht overbrengt, is op een natuurlijke manier in de supersnaartheorie aanwezig als een van de massaloze excitaties van een gesloten snaar. IJkbosonen zoals de krachtdeeltjes van het Standaard Model daarentegen, kunnen zowel trillende open als trillende gesloten snaren zijn. Aangezien een kwantumtheorie van alleen open snaren niet consistent blijkt te zijn, leidt de supersnaartheorie automatisch tot een unificatie van de zwaartekracht met de andere krachten van het Standaard Model. De (super)snaren interageren met elkaar door samen te smelten of af te splitsen, en de verscheidene krachten zijn hiervan gewoon verschillende manifestaties. De supersnaartheorie heeft ook een heel mooie symmetrie die supersymmetrie genoemd wordt, en welke een symmetrie is tussen de fermionen (elementaire materiedeeltjes) en de bosonen (krachtdeeltjes) in de theorie. Bovendien heeft de supersnaartheorie niet de oneindigheidsproblemen van vele kwantumveldentheorieën (van puntdeeltjes), zij is eindig orde per orde in perturbatietheorie.
Eén van de minder aantrekkelijke eigenschappen van de supersnaartheorie is dat er eigenlijk 5 verschillende supersnaartheorieën zijn, genaamd type IIA, type IIB, type I, $SO(32)$ heterotisch en $E_8 \times E_8$ heterotisch. Enkel type I is een theorie van open en gesloten snaren, alle anderen bevatten alleen gesloten snaren. Consistentie vereist dat de ijkgroep van type I $SO(32)$ is, terwijl er 2 consistente heterotische theorieën zijn, één met ijkgroep $SO(32)$ en één met ijkgroep $E_8 \times E_8$. De type II theorieën daarentegen hebben geen ijkgroep. De supersnaartheorieën zijn het eenvoudigst te beschrijven wanneer de snaren bewegen in een 10-dimensionale vlakke Minkowski ruimte, in welk geval de type II theorieën het maximale aantal van 32 supersymmetrieën hebben, en de andere supersnaartheorieën 16 supersymmetrieën hebben.
Alle supersnaartheorieën zijn alleen perturbatief gedefinieerd, als een (divergente) reeks-

*De zwaartekracht valt af volgens de inverse van het kwadraat van de afstand en wordt dus zwakker naarmate de afstand groter wordt. Toch domineert gravitatie op grote afstanden, omdat de zwakke en sterke kracht nog sneller afvallen met de afstand, en omdat er zowel positieve en negatieve elektromagnetische ladingen zijn en deze heffen elkaar doorgaans op op grote afstanden, terwijl er maar een soort gravitationele lading is, namelijk massa of energie.

ontwikkeling in de snaarkoppelingsconstante, welke de sterkte van de snaarinteracties bepaald. Midden jaren negentig kwamen we echter ook veel te weten over hun niet-perturbatief gedrag. Zo werden er bijvoorbeeld in type II en type I supersnaartheorie niet-perturbatieve objecten gevonden, genaamd D-branen, welke hypervlakken zijn waarop open snaren kunnen eindigen. De volledige type II theorieën bleken dus behalve gesloten snaren ook open snaren te bevatten, maar terwijl de gesloten snaren vrij kunnen bewegen doorheen de ruimte-tijd, moeten de open snaren op D-branen eindigen. Bovendien werden er ook allerlei relaties ontdekt tussen de verschillende supersnaartheorieën. T-dualiteit toonde ons dat een supersnaartheorie gecompactificeerd op een cirkel met kleine straal equivalent (duaal) kan zijn aan een supersnaartheorie gecompactificeerd op een cirkel met een grote straal. Bijvoorbeeld, type IIA is T-duaal aan type IIB, en de $SO(32)$ heterotische is T-duaal aan de $E_8 \times E_8$ heterotische supersnaar. Gelijkaardig aan T-dualiteit, leerde we ook dat type IIA (super)snaartheorie gecompactificeerd op een 6-dimensionale Calabi-Yau variëteit duaal is aan type IIB snaartheorie gecompactificeerd op een 6-dimensionale spiegel Calabi-Yau variëteit. Tenslotte werd ontdekt dat de sterke-koppelingslimiet van een supersnaartheorie equivalent kan zijn met de zwakke-koppelingslimiet van een andere supersnaartheorie, een dualiteit die S-dualiteit genoemd werd. Bijvoorbeeld, de $SO(32)$ heterotische snaar is S-duaal aan de type I snaar, waarbij de fundamentele snaar van de heterotische theorie door de S-dualiteit wordt afgebeeld op de D1-snaar (= D-braan met 1 ruimtedimensie) van de type I theorie. Verder is type IIB zelfduaal onder S-dualiteit, en gedragen de type IIA en de $E_8 \times E_8$ heterotische supersnaar zich nog eigenaardiger bij sterke koppeling, zij zijn S-duaal aan 11-dimensionale theorieën waarover weinig geweten is. Al deze dualiteiten leidden uiteindelijk tot de conclusie dat de 5 verschillende snaartheorieën het resultaat zijn van perturbatie-expansies rond verschillende vacua van één meer fundamentele theorie, genaamd M-theorie. Helaas is er nog geen volledig bevredigende definitie van deze theorie gevonden.

B.2 De AdS/CFT correspondentie en holografie

Een aantal jaren na de ontdekking van de dualiteiten tussen de 5 supersnaartheorieën werd er een andere verbazingwekkende dualiteit gevonden, genaamd de Anti-de Sitter/conforme veldentheorie (AdS/CFT) correspondentie. In zijn originele vorm was de AdS/CFT correspondentie een vermoede equivalentie tussen type IIB snaartheorie in $AdS_5 \times S^5$ en 4D $\mathcal{N} = 4$ Super Yang Mills theorie. Wat echt merkwaardig is aan dit vermoeden, is dat beweerd wordt dat een kwantumtheorie die gravitatie bevat (snaartheorie) equivalent beschreven kan worden door een ijktheorie (welke niet veel verschilt van deze die in het Standaard Model worden gebruikt). Bovendien is volgens de conjectuur een kwantumgravitietheorie in 5 niet-compacte dimensies equivalent aan een kwantumtheorie 'zonder' gravitatie in 4 dimensies. De AdS/CFT correspondentie is dus een concrete realisatie van het zo genoemde holografische principe, welke voornamelijk zegt dat de maximale hoeveelheid informatie die in een stukje ruimte-tijd kan worden opgeslagen, afhangt van de oppervlakte van de rand van het stukje ruimte-tijd (en niet van het volume van het stukje

ruimte-tijd zoals men naïef zou kunnen denken). Het holografisch principe ontstond uit werk op het gebied van zwarte gaten, van Hawking en anderen, dat toonde dat zwarte gaten zich gedragen als thermodynamische objecten wanneer kwantumeffecten in rekening worden gebracht. De hoeveelheid informatie van een zwart gat kan afgeleid worden uit haar entropie, welke proportioneel blijkt te zijn met de oppervlakte van de horizon van het zwarte gat. Observeren dat meer en meer informatie stoppen in een kleine regio van de ruimte-tijd uiteindelijk het ontstaan van een zwart gat tot gevolg heeft*, leidt dan tot het holografisch principe. Vele natuurkundigen geloven dat elke goede kwantumgravitatietheorie aan het holografische principe zal voldoen.
De AdS/CFT correspondentie kan gebruikt worden om een niet-perturbatieve definitie te geven van supersnaartheorie in een AdS achtergrond en kan ook gezien worden als een voorbeeld van een sterke-koppeling/zwakke-koppeling dualiteit. In de sterke-koppelingslimiet van de CFT, wordt de snaarkoppelingsconstante zwak en kan perturbatietheorie gebruikt worden in de snaartheorie, en omgekeerd, heeft de CFT een zwakke koppelingsconstante, wanneer de snaartheorie een sterke koppelingsconstante heeft. Dit maakt het moeilijk om de AdS/CFT correspondentie te bewijzen. Ze werd echter wel op verschillende wijzen getest, zodat er weinig twijfel is dat de conjectuur inderdaad correct is. De meeste van deze tests werden gedaan in de zogenaamde supergravitatie benadering. Behalve een aantal massaloze toestanden, heeft een supersnaar ook nog een oneindig aantal massieve toestanden. Bij voldoende lage energie kunnen de massieve toestanden echter genegeerd worden, en kan voor de massaloze velden een effectieve theorie worden neergeschreven. Als we ons beperken tot theorieën met ten hoogste 2 afgeleiden en geen rekening houden met kwantumcorrecties in de snaarkoppelingsconstante, dan blijkt de effectieve theorie een supergravitatietheorie te zijn, een (locaal) supersymmetrische versie van Einstein's gravitatietheorie. Omdat op dit moment het nog niet geweten is hoe supersnaartheorie in $AdS_5 \times S^5$ moet gekwantiseerd worden, worden de meeste berekeningen dan ook gedaan in de lage-energie-limiet waar de supergravitatietheorie betrouwbaar is.
De Anti-de Sitter ruimte is een maximaal symmetrische oplossing van de Einstein vergelijkingen met een negatieve kosmologische constante. Aangezien vele natuurkundigen verwachten dat kwantumgravitatie holografisch is, werden er gelijkaardige correspondenties voorgesteld voor de maximaal symmetrische oplossingen met kosmologische constante groter dan of gelijk aan nul, namelijk respectievelijk de de Sitter (dS) ruimte en de Minkowski ruimte. Deze correspondenties hebben echter een veel meer speculatief karakter, aangezien ze niet werden afgeleid uit snaartheorie, in tegenstelling met de AdS/CFT correspondentie die een duidelijke en heel natuurlijke oorsprong heeft in de snaartheorie.

*We verwachten de vorming van zwarte gaten op basis van de algemene relativiteitstheorie van Einstein, de klassieke gravitatietheorie. Het is niet uit te sluiten dat kwantumgravitatie op een of andere manier het ontstaan van zwarte gaten voorkomt, in welk geval het holografische principe niet hoeft te gelden.

B.3 Kosmologie

Eén van de voornaamste problemen van supersnaartheorie en hoge-energie fysica in het algemeen was en is het gebrek aan experimentele input in de laatste 20 jaar. Behalve bewijs voor kleine neutrino* massa's en een aantal hints voor ijkkoppelingunificatie†, waren de enige andere aanwijzingen van de natuur, in onze zoektocht naar een fundamentelere theorie dan het Standaard Model, afkomstig uit de kosmologie. In de late jaren negentig, ontdekten verschillende onderzoeksgroepen die supernova's observeerden tot ieders verbazing dat de uitdijing van ons heelal lijkt te versnellen. Sinds de aanvaarding van de Big Bang theorie, werd er altijd vanuit gegaan dat de zwaartekracht de uitdijing van het heelal zou vertragen. Het feit dat de uitdijing van het heelal niet vertraagt, maar in plaats daarvan versnelt, leert ons dat er een kracht moet zijn die op grote afstanden de zwaartekracht tegenwerkt. De meest populaire manier om deze waarneming te verklaren is op dit moment het veronderstellen van het bestaan van een vreemde vorm van energie die de volledige ruimte-tijd doordringt en een negatieve druk heeft, genaamd donkere energie. De eenvoudigste vorm van donkere energie, en tevens de vorm die het best bij de huidige waarnemingen past, is een positieve kosmologische constante.
Behalve donkere energie, wijzen de huidige waarnemingen ook op het bestaan van donkere materie, materie die geen elektromagnetische straling uitzendt, maar wel een meetbaar gravitationeel effect heeft op de zichtbare materie zoals sterren. Het meeste van die donkere materie moet noodzakelijk voorkomen in een vorm waar we vandaag niet vertrouwd mee zijn. Volgens het huidige standaard model van de kosmologie, het Λ-CDM model, welke we in hoofdstuk 2 beschrijven, bestaat ons heelal voor het grootste deel uit onbekende materie en energie. Hoewel alles wat we in ons dagelijks leven tegenkomen, gemaakt is uit een klein aantal elementaire deeltjes, is deze vorm van materie maar verantwoordelijk voor 5% van de energie/materie-inhoud van het heelal. Ongeveer 25 % komt in de vorm van donkere materie en een aanzienlijke 70 % komt in de vorm van donkere energie.
Tot op heden is er geen theorie die bovenstaande bizarre eigenschappen van het heelal kan verklaren. We begrijpen niet waarom ons heelal zoveel donkere energie bevat‡, noch waarom er zoveel donkere materie is. We weten zelf niet wat die donkere materie precies is. Echter, een goede kandidaat voor deze donkere materie is een zwak interagerend deeltje dat vaak voorspeld wordt door supersymmetrische uitbreidingen van het Standaard Model. Supersymmetrie verbetert ook problemen van het Standaard Model zoals het hiërarchie probleem en ijkkoppelingsunificatie§, en is ook heel natuurlijk vanuit een

*Neutrino's zijn fermionische deeltjes die heel zwak interageren met andere deeltjes. Ze interageren alleen via de zwakke en de gravitationele kracht. Miljarden neutrino's passeren elke seconde door je lichaam, maar slechts één of twee zullen door je lichaam worden gestopt gedurende je hele leven!

†IJkkoppelingunificatie is het idee dat de zwakke, sterke en elektromagnetische kracht verschillende manifestaties zijn van één fundamentelere (ijk)kracht.

‡Eigenlijk voorspellen de meeste kwantumveldentheorieën zelfs een kosmologische constante die een stuk groter is dan wat wordt waargenomen. Dit is het befaamde kosmologische constante probleem en is één van de grootste onopgeloste problemen in de hoge-energie fysica, maar dat is een heel ander verhaal. We zullen het in de thesis niet proberen op te lossen en zullen dit probleem dan ook volledig negeren!

§Het uitleggen van deze problemen zou ons te ver afleiden. Zie bijvoorbeeld [1] als je meer wil weten.

conceptueel oogpunt ('gelijkheid' tussen bosonen en fermionen). Er zijn dus vele redenen om te geloven dat supersymmetrie en supersnaartheorie, welke supersymmetrie impliceert en de beste kandidaat is voor een kwantumgravitatietheorie, stappen in de goede richting zijn. Misschien weten we het binnen enkele jaren, wanneer de Large Hadron Collider (LHC) volledig operationeel is, al zeker. Hoewel de LHC vooral gebouwd werd voor het detecteren van het Higgs deeltje, een massief boson voorspeld door het Standaard Model, is er de opwindende mogelijkheid dat het de eerste supersymmetrische partnerdeeltjes detecteert. De tijd zal het zeggen.

B.4 de Sitter ruimten in supergravitatie

De ontdekking dat de uitdijing van het heelal aan het versnellen is, leidde tot een energieke en nog steeds voortdurende zoektocht naar dS ruimten in supersnaartheorie en in de verwante lage-energie supergravitatietheorie, zowel als naar een analogon van de AdS/CFT correspondentie voor dS ruimten. Terwijl holografie voor ruimten met een positieve kosmologische constante in mysterie gehuld blijft, zijn er nu wel al verschillende manieren om dS vacua te construeren in supersnaartheorie voorgesteld, en zijn er verschillende supergravitatietheorieën met dS vacua gevonden. In de thesis beperken we ons tot supergravitatie, hoewel we een deel van onze motivatie halen uit het mogelijk bestaan van een onderliggende supersnaartheorie. Een gedetailleerd verslag van de zoektocht naar dS vacua in supergravitatie wordt gegeven in hoofdstuk 4, waarbij we ons vooral concentreren op de supergravitatietheorieën in 4 en 5 ruimte-tijd dimensies, aangezien dit de meest relevante dimensies zijn voor kosmologische en holografische toepassingen.
Voor het bestuderen van vacuumoplossingen van een supergravitatietheorie zijn alleen de scalaire velden van de theorie van belang (zie sectie B.5 voor meer uitleg). Het relevante gedeelte van elke supergravitatie actie is dus

$$S = \int d^D x \sqrt{-g} \left[\frac{1}{2\kappa^2} R - \frac{1}{2} a_{ij}(\phi) g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j - V(\phi) \right] , \tag{B.4.1}$$

met κ^2 de gravitationele koppelingsconstante, R de scalaire kromming van de D-dimensionale ruimte-tijd en ϕ^i een aantal scalaire velden gelabeled met de index i. De scalaire velden kunnen gezien worden als coördinaten van een Riemann'se variëteit, met a_{ij} de positief-definiete metriek. Zowel de scalaire metriek a_{ij} en de scalaire potentiaal V zijn geen willekeurige functies maar zijn beperkt door supersymmetrie. Wanneer er minstens 8 supersymmetrieën zijn, is de scalaire potentiaal volledig bepaald door supersymmetrie en de keuze van de ijkgroep. Bovendien is in 4D en 5D supergravitatietheorieën de scalaire variëiteit altijd een coset variëteit als de theorie meer dan 8 supersymmetrieën heeft, terwijl in 4D en 5D theorieën met exact 8 supersymmetrieën de scalaire variëteit een interessante meetkundige structuur heeft die de originele naam 'speciale meetkunde' heeft meegekregen. Kennis van de scalaire variëteit is nodig om de scalaire potentiaal te kunnen berekenen. Aangezien theorieën met exact 8 supersymmetrieën een speciale rol spelen in de zoektocht naar dS vacua, en omdat speciale meetkunde ook een belangrijke plaats

inneemt in hoofdstuk 5, geven we in hoofdstuk 4 ook een gedetailleerd overzicht van de speciale meetkunde in 4D en 5D supergravitatietheorieën.

B.5 Overzicht van de thesis

Geïnspireerd door het holografisch principe en de recente waarnemingen in kosmologie, is het hoofdthema van de thesis de zoektocht naar dS vacua in supergravitatie. De thesis is als volgt gestructureerd. Hoofdstuk 1 bevat de inleiding, welke we hierboven hebben vertaald, en een korte bespreking van de inhoud van de thesis. Vervolgens geven we in hoofdstuk 2 en hoofdstuk 3 de achtergrond in kosmologie en holografie die nodig is om de fysische context te kunnen appreciëren waarin de resultaten van hoofdstuk 4 en 5 kunnen of moeten gezien worden, en bekijken we enkele belangrijke resultaten betreffende stabiliteit in dS en AdS ruimten. Het hoofdthema van de thesis wordt behandeld in hoofdstuk 4, waarin we dS vacua in verschillende supergravitatie theorieën bestuderen. Dit hoofdstuk bevat zowel eigen werk [100, 117] als gerelateerde resultaten van andere auteurs. Het vijfde en tevens laatste hoofdstuk bevat vooral ons werk aangaande orientifold truncaties in 4D $\mathcal{N} = 2$ supergravitatie [134]. Het verband met het hoofdthema van de thesis wordt gelegd door op te merken dat orientifolds vaak een cruciaal ingrediënt zijn in flux compactificaties, welke de meest populaire methode is om dS vacua te construeren in supersnaartheorie. Na dit overzicht in vogelvlucht van de thesis, zullen we nu een meer gedetailleerde beschrijving geven, hoofdstuk per hoofdstuk.
In hoofdstuk 2, beginnen we met het introduceren van enkele nuttige concepten in (pseudo) Riemann'se meetkunde. We geven de definities van een Killing vector, een homogene ruimte, een isotropie, en een maximaal symmetrische ruimte, en vermelden enkele interessante theorema's. Vervolgens onderzoeken we de 3 Lorentziaanse maximaal symmetrische ruimten in detail. We vergelijken de dS ruimte met de AdS ruimte en de beter gekende Minkowski ruimte, gebruikmakend van verschillende coördinatensytemen om hun eigenschappen bloot te leggen. Daarna introduceren we de basisconcepten van de kosmologie, namelijk Robertson-Walker metriek, Hubble wet, Friedmann vergelijkingen, de toestandsvergelijking, etc., en we tonen hoe de dS ruimte past in dit theoretisch raamwerk. Dan vatten we de resultaten van de huidige waarnemingen samen en presenteren we het huidige standaard model van de kosmologie, namelijk het Λ-CDM model*. Voor de volledigheid besluiten we het hoofdstuk door kort het idee van inflatie te bekijken, aangezien dit een ander gebied is van de kosmologie waar de dS ruimte een belangrijke rol speelt.
In hoofdstuk 3, beginnen we met het bekijken van het typevoorbeeld van een holografische correspondentie, de AdS/CFT correspondentie. We vatten de oorspronkelijke argumenten van Juan Maldacena die leidden tot de vermoede equivalentie van supersnaartheorie in $AdS_5 \times S^5$ en 4D $\mathcal{N} = 4$ Super Yang-Mills theorie samen, en we bespreken verschillende veralgemeende AdS/CFT correspondenties. We tonen ook aan dat er veel kan geleerd worden door het bestuderen van de lage-energie supergravitatietheorie van de supersnaar-

*Λ is het symbool dat standaard wordt gebruikt voor de cosmologische constante en CDM staat voor 'cold dark matter', koude donkere materie dus.

theorie, en in deze context bekijken we ook de stabiliteit van AdS ruimten (onder scalaire fluctuaties). Vervolgens vatten we enkele voorstellen voor een dS/CFT correspondentie samen, en bestuderen we de stabiliteit van dS ruimten (onder scalaire fluctuaties). Het lezen van dit onderdeel zal duidelijk maken dat de situatie een stuk onduidelijker is dan in het AdS geval. Tenslotte geven we voor de volledigheid ook een kort overzicht van de status van holografie in de vlakke ruimte. Ook hier is de huidige toestand verre van duidelijk.
Hoofdstuk 4 bevat een gedetailleerde studie van dS vacuum oplossingen in 4D en 5D supergravitatie. Het meeste werk werd gedaan in supergravitatietheorieën waarbij het aantal supersymmetrieën gelijk is aan 4.2^n, met $n = 0, 1, 2$ of 3, aangezien deze theorieën meestal een duidelijkere band hebben met supersnaartheorie. Alleen in 4D is $n = 0$ mogelijk, en in dit geval is de potentiaal V niet volledig bepaald door supersymmetrie (en de keuze van de ijkgroep), maar kan er een vrij arbitraire potentiaal worden vooropgesteld. Om deze redenen beperken we ons in de thesis dan ook tot theorieën met 8 ($\mathcal{N} = 2$), 16 ($\mathcal{N} = 4$) of 32 ($\mathcal{N} = 8$) supersymmetrieën.
Net zoals in de algemene relativiteitstheorie, is het in supergravitatie gebruikelijk om naar de maximaal symmetrische oplossingen te verwijzen als vacua. Vaak worden echter ook de maximaal supersymmetrische oplossingen van supergravitatie vacua genoemd. Maximaal supersymmetrische oplossingen kunnen maximaal symmetrisch zijn, maar hoeven dat niet te zijn, en omgekeerd, kunnen maximaal symmetrische oplossingen maximaal supersymmetrisch zijn, maar is dat lang niet altijd zo. Zo is bijvoorbeeld een dS ruimte een maximaal symmetrische ruimte, maar kan het aangetoond worden dat een dS oplossing van supergravitatie nooit supersymmetrisch kan zijn. Wanneer we in de thesis spreken over vacua in supergravitatie bedoelen we altijd maximaal symmetrische oplossingen. Opdat ze D-dimensionale Lorentzinvariantie (+ translaties) zouden behouden, zijn deze vacuum oplossingen noodzakelijk configuraties met constante scalaire velden, $\phi^i = \phi_0^i = \text{cte}$, een dS, AdS of Minkowski metriek en alle andere supergravitatievelden gelijk aan nul. Om te voldoen aan de bewegingsvergelijkingen die uit de actie (B.4.1) kunnen worden afgeleid dienen de constante waarden van de scalairen extrema van de potentiaal V te zijn, i.e.

$$\left.\frac{\partial V}{\partial \phi^i}\right|_{\phi=\phi_0} = 0\,. \tag{B.5.1}$$

De kosmologische constante Λ wordt gegeven door $V|_{\phi=\phi_0}$, en als $\Lambda > 0$ hebben we een dS vacuum gevonden. Om na te gaan of het vacuum stabiel is onder scalaire fluctuaties, $\phi^i = \phi_0^i + \lambda^i$, berekenen we de massamatrix (van de scalaire fluctuaties λ^i),

$$V_j^i = a^{ik}\left.\frac{\partial^2 V}{\partial \phi^k \partial \phi^j}\right|_{\phi=\phi_0}\,, \tag{B.5.2}$$

met a^{ij} de inverse van de scalaire metriek a_{ij}. Een niet-positief-definiete massamatrix V_j^i signaleert een onstabiel vacuum (zie hoofdstuk 3 voor meer info). Zoals reeds meerdere malen vermeld werd, is de potentiaal V verre van arbitrair in supergravitatietheorie met 8 of meer supersymmetrieën en dit maakt het bestuderen van de potentiaal dan ook interessant. Elke supergravitatietheorie heeft een aantal ijkvelden A_μ^I en deze kunnen worden

gebruikt om globale symmetrieën van de theorie lokaal te maken. Zo zijn bijvoorbeeld bepaalde symmetrieën van de scalaire metriek ook globale symmetrieën van de supergravitatietheorie en deze kunnen dan onder bepaalde voorwaarden 'geijkt' worden. Eens de scalaire variëteit, welke ook sterk beperkt is door supersymmetrie (zie sectie B.4), en de ijkgroep worden vastgelegd, bepaalt supersymmetrie de potentiaal op een unieke manier. Verscheidene 4D en 5D supergravitatietheorieën met dS vacua worden in hoofdstuk 4 besproken. Zoals eerst werd opgemerkt door Kallosh et al in [12] zijn de meeste (gekende) dS vacua in supergravitatietheorieën onstabiel en zijn de massa's van de scalaire fluctuaties vaak gekwantiseerd in eenheden van de kosmologische constante. Zo hebben bijvoorbeeld vele 4D dS vacua een scalaire fluctuatie met massa $m^2 = -2\Lambda$. De enige gekende voorbeelden van stabiele dS vacua in supergravitatietheorieën met minstens 8 supersymmetrieën werden gevonden in supergravitatietheorieën met exact 8 supersymmetrieën en we bekijken in secties 4.1.3 en 4.2.2 alle gekende voorbeelden. Aangezien theorieën met 8 supersymmetrieën in deze zin speciaal zijn, wordt er heel wat plaats aan hen toegewijd en geven we ook een gedetailleerde presentatie van de aanwezige speciale meetkunde. In de werken [100, 117], besproken in secties 4.2.2 en 4.2.3, concentreerden we ons op 5D $\mathcal{N} = 2$ en 5D $\mathcal{N} = 4$ supergravitatie met de volgende resultaten als gevolg:

- We vonden de eerste en tot nog toe enige voorbeelden van stabiele dS vacua in 5D $\mathcal{N} = 2$ supergravitatie.
- We toonden aan dat de $U(1)$ Fayet-Iliopoulos potentiaal van 5D $\mathcal{N} = 2$ supergravitatie geen stabiele dS vacua heeft.
- In 5D $\mathcal{N} = 4$ supergravitatie vonden we alleen onstabiele dS vacua, telkens in theorieën met niet-compacte ijkgroepen.
- Echter, in 5D $\mathcal{N} = 2$ supergravitatie vonden we ook vele voorbeelden van compacte ijkingen die tot onstabiele dS vacua leiden.
- De 4D en 5D $\mathcal{N} = 2$ supergravitatietheorieën met gekende stabiele dS vacua hebben een aantal eigenaardige karakteristieken met elkaar gemeen. Beiden hebben niet-compacte ijkgroepen en Fayet-Iliopoulos termen, samen met de Roo-Wagemans hoeken in 4D en antisymmetrische tensors in 5D.
- Net zoals in de supergravitatietheorieën besproken in [12, 70, 87, 88, 94, 97], vonden we dat sommige (maar niet altijd alle) scalaire fluctuaties in dS vacua gekwantiseerde massa's hebben.
- Zoals verwacht, waren vele resultaten in 5D gelijkaardig aan resultaten in 4D.

De resultaten van secties 4.1 en 4.2 leiden tot volgende belangrijke vragen:

- Kunnen we stabiele dS vacua vinden in 4D en 5D $\mathcal{N} \geq 3$ supergravitatie, of is het mogelijk om te bewijzen dat bijvoorbeeld $\mathcal{N} = 8$ supergravitatie geen stabiele dS vacua heeft ?

- Kunnen we stabiele dS vacua vinden in $D > 5$ supergravitatie? Het kan interessant zijn om de 6D supergravitatietheorieën met 8 supersymmetrieën te bestuderen. Supergravitatietheorieën in $D > 6$ hebben noodzakelijk meer dan 8 supersymmetrieën en het is daarom minder waarschijnlijk dat zij stabiele dS vacua hebben, aangezien zij meer beperkt zijn door supersymmetrie.

- Is er een (diepe) reden voor de gekwantiseerde waarden van sommige scalaire massa's?

- Kunnen we meer voorbeelden vinden van stabiele dS vacua in 4D en 5D $\mathcal{N} = 2$ supergravitatie?

- Wat zijn nodige voorwaarden opdat een supergravitatietheorie stabiele dS vacua zou hebben?

- Wat zijn voldoende voorwaarden opdat een supergravitatietheorie stabiele dS vacua zou hebben?

De antwoorden op deze vragen moeten we helaas schuldig blijven. Verder onderzoek is vereist. Waarschijnlijk dienen een aantal van deze vragen beantwoord te worden vooraleer we kunnen zeggen welke rol er is weggelegd voor de dS vacua van supergravitatie.
Na de uitgebreide studie van dS vacua in 4D en 5D supergravitatie in de eerste twee secties van hoofdstuk 4, bekijken we in de laatste sectie de mogelijke 10D en 11D supergravitatietheorieën, en laten we zien hoe sommige van de gekende 4D en 5D dS vacua kunnen worden ingebed in 10D of 11D. In deze context bespreken we ook een welbekend no-go theorema dat compactificaties van 10D of 11D naar 4D of 5D dS space uitsluit onder bepaalde redelijke aannames. Eén manier om het no-go theorema te omzeilen is het toelaten van objecten met een negatieve spanning, zoals orientifold-vlakken in supersnaartheorie en dit brengt ons bij het laatste hoofdstuk.
In hoofdstuk 5, bespreken we eerst kort het nut van flux compactificaties in supersnaartheorie, en geven we enkele ingangen in de reeds uitgebreide literatuur. Het doel van deze sectie is het schetsen van de context, niet het geven van een introductie in dit reusachtig onderwerp. Nadat we de context duidelijk gemaakt hebben in de eerste sectie, geven we een gedetailleerde presentatie van ons werk [134] in orientifold truncaties in de tweede sectie. We studeren in detail truncaties van de hyperscalaire variëteit van 4D $\mathcal{N} = 2$ supergravitatie die plaatsvinden wanneer O3 en/of O7 orientifold-vlakken aanwezig zijn in Calabi-Yau compactificaties van de type IIB supersnaartheorie. Deze truncaties breken de helft van de supersymmetrieën (4 van de 8), en reduceren daarom de originele quaternionisch-Kähler hyperscalaire variëteiten naar Kähler variëteiten. Wanneer de quaternionisch-Kähler ruimte een speciale structuur heeft die 'heel speciaal' wordt genoemd, dan leidt de truncatie naar een afbeelding tussen 2 Kähler ruimten van dezelfde dimensie, zoals eerst werd geobserveerd door Ferrara e.a. in [148]. De 2 Kähler ruimten worden duale Kähler ruimten genoemd.
De resultaten van [134] kunnen als volgt worden samengevat:

- We toonden aan dat de orientifold truncaties inderdaad de quaternionisch-Kähler ruimten reduceren naar Kähler ruimten door expliciet de Kähler potentiaal te berekenen.

- We bekeken de getrunceerde isometrie-algebra. Voor de $(4n+4)$-dimensionale homogene heel speciale quaternionisch-Kähler ruimten, in het generieke geval, bestaat de $(2n+2)$-dimensionale Kähler variëteit die men verkrijgt na truncatie uit een $SU(1,1)/U(1)$ variëteit maal een $2n$-dimensionale Kähler variëteit $\mathcal{M}_2$, met n een natuurlijk getal. Elke $(4n+4)$-dimensionale (heel) speciale quaternionisch-Kähler ruimte is via de zogenaamde c-map ook gerelateerd aan een $2n$-dimensionale (heel) speciale Kähler ruimte $\mathcal{M}_1$. In navolging van [148] worden de Kähler ruimten $\mathcal{M}_1$ en $\mathcal{M}_2$ duaal genoemd. We vonden ook dat de Kähler ruimten $\mathcal{M}_2$ vlakke potentialen opleveren.

- In het geval de heel speciale quaternionisch-Kähler ruimten symmetrisch zijn, konden we aantonen dat de duale Kähler ruimten dezelfde cosetstructuur en dezelfde krommingstensor hebben.

- Voor de symmetrische ruimten $L(0,P)$ (zie tabel 5.3 in hoofdstuk 5 voor meer info) bestudeerden we een meer algemene truncatie, en vonden we dat er meestal geen $SU(1,1)/U(1)$ afsplitst.

- We trunceerden ook symmetrische speciale quaternionisch-Kähler ruimten die niet heel speciaal zijn (dit omvat ook het geval van minimale koppeling). Dit leidt opnieuw tot symmetrische Kähler variëteiten die niet factoriseren.

Voor de symmetrische speciale quaternionisch-Kähler ruimten hebben we in tabel 5.5 van hoofdstuk 5 een compleet overzicht gegeven van de Kähler ruimten die men krijgt na truncatie. In principe kunnen dezelfde methoden worden gebruikt om tot een gelijkaardige classificatie te komen voor de homogene niet-symmetrische speciale quaternionisch-Kähler ruimten.
Zowel hoofdstuk 4 als hoofdstuk 5 worden besloten met een korte samenvatting en een selectie van open problemen. De thesis wordt afgesloten met 2 appendices. In appendix A beschrijven we welke conventies we gebruiken. Appendix B bevat de Nederlandse samenvatting.

Bibliography

[1] L. Susskind, *The Gauge Hierarchy Problem, Technicolor, Supersymmetry, And All That.(Talk)*, Phys. Rept. **104** (1984) 181
J. R. Ellis, *Beyond the standard model for hillwalkers*, Proceedings of 1998 European High Energy School of Physics, St. Andrews, Scotland, 23 Aug - 5 Sep, 133-196, `hep-ph/9812235`

[2] S. Weinberg, *Gravitation and cosmology, principles and applications of the general theory of relativity*, John Wiley & Sons, Inc., 1972

[3] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973

[4] P. K. Townsend, *Black Holes*, Lecture notes given as part of the Cambridge University Mathematical Tripos, `gr-qc/9707012`

[5] M. Spradlin, A. Strominger and A. Volovich, *Les Houches lectures on de Sitter space*, Proceedings of Les Houches 2001 Summer School Unity of Fundamental Physics: Gravity, Gauge Theory and Strings, Les Houches, France, 30 Jul - 31 Aug, 423-453, `hep-th/0110007`

[6] E. Witten, *Quantum gravity in de Sitter space*, Proceedings of International School of Subnuclear Physics: 39th Course: New Fields and Strings in Subnuclear Physics, Erice, Italy, 29 Aug - 7 Sep 2001, `hep-th/0106109`

[7] O. Aharony, S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, Phys. Rept. **323** (2000) 183-386, `hep-th/9905111`

[8] M. Trodden and S. Carroll, *TASI lectures: introduction to cosmology*, Proceedings of TASI 2003: Recent Trends in String Theory, Boulder, Colorado, 1-27 Jun 2003, 703-793, `astro-ph/0401547`

[9] V. A. Rubakov, *Introduction to cosmology*, Proceedings of RTN Winter School on Strings, Supergravity and Gauge Theories, Trieste, Italy, 31 Jan - 4 Feb 2005, PoS RTN2005 (2005) 003

[10] O. Lahav and A. R. Liddle, *The cosmological parameters*, Phys. Lett. **B592** (2004) 1, `astro-ph/0406681`

[11] A. Linde, *Prospects of inflation*, Extended version of talk given at Nobel Symposium 2003: Cosmology and String Theory, Sigtunastiftelsen, Sweden, 14-19 Aug 2003, Phys. Scripta **T117** (2005) 40-48, `hep-th/0402051`

[12] R. Kallosh, A. Linde, S. Prokushkin and M. Shmakova, *Gauged supergravities, de Sitter space and cosmology*, Phys. Rev. **D65** (2002) 105016, `hep-th/0110089`

[13] J. R. Mould et al., *The Hubble Space Telescope Key Project on the extragalactic distance scale XXIV: the calibration of Tully-Fisher relations and the value of the Hubble constant*, Astrophys. Journ. **529** (2000) 698, `astro-ph/9909269`

[14] W. L. Freedman, *Determination of cosmological parameters*, Phys. Scripta **T85** (2000) 37-46, `astro-ph/9905222`

[15] C. L. Bennett et al., *First year Wilkinson Microwave Anisotropy Probe (WMAP) observations: preliminary maps and basic results*, Astrophys. J. Supp. **148** (2003) 1, `astro-ph/0302207`

[16] D. N. Spergel et al., *First year Wilkinson Microwave Anisotropy Probe (WMAP) observations: determination of cosmological parameters*, Astrophys. J. Supp. **148** (2003) 175, `astro-ph/0302209`

[17] A.V. Filippenko and A. G. Riess, *Results from the High-Z Supernova Search Team*, Phys. Rept. **307** (1998) 31-44, `astro-ph/9807008`
S. Perlmutter et al., *Cosmology from Type Ia Supernovae*, Bull. Am. Astron. Soc. **29** (1997) 1351, `astro-ph/9812473`

[18] A. D. Linde, *Particle Physics and Inflationary Cosmology*, Contemp. Concepts Phys. **5** (2005) 1, `hep-th/0503203`

[19] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231, `hep-th/9711200`

[20] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. B **428** (1998) 105, `hep-th/9802109`

[21] E. Witten, *Anti de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253, `hep-th/9802150`

[22] D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, *Correlation functions in the CFT(d)/AdS(d+1) correspondence*, Nucl. Phys. B **546** (1999) 96, `hep-th/9804058`
V. Balasubramanian, P. Kraus and A. Lawrence, *Bulk vs. boundary dynamics in Anti-de Sitter spacetime*, Phys. Rev. D **59** (1999) 046003, `hep-th/9805171`
V. Balasubramanian, P. Kraus, A. Lawrence and S. Trivedi, *Holographic probes of Anti-de Sitter spacetimes*, Phys. Rev. D **59** (1999) 104021, `hep-th/9808017`

I. R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, Nucl. Phys. B **556** (1999) 89, hep-th/9905104

[23] E. D'Hoker and D. Z. Freedman, *Supersymmetric gauge theories and the AdS/CFT correspondence*, Proceedings TASI 2001: Strings, Branes and EXTRA Dimensions, Boulder, Colorado, 3-29 Jun 2001, 3-158, hep-th/0201253
J. L. Petersen, *Introduction to the Maldacena conjecture on AdS/CFT*, Int. J. Mod. Phys. A **14** (1999) 3597, hep-th/9902131
P. Di Vecchia, *Large N gauge theories and AdS/CFT correspondence*, Proceedings ICTP Trieste Spring Workshop on Superstrings and Related Matters, Trieste, Italy, 22-30 Mar 1999, 1-70, hep-th/9908148

[24] J. de Boer, L. Maoz and A. Naqvi, *Some aspects of the AdS/CFT correspondence*, Proceedings of 73rd Meeting between Theoretical Physicists and Mathematicians: (A)ds-CFT Correspondence, Strasbourg, France, 11-13 Sep 2003, 33-57, hep-th/0407212

[25] G. 't Hooft, *Dimensional reduction in quantum gravity*, Salamfest 1993:0284-296, gr-qc/9310026
L. Susskind, *The world as a hologram*, J. Math. Phys. **36** (1995) 6377, hep-th/9409089

[26] M. B. Green, J. H. Schwarz and E. Witten, *Superstring theory*, Cambridge University Press, 1987
J. Polchinski, *String theory*, Cambridge University Press, 1998
B. Zwiebach, *A first course in string theory*, Cambridge University Press, 2004

[27] A. Sen, *An introduction to non-perturbative string theory*, Proceedings of a Newton Institute Euroconference On Duality And Supersymmetric Theories, Cambridge, England, 7-18 Apr 1997, 297-413, hep-th/9802051
C. V. Johnson, *D-brane primer*, Proceedings TASI 99: Strings, Branes, and Gravity, Boulder, Colorado, 31 May - 25 Jun 1999, 129-350, hep-th/0007170

[28] chapter 9 of R. Penrose and W. Rindler, *Spinors and space-time. Volume 2: Spinor and twistor methods in space-time geometry*, Cambridge University Press, 1986

[29] L. F. Abbott and S. Deser, *Stability of gravity with a cosmological constant*, Nucl. Phys. **B195** (1982) 76

[30] P. Breitenlohner and D. Z. Freedman, *Positive energy in anti-de Sitter backgrounds and gauge extended supergravity*, Phys. Lett. **115B** (1982) 197

[31] L. Mezincescu and P. K. Townsend, *Stability at a local maximum in higher dimensional anti de Sitter space and appications to supergravity*, Annals Phys. **160** (1985) 406

[32] E. Witten, *A new proof of the positive energy theorem*, Commun. Math. Phys. **80** (1981) 381
J. M. Nester, *A new gravitational energy expression with a simple positive proof*, Phys. Lett. **83A** (1981) 241

[33] G. W. Gibbons, C. M. Hull and N. Warner, *The stability of gauged supergravity*, Nucl. Phys. **B218** (1983) 173
L. Mezincescu, P. K. Townsend and P. van Nieuwenhuizen, *Stability of gauged d=7 supergravity and the definition of masslessness in* $(ads)_7$, Phys. Lett. **143B** (1984) 384

[34] W. Boucher, *Positive energy without supersymmetry*, Nucl. Phys. **B242** (1984) 282

[35] P. K. Townsend, *Positive energy and the scalar potential in higher dimensional (super)gravity theories*, Phys. Lett. **148B** (1984) 55

[36] B. de Wit and H. Nicolai, *The consistency of the* S^7 *truncation in* $d = 11$ *supergravity*, Nucl. Phys. **B281** (1987) 211
H. Nastase, D. Vaman and P. van Nieuwenhuizen, *Consistency of the* $AdS_7 \times S^4$ *reduction and the origin of self-duality in odd dimensions*, Nucl. Phys. **B581** (2000) 179, `hep-th/9911238`

[37] M. Cvetic, H. Lu, C. N. Pope, A. Sadrzadeh and T. A. Tran, *Consistent SO(6) reduction of type IIB supergravity on S(5)*, Nucl. Phys. **B586** (2000) 275, `hep-th/0003103`

[38] N. P. Warner, *Renormalization group flows from five-dimensional supergravity*, Class. Quant. Grav. **17** (2000) 1287, `hep-th/9911240`

[39] D. Anselmi, *The* $N = 4$ *quantum conformal algebra*, Nucl. Phys. **B541** (1999) 369, `hep-th/9809192`

[40] A. Strominger, *The dS/CFT correspondence*, JHEP **0110** (2001) 034, `hep-th/0106113`

[41] V. Balasubramanian, J. de Boer and D. Minic, *Exploring de Sitter space and holography*, Class. Quant. Grav. **19** (2002) 5655, `hep-th/0207245`

[42] T. Banks, *Cosmological breaking of supersymmetry or little Lambda goes back to the future II*, `hep-th/0007146`

[43] P. Pouliot, *Finite number of states, de Sitter space and quantum groups at roots of unity*, Class. Quant. Grav. **21** (2004) 145, `hep-th/0306261`
A. Guijosa and D. A. Lowe, *A new twist on dS/CFT*, Phys. Rev. D **69** (2004) 106008, `hep-th/0312282`
M. K. Parikh and E. P. Verlinde, *De Sitter space with finitely many states: A toy story*, Based in part on talk given at 10th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Gravitation and

Relativistic Field Theories (MG X MMIII), Rio de Janeiro, Brazil, 20-26 Jul 2003, hep-th/0403140
D. A. Lowe, *q-deformed de Sitter / conformal field theory correspondence*, Phys. Rev. D **70** (2004) 104002, hep-th/0407188
M. K. Parikh and E. P. Verlinde, *De Sitter holography with a finite number of states*, JHEP **0501** (2005) 054, hep-th/0410227

[44] C. M. Hull, *Timelike T-duality, de Sitter space, large N gauge theories and topological field theory*, JHEP **9807** (1998) 021, hep-th/9806146
M. Li, *Matrix model for de Sitter*, JHEP **0204** (2002) 005, hep-th/0106184
I. Sachs and S. N. Solodukhin, *Horizon holography*, Phys. Rev. D **64** (2001) 124023, hep-th/0107173
A. J. Tolley and N. Turok, *Quantization of the massless minimally coupled scalar field and the dS/CFT correspondence*, hep-th/0108119
S. Cacciatori and D. Klemm, *The asymptotic dynamics of de Sitter gravity in three dimensions*, Class. Quant. Grav. **19** (2002) 579, hep-th/0110031
P. de Medeiros, C. M. Hull, B. Spence and J. M. Figueroa-O'Farrill, *Conformal topological Yang-Mills theory and de Sitter holography*, JHEP **0208** (2002) 055, hep-th/0111190
M. Spradlin and A. Volovich, *Vacuum states and the S-matrix in dS/CFT*, Phys. Rev. D **65** (2002) 104037, hep-th/0112223
S. Ness and G. Siopsis, *dS/CFT correspondence in two dimensions*, Phys. Lett. B **536** (2002) 315, hep-th/0202096
L. Dyson, J. Lindesay and L. Susskind, *Is there really a de Sitter/CFT duality*, JHEP **0208** (2002) 045, hep-th/0202163
D. Kabat and G. Lifschytz, *De Sitter entropy from conformal field theory*, JHEP **0204** (2002) 019, hep-th/0203083
D. Klemm and L. Vanzo, *De Sitter gravity and Liouville theory*, JHEP **0204** (2002) 030, hep-th/0203268
L. Dyson, M. Kleban and L. Susskind, *Disturbing implications of a cosmological constant*, JHEP **0210** (2002) 011, hep-th/0208013
E. Abdalla, K. H. C. Castello-Branco and A. Lima-Santos, *Support of dS/CFT correspondence from space-time perturbations*, Phys. Rev. D **66** (2002) 104018, hep-th/0208065
M. K. Parikh, I. Savonije and E. P. Verlinde, *Elliptic de Sitter space: dS/Z(2)*, Phys. Rev. D **67** (2003) 064005, hep-th/0209120
J. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, JHEP **0305** (2003) 013, astro-ph/0210603
F. Leblond, D. Marolf and R. C. Myers, *Tall tales from de Sitter space. II: Field theory dualities*, JHEP **0301** (2003) 003, hep-th/0211025
N. Goheer, M. Kleban and L. Susskind, *The trouble with de Sitter space*, JHEP **0307** (2003) 056, hep-th/0212209
M. Alishahiha, A. Karch, E. Silverstein and D. Tong, *The dS/dS correspondence*,

AIP Conf. Proc. **743** (2005) 393, hep-th/0407125
T. Banks, *More thoughts on the quantum theory of stable de Sitter space*, hep-th/0503066
M. Alishahiha, A. Karch and E. Silverstein, *Hologravity*, JHEP **0506** (2005) 028, hep-th/0504056
A. Guijosa, D. A. Lowe and J. Murugan, *A prototype for dS/CFT*, Phys. Rev. D **72** (2005) 046001, hep-th/0505145
S. Ness and G. Siopsis, *Virasoro generators and the dS(3)/CFT(2) correspondence*, Class. Quant. Grav. **22** (2005) 3789, hep-th/0505146
T. Banks, L. Mannelli and W. Fischler, *Infrared divergences in dS/CFT*, hep-th/0507055

[45] V. Balasubramanian, *Accelerating universes and string theory*, Class. Quant. Grav. **21** (2004) S1337, hep-th/0404075
D. Klemm and L. Vanzo, *Aspects of quantum gravity in de Sitter spaces*, JCAP **0411** (2004) 006, hep-th/0407255

[46] V. Balasubramanian, J. de Boer and D. Minic, *Mass, entropy and holography in asymptotically de Sitter spaces*, Phys. Rev. D **65** (2002) 123508, hep-th/0110108
D. Kastor and J. Traschen, *A positive energy theorem for asymptotically de Sitter spacetimes*, Class. Quant. Grav. **19** (2002) 5901, hep-th/0206105

[47] R. Bousso, A. Maloney and A. Strominger, *Conformal vacua and entropy in de Sitter space*, Phys. Rev. **D65** (2002) 104039, hep-th/0112218

[48] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, *M theory as a matrix model: A conjecture*, Phys. Rev. D **55** (1997) 5112, hep-th/9610043
T. Banks, *TASI lectures on matrix theory*, Proceedings TASI 99: Strings, Branes, and Gravity, Boulder, Colorado, 31 May - 25 Jun 1999, 495-542, hep-th/9911068
W. Taylor, *M(atrix) theory: Matrix quantum mechanics as a fundamental theory*, Rev. Mod. Phys. **73** (2001) 419, hep-th/0101126

[49] L. Susskind, *Holography in the flat space limit*, Proceedings of 8th Canadian Conference on General Relativity and Relativistic Astrophysics, Montreal, Quebec, Canada, 10-12 Jun 1999, 98-112, hep-th/9901079

[50] J. Polchinski, *S-matrices from AdS spacetime*, hep-th/9901076

[51] G. Arcioni and C. Dappiaggi, *Exploring the holographic principle in asymptotically flat spacetimes via the BMS group*, Nucl. Phys. B **674** (2003) 553, hep-th/0306142
G. Arcioni and C. Dappiaggi, *Holography in asymptotically flat space-times and the BMS group*, Class. Quant. Grav. **21** (2004) 5655, hep-th/0312186
G. Arcioni and C. Dappiaggi, *Holography and BMS field theory*, AIP Conf. Proc. **751** (2005) 176, hep-th/0409313

[52] R. Bousso, *A Covariant Entropy Conjecture*, JHEP **9907** (1999) 004, hep-th/9905177
R. Bousso, *Holography in general space-times*, JHEP **9906** (1999) 028, hep-th/9906022
R. Bousso, *The holographic principle*, Rev. Mod. Phys. **74** (2002) 825, hep-th/0203101

[53] J. de Boer and S. N. Solodukhin, *A holographic reduction of Minkowski space*, Nucl. Phys. **B665** (2003) 545, hep-th/0303006
S. N. Solodukhin, *Reconstructing Minkowski space-time*, Proceedings of 73rd Meeting Between Theoretical Physicists And Mathematicians: (A)Ds-CFT Correspondence, Strasbourg, France, 11-13 Sep 2003, 123-163, hep-th/0405252

[54] R. D'Auria and S. Ferrara, *On Fermion masses, gradient flows and potential in supersymmetric theories*, JHEP **0105** (2001) 034, hep-th/0103153

[55] N. Alonso-Alberca and T. Ortin, *Supergravity vacua today*, Talk given at Spanish Relativity Meeting on Gravitation and Cosmology (ERE 2002), Mao, Menorca, Spain, 22-24 Sep 2002, gr-qc/0210039

[56] A. Van Proeyen, *Tools for supersymmetry*, Lectures given at Spring School on Quantum Field Theory: Supersymmetry and Superstrings, Calimanesti, Romania, 24-30 Apr 1998, hep-th/9910030

[57] S. Ferrara, P. Fre and L. Girardello, *Spontaneously Broken N=3 Supergravity*, Nucl. Phys. B **274** (1986) 600

[58] J. Lukierski and A. Nowicki, *All possible de Sitter superalgebras and the presence of ghosts*, Phys. Lett. **B151** (1985) 382
J. Lukierski and A. Nowicki, *Supersymmetry in the presence of positive cosmological constant*, preprint Wrocław, no. 609, March 1984
K. Pilch, P. van Nieuwenhuizen and M. F. Sohnius, *De Sitter superalgebras and supergravity*, Commun. Math. Phys. **98** (1985) 105
C. M. Hull, *Timelike T-duality, de Sitter space, large N gauge theories and topological field theory*, JHEP **9807** (1998) 021, hep-th/9806146
R. D'Auria, S. Ferrara, M. A. Lledó and V. S. Varadarajan, *Spinor algebras*, J. Geom. Phys. **40** (2001) 101–128, hep-th/0010124
S. Ferrara, *Spinors, superalgebras and the signature of space-time*, Proceedings of the 9th Marcel Grossmann meeting: On Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field Theories (volume 1), Rome, 2-8 July 2000, 68-83, hep-th/0101123
S. Ferrara and M. A. Lledó, *Considerations on super Poincaré algebras and their extensions to simple superalgebras*, Rev. Math. Phys. **14** (2002) 519, hep-th/0112177
R. D'Auria and S. Vaulà, *$D = 6$ $N = 2$ F(4)-supergravity with supersymmetric de Sitter background*, JHEP **0209** (2002) 057, hep-th/0203074

[59] B. de Wit and H. Nicolai, *N=8 Supergravity*, Nucl. Phys. B **208** (1982) 323

[60] F. Cordaro, P. Fre, L. Gualtieri, P. Termonia and M. Trigiante, *N = 8 gaugings revisited: An exhaustive classification*, Nucl. Phys. B **532** (1998) 245, hep-th/9804056

[61] C. M. Hull, *The Minimal Couplings And Scalar Potentials Of The Gauged N=8 Supergravities*, Class. Quant. Grav. **2** (1985) 343

[62] N. P. Warner, *Some Properties Of The Scalar Potential In Gauged Supergravity Theories*, Nucl. Phys. B **231** (1984) 250

[63] T. Fischbacher, *Mapping the vacuum structure of gauged maximal supergravities: An application of high-performance symbolic algebra*, Ph.D. Thesis, Potsdam, hep-th/0305176

[64] N. P. Warner, *Some New Extrema Of The Scalar Potential Of Gauged N=8 Supergravity*, Phys. Lett. B **128** (1983) 169

[65] C. M. Hull and N. P. Warner, *The Potentials Of The Gauged N=8 Supergravity Theories*, Nucl. Phys. B **253** (1985) 675

[66] C. M. Hull, *The Minimal Couplings And Scalar Potentials Of The Gauged N=8 Supergravities*, Class. Quant. Grav. **2** (1985) 343

[67] C. M. Hull, *Domain wall and de Sitter solutions of gauged supergravity*, JHEP **0111** (2001) 061, hep-th/0110048

[68] C. h. Ahn and K. s. Woo, *Domain wall and membrane flow from other gauged d = 4, n = 8 supergravity.Part I*, Nucl. Phys. B **634** (2002) 141, hep-th/0109010

[69] C. h. Ahn and K. s. Woo, *Domain wall from gauged d = 4, N = 8 supergravity. Part II*, JHEP **0311** (2003) 014, hep-th/0209128

[70] R. Kallosh, *Supergravity, M theory and cosmology*, Proceedings of Workshop on Conference on the Future of Theoretical Physics and Cosmology in Honor of Steven Hawking's 60th Birthday, Cambridge, England, 7-10 Jan 2002, 592-611, hep-th/0205315

[71] L. Andrianopoli, R. D'Auria, S. Ferrara and M. A. Lledo, *Gauging of flat groups in four dimensional supergravity*, JHEP **0207** (2002) 010, hep-th/0203206
C. M. Hull, *New gauged N = 8, D = 4 supergravities*, Class. Quant. Grav. **20** (2003) 5407, hep-th/0204156
B. de Wit, H. Samtleben and M. Trigiante, *On Lagrangians and gaugings of maximal supergravities*, Nucl. Phys. B **655** (2003) 93, hep-th/0212239

[72] B. de Wit, H. Samtleben and M. Trigiante, *Magnetic charges in local field theory*, JHEP **0509** (2005) 016, hep-th/0507289

[73] J. Schon and M. Weidner, *Gauged N = 4 supergravities*, hep-th/0602024

[74] P. Candelas, *Lectures on complex manifolds*, Proceedings Superstrings '87, Trieste, 1-88
T. Eguchi, P. B. Gilkey and A. J. Hanson, *Gravitation, Gauge Theories And Differential Geometry*, Phys. Rept. **66** (1980) 213
M. Nakahara, *Geometry, Topology and Physics*, IOP Publishing, 1990

[75] J. Bagger, *Supersymmetric Sigma Models*, Lectures given at Bonn-NATO Advanced Study Inst. on Supersymmetry, Bonn, West Germany, Aug 20-31, 1984, SLAC-PUB-3461

[76] B. Craps, F. Roose, W. Troost and A. Van Proeyen, *The definitions of special geometry*, Proceedings of second international Sakharov conference on physics, Moscow, Russia, 20-24 May 1996, 390-395, hep-th/9606073
B. Craps, F. Roose, W. Troost and A. Van Proeyen, *What is special Kaehler geometry?*, Nucl. Phys. B **503** (1997) 565, hep-th/9703082
B. Craps, F. Roose, W. Troost and A. Van Proeyen, *Special Kaehler geometry: Does there exist a prepotential?*, Proceedings NATO ASI on Strings, Branes And Dualities, Cargese, France, 26 May - 14 Jun 1997, 449-454, hep-th/9712092
P. Claus, K. Van Hoof and A. Van Proeyen, *A symplectic covariant formulation of special Kaehler geometry in superconformal calculus*, Class. Quant. Grav. **16** (1999) 2625, hep-th/9904066
A. Van Proeyen, *Special Kaehler geometry*, Contributed to 2nd Meeting on Quaternionic Structures in Mathematics and Physics, Rome, Italy, 6-10 Sep 1999, math.dg/0002122

[77] S. Ferrara, L. Girardello and M. Porrati, *Minimal Higgs Branch for the Breaking of Half of the Supersymmetries in N=2 Supergravity*, Phys. Lett. B **366** (1996) 155, hep-th/9510074
P. Fre, L. Girardello, I. Pesando and M. Trigiante, *Spontaneous $N = 2 \to N = 1$ local supersymmetry breaking with surviving compact gauge groups*, Nucl. Phys. B **493** (1997) 231, hep-th/9607032

[78] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fre and T. Magri, *$N = 2$ supergravity and $N = 2$ super Yang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map*, J. Geom. Phys. **23** (1997) 111, hep-th/9605032

[79] A. Van Proeyen, *$N = 2$ supergravity in $d = 4,5,6$ and its matter couplings*, Extended version of Lectures given during the semester "Supergravity, superstrings and M-theory" at Institut Henri Poincaré, Paris, November 2000, http://itf.fys.kuleuven.be/∼toine/LectParis.pdf

[80] E. Cremmer and A. Van Proeyen, *Classification Of Kahler Manifolds In N=2 Vector Multiplet Supergravity Couplings*, Class. Quant. Grav. **2** (1985) 445

[81] S. Ferrara and A. Van Proeyen, *A Theorem On N=2 Special Kahler Product Manifolds*, Class. Quant. Grav. **6** (1989) L243

[82] B. de Wit and A. Van Proeyen, *Broken sigma model isometries in very special geometry*, Phys. Lett. B **293** (1992) 94, `hep-th/9207091`

[83] E. Bergshoeff, S. Cucu, T. De Wit, J. Gheerardyn, R. Halbersma, S. Vandoren and A. Van Proeyen, *Superconformal N = 2, D = 5 matter with and without actions*, JHEP **0210** (2002) 045, `hep-th/0205230`

[84] D. V. Alekseevski and S. Marchiafava, *Quaternionic-like structures on a manifold: Note II. Autormorphism groups and their interrelations*, Rend. Mat. Acc. Lincei s. 9, v. 4:53-61 (1993)
E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren and A. Van Proeyen, *The map between conformal hypercomplex / hyper-Kaehler and quaternionic(-Kaehler) geometry*, Commun. Math. Phys. **262** (2006) 411, `hep-th/0411209`

[85] A. Van Proeyen, *Special geometries, from real to quaternionic*, Contributed to Workshop on Special Geometric Structures in String Theory, Bonn, Germany, 8-11 Sep 2001, `hep-th/0110263`

[86] J.A. Wolf, *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J. Math. Mech. **14** (1965), 1033-1047

[87] P. Fré, M. Trigiante and A. Van Proeyen, *Stable de Sitter vacua from $N = 2$ supergravity*, Class.Quant.Grav. **19** (2002) 4167, `hep-th/0205119`

[88] P. Fré, M. Trigiante and A. Van Proeyen, *$N = 2$ supergravity models with stable de Sitter vacua*, Class.Quant.Grav. **20** (2003) S487, `hep-th/0301024`

[89] S. J. J. Gates and B. Zwiebach, *Searching For All N=4 Supergravities With Global SO(4)*, Nucl. Phys. B **238** (1984) 99

[90] M. de Roo and P. Wagemans, *Gauge Matter Coupling In N=4 Supergravity*, Nucl. Phys. B **262** (1985) 644

[91] P. C. C. Wagemans, *Aspects Of N=4 Supergravity*, Ph.D. thesis, R.U. Groningen, 1990

[92] B. de Wit and A. Van Proeyen, *Potentials and symmetries of general gauged $N = 2$ supergravity-Yang Mills models*, Nucl. Phys. **B245** (1984) 89

[93] E. Cremmer, C. Kounnas, A. Van Proeyen, J. P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, *Vector Multiplets Coupled To N=2 Supergravity: Superhiggs Effect, Flat Potentials And Geometric Structure*, Nucl. Phys. B **250** (1985) 385

[94] M. de Roo, D. B. Westra and S. Panda, *De Sitter solutions in $N = 4$ matter coupled supergravity* JHEP **0302** (2003) 003, `hep-th/0212216`

[95] S. J. J. Gates and B. Zwiebach, *Gauged N=4 Supergravity Theory With A New Scalar Potential*, Phys. Lett. B **123** (1983) 200

[96] B. Zwiebach, *Gauged N=4 Supergravities And Gauge Coupling Constants*, Nucl. Phys. B **238** (1984) 367

[97] M. de Roo, D. B. Westra, S. Panda and M. Trigiante, *Potential and mass-matrix in gauged N = 4 supergravity*, JHEP **0311** (2003) 022, `hep-th/0310187`

[98] B. de Wit, H. Samtleben and M. Trigiante, *The maximal D = 5 supergravities*, Nucl. Phys. B **716** (2005) 215, `hep-th/0412173`

[99] M. Gunaydin, L. J. Romans and N. P. Warner, *Compact And Noncompact Gauged Supergravity Theories In Five-Dimensions*, Nucl. Phys. B **272** (1986) 598

[100] B. Cosemans and G. Smet, *Stable de Sitter vacua in N = 2, D = 5 supergravity*, Class. Quant. Grav. **22** (2005) 2359, `hep-th/0502202`

[101] M. Günaydin, G. Sierra and P. K. Townsend, *The geometry of $N = 2$ Maxwell-Einstein supergravity and Jordan algebras*, Nucl. Phys. **B242** (1984) 244

[102] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren and A. Van Proeyen, *$N = 2$ supergravity in five dimensions revisited*, Class.Quantum Grav. **21** (2004) 3015, `hep-th/0403045`

[103] M. Günaydin, G. Sierra and P. K. Townsend, *Gauging the $d = 5$ Maxwell–Einstein supergravity theories: more on Jordan algebras*, Nucl. Phys. **B253** (1985) 573

[104] B. de Wit and A. Van Proeyen, *Special geometry, cubic polynomials and homogeneous quaternionic spaces*, Commun. Math. Phys. **149** (1992) 307, `hep-th/9112027`

[105] M. Günaydin, G. Sierra and P. K. Townsend, *More on D = 5 Mawell-Einstein supergravity: symmetric spaces and kinks*, Class. Quantum Grav. **3** (1986) 763

[106] M. Günaydin and M. Zagermann, *The gauging of five-dimensional, $N = 2$ Maxwell-Einstein supergravity theories coupled to tensor multiplets*, Nucl. Phys. **B572** (2000) 131–150, `hep-th/9912027`

[107] A. Ceresole and G. Dall'Agata, *General matter coupled $\mathcal{N} = 2$, $D = 5$ gauged supergravity*, Nucl. Phys. **B585** (2000) 143–170, `hep-th/0004111`

[108] M. Günaydin, G. Sierra and P. K. Townsend, *Vanishing potentials in gauged N=2 supergravity: An application of Jordan algebras*, Phys. Lett. **B144** (1984) 41

[109] M. Günaydin and M. Zagermann, *The vacua of $5d$, $N = 2$ gauged Yang–Mills/Einstein/tensor supergravity: Abelian case*, Phys. Rev. **D62** (2000) 044028, `hep-th/0002228`

[110] M. Günaydin and M. Zagermann, *Gauging the full R-symmetry group in five-dimensional, N = 2 Yang–Mills/Einstein/tensor supergravity*, Phys. Rev. **D63** (2001) 064023, hep-th/0004117

[111] J. Ellis, M. Günaydin and M. Zagermann, *Options for gauge groups in five-dimensional supergravity*, JHEP **0111** (2001) 024, hep-th/0108094

[112] A. Ceresole, G. Dall'Agata, R. Kallosh and A. Van Proeyen, *Hypermultiplets, domain walls and supersymmetric attractors*, Phys. Rev. **D64** (2001) 104006, hep-th/0104056

[113] G. Dall'Agata, C. Herrmann and M. Zagermann, *General matter coupled N = 4 gauged supergravity in five dimensions*, Nucl. Phys. B **612** (2001) 123, hep-th/0103106

[114] M. Awada and P. K. Townsend, *N=4 Maxwell-Einstein Supergravity In Five-Dimensions And Its SU(2) Gauging*, Nucl. Phys. B **255** (1985) 617

[115] R. Gilmore, *Lie groups, Lie algebras and some of their applications*, John Wiley & Sons, 1974.

[116] A. Khavaev, K. Pilch and N. P. Warner, *New vacua of gauged N = 8 supergravity in five dimensions*, Phys. Lett. B **487** (2000) 14, hep-th/9812035

[117] Jan. Rosseel, G. Smet, *de Sitter vacua in 5D N=4 gauged supergravity*, in preparation.

[118] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, Phys. Lett. B **76** (1978) 409

[119] K. Bautier, S. Deser, M. Henneaux and D. Seminara, *No cosmological D = 11 supergravity*, Phys. Lett. B **406** (1997) 49, hep-th/9704131
S. Deser, *Uniqueness of D = 11 supergravity*, Proceedings of 6th Conference On Quantum Mechanics Of Fundamental Systems: Black Holes And The Structure Of The Universe, Santiago, Chile, 18-20 Aug 1997, 70-80, hep-th/9712064
S. Deser, *D = 11 supergravity revisited*, Relativity, Particle Physics, and Cosmology: Proceedings of the Richard Arnowitt Fest, College Station, Texas, 5-7 Apr 1998, 1-12, hep-th/9805205

[120] G. Dall'Agata, K. Lechner and D. P. Sorokin, *Covariant actions for the bosonic sector of D = 10 IIB supergravity*, Class. Quant. Grav. **14** (1997) L195, hep-th/9707044
G. Dall'Agata, K. Lechner and M. Tonin, *D = 10, N = IIB supergravity: Lorentz-invariant actions and duality*, JHEP **9807** (1998) 017, hep-th/9806140
G. Dall'Agata, K. Lechner and M. Tonin, *Action for IIB supergravity in 10 dimensions*, Proceedings of 2nd Conference On Quantum Aspects Of Gauge Theories, Supersymmetry And Unification, 21-26 Sep 1998, Corfu, Greece, 416-423, hep-th/9812170

[121] L. J. Romans, *Massive N=2a Supergravity In Ten-Dimensions*, Phys. Lett. B **169** (1986) 374.

[122] E. Bergshoeff, M. de Roo, M. B. Green, G. Papadopoulos and P. K. Townsend, *Duality of Type II 7-branes and 8-branes*, Nucl. Phys. B **470** (1996) 113, hep-th/9601150

[123] E. Bergshoeff, Y. Lozano and T. Ortin, *Massive branes*, Nucl. Phys. B **518** (1998) 363, hep-th/9712115
N. Alonso-Alberca and T. Ortin, *Gauged / massive supergravities in diverse dimensions*, Nucl. Phys. B **651** (2003) 263, hep-th/0210011

[124] P. S. Howe, N. D. Lambert and P. C. West, *A new massive type IIA supergravity from compactification*, Phys. Lett. B **416** (1998) 303, hep-th/9707139

[125] I. V. Lavrinenko, H. Lu and C. N. Pope, *Fibre bundles and generalised dimensional reductions*, Class. Quant. Grav. **15** (1998) 2239, hep-th/9710243

[126] D. Roest, *M-theory and gauged supergravities*, Fortsch. Phys. **53** (2005) 119, hep-th/0408175

[127] A. Chamblin and N. D. Lambert, *de Sitter space from M-theory?*, Phys. Lett. B **508** (2001) 369, hep-th/0102159

[128] A. Chamblin and N. D. Lambert, *Zero-branes, quantum mechanics and the cosmological constant*, Phys. Rev. D **65** (2002) 066002, hep-th/0107031

[129] D. Tsimpis, *Massive IIA supergravities*, JHEP **0510** (2005) 057, hep-th/0508214

[130] C. M. Hull and N. P. Warner, *Noncompact Gaugings From Higher Dimensions*, Class. Quant. Grav. **5** (1988) 1517

[131] C. M. Hull, *De Sitter space in supergravity and M theory*, JHEP **0111** (2001) 012, hep-th/0109213

[132] J. M. Maldacena and C. Nuñez, *Supergravity description of field theories on curved manifolds and a no go theorem*, Int. J. Mod. Phys. A **16** (2001) 822, hep-th/0007018

[133] G. W. Gibbons, *Aspects Of Supergravity Theories*, Print-85-0061 (CAMBRIDGE), Three lectures given at GIFT Seminar on Theoretical Physics, San Feliu de Guixols, Spain, Jun 4-11, 1984

[134] G. Smet and J. Van den Bergh, *O3/O7 orientifold truncations and very special quaternionic-Kaehler geometry*, Class. Quant. Grav. **22** (2005) 1, hep-th/0407233

[135] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, *Vacuum Configurations For Superstrings*, Nucl. Phys. B **258** (1985) 46

[136] M. Grana, *Flux compactifications in string theory: A comprehensive review*, Phys. Rept. **423** (2006) 91, hep-th/0509003

[137] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, *De Sitter vacua in string theory*, Phys. Rev. D **68** (2003) 046005, hep-th/0301240

[138] C. P. Burgess, R. Kallosh and F. Quevedo, *de Sitter string vacua from supersymmetric D-terms*, JHEP **0310** (2003) 056, hep-th/0309187

[139] J. J. Blanco-Pillado, R. Kallosh and A. Linde, *Supersymmetry and stability of flux vacua*, hep-th/0511042

[140] F. Denef, M. R. Douglas and B. Florea, *Building a better racetrack*, JHEP **0406** (2004) 034, hep-th/0404257
F. Denef, M. R. Douglas, B. Florea, A. Grassi and S. Kachru, *Fixing all moduli in a simple F-theory compactification*, hep-th/0503124
P. S. Aspinwall and R. Kallosh, *Fixing all moduli for M-theory on K3 x K3*, JHEP **0510** (2005) 001, hep-th/0506014

[141] C. Escoda, M. Gomez-Reino and F. Quevedo, *Saltatory de Sitter string vacua* JHEP **0311** (2003) 065, hep-th/0307160
R. Brustein and S. P. de Alwis, *Moduli potentials in string compactifications with fluxes: Mapping the discretuum*, Phys. Rev. D **69** (2004) 126006, hep-th/0402088
M. Becker, G. Curio and A. Krause, *De Sitter vacua from heterotic M-theory*, Nucl. Phys. B **693** (2004) 223 hep-th/0403027
J. P. Derendinger, C. Kounnas, P. M. Petropoulos and F. Zwirner, *Superpotentials in IIA compactifications with general fluxes*, Nucl. Phys. B **715** (2005) 211, hep-th/0411276
E. I. Buchbinder, *Raising anti de Sitter vacua to de Sitter vacua in heterotic M-theory*, Phys. Rev. D **70** (2004) 066008, hep-th/0406101
J. P. Derendinger, C. Kounnas, P. M. Petropoulos and F. Zwirner, *Fluxes and gaugings: $N = 1$ effective superpotentials*, Fortsch. Phys. **53** (2005) 926, hep-th/0503229
M. Davidse, F. Saueressig, U. Theis and S. Vandoren, *Membrane instantons and de Sitter vacua*, JHEP **0509** (2005) 065, hep-th/0506097
F. Saueressig, U. Theis and S. Vandoren, *On de Sitter vacua in type IIA orientifold compactifications*, Phys. Lett. B **633** (2006) 125, hep-th/0506181
J. P. Hsu, R. Kallosh and N. Sivanandam, *Features of de Sitter vacua in M-theory*, JHEP **0601** (2006) 123, hep-th/0510254

[142] E. Silverstein, *(A)dS backgrounds from asymmetric orientifolds*, Contributed to Strings 2001: International Conference, Mumbai, India, 5-10 Jan 2001, hep-th/0106209
A. Maloney, E. Silverstein and A. Strominger, *De Sitter space in noncritical string theory*, Proceedings of Workshop on Conference on the Future of Theoretical Physics and Cosmology in Honor of Steven Hawking's 60th Birthday, Cambridge, England,

7-10 Jan 2002, 570-591, hep-th/0205316
A. Saltman and E. Silverstein, *A new handle on de Sitter compactifications*, JHEP **0601** (2006) 139, hep-th/0411271

[143] S. Hosono, A. Klemm and S. Theisen, *Lectures on mirror symmetry*, Nucl. Phys. **B433** (1995) 501, hep-th/9403096

[144] S. Ferrara and S. Sabharwal, *Quaternionic manifolds for type II superstring vacua of Calabi-Yau spaces*, Nucl. Phys. **B332** (1990) 317

[145] R. Böhm, H. Günther, C. Herrmann, and J. Louis, *Compactification of type IIB string theory on Calabi-Yau threefolds*, Nucl. phys. **B569** (2000) 229, hep-th/9908007

[146] L. Andrianopoli, R. D'Auria and S. Ferrara, *Supersymmetry reduction of N-extended supergravities in four dimensions*, JHEP **0203** (2002) 025, hep-th/0110277
L. Andrianopoli, R. D'Auria and S. Ferrara, *Consistent reduction of $N = 2 \to N = 1$ four dimensional supergravity coupled to matter*, Nucl. Phys. **B628** (2002) 387, hep-th/0112192

[147] D. V. Alekseevsky, S. Marchiafava, *Hermitian and Kähler submanifolds of a Quaternionic Kähler manifold*, Osaka J. Math. 38 (2001) 869

[148] R. D'Auria, S. Ferrara and M. Trigiante, *c-map, very special quaternionic qeometry and dual Kähler spaces*, Phys. Lett. **B587** (2004) 138, hep-th/0401161

[149] R. D'Auria, S. Ferrara and M. Trigiante, *Homogeneous special manifolds, orientifolds and solvable coordinates*, Nucl. Phys. B **693** (2004) 261, hep-th/0403204

[150] C. N. Pope, *Kaluza-Klein theory*, http://faculty.physics.tamu.edu/pope

[151] M. B. Green, J. H. Schwarz and E. Witten, *Superstring theory*, Volume 2, Cambridge University Press, 1987
J. Polchinski, *String theory*, Volume 2, Cambridge University Press, 1998

[152] P. Candelas, *Lectures on complex manifolds*, Proceedings Superstrings '87, Trieste, 1-88
T. Hubsch, *Calabi-Yau manifolds: A Bestiary for physicists*, World Scientific Publishing, 1994

[153] P. Candelas and X. de la Ossa, *Moduli Space Of Calabi-Yau Manifolds*, Nucl. Phys. B **355** (1991) 455

[154] T. W. Grimm and J. Louis, *The effective action of N=1 Calabi-Yau orientifolds*, Nucl. Phys. B **699** (2004) 387, hep-th/0403067

[155] T. W. Grimm, *The effective action of type II Calabi-Yau orientifolds*, Fortsch. Phys. **53** (2005) 1179, hep-th/0507153

[156] B. Acharya, M. Aganagic, K. Hori and C. Vafa, *Orientifolds, mirror symmetry and superpotentials*, hep-th/0202208
I. Brunner and K. Hori, *Orientifolds and mirror symmetry*, JHEP **0411** (2004) 005, hep-th/0303135
I. Brunner, K. Hori, K. Hosomichi and J. Walcher, *Orientifolds of Gepner models*, hep-th/0401137

[157] M. Grana, T. W. Grimm, H. Jockers and J. Louis, *Soft supersymmetry breaking in Calabi-Yau orientifolds with D-branes and fluxes*, Nucl. Phys. **B690** (2004) 21, hep-th/0312232

[158] E. G. Gimon and J. Polchinski, *Consistency Conditions for Orientifolds and D-Manifolds*, Phys. Rev. D **54** (1996) 1667, hep-th/9601038
A. Sen, *Orientifold limit of F-theory vacua*, Nucl. Phys. Proc. Suppl. **68** (1998) 92 [Nucl. Phys. Proc. Suppl. **67** (1998) 81], hep-th/9709159
A. Dabholkar,*Lectures on orientifolds and duality*, Proceedings of ICTP Summer School in High-Energy Physics and Cosmology, Trieste, Italy, 2-11 Jul 1997, 128-191, hep-th/9804208

[159] B. de Wit, F. Vanderseypen, A. Van Proeyen, *Symmetry structure of special geometries*, Nucl. Phys. **B400** (1993) 463, hep-th/9210068

[160] E. Cremmer, S. Ferrara, C. Kounnas and D. V. Nanopoulos, *Naturally vanishing cosmological constant in N=1 supergravity*, Phys. Lett. **B133** (1983) 61
R. Barbieri, E. Cremmer and S. Ferrara, *Flat and positive potentials in N=1 supergravity*, Phys. Lett. **B163** (1985) 143
S. Cecotti, *Homogeneous Kahler Manifolds And Flat Potential N=1 Supergravities*, Phys. Lett. B **215** (1988) 489

[161] B. de Wit and A. Van Proeyen, *Symmetries of dual quaternionic manifolds*, Phys. Lett. **B252** (1990) 221

[162] B. de Wit and A. Van Proeyen, *Isometries of special manifolds*, Presented at Meeting on Quaternionic Structures, Trieste, Italy, Sep 1994, hep-th/9505097

[163] L. Andrianopoli, S. Ferrara and M. A. Lledo, *No-scale D = 5 supergravity from Scherk-Schwarz reduction of D = 6 theories*, JHEP **0406** (2004) 018, hep-th/0406018

[164] B. Cosemans, Productruimten in 5-dimensionale supergravitatie-Maxwell theorieën, Licentiaatsthesis natuurkunde, 2001-2002

[165] D.V. Alekseevskiĭ: *Compact quaternion spaces*, Funct. Anal. Applic. **2** (1968) 106-114

[166] C. LeBrun and S. Salamon, *Strong rigidity of positive quaternion-Kähler manifolds*, Invent. Math. **118**, no. 1 (1994) 109-132

[167] K. Galicki, *Course notes on quaternionic manifolds*, http://www.math.unm.edu/∼galicki/courses/pdf/Notes.pdf

[168] D.V. Alekseevskiĭ: *Classification of quaternionic spaces with a transitive solvable group of motions*, Math. USSR Izvestija **9**, No. 2 (1975) 297-339

[169] V. Cortés, *Alekseevskian spaces*, Diff. Geom. Appl. **6** (1996) 129-168