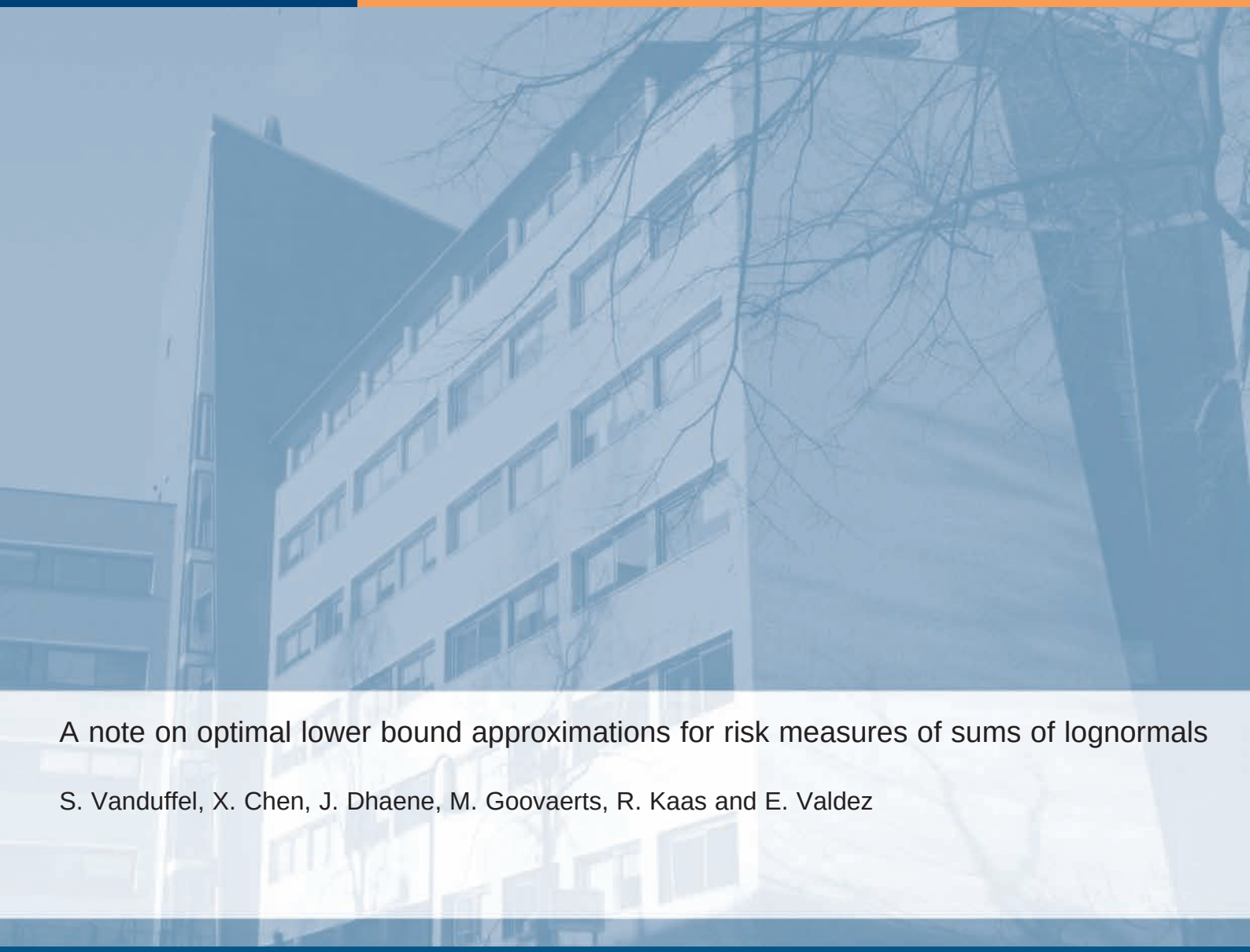


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A Note on Optimal Lower Bound Approximations for Risk Measures of Sums of Lognormals

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Abstract

This paper considers different approximations for computing either the distribution function or various risk measures associated with a discrete sum of non-independent lognormal random variables. In Dhaene et al. (2002a,b), convex upper and lower bounds for sums of non-independent lognormals were proposed. Of particular interest is the lower bound obtained by taking the conditional expectation of the original sum S with respect to a conditioning random variable Λ. This has proven to provide excellent approximations for various risk measures associated with S. The choice of this conditioning variable Λ plays a crucial role in arriving at a good approximation and intuitively, it could be chosen so that, in some sense, it is "close" to the original sum S. In the literature then, various choices for Λ have been proposed. For example, Dhaene et al (2002b) propose to determine Λ such that it can be viewed as a first-order approximation to the original sum S. In another development, Vanduffel et al (2005) argue that the best Λ is obtained when the variance of the resulting lower bound is as close as possible to the variance of the original S. In this paper, we notice that these choices for Λ are global in the sense that they lead to lower bounds for which the distribution function presents a global goodness-of-fit when compared with the original distribution function. In many financial and actuarial problems, however, we are usually only interested in the tails of the distribution function. Hence, in this paper, we derive lower bounds approximations considered to be 'locally' optimal.

Keywords: comonotonicity, lognormal, maximal variance.

1 Introduction

In Finance and Actuarial Science, one is often confronted with a discounted or compounded sum S which can be represented as follows:

$$S = \sum_{i=1}^{n} \alpha_i \; e^{Z_i}. \tag{1}$$

Here, the α_i's are assumed to be non-negative real numbers and $(Z_1, Z_2, ..., Z_n)$ is a random vector.

The accumulated value at time n of a series of future deterministic savings amounts α_i can be written in the form (1), where Z_i denotes the random accumulation factor over the period $[i, n]$. Also the present value of a series of future deterministic payments of α_i's can be written in the form of (1), where now the Z_i denote the random discount factor over the period $[0, i]$.

In an insurance context, the interest in such a sum S becomes apparent when setting provisions and computing required capitals: these problems ultimately boil down to the evaluation of a risk measure related to (1). As usual, a risk measure is defined as a mapping from the set of random variables to the set of real numbers $\mathbb{R}$.

In this paper, we will focus on some risk measures that are often used in practice. Firstly, we recall the p-quantile risk measure, often called the 'VaR' (Value-at-Risk) at level p in the financial and actuarial literature. For any p in $(0, 1)$, the p-quantile risk measure for a random variable X, denoted by $Q_p[X]$, is defined as

$$Q_p\left[X\right] = \inf\left\{x \in \mathbb{R} \mid F_X(x) \geq p\right\}, \qquad p \in (0, 1)\,, \tag{2}$$

where $F_X(x) = \Pr\left[X \leq x\right]$. Note that expression (2) can also be used to define $Q_0\left[X\right]$ and $Q_1\left[X\right]$. We then find that $Q_0(X) = -\infty$ and for the latter quantile we tacitly assume that $\inf \emptyset = +\infty$. Notice that the quantile function $Q_p\left[X\right]$ is a non-decreasing and left-continuous function of p. Finally, we also find for all $x \in \mathbb{R}$ and $p \in [0, 1]$ that

$$Q_p\left[X\right] \leq x \Leftrightarrow p \leq F_X(x). \tag{3}$$

Another very popular risk measure, which we are going to consider in this paper, is the Conditional Tail Expectation at level p, which we will denote by $\mathrm{CTE}_p\left[X\right]$. It is defined as

$$\mathrm{CTE}_p\left[X\right] = E\left[X \mid X > Q_p\left[X\right]\right], \qquad p \in (0, 1)\,. \tag{4}$$

Loosely speaking, it can also be interpreted as the VaR at level p augmented by the average exceeding of the claims X over that quantile, given that such exceeding occurs. Conditional Tail Expectations have been considered in Panjer (2002) and Landsman & Valdez (2003).

We also define the stop-loss premium with retention d of the random variable X to be $E[(X - d)_+]$, with the notation $(x - d)_+ = \max(x - d, 0)$. By using partial integration, we obtain

$$E[(X - d)_+] = \int_d^{\infty} \left(1 - F_X(x)\right) dx, \qquad -\infty < d < +\infty, \tag{5}$$

from which we see that the stop-loss premium with retention d can be considered as the weight of an upper tail of the c.d.f. (cumulative distribution function) of X.

We will also introduce the Conditional Left Tail Expectation, denoted by $\mathrm{CLTE}_p[X]$ and defined as

$$\mathrm{CLTE}_p[X] = \mathrm{E}\left[X \mid X < Q_p^+[X]\right], \qquad p \in (0,1). \tag{6}$$

One can prove that the following relations hold between $Q_p[X]$,$\mathrm{CTE}_p[X]$ and $\mathrm{CLTE}_p[X]$; see Dhaene et al (2004) for more details. For $p \in (0,1)$, we have that

$$\mathrm{CTE}_p[X] = Q_p[X] + \frac{1}{1 - F_X(Q_p[X])}\mathrm{E}\left[(X - Q_p[X])_+\right] \tag{7}$$

$$\mathrm{CLTE}_p[X] = -\mathrm{CTE}_{1-p}[-X] \tag{8}$$

As stated above, many financial problems require the evaluation of risk measures of S but it is not easy to obtain analytical expressions for these risk measures even when $(Z_1, Z_2, ..., Z_n)$ is a multivariate normal distributed random vector. Recently, so-called comonotonic approximations for the distribution function and the various risk measures related to the random variable (r.v.) S have been proposed. For an extensive overview on the theory of comonotonicity and its applications, we refer to the works of Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002a,b).

Especially, the comonotonic lower bound which is obtained by considering the conditional expectation of the original sum S with respect to a conditioning random variable Λ has proven to provide accurate and easy to compute approximations for several risk measures of S;.See for instance Huang, Milevsky & Wang (2004), Vanduffel, Dhaene & Goovaerts (2005) or Vanduffel, Hoedemakers & Dhaene (2005) for various illustrations. Intuitively, Λ should be chosen such that it is 'close' to the original random variable S. In the literature, various choices for Λ have been proposed. Dhaene et al (2000b) propose to determine Λ such that it can be seen as a first-order approximation for the original sum S. Vanduffel et al (2004) propose to select Λ such that the first-order approximation for the variance of the lower bound is maximized. From these papers, we start by noticing that these choices for Λ are 'global' in the sense that they lead to a lower bound for which the distribution function presents a global goodness-of-fit when compared to the original distribution function. However, there are many financial and actuarial problems where we may be only interested in the tails of the distribution function. Hence, in this paper, we derive lower bound approximations which can be considered to be 'locally' optimal in some sense.

The rest of this paper has been organized as follows. In Section 2, as a background overview, we present some general results of the comonotonic approximations. In section 3, we discuss 'global' optimal choices for the conditioning r.v. Λ. In particular, we describe the 'Taylor-based' and the 'maximal variance' lower bound approximations. In Section 4, we focus on choices for Λ that are 'locally' optimal, and we call these the 'maximal CTE' and the 'minimal CLTE' lower bound approximations. In Section 5, we compare these various lower bound approximations with numerical illustrations. Finally, we conclude in Section 6.

2 Comonotonic approximations: some general results

Let the random variable S be given by (1), where the α_i's are non-negative real numbers and the random vector $(Z_1, Z_2, ..., Z_n)$ has a multivariate Normal distrib-

ution. Consider the conditioning random variable Λ which is defined to be linear combinations of $Z_1, Z_2, ..., Z_n$ and is given by

$$\Lambda = \sum_{i=1}^{n} \gamma_i Z_i. \tag{9}$$

Moreover, consider the random variables S^l and S^c defined by

$$S^l \overset{\text{def}}{=} \mathrm{E}[S|\Lambda] \overset{d}{=} \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i] + \frac{1}{2}\left(1-r_i^2\right)\sigma_{Z_i}^2 + r_i \sigma_{Z_i} \Phi^{-1}(U)} \tag{10}$$

and

$$S^c = \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i] + \sigma_{Z_i} \Phi^{-1}(U)}, \tag{11}$$

respectively. Here U is a Uniform$(0,1)$ r.v. , Φ is the c.d.f. of the $N(0,1)$ distribution and r_i is the correlation between Z_i and Λ.

As demonstrated in Kaas, Dhaene & Goovaerts (2000), it follows that for the r.v.'s S, S^l and S^c, the following convex order relations hold:

$$S^l \leq_{cx} S \leq_{cx} S^c.$$

We have the following expressions for the risk measures of S^c for any $p \in (0,1)$:

$$Q_p\left[S^c\right] = \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i] + \sigma_{Z_i} \Phi^{-1}(p)}, \tag{12}$$

$$\mathrm{CTE}_p\left[S^c\right] = \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i] + \frac{1}{2}\sigma_{Z_i}^2} \frac{\Phi\left(\sigma_{Z_i} - \Phi^{-1}(p)\right)}{1-p}, \tag{13}$$

$$\mathrm{CLTE}_p\left[S^c\right] = \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i] + \frac{1}{2}\sigma_{Z_i}^2} \frac{1 - \Phi\left(\sigma_{Z_i} - \Phi^{-1}(p)\right)}{p}, \tag{14}$$

$$\mathrm{E}\left[S^c - d\right]_+ = \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i] + \frac{1}{2}\sigma_{Z_i}^2} \Phi\left(\sigma_{Z_i} - \Phi^{-1}(p)\right) - d(1-p). \tag{15}$$

Now provided all the correlation coefficients r_i's are non-negative, we also find, still in the lognormal case, that for $p \in (0,1)$:

$$Q_p\left[S^l\right] = \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i] + \frac{1}{2}\left(1-r_i^2\right)\sigma_{Z_i}^2 + r_i \sigma_{Z_i} \Phi^{-1}(p)}, \tag{16}$$

$$\mathrm{CTE}_p\left[S^l\right] = \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i] + \frac{1}{2}\sigma_{Z_i}^2} \frac{\Phi\left(r_i \; \sigma_{Z_i} - \Phi^{-1}(p)\right)}{1-p}, \tag{17}$$

$$\mathrm{CLTE}_p\left[S^l\right] = \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i] + \frac{1}{2}\sigma_{Z_i}^2} \frac{1 - \Phi\left(r_i \; \sigma_{Z_i} - \Phi^{-1}(p)\right)}{p}, \tag{18}$$

$$\mathrm{E}\left[S^l - d\right]_+ = \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i] + \frac{1}{2}\sigma_{Z_i}^2} \Phi\left(r_i \sigma_{Z_i} - \Phi^{-1}(p)\right) - d(1-p). \tag{19}$$

Notice that the expected values of the random variables S, S^l and S^c are all equal to

$$\mathrm{E}[S] = \mathrm{E}[S^l] = \mathrm{E}[S^c] = \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma^2_{Z_i}}, \tag{20}$$

whereas their variances are given by

$$Var[S] = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \; e^{\mathrm{E}[Z_i]+\mathrm{E}[Z_j]+\frac{1}{2}(\sigma^2_{Z_i}+\sigma^2_{Z_j})}(e^{Cov(Z_i,Z_j)} - 1), \tag{21}$$

$$Var[S^l] = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \; e^{\mathrm{E}[Z_i]+\mathrm{E}[Z_j]+\frac{1}{2}(\sigma^2_{Z_i}+\sigma^2_{Z_j})}(e^{r_i r_j \sigma_{Z_i}\sigma_{Z_j}} - 1), \tag{22}$$

and

$$Var[S^c] = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \; e^{\mathrm{E}[Z_i]+\mathrm{E}[Z_j]+\frac{1}{2}(\sigma^2_{Z_i}+\sigma^2_{Z_j})}(e^{\sigma_{Z_i}\sigma_{Z_j}} - 1), \tag{23}$$

respectively.

Finally, again for $p \in (0,1)$, we also have the following ordering relations:

$$\mathrm{CTE}_p\left[S^l\right] \;\leq\; \mathrm{CTE}_p\left[S\right] \leq \mathrm{CTE}_p\left[S^c\right], \tag{24}$$

$$\mathrm{E}\left[S^l - d\right]_+ \;\leq\; \mathrm{E}\left[S - d\right]_+ \leq \mathrm{E}\left[S^c - d\right]_+, \tag{25}$$

and

$$\mathrm{CLTE}_p\left[S^c\right] \leq \mathrm{CLTE}_p\left[S\right] \leq \mathrm{CLTE}_p\left[S^l\right]. \tag{26}$$

Note that such an ordering does not hold in general for $Q_p[S]$ and its approximations $Q_p\left[S^l\right]$ and $Q_p[S^c]$. In Vanduffel, Dhaene, Goovaerts & Kaas (2003), it is proven that the inequality $Q_p\left[S^l\right] \leq Q_p[S^c]$ holds, provided p is sufficiently large and at the same time, the correlations r_i's are all non-negative. Indeed, in this case, it is straightforward to show that for all $p \;\epsilon\; (0,1)$, the condition

$$p \geq \Phi\left[\frac{1}{2}\max\left\{(1+r_i)\,\sigma_{Z_i}\;; i = 1,\ldots,n\right\}\right] \equiv p^+ \tag{27}$$

implies that $Q_p\left[S^l\right] \leq Q_p[S^c]$. Similarly, one can prove that the condition

$$p \leq \Phi\left[\frac{1}{2}\min\left\{(1+r_i)\,\sigma_{Z_i}\;; i = 1,\ldots,n\right\}\right] \equiv p^- \tag{28}$$

implies that $Q_p[S^c] \leq Q_p\left[S^l\right]$. Note that $p^- \geq 0.5$.

Finally, we conclude that the cumulative distribution functions of S^l and S^c can only cross, which will occur at least once, in the region where their cumulative distribution functions take a value that is contained in the interval $[p^-,\; p^+]$. In the region where $p \leq p^-$, the cumulative distribution function of S^c will lay above the cumulative distribution function of S^l, while in the region $p \geq p^+$, the cumulative distribution function of S^c will lay below the cumulative distribution function of S^l.

3 Uniform choices for Λ

3.1 The 'Taylor-based' lower bound approach

Comparing variances is meaningful when comparing stop-loss premiums of convex ordered random variables, see e.g. Kaas, van Heerwaarden & Goovaerts (1994, p. 68). The following relation provides a link between the variances and stop-loss premiums:

$$\frac{1}{2}Var[X] = \int_{-\infty}^{\infty} (E[(X-t)_+] - (E[X]-t)_+)\, dt. \tag{29}$$

We deduce from this that if $X \leq_{cx} Y$,

$$\int_{-\infty}^{\infty} (E[(Y-t)_+] - (E[(X-t)_+]\, dt = \frac{1}{2}\{Var[Y] - Var[X]\} \tag{30}$$

Thus, if $X \leq_{cx} Y$, their stop-loss distance, i.e. the integrated absolute difference of their respective stop-loss premiums, equals half the variance difference between these two random variables. Hence, $\frac{1}{2}\{Var[Y] - Var[X]\}$ can be fairly interpreted as a measure for the total error that one makes when approximating the stop-loss premiums of Y by those of the convex smaller X. Since the integrand in the left hand side of (30) is non-negative, we also find that if $X \leq_{cx} Y$ whilst $Var[X] = Var[Y]$, then this means that X and Y must have equal stop-loss premiums and hence the same d.f. All this suggests that if we want to replace S by the convex smaller S^l, the best approximations will occur when Λ is such that $Var[S^l]$ is as large as possible or equivalently, since $Var[S] = Var[S^l] + \mathrm{E}[Var[S|\Lambda]]$, when $\mathrm{E}[Var[S|\Lambda]]$ is as small as possible. We notice that either of this criterion mean that Λ and S should be 'as alike as' possible. These ideas led Dhaene, Denuit, Kaas, Goovaerts & Vyncke (2002b) to propose the following choice for the parameters γ_i when computing the lower bound S^l:

$$\gamma_i = \alpha_i e^{\mathrm{E}[Z_i]}, \quad \text{for } i = 1, \cdots, n. \tag{31}$$

This choice makes indeed Λ a linear transformation of a first-order approximation to the sum S. This can be easily deduced from the following computation:

$$\begin{aligned} S &= \sum_{j=1}^{n} \alpha_j e^{\mathrm{E}[Z_j] + (Z_j - \mathrm{E}[Z_j])} \approx \sum_{j=1}^{n} \alpha_j e^{\mathrm{E}[Z_j]} [1 + Z_j - \mathrm{E}\,[Z_j]] \\ &\approx C + \sum_{j=1}^{n} \alpha_j e^{\mathrm{E}[Z_j]} Z_j \end{aligned} \tag{32}$$

where C is the appropriate constant. Intuitively then, one expects that for this choice of Λ, the expectation $\mathrm{E}[Var[S|\Lambda]]$ will be quite 'small' which goes hand in hand with a 'large' $Var[S^l]$. We will call this approximation the Taylor-based lower bound approach for approximating S.

3.2 The 'Maximal Variance' lower bound approach

The best approximations will be the ones where the variance of S^l is 'as large as possible' The Taylor-based approach assumes a rather intuitive approach to derive a

Λ that gives rise to a 'large' value for $Var[S^l]$. Vanduffel, Hoedemakers & Dhaene (2005) walk a slightly different path to derive Λ.They proved that the first order approximation of the variance of S^l will be maximized for the following choice of the parameters γ_i:

$$\gamma_i = \alpha_i e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma^2_{Z_i}}, \quad i = 1, \ldots, n. \tag{33}$$

Indeed, from (23), we find that

$$\begin{aligned} Var(S^l) &\approx \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \; e^{\mathrm{E}[Z_i]+\mathrm{E}[Z_j]+\frac{1}{2}(\sigma^2_{Z_i}+\sigma^2_{Z_j})}(r_i r_j \sigma_{Z_i}\sigma_{Z_j}) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \; e^{\mathrm{E}[Z_i]+\mathrm{E}[Z_j]+\frac{1}{2}(\sigma^2_{Z_i}+\sigma^2_{Z_j})} \left(\frac{Cov[Z_i,\Lambda]Cov[Z_j,\Lambda]}{Var(\Lambda)}\right) \\ &= \frac{(Cov(\sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} Z_i, \Lambda))^2}{Var(\Lambda)} \\ &= (Corr(\sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} Z_i, \Lambda))^2 \; Var(\sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} Z_i). \end{aligned} \tag{34}$$

Hence, the first order approximation of $Var(S^l)$ is maximized when Λ is given by

$$\Lambda = \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} Z_i. \tag{35}$$

Note that also in case the α_i's are not all positive, the choice (33) will optimize the first order approximation of the variance.

One can easily prove that the first-order approximation for $Var(S^l)$ with Λ given by (35) is equal to the first-order approximation of $Var(S)$. This observation gives an additional indication that this particular choice for Λ will provide a good fit However, it has to be noted that maximizing the variance of S^l does not mean that for every value $t > 0$, $E[(S^l - t)_+]$ will be close to $E[(S-t)_+]$ but numerical illustrations show, as indicated by formula (30) that this choice outperforms 'on average' the Taylor-based lower bound approach. We also emphasize that the conditioning r.v. Λ defined in (35) does not necessarily maximize the variance of S^l, but has to be understood as an approximation for the optimal Λ. Theoretically, one could use numerical procedures to determine the optimal Λ, but this would outweigh one of the main features of the convex bounds, namely that the quantiles, conditional tail expectations (and also other actuarial quantities of interest such as the stop-loss premiums) can easily be determined analytically. Having a readily available approximation that can be implemented easily is important from a practical point of view.

4 Specific choices for Λ

4.1 The 'Maximal CTE' lower bound approach

As mentioned in the introduction, in several applications, one usually focuses only on the tails of the distribution function. The provision or reserve to be established at time 0 for these future unit payment obligations can be determined as either $Q_p[S]$

or $CTE_p[S]$, with p sufficiently large. A provision, for example, equal to $Q_{0.95}[S]$, guarantees that the payments α_i's can be made with a probability of 0.95; see for instance Dhaene, Vanduffel, Goovaerts, Kaas & Vyncke (2004). Moreover, in the case of Asian options, one only needs to consider upper or lower tails of the distribution. Thus, it makes sense to consider choices for the conditioning r.v. Λ that are only optimal in the upper or lower tail of the distribution. In order to get an optimal Λ for the upper tail, note that from (24) it follows for all $p > 0$ it holds that $CTE_p[S^l] \leq CTE_p[S]$ and this suggests that taking Λ such that $CTE_p[S^l]$ is as 'large' as possible is a feasible choice for that case.

We now show that the following choice of the parameters γ_i maximizes a first-order approximation for the $\mathrm{CTE}_p[S]$:

$$\gamma_i = \alpha_i e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} \times e^{-\frac{1}{2}(r_i^{MV}\sigma_{Z_i}-\Phi^{-1}(p))^2}, \quad i = 1, \ldots, n, \tag{36}$$

where r_i^{MV} denotes the correlation between between Z_i and the 'Maximal Variance' $\Lambda = \sum_{i=1}^n \alpha_i \; e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} Z_i$.

Indeed, from (17), we find that as a first-order approximation, we have

$$\begin{aligned}\mathrm{CTE}_p\left[S^l\right] \approx & \sum_{i=1}^n \alpha_i \; e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} \frac{\Phi\left(r_i^{MV}\sigma_{Z_i} - \Phi^{-1}(p)\right)}{1-p} + \\ & \sum_{i=1}^n \alpha_i \; e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} \frac{\Phi'\left(r_i^{MV}\sigma_{Z_i} - \Phi^{-1}(p)\right)}{1-p}(r_i\sigma_{Z_i} - r_i^{MV}\sigma_{Z_i})\end{aligned} \tag{37}$$

Hence, the first-order approximation of $\mathrm{CTE}_p\left[S^l\right]$ is maximized when

$$\sum_{i=1}^n \alpha_i \; e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} \frac{\Phi'\left(r_i^{MV}\sigma_{Z_i} - \Phi^{-1}(p)\right)}{1-p}(r_i\sigma_{Z_i}) \tag{38}$$

is maximized. Now since

$$\begin{aligned}&\sum_{i=1}^n \alpha_i \; e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} \frac{\Phi'\left(r_i^{MV}\sigma_{Z_i} - \Phi^{-1}(p)\right)}{1-p} r_i\sigma_{Z_i} \\ &= \frac{Cov(\sum_{i=1}^n \alpha_i \; e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} \frac{\Phi'\left(r_i^{MV}\sigma_{Z_i} - \Phi^{-1}(p)\right)}{1-p} Z_i, \Lambda)}{\sigma(\Lambda)} \\ &= Corr(\sum_{i=1}^n \alpha_i \; e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} \frac{\Phi'\left(r_i^{MV}\sigma_{Z_i} - \Phi^{-1}(p)\right)}{1-p} Z_i, \Lambda) \\ &\times \sigma(\sum_{i=1}^n \alpha_i \; e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} \frac{\Phi'\left(r_i^{MV}\sigma_{Z_i} - \Phi^{-1}(p)\right)}{1-p} Z_i),\end{aligned} \tag{39}$$

it follows that the first-order approximation of $\mathrm{CTE}_p\left[S^l\right]$ is maximized when Λ is given by

$$\Lambda = \sum_{i=1}^n \alpha_i e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} \times e^{-\frac{1}{2}(r_i^{MV}\sigma_{Z_i}-\Phi^{-1}(p))^2} Z_i. \tag{40}$$

Note that in (36) one uses the results of the 'Maximal Variance' Λ to derive the 'Maximal CTE' Λ. One could now iteratively repeat the whole procedure by substituting in formula (36) r_i^{MV} by r_i^{CTE} where r_i^{CTE} denotes the correlation between between Z_i and the 'Maximal CTE' Λ..Intuitively one expects that such an iteration will further improve the quality of the 'Maximal CTE' Λ.

4.2 The 'Minimal CLTE' lower bound approach

There are also other applications where one focuses only on the lower tails of the distribution function. A decision-maker with a given target capital could for instance, be interested in the expected final wealth at time n, given that the target capital is not reached. From (26), in a similar reasoning as in the previous section, it follows that Λ should be chosen such that $\mathrm{CLTE}_p[S]$ is therefore minimized. Since $\mathrm{E}[S] = p\mathrm{CLTE}_p[S] + (1-p)\mathrm{CTE}_p[S]$, it follows that minimizing $\mathrm{CLTE}_p[S]$ provides the same solution for Λ as maximizing $\mathrm{CTE}_p[S]$. Therefore, the following choice of the parameters γ_i minimizes a first order approximation for the $\mathrm{CLTE}_p[S]$.

$$\gamma_i = \alpha_i e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}} \times e^{-\frac{1}{2}(r_i^{MV}\sigma_{Z_i}-\Phi^{-1}(p))^2}, \quad i = 1, \ldots, n. \tag{41}$$

Hence, the 'Minimal CLTE' lower bound approach coincides with the 'Maximal CTE' lower bound approach which corresponds to intuition. The same remarks as in case of the 'Maximal CTE' Λ can also be made here.

5 Comparing the approximations

Let us consider the random variables S_n and W_n which are defined, respectively, as the random present and compounded values of a series of n deterministic unit payment obligations due at times $1, 2, ..., n$:

$$S_n = \sum_{i=1}^{n} e^{-Y_1-Y_2-\cdots-Y_i} \stackrel{\mathrm{def}}{=} \sum_{i=1}^{n} e^{-Z_i} \tag{42}$$

and

$$W_n = \sum_{i=1}^{n} e^{Y_1+Y_2+\cdots+Y_i} \stackrel{\mathrm{def}}{=} \sum_{i=1}^{n} e^{Z_i}, \tag{43}$$

where the r.v. Y_i denotes the random return over the period $[i-1, i]$ and $e^{-(Y_1+Y_2+\cdots+Y_i)} = e^{-Z_i}$ is the random discount factor over the period $[0, i]$. We will assume that the periodic returns Y_i's are i.i.d. Normally distributed random variables with mean $\mu - \frac{\sigma^2}{2}$ and variance σ^2. Notice that both S_n and W_n are r.v.'s of the general type defined in (1).

As the time unit that we consider is long (1 year), assuming a Gaussian model for the returns seems to be appropriate, at least approximately, by the Central Limit Theorem; see also Levy (2004) and Cesari & Cremonini (2003).for empirical studies that confirm our theoretical setup.

In order to compute the comonotonic approximations for the quantiles and conditional tail expectations, notice that $E[Z_i]$, $\sigma^2_{Z_i}$ and r_i are given by

$$E[Z_i] = i(\mu - \frac{\sigma^2}{2}), \tag{44}$$

$$\sigma^2_{Z_i} = i\ \sigma^2 \tag{45}$$

and

$$r_i = \frac{\sum_{j=1}^{i}\sum_{k=j}^{n}\gamma_k}{\sqrt{i\sum_{j=1}^{n}\left(\sum_{k=j}^{n}\gamma_k\right)^2}}. \tag{46}$$

n	Method	$\sigma = 0.15$	$\sigma = 0.25$	$\sigma = 0.35$
20	UB	+10.98%	+14.17%	+12.98%
	LB_{TB}	-0.27%	-1.09%	-2.19%
	LB_{MV}	-0.14%	-0.36%	-0.59%
	LB_{MCTE}	**+0.01%**	**-0.05%**	**-0.37%**
	MC (±s.e)	24.4591 (2.16%)	59.6646 (2.90%)	198.0164 (3.27%)

Table 1: Approximations for the 0.95-conditional tail expectation of the discounted sum S_n for different volatilities (μ=0.075; yearly payments of 1).

n	Method	$\sigma = 0.15$	$\sigma = 0.25$	$\sigma = 0.35$
20	LB_{TB}	17.80	9.35	5.22
	LB_{MV}	17.82	9.48	5.51
	LB_{MLTE}	**17.75**	**9.21**	**5.09**

Table 2: Approximations for the 0.05-conditional left tail expectation of the compounded sum W_n for different volatilities (μ=0.075; yearly saving of 1).

respectively.

We present the results of our approximations in Tables 1, 2 and 3. In Table 1, we compare the different lower bound approximations for the 0.95-conditional tail expectation of S_n for different levels of the yearly volatility σ. Here, we fixed the number of yearly payments to $n = 20$ and the yearly expected return μ has been set equal to 0.075. The "Maximal CTE" lower bound approach turns out to fit the conditional tail expectations the best for all values of the parameter σ whereas the "Maximal Variance" lower bound (LB_{MV}) outperforms the "Taylor-based" lower bound (LB_{TB}) Table 2 compares lower bound approximations for the 0.05-conditional left tail expectation of W_n again for different levels of the yearly volatility σ fixing the number of yearly payments $n = 20$ and yearly expected return $\mu = 0.075$. Also in this case, the "Maximal CTE" lower bound which coincides with the "Minimal CLTE" lower bound, provides the best results. It is interesting to observe that in this case the "Maximal Variance" lower bound (LB_{MV}) approximation seems to be less accurate than the "Taylor-based" lower bound (LB_{TB}). This is because the "Maximal Variance" lower bound approach is more sensitive to the right, unbounded, tail of W_n when deriving its Λ and this is at the cost of loosing some accuracy in the left tail of W_n. This suggests that the "Taylor-based" lower bound (LB_{TB}) is often more appropriate in case of risk measures that focus on the left tail of the distribution such as CLTE whereas the "Maximal Variance" lower bound is better in case one focuses on the right tail of the distributions, finally providing the best overall fit.

Finally, we will also compare the comonotonic lower bound approximations for discrete arithmetic Asian option prices. We refer to Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R. & Vyncke, D., 2002(b) or Vanmaele, Deelstra, Liinev, Dhaene & Goovaerts (2006) for extensive reviews on how the prices of these instruments can be approximated using the comonotonic bounds. Consider a risky asset with a known price $S(0)$ at time i=0 and unknown prices $S(i)$ at times i=1,2,...,n. A discrete Asian option with maturity n, strike K and $n-j$ averaging dates is a financial instrument that generates at maturity n a pay-off that is equal to $\left[\frac{1}{n-j}\sum_{i=j}^{n-1} S(n-i) - K\right]_+$.

σ	K	MC (SE$\times 10^3$)	LB_{MCTE}	LB_{MV}	LB_{TB}	LB_{GA}
0.25	50	50.0506 (0.011%)	**-0.006%**	-0.007%	-0.006%	-0.007%
	80	24.7540 (0.023%)	**-0.024%**	-0.039%	-0.034%	-0.040%
	90	17.9405 (0.032%)	-0.047%	-0.059%	-0.052%	**-0.035%**
	100	12.4799 (0.047%)	**-0.032%**	-0.037%	-0.032%	-0.045%
	110	8.3887 (0.072%)	**-0.027%**	-0.027%	-0.032%	-0.068%
	200	0.1214 (2.389%)	**-0.449%**	-2.056%	-2.554%	-4.530%

Table 3: Lower bound approximations for Asian option prices for different strikes K (σ=0.25; r=0.04; S(0)=100; T=3; n=3).

In case averaging is done during the whole period $[0, n]$ at equidistant intermediate times $i = 1, 2, ..., n$, we find, using our notations, that the pay-off can also be represented by $\left[\frac{1}{n}S(0)W_n - K\right]_+$. Furthermore, in the absence of arbitrage opportunities and assuming a Black & Scholes market , the price of this instrument will be given by the $e^{-rn}E\left[\frac{1}{n}S(0)W_n - K\right]_+$.with r the risk free rate and $E[Z_i]$, $\sigma^2_{Z_i}$ now given by

$$E[Z_i] = i(r - \frac{\sigma^2}{2}), \tag{47}$$

$$\sigma^2_{Z_i} = i\ \sigma^2 \tag{48}$$

We will now assess the quality of the different lower bounds using the parameter setting from Vanmaele, Deelstra, Liinev, Dhaene & Goovaerts (2006, p.29). The time-unit is assumed to be one month and averaging is done over the whole period taking into account the monthly end prices of the underlying stock. Furthermore, monthly volatility $\sigma = \frac{0.25}{\sqrt{12}}$ and the monthly risk free $r = \frac{0.04}{12}$. in Table 3 we compare lower bound approximations for Asian option prices for different strike price K. The other parameters were fixed and are stated in the table. The last column, indicated by LB_{GA}, gives the result in case Λ is taken as the standardized logarithm of the geometric average $\sqrt[n]{\prod_{i=1}^{n} e^{Z_i}}$, the latter is a natural choice when approximating the d.f. of the arithmetic average. In line with previous results we find that the LB_{MCTE} approach significantly outperforms the other approximations. Since we focus on the right tail, the LB_{MV} approach outperforms the LB_{TB} approach, at least for out-of-the-money options when the strike K is larger than the current stock price. We notice that the quality of the LB_{GA} approach seems to decrease as K increases.

6 Concluding remarks

The stochastically discounted or compounded value of a series of cash flows is often a key quantity of importance in finance and actuarial science. Yet even for most realistic stochastic return models, it is often difficult to obtain analytic expressions for the risk measures involving these discounted sums. Following the works of Dhaene, et al. (2002a, 2002b), Dhaene, et al. (2004), and Vanduffel, et al. (2004), we show in this paper how to improve the lower bound approximations by suitably choosing for the conditioning variable Λ. It has already been documented in the literature that choosing this conditioning variable using either 'Taylor-based' or 'maximal variance' approach provides in some sense an overall goodness of fit. However, we further can improve the approximations if we concentrate on a local neighborhood of the distribution function such as the extreme lower or upper tails. Therefore, in these instances, we find that lower bound approximations for various risk measures can be improved if we use conditioning variables on the basis of a first-order approximation of the conditional tail expectation, if upper tails are concerned, and on a first-order approximation of the conditional left tail expectation, if lower tails are concerned.

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