Eigenvalues of fuzzy symmetric matrices

Karl Meerbergen
(joint work with Jeroen De Vlieger)

This talk reports work in progress on the eigenvalues of parameterized matrices.

Complex mathematical PDE models are used for simulating physical models. Most simulations are currently deterministic, i.e. all physical parameters (thickness, temperature, ...) are fixed. In practice, however, uncertainties about the precise values of parameters make deterministic solutions less attractive. In particular, one may want to validate a solution chosen by an automatic method, that minimizes a cost function. Often, the engineers still want to have an idea on the behaviour of the system near the optimal parameter values.

Uncertainties can be modeled in various ways. Stochastic methods are used for dealing with irreducible uncertainties due to the production process e.g. [5]. For an earlier stage in the design process, stochastic data about the parameters are not known. The parameter ranges can be large and a statistical analysis is not appropriate. The goal is to reduce the level of uncertainty. In order to study the impact of the parameters, recently, fuzzy numbers are used [4]. A sensitivity analysis of the parameter values (using derivatives) is only an option for small changes of the parameters. In this talk, we study the situation where parameters may vary significantly w.r.t. the chosen parameter values. The goal is to evaluate the variation of the solution among the given parameter values.

In the study of vibrations, sources of uncertainties can be geometric or material parameters, such as thickness, Young modulus, etc. An eigenvalue analysis is often performed in order to understand vibrations. The first eigenfrequency (smallest eigenvalue) should lie outside the frequency range for which the system is used, in order to avoid undesired vibrations. We therefore focus on the computation of the smallest eigenvalue. After finite element discretization, the algebraic eigenvalue problem takes the form $Ku = \lambda Mu$, where K and M are symmetric, and M is positive definite. In this talk, we only consider standard eigenvalue problems. As we now from the literature, $Ku = \lambda Mu$ can be written as a symmetric standard eigenvalue problem.

Pseudospectra can be used for a sensitivity analysis. The bounds provided by the pseudospectrum might be too large, since the specific parameter structure is not taken into account. When only one parameter changes, the structured pseudospectrum is useful. In this talk, we use fuzzy numbers as a generalization of a single parameter perturbation.

The notion fuzzy number is an extension of fuzzy set theory [6]. We use a tilde to denote fuzzy variables. We represent a fuzzy number $\widetilde{x}$ by its membership function $\mu_{\widetilde{x}}(x) : \mathbb{R} \to [0, 1]$. It defines the level of membership of x to the fuzzy set: $\mu_{\widetilde{x}}(x) = 1$ implies that x belongs to the set and $\mu_{\widetilde{x}}(x) = 0$ means that x does not belong to the set. For example, the concept "x is near 1" could be represented by the fuzzy number with the membership function that is one for $x = 1$, zero for $x < 0.5$ and $x > 1.5$ and values between 0 and 1 for $0.5 \leq x \leq 1.5$. The α cut is

defined as the set of x's for which $\mu_{\widetilde{x}}(x) \geq \alpha$. When all α-cuts are intervals, $\widetilde{x}$ is called a convex fuzzy number.

The extension principle is used to extend a (deterministic) function $f(x)$ with $x \in \mathbb{R}^N$ to fuzzy numbers. Let $\widetilde{x}$ be an N-vector of convex fuzzy numbers and f be continuous, then the extremes of the α-cuts of $\widetilde{y} = f(\widetilde{x})$ can be computed as:

$$[\widetilde{y}]_\alpha^- = \min_{x_i \in [\widetilde{x}_i]_\alpha} f(x_1, \ldots, x_n) \tag{1a}$$

$$[\widetilde{y}]_\alpha^+ = \max_{x_i \in [\widetilde{x}_i]_\alpha} f(x_1, \ldots, x_n) \; . \tag{1b}$$

There are three classes of methods for computing these extremes. The most popular one is the montecarlo method, which evaluates random samples. This method is not only expensive, it may also be inaccurate. The most accurate class solve the extremes by numerical optimization.

Fuzzy eigenvalues have been mentioned first for non-negative matrices [1][2], where all matrix entries can be independent fuzzy numbers. In this talk, we consider real symmetric matrices of the following form. Define

$$A(x) = B + \sum_{i=1}^{N} x_i E_i$$

where B, and E_i, $i = 1, \ldots, N$ are real symmetric. If $\widetilde{x}$ is a fuzzy number, we call $A(\widetilde{x})$ a fuzzy matrix. Following the extension principle, the eigenvalues of $A(\widetilde{x})$ are also fuzzy numbers.

We talk about the computation of the smallest or the largest eigenvalues (which is basically the same operation). Several approaches can be used for validating a selection of parameter values: the pseudo-spectrum computes the variation for all parameters together regardless the form of the perturbation. With the structure preserving pseudo-spectrum, we can handle one parameter (or a linear combination of parameters) at a time; this corresponds to the vertex method for fuzzy numbers, which can only be used in specific cases. In this talk, we analyse the cost needed for solving structured perturbations in the context of symmetric eigenvalue problems.

In order to solve the optimization problems (1) efficiently, we derive properties of the eigenvalues of A as a function of x where x lies in an α-cut of $\widetilde{x}$. Indeed, since function evaluations are eigenvalue computations, it may be important to reduce the number of function evaluations as much as possible. The key observation is that the largest eigenvalue of $A(x)$ is a convex function in x. This also holds when eigenvalues are multiple. If the elements of $\widetilde{x}$ are independent from each other, the α-cuts are hypercubes. The consequence is that the maximum is attained in one of the cornerpoints of the hypercube. This implies the solution of 2^N eigenvalue problems for each α-cut. The minimum is the local minimum of a (non-smooth) convex optimization problem. A similar conclusion holds for the smallest eigenvalue.

Details on the theory and the optimization algorithm can be found in [3].

References

[1] J. J. Buckley. Fuzzy eigenvalues and input-output analysis. *Fuzzy Sets Syst.*, 34(2):187–195, 1990.

[2] K.-P. Chiao. Generalized fuzzy eigenvalue problems. *Tamsui Oxford Journal of Mathematical Sciences*, 14:31–37, 1998.

[3] J. De Vlieger. *De berekening van de grootste vage eigenwaarde van onzekere symmetrische matrices.* Master thesis, K.U.Leuven, Department of Computer Science, Leuven, Belgium, 2009, 2009. (In English: The computation of the largest fuzzy eigenvalue of uncertain symmetric matrices).

[4] W. Oberkampf, S. DeLand, B. Rutherford, K. Diegert, and K. Alvin. A new methoddology for the estimation of the total uncertainty in computational simulation. In *proceedings of 40th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference*, pages 3061–3083, St. Louise, MO, USA, 1999.

[5] G.I. Schuëller. A state-of-the-art report on computational stochastic mechanics. *Probabilistic Engineering Mechanics*, 12(4):197–321, 1997.

[6] L.A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.