# Decay Rate Assignment through Multiple Spectral Values in Delay Systems

Islam Boussaada, Guilherme Mazanti, Silviu-Iulian Niculescu, and Wim Michiels

***Abstract*— This paper focuses on a spectral property for linear time-invariant (LTI) dynamical systems represented by delay-differential equations (DDEs) entitled *multiplicity-induced-dominancy* (MID), which consists, roughly speaking, in the spectral abscissa of the system being defined by a multiple spectral value. More precisely, we focus on the MID property for spectral values with over-order multiplicity, i.e., a multiplicity larger than the order of the DDE. We highlight the fact that a root of over-order multiplicity is necessarily a root of a particular polynomial, called the *elimination-produced polynomial*, and we address the MID property using a suitable factorization of the corresponding characteristic function involving special functions of Kummer type. Additional results and discussion are provided in the case of the n-th order integrator, in particular on the local optimality of a multiple root. The derived results show how the delay can be further exploited as a *control parameter* and are applied to some problems of stabilization of standard benchmarks with prescribed exponential decay.**

***Index Terms*— delay, characteristic function, exponential stability, Kummer functions, hypergeometric functions, Green–Hille transformation, partial pole placement.**

## I. Introduction

Since Hazen's paper [1] on the theory of servomechanisms in the 1930s, it is commonly accepted that the delays in systems' dynamics are at the origin of dynamic oscillations and instabilities. As a consequence, modeling delays, understanding the effects induced by them, and controlling delay systems represented a problem of recurring interest during the last century. More precisely, one of the ways to describe and explicitly take into account past information in processes and/or phenomena is to use mathematical models based on delay-differential equations (DDEs). For example, transport and propagation phenomena, signal transmission in communication networks, or age structure in population dynamics are typical classes of processes and/or phenomena where delays can be used to model time heterogeneity leading to DDEs. For further examples, we refer to [2]–[7] and the references therein. Finally, for an appropriate definition and related classification of DDEs, the reader is referred to [3], [8]–[11].

As pointed out in [12], a delay can induce stability (stabilizing effect) in some cases, and a lack of stability (destabilizing) in other cases. These stability issues have been extensively discussed in the open literature, and there exists a systematic methodological and numerical treatment for the stability analysis of most cases (see, e.g., [5], [13]–[18]). To the best of the authors' knowledge, in the control area, the beneficial effect of the delay in closed-loop appears in the 1970s within the framework of the approximation of the derivative action of PID controllers by a delay-difference operator [19], [20]. Moreover, the $\tau$-decomposition method, proposed a decade earlier in [21], enables the computation of the delay intervals guaranteeing asymptotic stability, explicitly showing that, in certain cases, augmentation of the delay leads to stability. In this context, the idea of using the *delay* as a *control parameter* came naturally. For instance, a chain of $n$ integrators can be stabilized by a controller including $n$ delays [22], [23]. If the main advantage of exploiting delays in the controllers is the simplicity of their implementation, their infinite-dimensional character, however, yields some unexpected behaviors of the corresponding closed-loop systems which imposes, as a consequence, some limitations in the choice of the parameters. For an overview of some of the methods and techniques, we refer to [5], [12], [24].

A classical approach in the stability analysis and stabilization of linear time-invariant (LTI) dynamical systems including delays is the application of spectral methods (see, e.g., [5]). The spectrum of a DDE can be characterized as the set of complex roots of its characteristic function, which presents itself under the form of a *quasipolynomial*[1]. These roots are usually referred to as *spectral values* or *characteristic roots* of the system. The analysis of quasipolynomials and, in particular, the location of their roots, is of fundamental importance for the spectral analysis of DDEs, and many works have addressed this question. For instance, the origin of a LTI DDE is exponentially stable if, and only if, the *spectral abscissa*[2] of the system, defined as the supremum of the real parts of the roots of its characteristic function, is negative. We refer the interested reader to [5] and [25, Chapter 3] (where these functions are referred to as *exponential polynomials*). In particular, an important fact about a quasipolynomial is that the multiplicity of any of its roots is upper bounded by some positive integer, known as the *degree* of the quasipolynomial, as stated, for instance, in [26, Part Three, Problem 206.2], [27], [28].

In the case of LTI systems represented by DDEs, recent works have highlighted a particularly interesting spectral

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[1] a finite sum of polynomials multiplied by exponentials

[2] also called the rightmost characteristic root

property, called *multiplicity-induced-dominancy (MID)*, which consists in conditions on the system's parameters under which a multiple spectral value corresponds to the spectral abscissa [29], [30]. The first analytic proof of this property has been proposed for first-order DDEs in [31], and it relies on an integral representation of the characteristic function and a contradiction argument. In particular, it appears that a characteristic root of maximal multiplicity (i.e., equal to the degree of the corresponding quasipolynomial) necessarily defines the spectral abscissa of the system. Such a systematic study of the links between roots of large multiplicity and the spectral abscissa was not sufficiently addressed in the literature until the early work [31], even though some hints in this direction are provided in [2] in the case of low-order systems. Since these works, the case of the assignment of a characteristic root with maximal multiplicity, called *generic MID property*, was recently addressed and completely characterized in [32] (generic retarded case) and in [33] (unifying retarded and neutral cases) for LTI DDEs including a *single delay*. As discussed in [32], [33], by exploiting the degree of quasipolynomials, this property opens an interesting perspective in control area through the so-called *partial pole placement* method, that is, imposing the multiplicity of a characteristic root of the closed-loop system by an appropriate choice of the controller gains guarantees the exponential stability of the closed-loop system with a prescribed decay rate.

The arguments used to prove the generic MID property in [32], [33] are based on some analytical properties of Kummer and Whittaker confluent hypergeometric functions, which cannot be extended straightforwardly to treat the non-generic case[3], a fact that represents a drawback of the method. However, as shown in [34], by way of different arguments that exploit the structure of the system, the MID property still holds in some cases with lower multiplicity, but there does not exist any systematic procedure to treat them.

The aim of this paper is to address these problems and to outline the ideas of a new method that could also encompass the MID with intermediate *over-order* multiplicities, i.e., multiplicities greater than the order of the DDE (see Section III-B below for a precise definition of the MID property and the notion of over-order multiplicity). This method exploits the factorization of the characteristic function, the particular structure and related properties of the integral term in the proposed factorization. More precisely, the contribution of the paper is threefold.

First, we provide conditions under which spectral values with the *lowest* over-order (algebraic) multiplicity are *dominant*, i.e., they have the largest real part among all spectral values. To prove the proposed results, we explicitly compute and exploit the properties of the so-called *elimination-produced polynomial*. To guarantee the dominancy of the multiple root, one makes use of the Green–Hille (integral) transformation introduced by Hille one century ago [35] for characterizing the location of the non-asymptotic zeros of Whittaker hypergeometric functions. It should be noted that these ideas complete the previous approaches based on the properties of Kummer hypergeometric functions to handle *generic MID* in the retarded and neutral cases (see, e.g., [32], [33]). The proposed method represents a novelty in the open literature. The underlying ideas were first explored recently by some of the authors in [36] in the characterization of the MID property of a root with generic multiplicity-minus-one[4], in which computations turned out to be simpler than in the present setting. Second, we show that the spectral abscissa function reaches a strict local minimum in the configuration corresponding to a root with the lowest over-order multiplicity. Finally, as a byproduct of the analysis, new insights on MID control of the dynamics of a chain of integrators and of a pendulum are proposed.

The remainder of the paper is organized as follows. Some prerequisites are proposed in Section II. A motivating example and the problem formulation are given in Section III. The main results are derived in Section IV, which also introduces the concept of elimination-produced polynomial and provides a representation of quasipolynomials in terms of a linear combination of two Kummer functions. Section V considers the application of the results to the partial pole placement for a chain of integrators, investigating also in this case the link between the proposed partial pole placement and the problem of minimizing the spectral abscissa. An illustrative example and some novelties and perspectives on the P3$\delta$ software are discussed in Section VI. Some concluding remarks end the paper.

*Notations.* Throughout the paper, the following notations are used: $\mathbb{N}^*$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, real, and complex numbers, respectively; we set $\mathbb{N} = \mathbb{N}^* \cup \{0\}$. The set of all integers is denoted by $\mathbb{Z}$ and, for $a, b \in \mathbb{R}$, we denote $[\![a, b]\!] = [a, b] \cap \mathbb{Z}$, with the convention that $[a, b] = \emptyset$ if $a > b$. For $\lambda \in \mathbb{C}$, $\Re(\lambda)$ and $\Im(\lambda)$ denote its real and imaginary parts, respectively. The open left (right) complex half-plane is the set $\mathbb{C}_-$ ($\mathbb{C}_+$) defined by $\mathbb{C}_- = \{\lambda \in \mathbb{C} \mid \Re(\lambda) < 0\}$ ($\mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}$).

## II. Prerequisites

### A. Spectral Properties of DDEs

Consider the LTI dynamical system described by the DDE

$$y^{(n)}(t) + \sum_{k=0}^{n-1} a_k y^{(k)}(t) + \sum_{k=0}^{m} \alpha_k y^{(k)}(t-\tau) = 0, \quad (1)$$

under appropriate initial conditions, where $y(\cdot)$ is a real-valued unknown function, $\tau > 0$ is the delay, and $a_0, \ldots, a_{n-1}, \alpha_0, \ldots, \alpha_m$ are real coefficients. The DDE (1) is said to be of *retarded (neutral) type* if $m < n$ ($m = n$). The goal of this section is to provide elementary results on the spectral properties of (1) that will be useful in the sequel. For a deeper discussion on DDEs, we refer to [5], [9].

Notice that (1) is a particular case of the time-delay system

$$\dot{\xi}(t) + B_\tau \dot{\xi}(t-\tau) = A_0 \xi(t) + A_\tau \xi(t-\tau); \quad (2)$$

indeed, if $y$ is a solution of (1), then the vector $\xi(t) = (y(t), y'(t), \ldots, y^{(n-1)}(t))^T \in \mathbb{R}^n$ satisfies (2) for suitable

[3] i.e., of multiplicity strictly smaller than the degree of the quasipolynomial

[4] i.e., the largest over-order multiplicity strictly smaller than the degree

real matrices $A_0$, $A_\tau$, $B_\tau \in \mathcal{M}_n(\mathbb{R})$ which can be constructed from (1), with $A_\tau$ and $B_\tau$ of rank one.

The characteristic function associated with (1) is the quasi-polynomial $\Delta\colon \mathbb{C} \to \mathbb{C}$ defined by

$$\Delta(\lambda) = P_0(\lambda) + P_\tau(\lambda)\mathrm{e}^{-\lambda\tau}, \tag{3}$$

where $P_0$ and $P_\tau$ are the polynomials with real coefficients

$$P_0(\lambda) = \lambda^n + \sum_{k=0}^{n-1} a_k\lambda^k, \quad P_\tau(\lambda) = \sum_{k=0}^{m} \alpha_k\lambda^k. \tag{4}$$

Roots of $\Delta$ are usually called *characteristic roots* or *spectral values* of (1), and they are infinite in number, except in the trivial case where $\Delta$ reduces to a polynomial. The exponential stability of the trivial solution of (1) holds true if, and only if, there exists $\gamma > 0$ such that $\Re(\lambda) \leq -\gamma$ for every root $\lambda$ of $\Delta$ (see, e.g., [5], [8]).

When the coefficient $\alpha_m$ is not zero, the *degree* of the quasipolynomial $\Delta$ from (3) is the integer $\deg(\Delta) = n+m+1$ (see for instance [25, p. 208] for the general definition of the degree of a quasipolynomial). As discussed in [27], this integer, which is larger than the degrees of the polynomials $P_0$ ($\deg(P_0) = n$) and $P_\tau$ ($\deg(P_\tau) = m$), is nothing but the integer appearing in the Pólya–Szegő bound from [26, Part Three, Problem 206.2][5], and also corresponds to the maximal (allowable) multiplicity that a characteristic root of (3)–(4) may have. In addition, a characteristic root reaching this bound is necessarily real.

*Remark 1:* On the imaginary axis, the characteristic roots of the quasipolynomial $\Delta$ defined by (3) admit a bounded frequency, i.e., a bounded imaginary part. Indeed, any imaginary root $\lambda_0 = i\,\omega_0$ of $\Delta$ necessarily satisfies

$$|P_0(i\,\omega_0)|^2 = |P_\tau(i\,\omega_0)|^2.$$

The function $\mathcal{F}$ defined by $\mathcal{F}(\omega) = |P_0(i\,\omega)|^2 - |P_\tau(i\,\omega)|^2$ is a polynomial on $\omega$ with real coefficients, and thus all its positive roots can be bounded in terms of its coefficients (see, for instance, [37]). However, this observation does not provide insights on frequency bounds for other roots, in particular (unstable) roots located in $\mathbb{C}_+$.

Despite the fact that the characteristic function of some DDE has an infinite number of characteristic roots, retarded systems, that is, (1) with $m < n$ or, equivalently, (2) with $B_\tau = 0$, admit finitely many roots on any vertical strip in the complex plane [9, Chapter 1, Lemma 4.1]. Several general results on the location of roots of (3) can be found in the literature, and we refer the interested reader to [38].

The next proposition collects two interesting properties, whose proofs can be found, respectively, in [5] and [39].

*Proposition 2:* Consider the DDE (1), the corresponding system (2), and their characteristic function $\Delta$ given by (3)–(4). Then the following properties hold:

1) If $m < n$ and $\lambda$ is a characteristic root of system (2) with $B_\tau = 0$, then it satisfies
$$|\lambda| \leq \|A_0 + A_\tau\,\mathrm{e}^{-\tau\lambda}\|, \tag{5}$$
where $\|\cdot\|$ is any induced matrix norm.
2) If $m = n$ and $\lim_{|\lambda|\to\infty} |P_\tau(\lambda)/P_0(\lambda)| < 1$, then the characteristic equation $\Delta$ defined by (4) has a finite number of roots in the right half-plane.

*Remark 3:* Inequality (5), combined with the triangular inequality, provides a generic *envelope curve* around the characteristic roots corresponding to system (2). In other words, the equality case in (5) defines a curve in the complex plane such that all characteristic roots of $\Delta$ are located to its left. We refer to [40] for further insights on spectral envelopes for retarded time-delay systems with a single delay.

In the sequel, we describe a procedure from [41], inferred from ideas used in the analysis of particular cases in [42]–[44], that, given $\lambda_0 \in \mathbb{R}$ and $\omega_0 > 0$, helps us finding appropriate conditions such that the characteristic roots of $\Delta$ with $\Re(\lambda) \geq \lambda_0$ necessarily satisfy $|\Im(\lambda)| \leq \omega_0$. In the case $\lambda_0 = 0$, $\omega_0$ represents a frequency bound for unstable characteristic roots.

Given $\lambda_0 \in \mathbb{R}$, let $\tilde{\Delta}(z) = \tau^n\Delta(\lambda_0 + \frac{z}{\tau})$, write $\tilde{\Delta}(z) = \tilde{P}_0(z) + \mathrm{e}^{-z}\tilde{P}_\tau(z)$ for some suitable polynomials $\tilde{P}_0$ and $\tilde{P}_\tau$, and note that any root $z$ of $\tilde{\Delta}$ satisfies the modulus condition

$$|\tilde{P}_0(x+i\omega)|^2\mathrm{e}^{2x} = |\tilde{P}_\tau(x+i\omega)|^2,$$

where $x = \Re(z)$ and $\omega = \Im(z)$. If $x \geq 0$, then $\mathrm{e}^{2x} \geq T_\ell(x)$, where, for $\ell \in \mathbb{N}$, the polynomial $T_\ell$ is the truncation of the Taylor expansion of $\mathrm{e}^{2x}$ at order $\ell$, i.e., $T_\ell(x) = \sum_{k=0}^{\ell} \frac{(2x)^\ell}{\ell!}$. Hence, any root $z = x + i\omega$ of $\tilde{\Delta}$ with non-negative real part satisfies $\mathcal{F}(x,\omega) \geq 0$, where $\mathcal{F}$ is the polynomial given by

$$\mathcal{F}(x,\omega) = |\tilde{P}_\tau(x+i\omega)|^2 - |\tilde{P}_0(x+i\omega)|^2 T_\ell(x).$$

In addition, $\mathcal{F}$ only depends on $\omega$ through $\omega^2$. Define the polynomial $H$ by setting $H(x,\Omega) = F(x,\sqrt{\Omega})$ for $\Omega \geq 0$. Hence, any root $z = x + i\omega$ of $\tilde{\Delta}$ with $x \geq 0$ satisfies

$$H(x,\Omega) \geq 0, \tag{6}$$

where $\Omega = \omega^2$. By exploiting the polynomial inequality (6), one can establish a bound on the imaginary parts of the roots of $\tilde{\Delta}$. The procedure described above and synthesized in Algorithm 1 (see Appendix) is adapted from [41], which consists in increasing the order of the Taylor expansion of $\mathrm{e}^{2x}$ until a suitable bound is found or a maximal truncation order is reached. For further details, we refer to [41]–[44].

### B. Hypergeometric Functions & Integral Transformations

The main ingredient of the proposed partial pole placement method is based on the properties of a particular class of hypergeometric functions, namely, *Kummer confluent hypergeometric function*, which, for $a, b \in \mathbb{C}$ such that $-b \notin \mathbb{N}$, is the entire function $\Phi(a,b,\cdot)\colon \mathbb{C} \to \mathbb{C}$ defined by the series

$$\Phi(a,b,z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{z^k}{k!}, \tag{7}$$

[5] Initially, the the Pólya–Szegő result gives a bound on the number of the quasipolynomial's roots inside some horizontal strip defined by $\mathbf{a} \leq \Im(\lambda) \leq \mathbf{b}$. By considering the particular case $\mathbf{a} = \mathbf{b}$, the Pólya–Szegő bound corresponds to the number of roots of the form $x + i\,\mathbf{a}$ for $x \in \mathbb{R}$, which is itself a bound on the multiplicity of any of such a roots.

where, for $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}$, $(\alpha)_k$ is the *Pochhammer symbol* for the *ascending factorial*, defined inductively as $(\alpha)_0 = 1$ and $(\alpha)_{k+1} = (\alpha + k)(\alpha)_k$. The series in (7) converges for every $z \in \mathbb{C}$ and, as presented in [45]–[47], the Kummer function satisfies the *Kummer differential equation*, that is,

$$z\frac{\partial^2 \Phi}{\partial z^2}(a,b,z) + (b-z)\frac{\partial \Phi}{\partial z}(a,b,z) - a\Phi(a,b,z) = 0. \quad (8)$$

Finally, for every $a, b, z \in \mathbb{C}$, such that $\Re(b) > \Re(a) > 0$, Kummer functions admit the integral representation [45]–[47]

$$\Phi(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 \mathrm{e}^{zt} t^{a-1}(1-t)^{b-a-1}\,\mathrm{d}t, \quad (9)$$

where $\Gamma$ denotes the Gamma function. This integral representation has been used in [32], [33], [48] to characterize the spectrum of some DDEs. Kummer functions exhibit a range of remarkable properties. In particular, they satisfy some recurrence relations, often called *contiguous relations* (see, for instance, [47]), which will be exploited in the sequel.

*Lemma 4 ([47, p. 325]):* Let $a, b, z \in \mathbb{C}$ with $a \neq b$, $z \neq 0$, and $-b \notin \mathbb{N}$. The following relations hold:

$$\begin{aligned}\Phi(a,b+1,z) &= \frac{-b\,(a+z)\;\Phi(a,b,z) + ab\,\Phi(a+1,b,z)}{z\,(a-b)},\\ \Phi(a+1,b+1,z) &= -\frac{-b\,\Phi(a+1,b,z) + b\,\Phi(a,b,z)}{z}.\end{aligned} \quad (10)$$

Kummer functions are strongly related to another interesting class of hypergeometric functions called *Whittaker functions*. For $k, l \in \mathbb{C}$ with $-2l \notin \mathbb{N}^*$, the *Whittaker function* $\mathcal{M}_{k,l}$ is defined for $z \in \mathbb{C}$ by

$$\mathcal{M}_{k,l}(z) = \mathrm{e}^{-\frac{z}{2}} z^{\frac{1}{2}+l} \Phi(\tfrac{1}{2}+l-k, 1+2l, z), \quad (11)$$

(see, e.g., [47]). Note that, if $\frac{1}{2} + l$ is not an integer, then the function $\mathcal{M}_{k,l}$ is a multi-valued complex function with branch point at $z = 0$. In addition, the nontrivial roots of $\mathcal{M}_{k,l}$ coincide with those of $\Phi(\frac{1}{2}+l-k, 1+2l, \cdot)$ and $\mathcal{M}_{k,l}$ satisfies the *Whittaker differential equation*

$$\varphi''(z) = \left(\frac{1}{4} - \frac{k}{z} + \frac{l^2 - \frac{1}{4}}{z^2}\right)\varphi(z). \quad (12)$$

Taking into account that $\mathcal{M}_{k,l}$ is a nontrivial solution of the second-order linear differential equation (12), then any nontrivial root of $\mathcal{M}_{k,l}$ is simple.

In the pioneering work by E. Hille [35], some *oscillation theorems* in the complex domain have been proposed. Among others, Hille studied the distribution of zeros of functions of a complex variable satisfying linear second-order homogeneous differential equations with variable coefficients, as is the case for the Whittaker function $\mathcal{M}_{k,l}$, which satisfies (12). In particular, Hille introduced an integral transformation called *Green–Hille transformation* ensuing from the differential equation and allowing the removal of regions in the complex plane that do not contain complex roots. To illustrate the idea, consider the general homogeneous second-order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[K(z)\,\frac{\mathrm{d}\varphi}{\mathrm{d}z}(z)\right] + G(z)\varphi(z) = 0, \quad (13)$$

where $z$ is the complex independent variable, and the functions $G$ and $K$ are assumed to be analytic in some region $\Theta$ such that $K$ does not vanish in that region. Equation (13) can be written in $\Theta$ as a first-order system by introducing the dependent variables $\varphi_1(z) = \varphi(z)$ and $\varphi_2(z) = K(z)\,\frac{d\varphi}{dz}(z)$. The Green–Hille transformation consists in multiplying the equation for $\varphi_1$ by $\overline{\varphi_2(z)}$, the one for $\varphi_2$ by $\overline{\varphi_1(z)}$, and integrating $z$ along a path in $\Theta$, which yields

$$\begin{aligned}\left[\overline{\varphi_1(z)}\,\varphi_2(z)\right]_{z_1}^{z_2} - \int_{z_1}^{z_2} |\varphi_2(z)|^2 \frac{\overline{\mathrm{d}z}}{\overline{K(z)}}\\ + \int_{z_1}^{z_2} |\varphi_1(z)|^2 G(z)\,\mathrm{d}z = 0,\end{aligned} \quad (14)$$

where $z_1, z_2 \in \Theta$ and both integrals are taken along the same arbitrary smooth path in $\Theta$ connecting $z_1$ to $z_2$.

The following result, which is proved in [48] using the Green–Hille transformation from [35], gives insights on the distribution of the nonasymptotic zeros of Kummer hypergeometric functions with real arguments $a$ and $b$.

*Proposition 5 ([48]):* Let $a$, $b \in \mathbb{R}$ be such that $b \geq 2$.

1) If $b = 2a$, then all nontrivial roots $z$ of $\Phi(a,b,\cdot)$ are purely imaginary;
2) If $b > 2a$ (resp., $b < 2a$), then all nontrivial roots $z$ of $\Phi(a,b,\cdot)$ satisfy $\Re(z) > 0$ (resp., $\Re(z) < 0$);
3) If $b \neq 2a$, then all nontrivial roots $z$ of $\Phi(a,b,\cdot)$ satisfy

$$(b-2a)^2\Im(z)^2 - (4a(b-a)-2b)\,\Re(z)^2 > 0.$$

## III. Motivating Example and Problem Formulation

The problem of stabilization of a chain of integrators is considered in [22], where it is shown that a single integrator can be stabilized by a single delay state-feedback. Indeed, a positive gain guarantees the closed-loop stability of the system free of delay, and, by continuity, there exists a (sufficiently small) delay in the output preserving the stability of the closed-loop system. However, the situation is completely different for a chain of integrators of order $n$ when $n > 1$.

### A. Controlling the Double Integrator

Let us revisit the standard control problem $y''(t) = u(t)$ with a delayed output feedback controller $u(t) = -\alpha\, y(t-\tau)$. The closed-loop characteristic quasipolynomial writes as

$$\Delta(\lambda) = \lambda^2 + \alpha\,\mathrm{e}^{-\tau\lambda}. \quad (15)$$

With the notations of (3), one has $P_0(\lambda) = \lambda^2$ and $P_\tau(\lambda) = \alpha$. Thus, for $\alpha \neq 0$, the degree of the quasipolynomial (15) is $\deg(\Delta) = 3$. However, one can show that, due to the particular structure of $\Delta$, it cannot admit roots of multiplicity 3, and the largest multiplicity that a root $\lambda_0$ of (15) can have is 2. Such multiplicity is attained if, and only if,

$$\alpha = -4\,\frac{\mathrm{e}^{-2}}{\tau^2},\ \lambda_0 = -\frac{2}{\tau}. \quad (16)$$

It was shown in [22], [23] that, for the problem of controlling the chain of $n$ integrators $y^{(n)}(t) = u(t)$ with a delayed controller $u(t) = -\sum_{k=1}^{N} \alpha_k y(t-\tau_k)$ with $0 \leq \tau_1 < \cdots < \tau_N$, a necessary and sufficient condition for stabilizability is

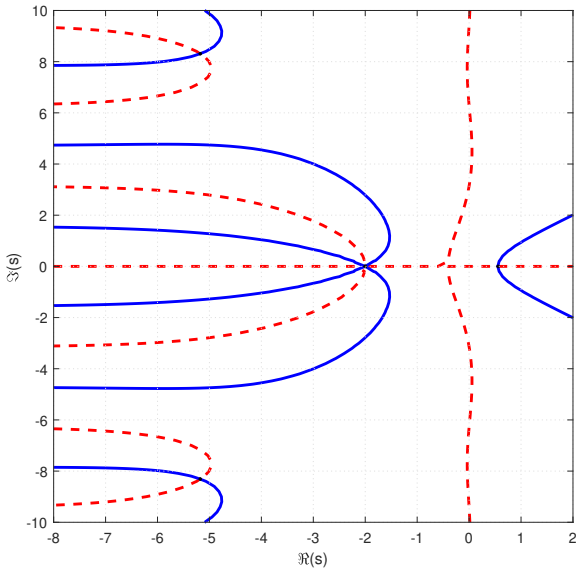



Fig. 1. Illustration of the roots of (17): solid blue curves represent the region $\Re(\Delta(\lambda)) = 0$, dashed red curves represent the region $\Im(\Delta(\lambda)) = 0$, so that the roots of $\Delta$ correspond to the intersections between solid blue curves and dashed red curves. This quasipolynomial $\Delta$ admits a double root at $\lambda_0 = -2$, which is not its rightmost root, since $\Delta$ also admits a real root $\lambda_1 \approx 0.557$. The figure was produced using the QPmR toolbox from [49].

that $N \geq n$. Since (15) contains only a single delay, we deduce that it admits at least one spectral value with a positive real part. Consequently, $\lambda_0 = -\frac{2}{\tau}$, while being a multiple root, is not dominant. Indeed, consider (15)–(16) with $\tau = 1$, that is,

$$\Delta(\lambda) = \lambda^2 - 4\mathrm{e}^{-(\lambda+2)}. \tag{17}$$

As illustrated in Figure 1, the dominancy property is lost since $\lambda_1 \approx 0.557$ is also a root of the function (17). This is justified by the sparsity of (17), i.e., the delay-free polynomial $P_0$ has some null coefficients. In accordance with the previous observation, consider now the same problem but using a delayed PD controller, which, in frequency domain, gives the closed-loop characteristic function:

$$\Delta(\lambda) = \lambda^2 + (\alpha_1\lambda + \alpha_0)\ \mathrm{e}^{-\lambda\tau}, \tag{18}$$

which is a quasipolynomial of degree 4. Following [29, Theorem 4.2] it has been shown that, for an arbitrary positive delay $\tau$, the quasipolynomial (18) admits a real spectral value at $\lambda = \lambda_\pm$ with algebraic multiplicity 3 if, and only if,

$$\lambda_\pm = \frac{-2 \pm \sqrt{2}}{\tau}, \tag{19}$$

and the system parameters satisfy:

$$\alpha_0 = \frac{6 + 10\,\lambda_\pm\,\tau}{\tau^2}\ \mathrm{e}^{\lambda_\pm\tau}, \quad \alpha_1 = \frac{2 + 2\,\lambda_\pm\,\tau}{\tau}\ \mathrm{e}^{\lambda_\pm\tau}. \tag{$\star_\pm$}$$

Furthermore, it has been shown in [29, Theorem 4.2] that the MID property is valid for $\lambda_+$, that is, the triple spectral value corresponds to the spectral abscissa only if ($\star_+$) is satisfied. It has also been emphasized in [29] that the multiple spectral value at $\lambda_-$ is always dominated by a single real root. In conclusion, independently from the chosen delay $\tau > 0$, the closed-loop solution is always exponentially stable with a decay rate corresponding to $\lambda_+ < 0$.

As emphasized in the above example, in such cases, multiple spectral values are not necessarily dominant and a deeper investigation to understand when such roots are dominant is needed, which is the aim of this paper.

### B. Problem Formulation

Consider the DDE (1) and its characteristic function $\Delta$ given by (3)–(4). As recalled in Section II-A, the degree of the characteristic function $\Delta$ is $\deg(\Delta) = n + m + 1$.

We say that a characteristic root $\lambda_0$ of $\Delta$ satisfies the *MID property* if (i) its *algebraic multiplicity* (denoted by $M(\lambda_0)$) is *larger than one*, and (ii) it is *dominant*, meaning that all the characteristic roots $\lambda_\sigma$ of $\Delta$ satisfy $\Re(\lambda_\sigma) \leq \Re(\lambda_0)$. Such a root $\lambda_0$ corresponds to the rightmost root of the spectrum and defines the *spectral abscissa* of $\Delta$. In addition, we say that the root $\lambda_0$ is *over-order* if its algebraic multiplicity is strictly greater than the order $n$, which corresponds to the degree of the characteristic function in the delay-free case. In this situation, we refer to the MID property as *over-order MID*. In the case $M(\lambda_0) = \deg(\Delta)$, it was shown in [32] (case $m = n-1$) and [33] (general case $m \leq n$) that $\lambda_0$ satisfies the MID property. This "limit" case corresponding to the maximal possible multiplicity is also called *generic MID* (or GMID for short). Finally, with the notions and notations above, the *lowest over-order (algebraic) multiplicity* corresponds to the case when the multiplicity of $\lambda_0$ is $M(\lambda_0) = n + 1$.

The problem addressed in this paper can be formulated as follows: *finding, on the one hand, conditions on the parameters of the dynamical system* (1) *ensuring that a characteristic root $\lambda_0$ has over-order (intermediate) algebraic multiplicity*[6]*, and determining, on the other hand, if such a root satisfies the MID property*. For the sake of brevity, our focus will be to infer *appropriate conditions guaranteeing that $\lambda_0$ has multiplicity $M(\lambda_0) = n+1$*, which corresponds to the smallest possible over-order multiplicity, and that it satisfies the *MID property*, in the particular case where $m = n - 1$. It should be mentioned that this configuration has been investigated in [34], in the particular case where the delay-free polynomial $P_0$ of $\Delta$ is real-rooted. The approach proposed hereafter relaxes the former assumption.

In the motivating example above, we have that $\deg(\Delta) = 4$. Since $n = 2$, the only possible intermediate multiplicity is $M(\lambda_0) = 3$, which, in our terminology, coincides with the lowest over-order (algebraic) multiplicity.

We shall study the MID property in this paper only for *real* characteristic roots $\lambda_0$. The main motivation for this restriction is that it is known that, if a function $\Delta$ of the form (3)–(4) has real coefficients, then a root $\lambda_0$ satisfying the MID property *with maximal multiplicity* $M(\lambda_0) = n + m + 1$ is necessarily real (see, e.g., [28, Corollaries 1 and 2] and [33, Remark 15]). Such a result has also been extended to roots with smaller over-order multiplicities in some particular cases, including

[6] that is, the multiplicity $M(\lambda_0)$ of $\lambda_0$ verifies $n + 1 \leq M(\lambda_0) \leq n + m$

roots with the smallest over-order multiplicity $n+1$ (see, e.g., [50], [51]).

## IV. Main Results

Our first results, presented in Section IV-A, provide a *necessary condition* for the existence of a root of $\Delta$ with the lowest over-order multiplicity, in terms of the so-called *elimination-produced polynomial*. Exploiting this result, *necessary and sufficient conditions* for the existence of such a root are derived in Section IV-B, in terms of the elimination-produced polynomial and a suitable factorization of $\Delta$. We then highlight the links between quasipolynomials $\Delta$ with a root of over-order multiplicity and Kummer confluent hypergeometric functions in Section IV-C. Finally, *sufficient conditions* guaranteeing the dominance of the over-order root are derived in Section IV-D.

### A. Elimination-Produced Polynomial

In the sequel, we provide an appropriate *necessary condition* for a given real number $\lambda_0$ to be a root of multiplicity at least $n+1$ of a given quasipolynomial $\Delta$ under the form (3)–(4) with $m = n-1$. The necessary condition consists in stating that $\lambda_0$ must be a root of a polynomial, known as the *elimination-produced polynomial*.

The main ideas underlying the construction of the *elimination-produced polynomial* can be resumed as follows: Imposing that a real number $\lambda_0$ is a root of $\Delta$ of multiplicity at least $n+1$ amounts to imposing that the following conditions hold simultaneously:

$$\Delta(\lambda_0) = \Delta'(\lambda_0) = \cdots = \Delta^{(n)}(\lambda_0) = 0. \tag{20}$$

On the other hand, if we consider, in (3)–(4), that the coefficients $a_0, \ldots, a_{n-1}$ of $P_0$ are "fixed" and known and the coefficients $\alpha_0, \ldots, \alpha_{n-1}$ of $P_\tau$ are "free" and available for choice, then (20) imposes $n+1$ linear equality constraints on the $n$ free parameters $\alpha_0, \ldots, \alpha_{n-1}$. While $n$ of those constraints should be sufficient to determine the values of the $n$ free parameters $\alpha_0, \ldots, \alpha_{n-1}$ in terms of the fixed parameters $a_0, \ldots, a_{n-1}$, the delay $\tau$, and the root $\lambda_0$, the additional constraint will express a relation that must be satisfied between $a_0, \ldots, a_{n-1}$, the delay $\tau$, and the root $\lambda_0$ in order for the multiplicity $n+1$ to be attained. This relation is precisely the elimination-produced polynomial, as described below:

*Proposition 6:* Consider the quasipolynomial $\Delta$ from (3)–(4) with $m = n-1$. If the real number $\lambda_0$ is a root of multiplicity at least $n+1$ of $\Delta$, then

$$\mathbb{P}(\lambda_0, \tau) = 0,$$

where $\mathbb{P}$ is the *elimination-produced polynomial*, defined by

$$\mathbb{P}(\lambda, \tau) = \sum_{k=0}^{n} \binom{n}{k} P_0^{(k)}(\lambda) \tau^{n-k}. \tag{21}$$

*Proof:* Note that $\lambda_0$ is a root of $\Delta(\cdot)$ with multiplicity at least $n+1$ if, and only if, it is a root of $\hat{\Delta}\colon \lambda \mapsto \mathrm{e}^{\lambda\tau}\Delta(\lambda)$ with the same multiplicity. In particular, we have $\hat{\Delta}^{(n)}(\lambda_0) = 0$. Since $\hat{\Delta}(\lambda) = \mathrm{e}^{\lambda\tau}P_0(\lambda) + P_1(\lambda)$ and $P_1$ is a polynomial of degree $n-1$, we deduce that

$$\hat{\Delta}^{(n)}(\lambda) = \mathrm{e}^{\lambda\tau} \sum_{k=0}^{n} \binom{n}{k} P_0^{(k)}(\lambda) \tau^{n-k},$$

yielding the conclusion since $\mathrm{e}^{\lambda_0\tau} \neq 0$. ∎

Before turning to our next result, we recall the statement of the Hermite–Poulain Theorem on roots of polynomials. Its proof can be found, for instance, in [52, Theorem 7.3.3].

*Theorem 7 (Hermite–Poulain):* Let $h(x) = c_0 + c_1 x + \ldots + c_n x^n$ be a real-rooted polynomial. If $f(x)$ is a polynomial with real coefficients, then the polynomial

$$F(x) = c_0 f(x) + c_1 f'(x) + \ldots + c_n f^{(n)}(x)$$

has at least as many real roots as $f(x)$ has.

To deduce a link between the number of real roots of $\mathbb{P}(\cdot, \tau)$ and that of $P_0(\cdot)$, we exploit the Hermite–Poulain Theorem.

*Proposition 8:* Let $P_0$ be a polynomial of degree $n$ with real coefficients and $\mathbb{P}$ be defined from $P_0$ as in (21). Then, for every $\tau \in \mathbb{R}$, the polynomial $\lambda \mapsto \mathbb{P}(\lambda, \tau)$ has at least as many real roots as $P_0$ (counted with their multiplicities).

*Proof:* This is an immediate consequence of (21) and the Hermite–Poulain Theorem applied to the real-rooted polynomial $h$ given by $h(x) = (x+\tau)^n = \sum_{k=0}^{n} \binom{n}{k} \tau^{n-k} x^k$ and to $f = P_0$. ∎

### B. Necessary and Sufficient Conditions for the Lowest Over-Order Multiplicity

We now provide a characterization of the situations under which a quasipolynomial $\Delta$ under the form (3)–(4) with $m = n-1$ admits a root with an over-order multiplicity. Given a delay $\tau > 0$, we denote by $\mathcal{V}$ the set of all functions $\Delta$ of the form $\Delta(\lambda) = P_0(\lambda) + \mathrm{e}^{-\lambda\tau}P_\tau(\lambda)$ with $P_0$ and $P_\tau$ given by (4) and $m = n-1$, i.e.,

$$\mathcal{V} = \left\{ \Delta\colon \mathbb{C} \to \mathbb{C} \;\middle|\; \exists\, a = (a_0, \ldots, a_{n-1}) \in \mathbb{R}^n, \; \exists\, \alpha = (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{R}^n \text{ such that } \Delta(\lambda) = \lambda^n + \sum_{k=0}^{n-1} a_n \lambda^k + \mathrm{e}^{-\lambda\tau} \sum_{k=0}^{n-1} \alpha_k \lambda^k \right\}.$$

Note that $\mathcal{V}$ is a real vector space with $\dim \mathcal{V} = 2n$, which is a subspace of the space of all entire complex functions, seen as a real vector space. In addition, $\mathcal{V}$ can be canonically identified with $\mathbb{R}^{2n}$ by identifying a quasipolynomial $\Delta$ with its coefficients $a_0, \ldots, a_{n-1}, \alpha_0, \ldots, \alpha_{n-1}$. In this setting, our first result provides a factorization of $\Delta$ in terms of such a root and an appropriate integral expression.

*Proposition 9:* Consider the quasipolynomial $\Delta$ from (3)–(4) with $m = n-1$. The real number $\lambda_0$ is a root of multiplicity at least $n+1$ of $\Delta$ if, and only if, there exists a polynomial $p$ of degree $n-1$ with $p(0) = 1$ such that

$$\Delta(\lambda) = \tau(\lambda - \lambda_0)^{n+1} \int_0^1 (1-t) p(t) \mathrm{e}^{-t(\lambda-\lambda_0)\tau} \, \mathrm{d}t. \tag{22}$$

*Proof:* Denote by $\mathcal{V}_{\lambda_0}$ the subset of $\mathcal{V}$ of those functions $\Delta$ admitting $\lambda_0$ as a root of multiplicity at least $n+1$, i.e.,

$$\mathcal{V}_{\lambda_0} = \left\{ \Delta \in \mathcal{V} \;\middle|\; \Delta^{(k)}(\lambda_0) = 0 \text{ for all } k \in \{0, \dots, n\} \right\}.$$

Each equation $\Delta^{(k)}(\lambda_0) = 0$, $k \in \{0, \dots, n\}$, defines a hyperplane in $\mathcal{V}$, and, when identifying $\mathcal{V}$ with the Euclidean space $\mathbb{R}^{2n}$, the normal vectors to all such hyperplanes are linearly independent. Hence $\mathcal{V}_{\lambda_0}$ is a subspace of $\mathcal{V}$ of codimension $n+1$, i.e., $\dim \mathcal{V}_{\lambda_0} = n-1$. Introduce now $\mathcal{W}_{\lambda_0}$ as the space of all functions $\Delta$ of the form (22) for some polynomial $p$ of degree $n-1$ with $p(0) = 1$. The set $\mathcal{W}_{\lambda_0}$ is an affine subspace of the space of all entire complex functions, with $\dim \mathcal{W}_{\lambda_0} = n-1$.

As a first step, we will prove that $\mathcal{W}_{\lambda_0} \subseteq \mathcal{V}$, i.e., that every function $\Delta$ of the form (22) is indeed a quasipolynomial of the form (3)–(4). To do so, we first observe that, by an immediate inductive integration by parts, we have (see also [32, Proposition 2.1])

$$\int_0^1 q(t) \mathrm{e}^{-zt} \, \mathrm{d}t = \sum_{k=0}^{d} \frac{q^{(k)}(0) - q^{(k)}(1)\mathrm{e}^{-z}}{z^{k+1}} \tag{23}$$

for every $z \in \mathbb{C} \setminus \{0\}$, $d \in \mathbb{N}$, and $q$ a polynomial of degree $d$. Next, let $\Delta \in \mathcal{W}_{\lambda_0}$ and $p$ be a polynomial of degree $n-1$ with $p(0) = 1$ be such that $\Delta$ is given by (22). Define $q(t) = (1-t)p(t)$ and notice that $q(1) = 0$. By using (23), we deduce:

$$\begin{aligned} \Delta(\lambda) &= \tau(\lambda - \lambda_0)^{n+1} \sum_{k=0}^{n} \frac{q^{(k)}(0) - q^{(k)}(1)\mathrm{e}^{-\tau(\lambda-\lambda_0)}}{\tau^{k+1}(\lambda-\lambda_0)^{k+1}} \\ &= (\lambda-\lambda_0)^n + \sum_{k=0}^{n-1} \frac{q^{(n-k)}(0)}{\tau^{n-k}} (\lambda-\lambda_0)^k \\ &\quad - \mathrm{e}^{-\tau(\lambda-\lambda_0)} \sum_{k=0}^{n-1} \frac{q^{(n-k)}(1)}{\tau^{n-k}} (\lambda-\lambda_0)^k, \end{aligned} \tag{24}$$

so that $\Delta \in \mathcal{V}$, as required.

We now notice that $\mathcal{W}_{\lambda_0} \subseteq \mathcal{V}_{\lambda_0}$, since, for any $\Delta$ given by (22), $\lambda_0$ is clearly a root of multiplicity at least $n+1$ of $\Delta$. Moreover, $\mathcal{W}_{\lambda_0}$ and $\mathcal{V}_{\lambda_0}$ are both affine spaces with the same dimension, so that $\mathcal{W}_{\lambda_0} = \mathcal{V}_{\lambda_0}$, yielding the conclusion. ∎

*Remark 10:* The integral representation in Proposition 9 is different from the one proposed in [36] since the latter considers the case of multiplicity $M(\lambda_0) = n+m = 2n-1$. In fact, the polynomial $\hat{p}$ obtained in [36, Theorem 9] can be written as $\hat{p}(t) = t^{m-1}(1-t)^{n-2}(1-At)$ where $A$ is a specified real number. It is easy to observe that for all $n \geq 2$, $\deg \hat{p} = n+m-2 = 2n-3$ while $\deg(p) = n-1$ with $p$ given by (22). Interestingly, the polynomials $p$ and $\hat{p}$ have degree one and coincide when $n = 2$, and the only over-order multiplicities are 3 and 4.

*Remark 11:* Note that (24) provides explicit expressions for the polynomials $P_0$ and $P_\tau$ from (4) in terms of the polynomial $q$ introduced in the above proof. More precisely, we have

$$P_0(\lambda) = (\lambda-\lambda_0)^n + \sum_{k=0}^{n-1} \frac{q^{(n-k)}(0)}{\tau^{n-k}} (\lambda-\lambda_0)^k,$$

$$P_\tau(\lambda) = -\mathrm{e}^{\tau\lambda_0} \sum_{k=0}^{n-1} \frac{q^{(n-k)}(1)}{\tau^{n-k}} (\lambda-\lambda_0)^k.$$

Since $q(t) = (1-t)p(t)$, one may also provide similar expressions of $P_0$ and $P_\tau$ in terms of $p$. Indeed, we have

$$\begin{aligned} P_0(\lambda) &= (\lambda-\lambda_0)^n \\ &\quad + \sum_{k=0}^{n-1} \frac{p^{(n-k)}(0) - (n-k)p^{(n-k-1)}(0)}{\tau^{n-k}} (\lambda-\lambda_0)^k, \end{aligned} \tag{25a}$$

$$P_\tau(\lambda) = \mathrm{e}^{\tau\lambda_0} \sum_{k=0}^{n-1} \frac{(n-k)p^{(n-k-1)}(1)}{\tau^{n-k}} (\lambda-\lambda_0)^k. \tag{25b}$$

Let us now identify the link between the polynomial $p$ from (22) and the elimination-produced polynomial $\mathbb{P}$ from (21).

*Proposition 12:* Consider the quasipolynomial $\Delta$ from (3)–(4) with $m = n-1$. The real number $\lambda_0$ is a root of multiplicity at least $n+1$ of $\Delta$ if, and only if, $\mathbb{P}(\lambda_0, \tau) = 0$ and

$$\Delta(\lambda) = \frac{\tau}{n!}(\lambda-\lambda_0)^{n+1} \int_0^1 \mathbb{P}(\lambda_0, \tau t) \mathrm{e}^{-t(\lambda-\lambda_0)\tau} \, \mathrm{d}t, \tag{26}$$

where $\mathbb{P}$ is the elimination-produced polynomial (21).

*Proof:* First, assume that $\mathbb{P}(\lambda_0, \tau) = 0$ and that $\Delta$ is given by (26). In step with (21), the function $t \mapsto \frac{1}{n!}\mathbb{P}(\lambda_0, \tau t)$ is a polynomial in $t$ of degree $n$ with $\frac{1}{n!}\mathbb{P}(\lambda_0, 0) = \frac{P_0^{(n)}(\lambda_0)}{n!} = 1$ and $\frac{1}{n!}\mathbb{P}(\lambda_0, \tau) = 0$, so that $\frac{1}{n!}\mathbb{P}(\lambda_0, \tau t) = (1-t)p(t)$ for some polynomial $p$ of degree $n-1$ with $p(0) = 1$. Hence, by Proposition 9, $\lambda_0$ is a root of multiplicity at least $n+1$ of $\Delta$.

Conversely, assume that $\lambda_0$ is a root of multiplicity at least $n+1$ of $\Delta$. Proposition 6 states that $\mathbb{P}(\lambda_0, \tau) = 0$ and, by Proposition 9, there exists a polynomial $p$ of degree $n-1$ with $p(0) = 1$ such that $\Delta$ is given by (22). In addition, due to Remark 11, we have

$$P_0(\lambda) = \sum_{k=0}^{n} \frac{q^{(n-k)}(0)}{\tau^{n-k}} (\lambda-\lambda_0)^k, \tag{27}$$

where $q$ is the polynomial defined by $q(t) = (1-t)p(t)$. On the other hand, since $P_0$ is a polynomial of degree $n$, it coincides with its Taylor expansion of order $n$ at $\lambda_0$, i.e.,

$$P_0(\lambda) = \sum_{k=0}^{n} \frac{P_0^{(k)}(\lambda_0)}{k!} (\lambda-\lambda_0)^k. \tag{28}$$

Using the uniqueness of the Taylor expansion at a given point and combining (27) and (28), we deduce that $P_0^{(k)}(\lambda_0) = \frac{k!}{\tau^{n-k}} q^{(n-k)}(0)$. Hence, by (21), we have

$$\begin{aligned} \mathbb{P}(\lambda_0, \tau t) &= \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{\tau^{n-k}} q^{(n-k)}(0) (\tau t)^{n-k} \\ &= n! \sum_{k=0}^{n} q^{(n-k)}(0) \frac{t^{n-k}}{(n-k)!} = n! q(t), \end{aligned}$$

since the last sum is the Taylor expansion of $q$ at 0 of order $n$ and $q$ is a polynomial of degree $n$. Consequently, $q(t) = \frac{1}{n!}\mathbb{P}(\lambda_0, \tau t)$ and $\Delta$ is given by (26). ∎

### C. Some Insights on Linear Combinations of Kummer Functions

Here, our goal is to establish links between a quasipolynomial $\Delta$ of the form (3)–(4) with $m = n-1$ admitting

a root of multiplicity at least $n+1$ and Kummer confluent hypergeometric functions. A first connection is stated below:

*Proposition 13:* Consider the quasipolynomial $\Delta$ from (3)–(4) with $m=n-1$. The real number $\lambda_0$ is a root of multiplicity at least $n+1$ of $\Delta$ if, and only if, $\mathbb{P}(\lambda_0,\tau)=0$ and

$$\Delta(\lambda)=(\lambda-\lambda_0)^{n+1}\sum_{k=0}^{n-1}\sigma_k\,\Phi(k+1,\,k+3,\,-\tau\,(\lambda-\lambda_0)) \quad (29)$$

where, for $k\in[\![0,n-1]\!]$, we have

$$\sigma_k=\frac{-\tau}{(k+1)\,(k+2)\,n!}\sum_{j=0}^{n-k-1}\binom{n}{j}P_0^{(j)}(\lambda_0)\,\tau^{n-j}. \quad (30)$$

*Proof:* Note that, owing to Proposition 12, it suffices to show that, if $\mathbb{P}(\lambda_0,\tau)=0$, then (26) can be rewritten as (29).

Let $q(t)=\frac{\tau}{n!}\mathbb{P}(\lambda_0,\tau t)$, then $q(1)=0$ since $\mathbb{P}(\lambda_0,\tau)=0$. As a result, the polynomial $q$ can be factorized as $q(t)=(1-t)p(t)$, where $p$ is a polynomial of degree $n-1$. We write $p(t)=\sum_{k=0}^{n-1}\widetilde{\sigma}_k t^k$ for some real coefficients $\widetilde{\sigma}_0,\dots,\widetilde{\sigma}_{n-1}$, and thus (26) can be rewritten as

$$\Delta(\lambda)=(\lambda-\lambda_0)^{n+1}\sum_{k=0}^{n-1}\widetilde{\sigma}_k\int_0^1 t^k(1-t)\mathrm{e}^{-t(\lambda-\lambda_0)\tau}\,\mathrm{d}t.$$

Thanks to (9), the above equation takes the form (29) after setting $\sigma_k=\frac{\widetilde{\sigma}_k}{(k+1)(k+2)}$ for $k\in[\![0,n-1]\!]$.

In order to conclude, it suffices to compute $\widetilde{\sigma}_k$ for $k\in[\![0,n-1]\!]$. To do that, from (21), we infer that

$$q(t)=\frac{\tau}{n!}\sum_{k=0}^{n}\binom{n}{k}P_0^{(k)}(\lambda_0)\tau^{n-k}t^{n-k},$$

and, since $q(t)=(1-t)p(t)$, we also have

$$q(t)=\widetilde{\sigma}_0+\sum_{k=1}^{n-1}(\widetilde{\sigma}_k-\widetilde{\sigma}_{k-1})t^k-\widetilde{\sigma}_{n-1}t^n.$$

Hence, equating the coefficients of monomials of the same degree in the two above expressions of $q$, we deduce, for every $k\in[\![0,n-1]\!]$, that

$$\widetilde{\sigma}_k=-\frac{\tau}{n!}\sum_{j=0}^{n-1-k}\binom{n}{j}P_0^{(j)}(\lambda_0)\tau^{n-j},$$

which concludes the proof. ∎

Note that (29) factorizes $\Delta$ in terms of a linear combination of $n$ Kummer functions with real coefficients. One can also express $\Delta$ as a combination of two Kummer functions if one allows for rational functions as coefficients.

*Proposition 14:* Consider the quasipolynomial $\Delta$ from (3)–(4) with $m=n-1$. The real number $\lambda_0$ is a root of multiplicity at least $n+1$ of $\Delta$ if, and only if, $\mathbb{P}(\lambda_0,\tau)=0$ and

$$\begin{aligned}\Delta(\lambda)=\beta(\lambda)\,\Phi(0,\,1,\,-\tau\,(\lambda-\lambda_0))\\+\gamma(\lambda)\,\Phi(1,\,1,\,-\tau\,(\lambda-\lambda_0)),\end{aligned} \quad (31)$$

with

$$\beta(\lambda)=-(\lambda-\lambda_0)^{n+1}\sum_{k=0}^{n-1}\frac{\sigma_k(k+2)!(k+1-\tau(\lambda-\lambda_0))}{(\tau(\lambda-\lambda_0))^{k+2}} \quad (32)$$

and

$$\gamma(\lambda)=(\lambda-\lambda_0)^{n+1}\sum_{k=0}^{n-1}\sigma_k\left(\frac{(k+1)^2(k+2)}{(\tau(\lambda-\lambda_0))^2}+\sum_{r=0}^{k-1}\frac{(k+2)!(k+1-\tau(\lambda-\lambda_0))}{r!(\tau(\lambda-\lambda_0))^{k+2-r}}\right), \quad (33)$$

where $\sigma_0,\dots,\sigma_{n-1}$ are defined as in (30).

*Proof:* According to Proposition 13, it suffices to show that (29) is equivalent to (31). This can be done by exploiting the Kummer functions' contiguous relations recalled in Lemma 4. As a matter of fact, using (10), one obtains, for every $k\in[\![0,n-1]\!]$ and $z\in\mathbb{C}\setminus\{0\}$, that

$$\begin{aligned}\Phi(k+1,k+3,z)&=-\frac{k+2}{z}\Phi(k,k+2,z)\\&\quad+\frac{k+2}{z}\Phi(k+1,k+2,z),\\\Phi(k,k+2,z)&=\frac{(k+1)(k+z)}{z}\Phi(k,k+1,z)\\&\quad-\frac{k(k+1)}{z}\Phi(k+1,k+1,z),\\\Phi(k+1,k+2,z)&=-\frac{k+1}{z}\Phi(k,k+1,z)\\&\quad+\frac{k+1}{z}\Phi(k+1,k+1,z).\end{aligned}$$

Next, we remark that, by (7), for every $a\in\mathbb{C}$ with $-a\notin\mathbb{N}$, we have $\Phi(a,a,z)=\Phi(1,1,z)=\mathrm{e}^z$. As a result,

$$\begin{aligned}&\Phi(k+1,k+3,z)\\&\quad=-\frac{(k+1)(k+2)(z+k+1)}{z^2}\Phi(k,k+1,z)\\&\qquad+\frac{(k+1)^2(k+2)}{z^2}\Phi(1,1,z).\end{aligned} \quad (34)$$

Again, (10) entails, for every $j\in\mathbb{N}^*$,

$$\Phi(j,j+1,z)=-\frac{j}{z}\Phi(j-1,j,z)+\frac{j}{z}\Phi(1,1,z),$$

and, by an immediate inductive argument,

$$\begin{aligned}\Phi(j,j+1,z)&=\frac{j!}{(-z)^j}\Phi(0,1,z)\\&\quad-\left(\sum_{r=0}^{j-1}\frac{j!}{r!(-z)^{j-r}}\right)\Phi(1,1,z).\end{aligned} \quad (35)$$

Combining the above with (34), we deduce that

$$\begin{aligned}\Phi(k+1,k+3,z)&=-\frac{(k+2)!(z+k+1)}{(-z)^{k+2}}\Phi(0,1,z)\\&\quad+\left(\sum_{r=0}^{k-1}\frac{(k+2)!(z+k+1)}{r!(-z)^{k+2-r}}\right)\Phi(1,1,z)\\&\quad+\frac{(k+1)^2(k+2)}{z^2}\Phi(1,1,z),\end{aligned} \quad (36)$$

and the conclusion follows by inserting (36) in (29). ∎

*Remark 15:* The formula for the Kummer function $\Phi(j,j+1,z)$, $j\in\mathbb{N}^*$, in (35) was derived by using an inductive argument and the fact that $\Phi(a,b,z)$ $(a,b,z\in\mathbb{C})$ can be expressed

as a linear combination of the contiguous Kummer functions $\Phi(a\pm1,b,z)$ and $\Phi(a,b\pm1,z)$, using also $\Phi(1,1,z)=e^z$, and $\Phi(0,1,z)=1$. Similar relations can be found in [53].

Beyond the standard contiguous relation, to the best of the authors' knowledge, there does not exist any result describing the distribution of the nonasymptotic zeros of linear combinations of Kummer functions. The next lemma provides a partial step towards that goal, by providing a non-autonomous second-order differential equation admitting a given linear combination of Kummer functions as a solution.

*Lemma 16:* Let $\widetilde{\beta}$ and $\widetilde{\gamma}$ be two meromorphic functions. Then, the complex function $F$ defined by

$$F(z)=\widetilde{\beta}(z)\,\Phi(0,1,z)+\widetilde{\gamma}(z)\,\Phi(1,1,z), \tag{37}$$

with $\widetilde{\beta}\,(z)\,\widetilde{\gamma}'\,(z)+\widetilde{\gamma}\,(z)\left(\widetilde{\beta}\,(z)\,\tau-\widetilde{\beta}'\,(z)\right)\neq 0$ satisfies the following second-order differential equation

$$F''(z)+Q(z)F'(z)+R(z)F(z)=0, \tag{38}$$

where $Q$ and $R$ are given by

$$\begin{cases} Q(z)=\dfrac{\vartheta(z)\widetilde{\gamma}\,(z)-\widetilde{\gamma}''\,(z)\,\widetilde{\beta}\,(z)}{\widetilde{\beta}\,(z)\,\widetilde{\gamma}'\,(z)+\widetilde{\gamma}\,(z)\left(\widetilde{\beta}\,(z)\,\tau-\widetilde{\beta}'\,(z)\right)}, \\ R(z)=\dfrac{\left(-\widetilde{\beta}\,(z)\,\tau+\widetilde{\beta}'\,(z)\right)\widetilde{\gamma}''\,(z)-\widetilde{\gamma}'\,(z)\,\vartheta(z)}{\widetilde{\beta}\,(z)\,\widetilde{\gamma}'\,(z)+\widetilde{\gamma}\,(z)\left(\widetilde{\beta}\,(z)\,\tau-\widetilde{\beta}'\,(z)\right)}, \end{cases} \tag{39}$$

with $\vartheta(z)=\left(\widetilde{\beta}''\,(z)+\tau\,\left(\widetilde{\beta}\,(z)\,\tau-2\,\widetilde{\beta}'\,(z)\right)\right)$.

Lemma 16 may be proved by using that $\frac{\partial\Phi}{\partial z}(a,b,z)=\frac{a}{b}\Phi(a+1,b+1,z)$, which follows immediately from (7), and exploiting the contiguous relations from Lemma 4. In what follows, we shall refer to functions $F$ of the form (37) as *Kummer-type functions*.

Note that Whittaker functions are defined in terms of Kummer functions in (11) by applying the multiplicative factor $\mathrm{e}^{-\frac{z}{2}}z^{\frac{1}{2}+l}$, thanks to which the Whittaker differential equation (12) has no first-order term. We now proceed similarly from Kummer-type functions in order to define *Whittaker-type functions*. The next lemma can be shown by straightforward computations.

*Lemma 17:* Let $\widetilde{\beta},\widetilde{\gamma}$ be two meromorphic functions, $F$ be the function defined in (37), and $Q$ and $R$ be given by (39). Let $\mathcal{Q}$ be a primitive of $\frac{Q}{2}$ and define the function $W$ by

$$W(z)=\mathrm{e}^{\mathcal{Q}(z)}F(z). \tag{40}$$

Then, $W$ satisfies the second-order differential equation

$$W''(z)+G(z)W(z)=0, \tag{41}$$

where

$$G(z)=R(z)-\frac{(Q(z))^2}{4}-\frac{1}{2}Q'(z). \tag{42}$$

In the sequel, we refer to functions $W$ of the form (40) as *Whittaker-type functions*.

### D. MID Validity for the Lowest Over-order Multiplicity

Now, we shall use the results of Section IV-C relating quasipolynomials with roots of over-order multiplicity and Kummer and Whittaker functions in order to provide *sufficient conditions* under which the MID property is valid for characteristic roots of $\Delta$ of multiplicity at least $n+1$.

*Theorem 18:* Consider the quasipolynomial $\Delta$ from (3)–(4) with $m=n-1$, and assume that $\Delta$ admits a real root $\lambda_0$ of multiplicity at least $n+1$. Let $\beta$ and $\gamma$ be the meromorphic functions defined in (32) and (33), respectively, and define the meromorphic functions $\widetilde{\beta}$ and $\widetilde{\gamma}$ by

$$\widetilde{\beta}(z)=\beta\left(\lambda_0-\frac{z}{\tau}\right),\quad \widetilde{\gamma}(z)=\gamma\left(\lambda_0-\frac{z}{\tau}\right).$$

Let $F$, $Q$, $R$, and $G$ be defined by (37), (39) and (42), respectively. Assume that, for every $t\in(0,1)$ and every root $z$ of $F$ in $\mathbb{C}_-$, we have $\Re[zG(tz)]\geq 0$. Then, $\lambda_0$ is a dominant root of $\Delta$, i.e., $\lambda_0$ satisfies the MID property.

*Remark 19:* A result similar to Theorem 18 was already shown in [36, Theorem 10] for the case of roots of multiplicity $n+m$. The proof of the former can be obtained by an easy adaptation of that of the latter, and we detail this argument here for the sake of completeness.

*Proof:* We deduce from Proposition 14 that

$$\Delta(\lambda)=F(-\tau(\lambda-\lambda_0)). \tag{43}$$

In particular, the result is thereby proved if we show that all roots of the Kummer-type function $F$ have nonnegative real part. To do so, we consider the Whittaker-type function $W(\cdot)$ defined from $F$ as in (40). Note that the differential equation (41) satisfied by $W$ is of the form (13), with $K(z)=1$. As a consequence, one can apply Hille's method to (41). By taking $z_1=0$ and $z_2$ equal to a root $z_*$ of $F(\cdot)$ in (14), we obtain:

$$\int_0^{z_*}|W'(z)|^2\overline{\mathrm{d}z}=\int_0^{z_*}|W(z)|^2G(z)\,\mathrm{d}z.$$

We choose as integration path the line segment from 0 to $z_*$. Hence

$$\overline{z_*}\int_0^1|W'(tz_*)|^2\,\mathrm{d}t=z_*\int_0^1|W(tz_*)|^2G(tz_*)\,\mathrm{d}t.$$

Taking the real part, we get

$$x_*\int_0^1|W'(tz_*)|^2\,\mathrm{d}t=\int_0^1|W(tz_*)|^2\Re\left[z_*G(tz_*)\right]\mathrm{d}t, \tag{44}$$

where $x_*=\Re(z_*)$ and $y_*=\Im(z_*)$.

Assume now, by contradiction, that $F(\cdot)$ admits a root with negative real part, and take $z_*$ in (44) as equal to this root. The left-hand side of (44) is negative, however its right-hand side is nonnegative by assumption, yielding the desired contradiction. Hence, all roots of $F$ have nonnegative real parts, entailing the conclusion thanks to (43). ■

*Remark 20:* As standard pole placement methods, the *partial pole placement* approach proposed for controlling dynamical systems is subject to the same sensitivity issues with respect to parameters' changes. However, the computational effort of the control scheme is low, emphasizing the interest

in using such an approach for infinite-dimensional systems. As discussed in [5], the spectral abscissa is a continuous function and, under small variations of the parameters, the stability of the closed-loop system is conserved. Finally, there exist several results describing the asymptotic behavior of multiple characteristic roots subject to parameter variation. In particular, in the case when the delay is the parameter subject to changes, we refer to [54]–[56] (see also [57] for further discussions).

## V. Prescribed Stabilization of the Chain of $n$ Integrators

This section focuses on the problem of stabilization of a chain of $n$ integrators with a prescribed exponential decay. In this particular configuration, we are able to establish necessary and sufficient conditions for the MID property to hold for a root with the lowest over-order multiplicity. The section closes on the presentation of a link between multiple roots and a local minimizer for the spectral abscissa.

### A. MID Property for the Chain of n Integrators

Let us consider the chain of $n$ integrators, i.e., we consider the function $\Delta$ from (3)–(4) with $P_0(\lambda) = \lambda^n$. In this case, we have the following characterization of the elimination-produced polynomial.

*Proposition 21:* Consider the quasipolynomial $\Delta$ from (3)–(4) in the case $P_0(\lambda) = \lambda^n$ and $m = n-1$. The elimination-produced polynomial $\mathbb{P}$ defined by (21) is given by

$$\mathbb{P}(\lambda, \tau) = \widehat{\mathbb{P}}(\lambda\tau), \tag{45}$$

where $\widehat{\mathbb{P}}$ is the polynomial defined by

$$\widehat{\mathbb{P}}(s) = \sum_{k=0}^{n} k! \binom{n}{k}^2 s^{n-k}. \tag{46}$$

In addition, all roots of $\widehat{\mathbb{P}}$ are negative real numbers.

*Proof:* Since $P_0(\lambda) = \lambda^n$, we have $P_0^{(k)}(\lambda) = \frac{n!}{(n-k)!}\lambda^{n-k}$ and thus, inserting these expressions into (21), we deduce (45)–(46).

Furthermore, since $P_0$ has $n$ real roots, all equal to 0, it follows from Proposition 8 that the polynomial $\lambda \mapsto \mathbb{P}(\lambda, \tau)$ is real rooted, and hence so is $\widehat{\mathbb{P}}$. Since all coefficients of $\widehat{\mathbb{P}}$ are positive, we infer that all its real roots are necessarily negative, yielding the conclusion. ∎

Recall that, by Proposition 12, when $\Delta$ admits a root of multiplicity at least $n+1$, the elimination-produced polynomial $\mathbb{P}$ appears in the factorization (26) under the form $\mathbb{P}(\lambda_0, \tau t)$. We now study the behavior of this expression, seen as a function of $t$ on the interval $(0, 1)$.

*Proposition 22:* Consider the quasipolynomial $\Delta$ from (3)–(4) in the case $P_0(\lambda) = \lambda^n$ and $m = n-1$, let $\widehat{\mathbb{P}}$ be defined by (46), and define $q(t) = \frac{1}{n!}\widehat{\mathbb{P}}(\lambda_0\tau t)$. Assume, in addition, that $\lambda_0\tau = c_0$, where $c_0$ is the rightmost root of $\widehat{\mathbb{P}}$. Then, for every $t \in (0, 1)$, we have $q(t) > 0$ and $q'(t) < 0$.

*Proof:* By Proposition 21, the polynomial $\widehat{\mathbb{P}}$ is real rooted and all its roots are negative, hence its rightmost root $c_0$ is negative. In addition, the coefficient of the leading monomial of $\widehat{\mathbb{P}}$ is positive, thus we also have that $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{P}}'$ are positive in $(c_0, +\infty)$, and in particular in $(c_0, 0)$. As $\lambda_0\tau = c_0 < 0$, this entails that $q$ is positive and its derivative is negative in $(0, 1)$, as required. ∎

Note that the factorization (26) involves an integral of the form

$$F(s) = \int_0^1 f(t)\mathrm{e}^{-st}\,\mathrm{d}t, \tag{47}$$

which can be seen as the Laplace transform of a (real) function $f$ with support included in $[0, 1]$. The study of the distribution of the zeros of functions $F$ under the form (47) is related to a wide range of problems related to Physics and Engineering, and goes back to the pioneering works by Hardy [58], Pólya [59], and Titchmarsh [60] in the first decades of the 20th century. In particular, we have the following result from [59] (see also [61, Part Five, Chapter 3, Problem 177, page 66]).

*Theorem 23 (G. Pólya, 1918):* Let $f$ be a positive and continuously differentiable function defined in the interval $[0, 1]$ and satisfying $f'(t) < 0$ for every $t \in [0, 1]$. Consider the function $F\colon \mathbb{C} \to \mathbb{C}$ defined in terms of $f$ as in (47). Then, all the zeros of $F$ lie in the open left half-plane $\mathbb{C}_-$.

As a consequence of Theorem 23, we infer the following result on the dominance of a root $\lambda_0$ of multiplicity $n+1$ in the case of the $n$-th order integrator.

*Theorem 24:* Consider the quasipolynomial $\Delta$ from (3)–(4) in the case $P_0(\lambda) = \lambda^n$ and $m = n-1$, and let $\widehat{\mathbb{P}}$ be the polynomial defined in (46). Assume that $\Delta$ admits a root $\lambda_0$ of multiplicity $n+1$ and that $\lambda_0\tau$ is the rightmost root of $\widehat{\mathbb{P}}$. Then $\lambda_0$ is the rightmost root of $\Delta$.

*Proof:* Owing to Propositions 12, 21, and 22, we have the following factorization,

$$\Delta(\lambda) = \tau(\lambda - \lambda_0)^{n+1} \int_0^1 q(t)\mathrm{e}^{-t(\lambda-\lambda_0)\tau}\,\mathrm{d}t, \tag{48}$$

where $q$ is the polynomial defined in the statement of Proposition 22. Since $q$ is positive and $q'$ is negative in $(0, 1)$, it follows from Theorem 23 that all roots of $s \mapsto \int_0^1 q(t)\mathrm{e}^{ts}\,\mathrm{d}t$ have positive real parts, and, consequently, all roots of $\Delta$ different from $\lambda_0$ have real parts strictly less than $\lambda_0$, as required. ∎

*Remark 25:* Note that, if $\lambda_0$ is a root of multiplicity $n+1$ of $\Delta$, then $\lambda_0$ is one of the $n$ roots of the elimination-produced polynomial $\mathbb{P}(\cdot, \tau)$. The previous theorem ensures that, by selecting $\lambda_0$ as the rightmost root of $\mathbb{P}(\cdot, \tau)$, it will also be a dominant root of $\Delta$.

### B. A Link with an Optimization Problem

An intriguing research question concerns whether and when the assignment of a root satisfying the MID property can be recast in terms of minimizing the spectral abscissa function. We shed light on this question by considering a chain of $n$ integrators with input delay, controlled with static state feedback. We start with a theorem.

*Theorem 26:* Consider the characteristic function

$$\Delta(\lambda;\ \alpha_0, \ldots, \alpha_{n-1}) = \lambda^n + (\alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0)\mathrm{e}^{-\lambda\tau}.$$

and the choice of gain parameters $(\alpha_0,\dots,\alpha_{n-1})=(\alpha_0^*,\dots,\alpha_{n-1}^*)$, which are determined by (25) and assign a rightmost root $\lambda_0$ of multiplicity $n+1$. Then in any direction in the parameter space, taken from $(\alpha_0^*,\dots,\alpha_{n-1}^*)$, the spectral abscissa function $c\colon \mathbb{R}^n\to\mathbb{R}$,

$$(\alpha_0,\dots\alpha_{n-1})\mapsto c(\alpha_0,\dots,\alpha_{n-1})=\max_{\lambda\in\mathbb{C}}\{\Re(\lambda)\mid \Delta(\lambda;\ \alpha_0,\dots,\alpha_{n-1})=0\},\tag{49}$$

is strictly increasing.

*Proof:* We denote by $(d_0,\dots,d_{n-1})\in\mathbb{R}^n$, $(d_0,\dots,d_{n-1})\neq(0,\dots,0)$, the considered direction, and by $\epsilon\in\mathbb{R}_+$ a perturbation parameter that determines the step taken in the considered direction. For fixed $(d_0,\dots,d_{n-1})$, we then analyze the zeros of the perturbed quasi-polynomial

$$H(\lambda;\ \epsilon)=\Delta(\lambda)+\epsilon(d_{n-1}\lambda^{n-1}+\dots+d_1\lambda+d_0)\mathrm{e}^{-\lambda\tau}$$

as a function of $\epsilon$, around $\epsilon=0$. Due to the continuity of the spectral abscissa function with respect to parameter $\epsilon$ and the property that $\lambda_0$ is the unique rightmost characteristic root for $\epsilon=0$, for sufficiently small $\epsilon>0$ the spectral abscissa is induced by one of the roots that emerge from the splitting of the $n+1$-th order root $\lambda_0$ as $\epsilon$ is increased from zero.

We can always write $H$ in the form

$$H(\lambda;\ \epsilon)=\Delta(\lambda)+\epsilon\big[f_0+f_1(\lambda-\lambda_0)+\dots+f_{n-1}(\lambda-\lambda_0)^{n-1}\big]\mathrm{e}^{-\lambda\tau},$$

where the $n$-tuple $(f_0,\dots,f_{n-1})$ is induced by $(d_0,\dots,d_{n-1})$. Let $j\in\{0,\dots,n-1\}$ be the index such that

$$f_0=\dots=f_{j-1}=0,\ f_j\neq 0.$$

Then we can factorize

$$H(\lambda;\ \epsilon)=(\lambda-\lambda_0)^j\Big(\hat H(\lambda)+\epsilon\big[f_j+f_{j+1}(\lambda-\lambda_0)+\dots+f_{n-1}(\lambda-\lambda_0)^{n-1-j}\big]\mathrm{e}^{-\lambda\tau}\Big),$$

where the entire function $\hat H$ is the analytic extension of the function $\lambda\mapsto\frac{H(\lambda;\ 0)}{(\lambda-\lambda_0)^j}$, which has a removable singularity at $\lambda_0$. Note that, due to the deflation, $\hat H$ has a zero with multiplicity $n+1-j$ at $\lambda_0$. The roots of

$$\hat H(\lambda)+\epsilon\big[f_j+f_{j+1}(\lambda-\lambda_0)+\dots+f_{n-1}(\lambda-\lambda_0)^{n-1-j}\big]\mathrm{e}^{-\lambda\tau}=0\tag{50}$$

satisfy the complete regular splitting property at $\lambda=\lambda_0$ and $\epsilon=0$ because $\frac{\partial\hat H}{\partial\epsilon}(\lambda_0,\ 0)=f_j\neq 0$ (see [62], [63]). Consequently, they can be expanded as Puiseux series in powers of $\epsilon^{\frac{1}{n+1-j}}$ as

$$\lambda_i(\epsilon)=\lambda_0+\left(-(n+1-j)!\frac{\frac{\partial\hat H}{\partial\epsilon}(\lambda_0,\ 0)}{\frac{\partial^{n+1-j}\hat H}{\partial\lambda^{n+1-j}}(\lambda_0,\ 0)}\right)^{\frac{1}{n+1-j}}\mathrm{e}^{i\frac{2\pi i}{n+1-j}}\ \epsilon^{\frac{1}{n+1-j}}+\mathcal{O}\left(\epsilon^{\frac{2}{n+1-j}}\right),\quad i=[\![1,n+1-j]\!].\tag{51}$$

The additional $j$ rightmost roots of $H$ are invariant with respect to $\epsilon$, that is, $\lambda_i(\epsilon)=\lambda_0,\ i\in[\![n+2-j,n+1]\!]$. In what follows we can distinguish between two cases: $j<n-1$ and $j=n-1$.

If $j<n-1$, then $n+1-j>2$ roots split according to (51). It follows that there is a real number $\bar\epsilon>0$ such that

$$\left(\max_{i\in\{1,\dots,n+1-j\}}\Re(\lambda_i(\epsilon))\right)>\Re(\lambda_0),\ \forall\epsilon\in(0,\ \bar\epsilon),$$

hence, the spectral abscissa is strictly increasing at $\epsilon=0$;

If $j=n-1$, then the form of expansion (51) may be inconclusive as two roots can possibly split along the imaginary axis. To proceed, we employ the factorization (48) and rewrite (50) as

$$\underbrace{\tau M(\lambda\tau)(\lambda-\lambda_0)^2}_{\hat H(\lambda)}+\epsilon f_{n-1}\mathrm{e}^{-\lambda\tau}=0,\tag{52}$$

with

$$M(\lambda)=\int_0^1 q(t)\mathrm{e}^{-t(\lambda-\lambda_0\tau)}\,\mathrm{d}t,$$

which implies $M(\lambda_0\tau)>0$ and $M'(\lambda_0\tau)<0$. In case $f_{n+1}<0$, then the double root of $\lambda_0$ of (52) splits according to

$$\lambda_\pm(\epsilon)=\lambda_0\pm\sqrt{\frac{-f_{n-1}\mathrm{e}^{-\lambda_0\tau}}{M(\lambda_0\tau)}}\sqrt{\frac{\epsilon}{\tau}}+\mathcal{O}(\epsilon),\tag{53}$$

whose derivation relies on interpreting (52), after pre-multiplication by $\tau$ as an equation in the argument $\lambda\tau$. In case $f_{n+1}>0$, then the double root $\lambda_0$ splits according to

$$\lambda_\pm(\epsilon)=\lambda_0\pm\sqrt{\frac{f_{n-1}\mathrm{e}^{-\lambda_0\tau}}{M(\lambda_0\tau)}}\sqrt{\frac{\epsilon}{\tau}}\,i+\frac{f_{n-1}\mathrm{e}^{-\lambda_0\tau}}{2M(\lambda_0\tau)}\left(1+\frac{M'(\lambda_0\tau)}{M(\lambda_0\tau)}\right)\epsilon+\mathcal{O}\left(\epsilon^{\frac{3}{2}}\right).\tag{54}$$

The coefficient of $\epsilon$ is positive since $M(\lambda_0\tau)>0$ and

$$1+\frac{M'(\lambda_0\tau)}{M(\lambda_0\tau)}=1-\frac{\int_0^1 tq(t)\,dt}{\int_0^1 q(t)\,dt}>0,$$

since $q$ is positive on $[0,1]$ and the 'weight' function $t$ in the top integral is smaller than one. Hence, if $\epsilon$ is increased from zero, $n-1$ characteristic roots are invariant, while two others split along the imaginary axis but bend towards the open right half plane. The proof is completed. ■

Observe that the (local) growth of the spectral abscissa in a given direction, characterized by (51), (53), and (54), can always be bounded from below by a strictly increasing linear function. On the one hand, this argument is mathematically not sufficient to conclude that $(\alpha_0^*,\dots,\alpha_{n-1}^*)$ is a strict local minimizer of the spectral abscissa function, given that this function is non-smooth. To see this, note that the function $s\colon\mathbb{R}^2\mapsto\mathbb{R}$, defined by

$$s(\alpha_0,\alpha_1)=\sqrt{|\alpha_1-\alpha_0^2|}-\alpha_0^4-\alpha_1^4,$$

satisfies (with $\epsilon\geq 0$)

$$s(\epsilon d_0,\epsilon d_1)=\begin{cases}\sqrt{|d_1|}\sqrt{\epsilon}+\mathcal{O}(\epsilon), & d_1\neq 0,\\ |d_0|\epsilon+\mathcal{O}\left(\epsilon^2\right), & d_1=0,\end{cases}$$

but $(0,0)$ is no local minimizer since $s\left(\epsilon,\epsilon^2\right) < 0$ for $\epsilon > 0$. On the other hand, for the second-order integrator, the assertion of Theorem 26 can be strengthened significantly.

*Theorem 27:* For $n = 2$, the spectral abscissa function (49) exhibits a strict global minimizer in $(\alpha_0,\alpha_1) = (\alpha_0^*,\alpha_1^*)$, determined by (25).

*Proof:* The argument is by contradiction. Assume that there exists $(\hat{\alpha}_0,\hat{\alpha}_1) \neq (\alpha_0^*,\alpha_1^*)$ for which we have $c\left(\hat{\alpha}_0,\hat{\alpha}_1\right) \leq c\left(\alpha_0^*,\alpha_1^*\right)$. For $n = 2$, the spectral abscissa function is quasi-convex, see [64, Proposition 1]. This implies $c\left(\alpha_0^* + \theta(\hat{\alpha}_0 - \alpha_0^*), \alpha_1^* + \theta(\hat{\alpha}_1 - \alpha_1^*)\right) \leq c\left(\alpha_0^*,\alpha_1^*\right)$ for all $\theta \in [0,\ 1]$, which contradicts Theorem 26. ■

For $n = 2,3,\ldots,10$, we have done extensive numerical experiments with the package TDS-CONTROL [65], whose stabilization routine is based on minimizing the spectral abscissa. For all considered initial values for the controller gains, the optimization algorithm converged to $\left(\alpha_0^*,\ldots,\alpha_{n-1}^*\right)$. This suggests that Theorem 27 can be generalized to $n > 2$, but a proof is currently lacking.

## VI. Prescribed Pendulum Stabilization and Partial Pole Placement

### A. GMID, Intermediate MID & Pendulum Control

Let us revisit the classical control problem of the stabilization of the friction-free pendulum [66], whose dynamics are governed by the following second-order differential equation:

$$\ddot{\theta}(t) + \frac{g}{L}\sin(\theta(t)) = u(t), \tag{55}$$

where $\theta(t)$ is the angular displacement of the pendulum at time $t$ with respect to the stable equilibrium position, $L$ the pendulum length, $g$ the gravitational acceleration, and $u(t)$ the control input, which stems from an applied external torque. We follow in this section the control strategy proposed in [36].

Consider the standard delayed PD controller

$$u(t) = -k_p\,\theta(t-\tau) - k_d\,\dot{\theta}(t-\tau), \tag{56}$$

with $(k_p,k_d) \in \mathbb{R}^2$. The linear stability of the closed-loop system is given by the location of the roots of the quasipolynomial

$$\Delta(\lambda) = \lambda^2 + \frac{g}{L} + (k_d\lambda + k_p)\,e^{-\lambda\tau}. \tag{57}$$

Notice that $\deg(\Delta) = 4$. Hence, if one exploits the GMID property proved in [32] (i.e., if we consider a root of (57) with multiplicity 4, it follows that the only admissible quadruple root is $\lambda_0 = -\sqrt{2g/L}$ which is necessarily the corresponding spectral abscissa and is achieved if the $(k_p,k_d,\tau)$ satisfy $k_d = -e^{-2}\sqrt{2g/L}$, $\quad k_p = -5\,e^{-2}g/L$, $\quad \tau = \sqrt{2L/g}$.

As emphasized in [33], the GMID does not allow any degree of freedom in assigning $\lambda_0$, i.e., there is a single possible choice of $\lambda_0$ ensuring that it is a root of multiplicity 4. In order to allow for some additional degrees of freedom when assigning $\lambda_0$, one can relax such a constraint by forcing the root $\lambda_0$ to have a multiplicity lower than the maximal one. Consider for instance, the delay as a *free tuning parameter*. This is the subject of the next result, extracted from [36], which considers a root $\lambda_0$ of multiplicity 3, which is nothing else than the lowest over-order multiplicity for (57).

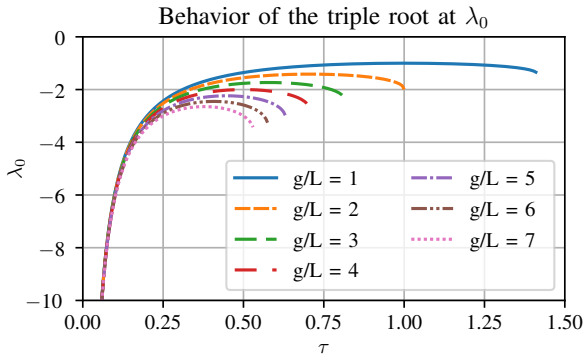



Fig. 2. The behavior of the triple root (spectral abscissa) of (57) at $\boldsymbol{\lambda = \lambda_0}$ given by (58) as a function of the tuning ("free") delay parameter $\boldsymbol{0 < \tau < \sqrt{2\,L/g}}$ for $\boldsymbol{g/L \in \{1,\ldots,7\}}$. Clearly, increasing the ratio $\boldsymbol{g/L}$ decreases the assignment region as well as the delay margin. Figure extracted from [36].

*Proposition 28 ([36]):* For any $0 < \tau < \sqrt{2\,L/g}$, let

$$\lambda_0 = \frac{-2 + \sqrt{-\frac{g\,\tau^2}{L} + 2}}{\tau}. \tag{58}$$

The delayed PD controller (56) with

$$k_d = \frac{2\,(\tau\lambda_0 + 1)\,\mathrm{e}^{\tau\lambda_0}}{\tau},\ k_p = \frac{2\left(5L\tau\lambda_0 + g\,\tau^2 + 3L\right)\mathrm{e}^{\tau\lambda_0}}{\tau^2 L} \tag{59}$$

and $\lambda_0\,\tau \geq -1$, locally exponentially stabilizes the dynamical system (55). More precisely, $\lambda_0$ is a root of multiplicity at least 3 of (57) and it is dominant, i.e., the lowest over-order (intermediate) MID property holds true.

The proof of this result can be found in [36], which is concerned with the over-order MID property with multiplicity $M(\lambda_0) = n + m$. In our case, this multiplicity coincides with the lowest over-order multiplicity of (57). It is also a direct consequence of the results of the present paper since, in the pendulum case under consideration, the only intermediate multiplicity corresponds to the lowest over-order multiplicity. It should be mentioned that Proposition 28 can also be proven by using the argument principle as it is done in [29].

### B. P3δ Software

Partial pole placement via delay action (P3$\delta$) is an intuitive `Python` software [67]–[69] which enables the design of stabilizing feedback control laws exploiting the delay effect on the closed-loop dynamics. The pole placement methods exploited in P3$\delta$ rely on two design strategies, the multiplicity-induced-dominancy (MID) presented in this paper and the *coexistent-real-roots-inducing-dominancy* (CRRID), presented, for instance in [70], and which consists in conditions on the system's parameters guaranteeing the *dominance of coexistent real spectral values*. When using the MID strategy on P3$\delta$, two options are proposed: the GMID-based design and the control-oriented MID-based design. The latter exploits the over-order intermediate multiplicity $M = n + 1$, offering sufficient freedom in parameters' choice. The present work represents the theoretical certification of the latter. Notice that both strategies adopted in P3$\delta$ control design, MID and CRRID,

allow prescribing the exponential decay rate of the closed-loop system.

## VII. Conclusion

This paper discusses the spectral properties of linear time-invariant dynamical systems represented by delay-differential equations. It exploits links between spectral values of an over-order intermediate admissible multiplicity for a quasipolynomial and the distribution of zeros of linear combinations of Kummer confluent hypergeometric functions. It proposes a delay-based control design methodology enabling the closed-loop system's solution to obey a prescribed exponential decay rate, opening perspectives in concrete applications including, among others, vibration control (see, e.g., [71], [72]). Finally, the proposed methodology is illustrated through the stabilization problem of the pendulum as well as the stabilization of chains of integrators.

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## APPENDIX: FREQUENCY BOUND ALGORITHM



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**Algorithm 1:** Estimation of a frequency bound for time-delay differential equations with a single delay

**Input:** $\tilde{\Delta}(z) = \tilde{P}_0(z) + \tilde{P}_\tau(z)\,\mathrm{e}^{-z}$; // Normalized quasipolynomial

**Input:** $\omega_0 > 0$; // Desired frequency bound

**Input:** maxOrd; // Maximal truncation order

// Initialization

ord $= 0$; // ord: order of truncation of the Taylor expansion of $\mathrm{e}^{2x}$;

Bound $=$ false;

**while** *(not* Bound*) and (*ord $\leq$ maxOrd*)* **do**

  Set $\mathcal{F}(x,\omega) = |\tilde{P}_\tau(x+i\omega)|^2 - |\tilde{P}_0(x+i\omega)|^2 T_{\mathrm{ord}}(x)$; // $T_{\mathrm{ord}}(x)$: Taylor expansion of $\mathrm{e}^{2x}$ of order $=$ ord

  Set $H(x,\Omega) = \mathcal{F}(x,\sqrt{\Omega})$; // $H$ is a polynomial

  Set $\Omega_k(x)$ as the $k$-th real root of $H(x,\cdot)$;

  **if** $\sup_{x\geq 0} \max_k \Omega_k(x) \leq \omega_0^2$ **then**

    Bound $=$ true;

  ord $=$ ord $+ 1$;

**Output:** Frequency bound: If Bound is true, then $|\omega| \leq \omega_0$ for every root $z = x + i\omega$ of $\tilde{\Delta}$ with $x \geq 0$;



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