A DENSITY PROBLEM FOR ORTHOGONAL RATIONAL FUNCTIONS

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ABSTRACT

Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of points in the open unit disk in the complex plane and let

$$\mathbb{B}_0 = 1 \quad \text{and} \quad \mathbb{B}_n(z) = \prod_{k=0}^{n} \frac{\overline{\alpha_k}}{|\alpha_k|} \frac{\alpha_k - z}{1 - \overline{\alpha_k} z}, \quad n = 1, 2, \ldots,$$

($\frac{\overline{\alpha_k}}{|\alpha_k|} = -1$ when $\alpha_k = 0$). We put

$$\mathcal{L} = \text{span}\{\mathbb{B}_n : n = 0, 1, 2, \ldots\}$$

and we consider the following "moment" problem:

Given a positive-definite Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{L} \times \mathcal{L}$, find a non-decreasing function μ on $[-\pi, \pi]$ (or a positive Borel measure μ on $[-\pi, \pi)$) such that

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\mu(\theta) \quad \text{for} \quad f, g \in \mathcal{L}.$$

We give a necessary and sufficient condition (called "N-extremality") on a solution μ of the moment problem in order that $\mathcal{L}$ is dense in L^2_μ.

KEYWORDS: orthogonal rational functions, moment problem, nested disks, N-extremal.

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1. Introduction. In [2] it is shown that a solution μ of the Hamburger moment problem is N-extremal if and only if the set of polynomials is dense in L^2_μ. The purpose of the present paper is to prove a similar statement for a moment problem that arises in the study of certain orthogonal rational functions. In both cases the results are based on nested disks-theories related to the coresponding moment problems. For orthogonal rational functions the nested disks-theory is developed in [6].

Let

$$T = \{z \in \mathbb{C} : |z| = 1\}, \quad D = \{z \in \mathbb{C} : |z| < 1\}, \quad E = \{z \in \mathbb{C} : |z| > 1\}$$

and let α_n, $n = 0, 1, 2, \ldots$ be given points in D with $\alpha_0 = 0$. The Blaschke factors ζ_n are given by

$$\zeta_n(z) = \frac{\overline{\alpha_n}}{|\alpha_n|} \cdot \frac{\alpha_n - z}{1 - \overline{\alpha_n} z}, \quad n = 0, 1, 2, \ldots$$

where by convention

$$\frac{\overline{\alpha_n}}{|\alpha_n|} = -1 \quad \text{when} \quad \alpha_n = 0.$$

The (finite) Blaschke products are

$$\mathbb{B}_n(z) = \prod_{k=1}^{n} \zeta_k(z), \quad n = 1, 2, \ldots \quad \text{and} \quad \mathbb{B}_0(z) = 1.$$

It will be convenient to write

$$\mathbb{B}_{n\setminus k} = \mathbb{B}_n / \mathbb{B}_k, \quad k = 0, 1, \ldots, n; \quad n = 0, 1, \ldots$$

We define the linear spaces $\mathcal{L}_n$, $n = 0, 1, 2, \ldots$ and $\mathcal{L}$ by

$$\mathcal{L}_n = \text{span}\{\mathbb{B}_m : m = 0, 1, \ldots, n\} \quad \text{and} \quad \mathcal{L} = \bigcup_{n=0}^{\infty} \mathcal{L}_n.$$

Clearly $\mathcal{L}_n$ consists of the functions that may be written as

$$\frac{p_n(z)}{\pi_n(z)}$$

where

$$\pi_n(z) = \prod_{k=1}^{n} (1 - \overline{\alpha_k} z), \quad n = 1, 2, \ldots \quad \text{and} \quad \pi_0(z) = 1$$

and p_n belongs to Π_n, the set of polynomials of degree at most n. The substar conjugate f_* of a function f is defined as

$$f_*(z) = \overline{f(1/\overline{z})}.$$

For $f \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ the superstar conjugate f^* will be

$$f^*(z) = \mathbb{B}_n(z) f_*(z).$$

If $f \in \mathcal{L}_0$, then $f^* = f_*$.
The linear spaces $\mathcal{L}_{n*}$, $n = 0, 1, 2, \ldots$, and $\mathcal{L}_*$ are defined as

$$\mathcal{L}_{n*} = \{f_* : f \in \mathcal{L}_n\} \quad \text{and} \quad \mathcal{L}_* = \{f_* : f \in \mathcal{L}\}.$$

Then we have

$$\mathcal{L}_{n*} = \text{span}\left\{\frac{1}{\mathbb{B}_m} : m = 0, 1, \ldots, n\right\} = \text{span}\left\{\frac{1}{\omega_m} : m = 0, 1, \ldots, n\right\}$$

where

$$\omega_m(z) = \prod_{k=1}^{m} (z - \alpha_k), \quad \text{and} \quad \omega_0(z) = 1.$$

Furthermore we assume that M is a linear functional on $\mathcal{L}+\mathcal{L}_*$ such that for $f \in \mathcal{L}$ we have

$$M(f_*) = \overline{M(f)}, \quad \text{and} \quad M(ff_*) > 0 \quad \text{if} \quad f \neq 0.$$

The functional M induces an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{L} \times \mathcal{L}$ by

$$\langle f, g \rangle = M(fg_*), \quad f, g \in \mathcal{L}.$$

Also for $f, g \in \mathcal{L}_*$ we may define $\langle f, g \rangle = M(fg_*)$. Then we get

$$\langle f, g \rangle = \langle g_*, f_* \rangle \quad \text{for} f, g \in \mathcal{L}.$$

As $\overline{\langle g, f \rangle} = \overline{M(gf_*)} = M(fg_*) = \langle f, g \rangle$ for $f, g \in \mathcal{L}$ and $\langle f, f \rangle = M(ff_*) > 0$ for $f \in \mathcal{L}$, $f \neq 0$, the inner product is Hermitian and positive-definite on $\mathcal{L} \times \mathcal{L}$.

In this paper we consider the following "moment" problem:

Given the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{L} \times \mathcal{L}$ (or the linear functional M on $\mathcal{L}+\mathcal{L}_*$), find a non-decreasing function μ on $[-\pi, \pi]$ (or a positive Borel measure μ on $(-\pi, \pi]$) such that

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})} d\mu(\theta) \quad \text{for} \quad f, g \in \mathcal{L}$$

$$(\text{or} \quad M(f) = \int_{-\pi}^{\pi} f(e^{i\theta}) d\mu(\theta) \quad \text{for} \quad f \in \mathcal{L}+\mathcal{L}_*).$$

In particular we give necessary and sufficient conditions for the density of $\mathcal{L}$ in L^2_μ. When

$$\sum_{n=1}^{\infty}(1 - |\alpha_n|) = \infty.$$

and μ is a solution of the moment problem then $\mathcal{L}$ is always dense in L^2_μ. This follows from the "closure criterion" discussed in Addendum A.2 of [1].

2. Orthogonal rational functions. In our approach orthogonal rational functions will play an important role. Let the sequence $\{\phi_n\}_{n=0}^{\infty}$ in $\mathcal{L}$ be obtained by orthonormalization of the sequence $\{\mathbb{B}_n\}_{n=0}^{\infty}$ with respect to the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{L} \times \mathcal{L}$, i.e.

$$\phi_n \in \mathcal{L}_n \quad \text{and} \quad \langle \phi_n, \phi_n \rangle = 1, \quad n = 0, 1, 2, \ldots$$

and

$$\langle f, \phi_n \rangle = 0 \quad \text{for} \quad f \in \mathcal{L}_{n-1}, \quad n = 1, 2, \ldots.$$

It follows easily that

$$\langle f, \phi_n^* \rangle = 0 \quad \text{for} \quad f \in \mathcal{L}_n(\alpha_n), \quad n = 1, 2, \ldots,$$

because $\mathbb{B}_n f_* \in \mathcal{L}_{n-1}$ for such f. Each ϕ_n can be written as

$$\phi_n(z) = \sum_{k=0}^{n} b_k^{(n)} \mathbb{B}_k(z).$$

Here the non-zero number $b_n^{(n)}$ is called the leading coefficient of ϕ_n. We assume that the ϕ_n are chosen such that $b_n^{(n)} > 0$ and we write $\kappa_n = b_n^{(n)}$. It is easily shown that

$$\kappa_n = \overline{\phi_n^*(\alpha_n)} = \phi_n^*(\alpha_n).$$

Using the uniqueness of the reproducing kernel

$$\sum_{k=0}^{n} \phi_k(z)\overline{\phi_k(w)}$$

for the inner product space $\mathcal{L}_n$ it can be shown, see for instance [3], that the following Christoffel-Darboux formula holds

$$\sum_{k=0}^{n-1} \phi_k(z)\overline{\phi_k(w)} = \frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \phi_n(z)\overline{\phi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}}, \tag{2.1}$$

The ϕ_n and ϕ_n^* satisfy the recurrence relations

$$\phi_n(z) = \epsilon_n \frac{z-\alpha_{n-1}}{1-\overline{\alpha_n}z}\frac{\kappa_n}{\kappa_{n-1}}\phi_{n-1}(z) + \delta_n \frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z}\frac{\kappa_n}{\kappa_{n-1}}\phi_{n-1}^*(z), \quad n=1,2,\ldots \tag{2.2}$$

and (superstar conjugation)

$$\phi_n^*(z) = -\frac{\overline{\alpha_n}}{|\alpha_n|}\overline{\delta_n}\frac{z-\alpha_{n-1}}{1-\overline{\alpha_n}z}\frac{\kappa_n}{\kappa_{n-1}}\phi_{n-1}(z) - \frac{\overline{\alpha_n}}{|\alpha_n|}\overline{\epsilon_n}\frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z}\frac{\kappa_n}{\kappa_{n-1}}\phi_{n-1}^*(z), \quad n=1,2,\ldots \tag{2.3}$$

with $\phi_0 = \phi_0^* = \kappa_0$. Here

$$\epsilon_n = -\frac{\overline{\alpha_n}}{|\alpha_n|}\frac{1-\overline{\alpha_{n-1}}\alpha_n}{1-|\alpha_{n-1}|^2}\frac{\overline{\phi_n^*(\alpha_{n-1})}}{\kappa_n}, \tag{2.4}$$

$$\delta_n = \frac{1-\alpha_{n-1}\overline{\alpha_n}}{1-|\alpha_{n-1}|^2}\frac{\phi_n(\alpha_{n-1})}{\kappa_n}. \tag{2.5}$$

The ϵ_n and δ_n are related by

$$|\epsilon_n|^2 - |\delta_n|^2 = \frac{\kappa_{n-1}^2}{\kappa_n^2}\frac{1-|\alpha_n|^2}{1-|\alpha_{n-1}|^2}. \tag{2.6}$$

In particular this implies

$$|\epsilon_n| > |\delta_n|.$$

For later use we mention the superstar conjugate with respect to z and w of the Christoffel-Darboux fomula (2.1),

$$\frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \phi_n(z)\overline{\phi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}} = \sum_{k=0}^{n-1}\mathbb{B}_{(n-1)\setminus k}(z)\overline{\mathbb{B}_{(n-1)\setminus k}(w)}\phi_k^*(z)\overline{\phi_k^*(w)}. \tag{2.7}$$

Proofs of (2.2)-(2.7) can be found in [3],[4], [5] and [6].

3. Associated functions. Next to the orthogonal functions ϕ_n we consider the associated functions ψ_n defined by

$$\psi_0(z) = -\frac{1}{\kappa_0}, \quad (\psi_0(z) = -M(\phi_0)),$$

and

$$\psi_n(z) = M(D(t,z)[\phi_n(z) - \phi_n(t)]), \quad n=1,2,\ldots.$$

Here M is acting on t and

$$D(t,z) = \frac{t+z}{t-z}.$$

Obviously $\psi_n \in \mathcal{L}_n$ for $n=0,1,2,\ldots$. For the superstar conjugates of the ψ_n we have

$$\psi_0^*(z) = -\frac{1}{\kappa_0}$$

and

$$\psi_n^*(z) = M\left(D(t,z)\left[\frac{\mathbb{B}_n(z)}{\mathbb{B}_n(t)}\phi_n^*(t) - \phi_n^*(z)\right]\right), \quad n=1,2,\ldots.$$

The functions ψ_n and ψ_n^* satisfy the recurrences

$$\psi_n(z) = \epsilon_n \frac{z-\alpha_{n-1}}{1-\overline{\alpha_n}z}\frac{\kappa_n}{\kappa_{n-1}}\psi_{n-1}(z) - \delta_n \frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z}\frac{\kappa_n}{\kappa_{n-1}}\psi_{n-1}^*(z), \quad n=1,2,\dots \tag{3.1}$$

and (superstar conjugation)

$$\psi_n^*(z) = \frac{\overline{\alpha_n}}{|\alpha_n|}\overline{\delta_n}\frac{z-\alpha_{n-1}}{1-\overline{\alpha_n}z}\frac{\kappa_n}{\kappa_{n-1}}\psi_{n-1}(z) - \frac{\overline{\alpha_n}}{|\alpha_n|}\overline{\epsilon_n}\frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z}\frac{\kappa_n}{\kappa_{n-1}}\psi_{n-1}^*(z), \quad n=1,2,\dots \tag{3.2}$$

A proof of the recurrence formulas (3.1) and (3.2) can be found in [3]. Another proof is given in [5]. The pair $(\psi_n, -\psi_n^*)$ satisfies the same recurrence as the pair (ϕ_n, ϕ_n^*). The initial values are $(\phi_0, \phi_0^*) = \kappa_0(1,1)$ and $(\psi_0, -\psi_0^*) = (-1/\kappa_0)(1,-1)$.

4. Analogues of the Liouville-Ostrogradskii formula (determinant formula) and Green's formula. In the previous section we have seen that the pairs $(\phi_n(z), \phi_n^*(z))$ and $(\psi_n(z), -\psi_n^*(z))$ satisfy the recurrence

$$\frac{\kappa_{n-1}}{\kappa_n}X_n(z) = \epsilon_n A_n(z)X_{n-1}(z) + \delta_n B_n(z)X_{n-1}^\dagger(z), \quad n=1,2,\dots, \tag{4.1a}$$

$$\frac{\kappa_{n-1}}{\kappa_n}X_n^\dagger(z) = \tau_n[\overline{\delta_n}A_n(z)X_{n-1}(z) + \overline{\epsilon_n}B_n(z)X_{n-1}^\dagger(z)], \quad n=1,2,\dots \tag{4.1b}$$

where

$$\tau_n = -\frac{\overline{\alpha_n}}{|\alpha_n|}, \quad A_n(z) = \frac{z-\alpha_{n-1}}{1-\overline{\alpha_n}z}, \quad B_n(z) = \frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z}, \quad n=1,2,\dots.$$

Suppose that the pair $(x_n(z), x_n^\dagger(z))$ satisfies (4.1) and suppose that the pair $(y_n(w), y_n^\dagger(w))$ satisfies (4.1) with z replaced by w. In [6] it has been shown that

$$\frac{x_n^\dagger(z)y_n(w) - x_n(z)y_n^\dagger(w)}{1-\frac{\zeta_n(z)}{\zeta_n(w)}} - \mathbb{B}_n(w)\frac{x_0^\dagger(z)y_0(w) - x_0(z)y_0^\dagger(w)}{1-\frac{\zeta_0(z)}{\zeta_0(w)}} = -\sum_{k=0}^{n-1} x_k(z)\mathbb{B}_{n\setminus k}(w)y_k^\dagger(w).$$

For $z=w$ we get the determinant formula

$$x_n^\dagger(z)y_n(z) - x_n(z)y_n^\dagger(z) = \frac{1-|\alpha_n|^2}{1-\overline{\alpha_n}z}\frac{z\mathbb{B}_n(z)}{z-\alpha_n}(x_0^\dagger(z)y_0(z) - x_0(z)y_0^\dagger(z)).$$

In particular for

$$x_n(z) = \phi_n(z), \quad x_n^\dagger(z) = \phi_n^*(z), \quad y_n(z) = \psi_n(z), \quad y_n^\dagger(z) = -\psi_n^*(z)$$

we obtain the analogue of the Liouville-Ostrogradskii formula

$$\phi_n^*(z)\psi_n(z) + \phi_n(z)\psi_n^*(z) = \frac{1-|\alpha_n|^2}{1-\overline{\alpha_n}z}\frac{-2z\mathbb{B}_n(z)}{z-\alpha_n} \tag{4.2}$$

as a special case of the determinant formula.

In [6] the analogue of Green's formula is derived in a similar way. This time we obtain

$$\frac{x_n^\dagger(z)\overline{y_n^\dagger(w)} - x_n(z)\overline{y_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}} - \frac{x_0^\dagger(z)\overline{y_0^\dagger(w)} - x_0(z)\overline{y_0(w)}}{1-z\overline{w}} = \sum_{k=0}^{n-1} x_k(z)\overline{y_k(w)}. \tag{4.3}$$

Notice that $1-\zeta_0(z)\overline{\zeta_0(w)} = 1-z\overline{w}$.

We only mention the following special cases of Green's formula. For

$$x_n(z) = \phi_n(z), \quad x_n^\dagger(z) = \phi_n^*(z), \quad y_n(w) = \psi_n(w), \quad y_n^\dagger(w) = -\psi_n^*(w)$$

we get

$$\frac{\phi_n^*(z)\overline{\psi_n^*(w)} + \phi_n(z)\overline{\psi_n(w)}}{1 - \zeta_n(z)\overline{\zeta_n(w)}} + \frac{2}{1 - z\overline{w}} = -\sum_{k=0}^{n-1} \phi_k(z)\overline{\psi_k(w)}. \tag{4.4}$$

For

$$x_n(z) = \psi_n(z), \quad x_n^\dagger(z) = -\psi_n^*(z), \quad y_n(w) = \psi_n(w), \quad y_n^\dagger(w) = -\psi_n^*(w)$$

we get a "Christoffel-Darboux" formula for the associated functions

$$\frac{\psi_n^*(z)\overline{\psi_n^*(w)} - \psi_n(z)\overline{\psi_n(w)}}{1 - \zeta_n(z)\overline{\zeta_n(w)}} = \sum_{k=0}^{n-1} \psi_k(z)\overline{\psi_k(w)},$$

and if $z = w$

$$\frac{|\psi_n^*(z)|^2 - |\psi_n(z)|^2}{1 - |\zeta_n(z)|^2} = \sum_{k=0}^{n-1} |\psi_k(z)|^2. \tag{4.5}$$

The superstar conjugate of (4.5) reads

$$\frac{|\psi_n^*(z)|^2 - |\psi_n(z)|^2}{1 - |\zeta_n(z)|^2} = \sum_{k=0}^{n-1} |\mathbb{B}_{(n-1)\setminus k}(z)|^2 |\psi_k^*(z)|^2.$$

5. Nested disks. Let

$$D_0 = \{z \in D : z \neq \alpha_j,\ j = 0, 1, 2, \ldots\} \quad \text{and} \quad E_0 = \{z \in E : z \neq 1/\overline{\alpha_j},\ j = 1, 2, \ldots\}.$$

For fixed $z \in D_0 \cup E_0$ the values of

$$s = R_n(z, w) = \frac{\psi_n(z) - w\psi_n^*(z)}{\phi_n(z) + w\phi_n^*(z)}$$

describe a circle, say $K_n(z)$, if w varies in T. Indeed, by the Christoffel-Darboux formula (2.1) and formula (4.5) we have

$$0 < \frac{||\psi_n(z)| - |\psi_n^*(z)||}{|\phi_n(z)| + |\phi_n^*(z)|} \leq |s| \leq \frac{|\psi_n(z)| + |\psi_n^*(z)|}{||\phi_n(z)| - |\phi_n^*(z)||} < \infty.$$

As $w \in T$, the equation of $K_n(z)$ is

$$|\psi_n^*(z) + s\phi_n^*(z)| = |\psi_n(z) - s\phi_n(z)|. \tag{5.1}$$

Since the pairs$(\phi_n(z), \phi_n^*(z))$ and $(\psi_n(z), -\psi_n^*(z))$ are (independent) solutions of the recurrency (4.1) also the pair $(\psi_n(z) - s\phi_n(z), -\psi_n^*(z) - s\phi_n^*(z))$ is a solution of (4.1). Hence by the analogue of Green's formula (4.3) we have for the given z

$$\frac{|\psi_n^*(z) + s\phi_n^*(z)|^2 - |\psi_n(z) - s\phi_n(z)|^2}{1 - |\zeta_n(z)|^2} - \frac{|\frac{1}{\kappa_0} - s\kappa_0|^2 - |\frac{1}{\kappa_0} + s\kappa_0|^2}{1 - |z|^2} = \sum_{k=0}^{n-1} |\psi_k(z) - s\phi_k(z)|^2.$$

Since the first term in the left hand side of this equation is zero, the equation of the circle $K_n(z)$ is

$$\sum_{k=0}^{n-1} |\psi_k(z) - s\phi_k(z)|^2 = \frac{2(s + \overline{s})}{1 - |z|^2}. \tag{5.2}$$

Clearly the circular disk $\Delta_n(z)$ corresponding to $K_n(z)$ is given by

$$\sum_{k=0}^{n-1} |\psi_k(z) - s\phi_k(z)|^2 \leq \frac{2(s+\overline{s})}{1-|z|^2}. \tag{5.3}$$

It follows from (5.2) that

$$K_n(z) \subset \{s \in \mathbb{C} : \Re s > 0\} \quad \text{if} \quad z \in D_0$$

and

$$K_n(z) \subset \{s \in \mathbb{C} : \Re s < 0\} \quad \text{if} \quad z \in E_0.$$

Centre $c_n(z)$ and radius $r_n(z)$ of $K_n(z)$ follow easily from (5.1). We have

$$c_n(z) = -\frac{\psi_n^*(z)\overline{\phi_n^*(z)} + \psi_n(z)\overline{\phi_n(z)}}{|\phi_n^*(z)|^2 - |\phi_n(z)|^2},$$

$$r_n(z) = \left| \frac{\psi_n^*(z)\phi_n(z) + \phi_n^*(z)\psi_n(z)}{|\phi_n^*(z)|^2 - |\phi_n(z)|^2} \right|.$$

Using (4.4) and (2.1) with $z = w$ we get

$$c_n(z) = \frac{\frac{2}{1-|z|^2} + \sum_{k=0}^{n-1} \psi_k(z)\overline{\phi_k(z)}}{\sum_{k=0}^{n-1} |\phi_k(z)|^2}.$$

Using (4.2) and (2.1) with $z = w$ we get

$$r_n(z) = \left| \frac{\frac{1-|\alpha_n|^2}{1-\overline{\alpha_n}z} \cdot \frac{-2z\mathbb{B}_n(z)}{z-\alpha_n}}{(1-|\zeta_n(z)|^2) \cdot \sum_{k=0}^{n-1} |\phi_k(z)|^2} \right| = \frac{2|z|}{|1-|z|^2|} \cdot \frac{|\mathbb{B}_{n-1}(z)|}{\sum_{k=0}^{n-1} |\phi_k(z)|^2}. \tag{5.4}$$

Formula (5.3) implies that the $\Delta_n(z)$ are nested disks,

$$\Delta_n(z) \subset \Delta_{n+1}(z), \quad n = 1, 2, \ldots.$$

Moreover, by (5.2), the circles $K_n(z)$ and $K_{n+1}(z)$ touch if z is not a zero of ϕ_n or if both $\phi_n(z)$ and $\psi_n(z)$ are zero.

The intersection of the disks $\Delta_n(z)$ is denoted as $\Delta_\infty(z)$. Clearly $\Delta_\infty(z)$ is a circular disk (with a positive radius) or $\Delta_\infty(z)$ is a point. The limiting circle, which may reduce to a point, is denoted as $K_\infty(z)$.

In the sequel we say that $\Delta_\infty(z)$ is a "disk" if $\Delta_\infty(z)$ is not a single point. Thus by a disk we mean a disk with a positive radius. Similarly we say that $K_\infty(z)$ is a "circle" if $K_\infty(z)$ does not reduce to a single point. The inequality for $\Delta_\infty(z)$ is

$$\sum_{k=0}^{\infty} |\psi_k(z) - s\phi_k(z)|^2 \leq \frac{2(s+\overline{s})}{1-|z|^2}. \tag{5.5}$$

As we have nested disks, (5.4) implies that the sequence

$$\left(\frac{|\mathbb{B}_n(z)|}{\sum_{k=0}^{n} |\phi_k(z)|^2} \right)_{n=0}^{\infty}$$

is non-increasing, (obvious for $|z| < 1$), and $\Delta_\infty(z)$ is a point if and only if this sequence tends to zero as $n \to \infty$.

When $z_0 \in D_0 \cup E_0$ and the (infinite) Blaschke product

$$\mathbb{B}(z) = \prod_{k=1}^{\infty} \frac{\overline{\alpha_k}}{|\alpha_k|} \frac{\alpha_k - z}{1 - \overline{\alpha_k}z} \tag{5.6}$$

diverges, then $\Delta_\infty(z)$ is a point. For $z \in D_0$ this follows from (5.4) because $\mathbb{B}_n(z) \to 0$ as $n \to \infty$ for such z. When $z \in E_0$, we use (2.7) with $z = w$,

$$\frac{|\phi_n^*(z)|^2 - |\phi_n(z)|^2}{1 - |\zeta_n(z)|^2} = \sum_{k=0}^{n-1} |\mathbb{B}_{(n-1)\setminus k}(z)|^2 |\phi_k^*(z)|^2$$

to obtain from (5.4) that

$$r_n(z) = \frac{2|z|}{|1-|z|^2|} \cdot \frac{1}{|\mathbb{B}_{n-1}(z)| \sum_{k=0}^{n-1} |\mathbb{B}_k(z)|^{-2} |\phi_k^*(z)|^2}.$$

This implies

$$r_n(z) \le \frac{2|z|}{|1-|z|^2|} \cdot \frac{1}{|\mathbb{B}_{n-1}(z)||\phi_0^*(z)|^2}.$$

As $z \in E_0$, we have $\mathbb{B}_n(z) \to \infty$ as $n \to \infty$, so it follows that $r_n(z) \to 0$ as $n \to \infty$. Hence also in this case $\Delta_\infty(z)$ is a point. Thus, when $z_0 \in D_0 \cup E_0$ and $\Delta_\infty(z)$ is a disk, then the Blaschke product (5.6) converges.

The following theorem is proved in [6].

THEOREM 5.1. (INVARIANCE). Let $z_0 \in D_0 \cup E_0$ be such that $\Delta_\infty(z_0)$ is a disk. Then $\Delta_\infty(z)$ is a disk for every $z \in D_0 \cup E_0$ and

$$\sum_{k=0}^{\infty} |\phi_k(z)|^2 \quad \text{and} \quad \sum_{k=0}^{\infty} |\psi_k(z)|^2$$

converge uniformly on every compact subset of $D_0 \cup E_0$.

We now may speak of an alternative:

either $\Delta_\infty(z)$ is a disk for every $z \in D_0 \cup E_0$,

or $\Delta_\infty(z)$ is a point for every $z \in D_0 \cup E_0$.

Thus we may say that Δ_∞ (or K_∞) is a disk (circle) or Δ_∞ (or K_∞) is a point without specifying any $z \in D_0 \cup E_0$.

6. The moment problem. Let $\mathcal{M}$ be the set of all the solutions of the moment problem mentioned in section 1. Recall that a solution μ is a non-decreasing real valued function on $[-\pi, \pi]$ such that

$$\int_{-\pi}^{\pi} R(t) d\mu(\theta) = M(R) \quad \text{for} \quad R \in \mathcal{L} + \mathcal{L}_*, \quad (t = e^{i\theta}).$$

In [6] it is observed that $\mathcal{M} \neq \emptyset$. Two solutions μ_1 and μ_2 are considered to be equal if they determine the same continuous linear functional on $C(T)$, i.e. if

$$\int_{-\pi}^{\pi} f(t) d\mu_1(\theta) = \int_{-\pi}^{\pi} f(t) d\mu_2(\theta) \quad \text{for all} \quad f \in C(T).$$

This means that μ_1 and μ_2 determine the same regular countably additive measure on the σ-field of the Borel sets in $(-\pi, \pi]$. This is just the case if there is a constant C such that $\mu_1(\theta) - \mu_2(\theta) = C$ at all θ where $\mu_1 - \mu_2$ is continuous. (Consider the Fourier series for $\mu_1 - \mu_2$).

It follows from the positivity of the functional M that every solution μ of the moment problem has infinitely many points of increase. (The corresponding Borel measure has infinite support).

For $\mu \in \mathcal{M}$ we define

$$F_\mu(z) = \int_{-\pi}^{\pi} \frac{t+z}{t-z} d\mu(\theta), \quad (t = e^{i\theta}).$$

THEOREM 6.1. For fixed $z \in D_0 \cup E_0$ we have

$$\{F_\mu(z) : \mu \in \mathcal{M}\} = \Delta_\infty(z).$$

COROLLARY 6.2. In the case of a limiting disk, for each $s \in \Delta_\infty(z)$, $(z \in D_0 \cup E_0)$, there is a $\mu \in \mathcal{M}$ such that $s = F_\mu(z)$. In this case the moment problem has more than one solution.

COROLLARY 6.3. In the case of a limiting point the moment problem has a unique solution.

Proofs of theorem 6.1 and of corollary 6.3 can be found in [6].

7. N-extremal solutions of the moment problem. As in Akhiezer's book [2] we say that a solution $\mu \in \mathcal{M}$ is N-extremal at the point $z \in D_0 \cup E_0$ if $s = F_\mu(z)$ belongs to the boundary $K_\infty(z)$ of $\Delta_\infty(z)$. If $\Delta_\infty(z)$ is a disk, then $\mu \in \mathcal{M}$ is N-extremal at $z \in D_0 \cup E_0$ if and only if

$$\sum_{k=0}^{\infty} |\psi_k(z) - s\phi_k(z)|^2 = 2\frac{s+\overline{s}}{1-|z|^2}. \tag{7.1}$$

If $\Delta_\infty(z)$ is a point, say s, then $s \in \Delta_n(z)$ for each n, so

$$0 \le 2\frac{s+\overline{s}}{1-|z|^2} - \sum_{k=0}^{n-1} |\psi_k(z) - s\phi_k(z)|^2 \le r_n^2 \cdot \sum_{k=0}^{n-1} |\phi_k(z)|^2 \to 0 \quad \text{as} \quad n \to \infty,$$

where r_n is the radius of $K_n(z)$,

$$r_n = \frac{2|z|}{|1-|z|^2|} \, \frac{|\mathbb{B}_{n-1}(z)|}{\sum_{k=0}^{n-1} |\phi_k(z)|^2}.$$

Hence also in this case we have (7.1).

THEOREM 7.1. Let $\mu \in \mathcal{M}$. Then
a) If μ is N-extremal at some point $z \in D_0 \cup E_0$, then $\mathcal{L}$ is dense in L^2_μ
and
b) If $\mathcal{L}$ is dense in L^2_μ, then μ is N-extremal at every point $z \in D_0 \cup E_0$.

Proof. Let $z \in D_0 \cup E_0$. Put

$$s = F_\mu(z) = \int_{-\pi}^{\pi} \frac{t+z}{t-z} d\mu(\theta), \quad (t = e^{i\theta})$$

and let

$$f_z(t) = \frac{1+\overline{z}t}{1-\overline{z}t}.$$

Then

$$f_z(t) = \overline{\left(\frac{t+z}{t-z}\right)} = -\overline{f_{1/\overline{z}}(t)}.$$

Let

$$\sum_{k=0}^{\infty} \gamma_k \phi_k$$

be the generalized Fourier series for f_z. Then

$$\gamma_k = \int_{-\pi}^{\pi} f_z(t)\overline{\phi_k(t)} d\mu(\theta), \quad k = 0, 1, 2, \ldots,$$

so

$$\gamma_0 = \kappa_0 \int_{-\pi}^{\pi} f_z(t) d\mu(\theta) = \kappa_0 \overline{s}$$

and

$$\begin{aligned} \gamma_k &= \overline{\int_{-\pi}^{\pi} \frac{t+z}{t-z}(\phi_k(t) - \phi_k(z)) d\mu(\theta)} + \overline{\int_{-\pi}^{\pi} \frac{t+z}{t-z}\phi_k(z) d\mu(\theta)} \\ &= \overline{-\psi_k(z) + s\phi_k(z)}, \quad k = 1, 2, \ldots \end{aligned}$$

and Bessel's inequality for f_z reads

$$\sum_{k=0}^{\infty} |\gamma_k|^2 \leq \int_{-\pi}^{\pi} \left| \frac{t+z}{t-z} \right|^2 d\mu(\theta).$$

¿From

$$\left| \frac{t+z}{t-z} \right|^2 = -1 + \frac{1+|z|^2}{1-|z|^2}(f_z(t) + \overline{f_z(t)})$$

we get

$$\int_{-\pi}^{\pi} \left| \frac{t+z}{t-z} \right|^2 d\mu(\theta) = -\int_{-\pi}^{\pi} d\mu(\theta) + \frac{1+|z|^2}{1-|z|^2}(s+\overline{s}) = \frac{-1}{\kappa_0^2} - (s+\overline{s}) + \frac{2(s+\overline{s})}{1-|z|^2}.$$

Notice that $\phi_0(z) = \kappa_0 > 0$ and $\psi_0(z) = -1/\kappa_0$. Thus we also have

$$|\psi_0(z) - s\phi_0(z)|^2 = |\frac{1}{\kappa_0} + s\kappa_0|^2 = \frac{1}{\kappa_0^2} + s + \overline{s} + |s|^2\kappa_0^2 = \frac{1}{\kappa_0^2} + s + \overline{s} + |\gamma_0|^2.$$

Hence Bessel's inequality becomes

$$\sum_{k=0}^{\infty} |\psi_k(z) - s\phi_k(z)|^2 - \frac{1}{\kappa_0^2} - (s+\overline{s}) \leq \frac{-1}{\kappa_0^2} - (s+\overline{s}) + \frac{2(s+\overline{s})}{1-|z|^2},$$

so

$$\sum_{k=0}^{\infty} |\psi_k(z) - s\phi_k(z)|^2 \leq \frac{2(s+\overline{s})}{1-|z|^2}, \tag{7.2}$$

the inequality for $\Delta_\infty(z)$. Hence

$$\mu \textit{ is } N-\textit{extremal at } z \iff s \in K_\infty(z) \iff \textit{equality sign in } (7.2) \iff f_z \in \mathrm{cl}\mathcal{L}.$$

Here $\mathrm{cl}\mathcal{L}$ is the closure of $\mathcal{L}$ in the Hilbert space L^2_μ. As so far z is arbitrary in $D_0 \cup E_0$, this proves part b).

In order to prove part a) we assume that z is a fixed point in $D_0 \cup E_0$ with the property that μ is N-extremal at z. From

$$F_\mu(1/\overline{z}) = -\overline{F_\mu(z)}$$

it follows that

$$\Delta_\infty(1/\overline{z}) = -\overline{\Delta_\infty(z)}$$

and in particular for the boundary points we get

$$K_\infty(1/\overline{z}) = -\overline{K_\infty(z)}.$$

Hence μ is also N-extremal at $1/\overline{z}$ and $f_{1/\overline{z}} \in \mathrm{cl}\mathcal{L}$. As $1 \in \mathcal{L}$ this implies that both

$$\frac{1}{t-z} \quad \text{and} \quad \frac{1}{1-\overline{z}t}$$

belong to $\mathrm{cl}\mathcal{L}$. We will show that

$$\frac{1}{(t-z)^k} \quad \text{and} \quad \frac{1}{(1-\overline{z}t)^k}$$

are in $\mathrm{cl}\mathcal{L}$ for $k = 1, 2, \ldots$.

For each n and k and each polynomial $p \in \Pi_n$ there are a constant A and a polynomial $q \in \Pi_{n-1}$ such that

$$\frac{1}{t-z}\left(\frac{1}{(t-z)^k} - \frac{p(t)}{\pi_n(t)}\right) = \frac{1}{(t-z)^{k+1}} - \frac{A}{t-z} - \frac{q(t)}{\pi_n(t)},$$

(in fact $A = p_n(z)/\pi_n(z)$ and $q(t) = (p(t) - A\pi_n(t))/(t-z)$), so

$$\left\| \frac{1}{(t-z)^{k+1}} - \frac{A}{t-z} - \frac{q(t)}{\pi_n(t)} \right\| \leq \frac{1}{|1-|z||} \left\| \frac{1}{(t-z)^k} - \frac{p(t)}{\pi_n(t)} \right\|.$$

As q/π_n and p/π_n are in $\mathcal{L}$, it follows by mathematical induction that

$$\frac{1}{(t-z)^k} \in \mathrm{cl}\mathcal{L}, \quad k = 1, 2, \ldots.$$

In a similar way we obtain

$$\frac{1}{(1-\overline{z}t)^k} \in \mathrm{cl}\mathcal{L}, \quad k = 1, 2, \ldots.$$

Now let $g \in L^2_\mu$ such that

$$\int_{-\pi}^{\pi} \phi(t)\overline{g(t)}d\mu(\theta) = 0 \tag{7.3}$$

for all $\phi \in \mathcal{L}$. In order to show that $\mathcal{L}$ is dense in L^2_μ it is sufficient to show that $g(e^{i\theta}) = 0$ almost everywhere on $[-\pi, \pi]$. To do so we put

$$\nu(x) = \int_{-\pi}^{x} \overline{g(t)}d\mu(\theta), \quad -\pi < x \leq \pi.$$

Then ν is of bounded variation on $(-\pi, \pi]$ since $g \in L^2_\mu$ implies that $g \in L^1_\mu$, so

$$H(\zeta) = \int_{-\pi}^{\pi} \frac{d\nu(\theta)}{t-\zeta}, \quad (t = e^{i\theta}),$$

is analytic in $\mathbb{C} \setminus T$. As (7.3) also holds for $\phi \in \mathrm{cl}\mathcal{L}$ it follows that

$$\int_{-\pi}^{\pi} \frac{d\nu(\theta)}{(t-z)^k} = 0, \quad \int_{-\pi}^{\pi} \frac{d\nu(\theta)}{(1-\overline{z}t)^k} = 0 \quad k = 1, 2, \ldots.$$

But then $H(\zeta) \equiv 0$ on $\mathbb{C} \setminus T$ as H and all its derivatives vanish at z and at$1/\overline{z}$. Considering the power series of the function H around 0 and around ∞ we see that

$$\int_{-\pi}^{\pi} t^k d\nu(\theta) = 0, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{7.4}$$

Since ν defines a continuous linear functional on $C(T)$ by

$$\phi \longmapsto \int_{-\pi}^{\pi} \phi(t)d\nu(\theta), \quad \phi \in C(T)$$

and the trigonometric polynomials are dense in $C(T)$ it follows from (7.4) that this functional is zero. In particular this implies $\nu(x) = 0$ for all $x \in (-\pi, \pi]$. Hence $g(e^{i\theta}) = 0$ almost everywhere on $(-\pi, \pi]$. □

Theorem 7.1 implies that a solution μ of the moment problem is N-extremal at all points $z \in D_0 \cup E_0$ when μ is N-extremal at at least one point $z \in D_0 \cup E_0$. In this case we say that μ is N-extremal, without specifying any $z \in D_0 \cup E_0$. This leads to

THEOREM 7.2. Let $\mu \in \mathcal{M}$. Then the following are equivalent:
a) μ is N-extremal,
b) $\mathcal{L}$ is dense in L^2_μ.

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