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On the Normal Form of Linear Time-Delay Systems with Application to Output Feedback Stabilization ⋆

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Abstract

This paper mainly investigates the normal form of linear systems with non-commensurate state, input and output delays. This work begins with introducing delay operators, so that the linear time-delay systems can be associated with a system over a multivariate polynomial ring. A design procedure for constructing a null-space matrix and its left inverse matrix is proposed based on the equivalence of polynomial matrices, such that under the assumption of a pure relative degree, a novel construction method of the state transformation matrix, referred to as the unimodular completion method, can be given. Then the system can be transformed into a normal form with isolated zero dynamics, which contributes to exploring some properties of the system, including invariant zeros, spectral controllability, spectral observability and system inversion. As an application of the normal form, the output feedback stabilization problem is addressed. Numerical examples are given to illustrate the validity of the obtained results.

Key words: linear time-delay systems; multivariate polynomial ring; equivalence of polynomial matrices; normal form; output feedback stabilization.

1 Introduction

The type of relative degree, for which the coupling matrix is invertible, called regular relative degree in the case of linear delay-free systems and pure relative degree in the case of linear delay systems, is of great account. The conditions of regular (pure) relative degree are restrictive, thus it does not always exist, while output transformations can be constructed to induce the regular relative degree of the transformed system [37]. Other studies on the regular (pure) relative degree can be found in [9,10]. The regular (pure) relative degree is useful for solving the input-output decoupling problem [33], the tracking problem [7], the feedback stabilization problem [38]. More importantly, the existence of regular (pure) relative degree is necessary for the existence of a normal form [20].

The normal form is instrumental in solving various control problems, such as the feedback stabilization [3], the output tracking [37], the system inversion [13], the observer design [11]. Besides, the zero dynamics and invariant zeros can be exposed by the normal form [2]. For linear delay-free systems, the theory at the basis of the relative degree and the normal form is mature. Scholars have turned the spotlight on linear time-delay systems, which widely appeared in engineering practice

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[22,34,36]. From the conditions for a pure relative degree, it follows that the normal form of linear time-delay systems must be investigated in the framework of polynomial rings. By delay operators, linear time-delay systems can be associated with systems over a polynomial ring. For a linear system with commensurate delays, the associated system is over the univariate polynomial ring. Many properties of the univariate polynomial ring and matrices over this ring have been investigated thoroughly, including the division algorithm and the equivalence between a matrix and its Smith form, which provide favorable preparations for the study of normal forms. Based on the properties of polynomial matrices, the existence of an invertible state transformation matrix was proved in [12,13], so that the system can be transformed into a normal form with zero dynamics. A geometric condition weaker than pure relative degree was proposed in [2] to transform the system into a general normal form, so that the zero dynamics can be revealed under additional conditions. More generally, a linear system with non-commensurate delays corresponds to a system over the multivariate polynomial ring. However, it is not trivial to extend the research schemes on an univariate polynomial ring to a multivariate polynomial ring, caused by their different algebraic structures. In fact, the theory of polynomial matrices, even bivariate polynomial matrices, is incomplete or even lacking [35]. Thus the derivation of the normal form of linear systems with non-commensurate delays suffers huge challenges. It was pointed out in [19] that the invertibility requirements of coupling matrix and state transformation matrix are major obstacles to build a normal form. The normal form of the system was established in [31], where the existence of an invertible state transformation matrix was directly obtained by the Quillen-Suslin theorem [27], but no construction method for the state transformation matrix was given.

The output feedback stabilization problem is one of the most important control problems for linear time-delay systems [23]. The output feedback stabilization problem can be directly solved by the pole placement method [8]. A controller gain synthesis method was presented in [1] to design a strongly stabilizing proportional-integral-derivative (PID) output feedback controller by minimizing the robust spectral abscissa of the closed-loop system as a function of controller parameters. In addition, the matrix transformation method in [16], the algebraic method in [17], and the linear matrix inequality method [21] were proven to be effective for solving the output feedback stabilization problem. It was reported in [38] that for linear delay-free systems, a proportional-derivative (PD) high-gain output feedback controller can be designed based on a Lyapunov approach to stabilize a minimum-phase system, which, however, is no longer applicable to systems over polynomial rings.

Utilizing the representations of the associated systems over a multivariate polynomial ring, the normal forms of linear systems with non-commensurate delays and the application to output feedback stabilization are investigated in this paper, with emphasis on developing a novel unimodular completion method to construct a state transformation matrix. Compared with the existing literature, the contributions of this article are as follows. Firstly, for a given right invertible matrix, we propose a design procedure to construct a null-space matrix and its left inverse based on the equivalence of polynomial matrices. Secondly, using the result for the null-space matrix, an unimodular completion method is proposed to construct an unimodular state transformation matrix so that the system can be transformed into a normal form with isolated zero dynamics. Then, with the normal form, some fundamental properties of the system are studied. Finally, a new PD high-gain output feedback controller is designed by using the normal form, so that an output feedback stabilization problem can be solved.

The remainder of this article is as follows. Preliminaries are introduced in section 2. The main results are given in section 3. The output feedback stabilization problem is investigated in section 4. The conclusions of this article are drawn in section 5.

Notations: Let $A^{\mathrm{T}}/A^{-1}/\det(A)$ denote the transpose/inverse/determinant of matrix A. Let $y^{(k)}$ represent the k-th time derivative of a function y. Let $\Re(s)/\Im(s)$ denote the real part/imaginary part of complex number s. Denote the real number field/complex number field by $\mathbf{R}/\mathbf{C}$. Denote the set of n-dimensional vectors with elements in $\mathbf{R}/\mathbf{C}$ by $\mathbf{R}^n/\mathbf{C}^n$. Denote the set of $p \times q$ real-valued matrices by $\mathbf{R}^{p\times q}$. Let $I_p \in \mathbf{R}^{p\times p}$ denote an identity matrix, and let $0_{p,q}$ be an $p \times q$ zero matrix. Let $\mathbf{R}[d]$ be the multivariate polynomial ring with real coefficients in algebraic variables $d = (d_1, d_2, \ldots, d_h) \in \mathbf{C}^h$. Let $\mathbf{R}[z, d]$ be the multivariate polynomial ring with real coefficients in variables $(z, d_1, \ldots, d_h) \in \mathbf{C}^{h+1}$. Let $\mathbf{R}^n[d]$ represent the set of n-dimensional vectors with elements in $\mathbf{R}[d]$. Let $\mathbf{R}^{p\times q}[d]$ stand for the set of all $p \times q$ matrices whose entries are in $\mathbf{R}[d]$. Let $\mathbf{R}_l^{p\times q}[d]$ represent the set of matrices with normal rank l in $\mathbf{R}^{p\times q}[d]$, where the normal rank of a matrix $F(d) \in \mathbf{R}^{p\times q}[d]$ is defined by $\operatorname{nrank}(F(d)) = \max_{(d_1,d_2,\ldots,d_h)\in\mathbf{C}^h}\{\operatorname{rank}(F(d))\}$. Denote the direct sum of matrices A and B by $A \oplus B$.

2 Preliminaries

Consider the linear system with non-commensurate state, input and output delays simultaneously

$$\begin{cases} \dot{x}(t) = \sum\limits_{i=0}^{h} A_i x(t-\tau_i) + \sum\limits_{i=0}^{h} B_i u(t-\tau_i), \\ y(t) = \sum\limits_{i=0}^{h} C_i x(t-\tau_i), \end{cases} \tag{1}$$

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where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and $y \in \mathbf{R}^m$ denote the state, input and output vectors, respectively, $A_i \in \mathbf{R}^{n\times n}$, $B_i \in \mathbf{R}^{n\times m}$, $C_i \in \mathbf{R}^{m\times n}$, $i = 0, 1, \ldots, h$ are coefficient matrices, and $0 = \tau_0 < \tau_1 < \cdots < \tau_h$ represent time delays. Note that the existence of delays causes the system (1) to become infinite-dimensional. Let δ_i be the delay (shift) operator corresponding to τ_i defined by $f(t) \mapsto f(t-\tau_i)$, and denote the Laplace transform of δ_i by $\delta_i(s) = \mathrm{e}^{-s\tau_i}$, where s is the Laplace variable. Then system (1) can be associated with a finite-dimensional system over rings. In fact, by formally replacing the delay operator δ_i with an algebraic variable d_i (replacing the identity operator δ_0 with $d_0 = 1$), system (1) can be written as

$$\begin{cases} \dot{x}(t) = A(d)x(t) + B(d)u(t), \\ y(t) = C(d)x(t), \end{cases} \tag{2}$$

where $A(d) = \sum_{i=0}^{h} A_i d_i \in \mathbf{R}^{n\times n}[d]$, $B(d) = \sum_{i=0}^{h} B_i d_i \in \mathbf{R}^{n\times m}[d]$ and $C(d) = \sum_{i=0}^{h} C_i d_i \in \mathbf{R}^{m\times n}[d]$ are coefficient matrices. Since the field $\mathbf{R}$ is a subring of $\mathbf{R}[d]$, we have $x(t) \in \mathbf{R}^n[d]$, $u(t) \in \mathbf{R}^m[d]$ and $y(t) \in \mathbf{R}^m[d]$. Denote the i-th component of u and y by u_i and y_i respectively, also let $B(d) = [b_1(d)\ b_2(d)\ \cdots\ b_m(d)]$ and $C(d) = [c_1^{\mathrm{T}}(d)\ c_2^{\mathrm{T}}(d)\ \cdots\ c_m^{\mathrm{T}}(d)]^{\mathrm{T}}$. As pointed out in [6], systems (1) and (2) present the same signal flow graph. Thus control problems regarding the input-output behavior of system (1), such as the system inversion [4] and output tracking [7], can be expressed in terms of the input-output behavior of system (2), and may be solved in the framework of systems over rings. In turn, solutions to these control problems found in this framework can be brought back to the delay-differential framework. In the rest of this paper, we will omit the variable d in some places for brevity. Firstly, some vital notions and results in systems theory will be given below.

Definition 1 *[6] The ordered set $\{r_1, r_2, \ldots, r_m\}$, $1 \le r_i \le n$ is called the relative degree of system (2) if $c_i(d)A^k(d)B(d) = 0$, $k = 0, 1, \ldots, r_i - 2$ and $c_i(d)A^{r_i-1}(d)B(d) \neq 0$, $i = 1, 2, \ldots, m$. The relative degree $\{r_1, r_2, \ldots, r_m\}$ is said to be pure if the coupling matrix $\Gamma(d) \in \mathbf{R}^{m\times m}[d]$ is invertible, namely, $\det(\Gamma(d)) \in \mathbf{R}\backslash\{0\}$ is a nonzero constant, where $\Gamma(d) = \left[\left(c_1(d)A^{r_1-1}(d)B(d)\right)^{\mathrm{T}} \ \cdots\ \left(c_m(d)A^{r_m-1}(d)B(d)\right)^{\mathrm{T}}\right]^{\mathrm{T}}$.*

The condition of pure relative degree is restrictive since in general we have $\det(\Gamma(d)) \in \mathbf{R}[d]$. The relationship between input and output of system (2) can be characterized by the notion of relative degree. In fact, if system (2) has relative degree $\{r_1, r_2, \ldots, r_m\}$, then taking the time derivative of component y_i gives

$$y_i^{(k)} = c_i(d)A^k(d)x, \quad k = 0, 1, \ldots, r_i - 1, \tag{3}$$
$$y_i^{(r_i)} = c_i(d)A^{r_i}(d)x + c_i(d)A^{r_i-1}(d)B(d)u. \tag{4}$$

Definition 2 *[12] $z \in \mathbf{C}$ is called an invariant zero of system (1) if z reduces the rank of Rosenbrock matrix*

$$R(s) = \begin{bmatrix} A(d) - sI_n & B(d) \\ C(d) & 0 \end{bmatrix} \in \mathbf{C}^{(n+m)\times(n+m)},$$

where $d = (\mathrm{e}^{-s\tau_1}, \ldots, \mathrm{e}^{-s\tau_h})$, namely, $\operatorname{rank}(R(z)) < \operatorname{nrank}(R(s)) = \max_{s\in\mathbf{C}} \{\operatorname{rank}(R(s))\}$.

System (1) is called spectral controllable/observable if all its eigenvalues are controllable/observable [30,24]. Criteria of spectral controllability and spectral observability that are easy to check are given below.

Lemma 1 *[30,24] System (1) is spectrally controllable if and only if $\operatorname{rank}[sI_n - A(d)\ \ B(d)] = n$, $\forall s \in \mathbf{C}$, where $d = (\mathrm{e}^{-s\tau_1}, \ldots, \mathrm{e}^{-s\tau_h})$. Likewise, system (1) is spectrally observable if and only if $\operatorname{rank}\left[sI_n - A^{\mathrm{T}}(d)\ \ C^{\mathrm{T}}(d)\right]^{\mathrm{T}} = n$, $\forall s \in \mathbf{C}$.*

Definition 3 *[5] If $\det(zI_n - A(d)) \in \mathcal{H}$, where $\mathcal{H} = \{f(z, d_1, \ldots, d_h) \in \mathbf{R}[z, d] \mid f(\gamma, \mathrm{e}^{-\gamma\tau_1}, \ldots, \mathrm{e}^{-\gamma\tau_h}) \neq 0 \text{ for all } \gamma \in \mathbf{C} \text{ with } \Re(\gamma) \ge 0\}$ is called the Hurwitz set, then system (2) is said to be $\mathcal{H}$-stable.*

It should be noted that the $\mathcal{H}$-stability of system (2) corresponds to the exponential stability of system (1). Before exploring the normal form of system (2), some definitions and properties on polynomial matrices with elements in ring $\mathbf{R}[d]$ are necessary to be introduced. A matrix $F(d) \in \mathbf{R}^{p\times q}[d]$, $p \le q$ is said to be of normal full rank if there exists an $p \times p$ minor that is not identically zero, or equivalently, $p = \operatorname{nrank}(F(d)) = \max_{(d_1,d_2,\ldots,d_h)\in\mathbf{C}^h} \{\operatorname{rank}(F(d))\}$ [14]. In addition, a matrix $F(d) \in \mathbf{R}^{p\times q}[d], p \le q$ is called right invertible (right unimodular) if there exists a matrix $F^+(d) \in \mathbf{R}^{q\times p}[d]$ such that $F(d)F^+(d) = I_p$ [2]. A left invertible (left unimodular) matrix is defined in a similar way. A square matrix is called invertible (unimodular) if it is both left invertible and right invertible, or equivalently, its determinant is a nonzero constant in $\mathbf{R}$. The equivalence between polynomial matrices refers to the unimodular equivalence [15]. In fact, matrices $F(d) \in \mathbf{R}^{p\times q}[d]$ and $F_1(d) \in \mathbf{R}^{p\times q}[d]$ are said to be equivalent if there exist invertible matrices $U(d) \in \mathbf{R}^{p\times p}[d]$ and $V(d) \in \mathbf{R}^{q\times q}[d]$ such that $F_1(d) = U(d)F(d)V(d)$.

Definition 4 *[15] Let $F(d) \in \mathbf{R}^{p\times q}[d]$ be of normal rank l, and let $a_{i1}, a_{i2}, \ldots, a_{i,\alpha_i}$, where $\alpha_i = \frac{p!}{i!(p-i)!}\frac{q!}{i!(q-i)!}$, $i \in \{1, 2, \ldots, l\}$, be all $i \times i$ minors of $F(d)$. Then the monic greatest common divisor of $a_{i1}, a_{i2}, \ldots, a_{i,\alpha_i}$, denoted as $D_i(F)$, is called the i-th order determinant factor of $F(d)$. Also, $b_{i1} = \frac{a_{i1}}{D_i(F)}, b_{i2} = \frac{a_{i2}}{D_i(F)}, \ldots, b_{i,\alpha_i} = \frac{a_{i,\alpha_i}}{D_i(F)}$ is called the reduced i-th order minors of $F(d)$.*

Definition 5 *$N(d) \in \mathbf{R}_{q-l}^{q\times(q-l)}[d]$ is called a null-space*

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matrix of matrix $F(d) \in \mathbf{R}_l^{p\times q}[d]$ *if its columns can span the null space of* $F(d)$*, that is,* $F(d)N(d) = 0$.

For a right invertible matrix $F(d) \in \mathbf{R}_p^{p\times q}[d]$, $p \le q$, next we discuss the existence of its null-space matrix. In particular, for $F \in \mathbf{R}^{p\times q}$, a left invertible null-space matrix exists, characterized by a basis of solutions of equation $Fz = 0$. If $F(d) \in \mathbf{R}_p^{p\times q}[d]$ is an univariate polynomial matrix, there exists a left invertible null-space matrix $N(d) \in \mathbf{R}^{q\times(q-p)}[d]$. In fact, there are invertible matrices $U(d) \in \mathbf{R}^{p\times p}[d]$ and $V(d) \in \mathbf{R}^{q\times q}[d]$ such that $F(d)$ can be reduced to its Smith form, that is, $U(d)F(d)V(d) = [I_p\ 0]$. By partitioning $V(d)$ as $[V_1(d)\ V_2(d)]$, where $V_1(d) \in \mathbf{R}^{q\times p}[d]$ and $V_2(d) \in \mathbf{R}^{q\times(q-p)}[d]$, we have $F(d)\,[V_1(d)\ \ V_2(d)] = \left[U^{-1}(d)\ \ 0\right]$. Thus, $V_2(d)$ can be treated as a null-space matrix of $F(d)$, which is left invertible. In the general case of $h \ge 2$, the equivalence between a polynomial matrix and its Smith form is much more complicated. A recent result can be stated as follows.

Lemma 2 *[18] Let* $F(d) \in \mathbf{R}_l^{p\times q}[d]$ *and* $D_l(F) \in \mathbf{R}[d_1]$*. Then* $F(d)$ *is equivalent to its Smith form if and only if all the reduced* i*-th order minors of* $F(d)$ *generate* $\mathbf{R}[d]$*,* $i = 1, 2, \ldots, l$.

Inspired by the equivalence between polynomial matrices, the following result is obtained when $h \ge 2$, necessary for constructing the state transformation matrix and further establishing the normal form of system (2).

Theorem 1 *Let* $F(d) \in \mathbf{R}_p^{p\times q}[d]$*,* $p \le q$ *be right invertible. Then there exists a left invertible null-space matrix* $N(d) \in \mathbf{R}^{q\times(q-p)}[d]$ *such that* $F(d)N(d) = 0$.

Theorem 1 is proven step-wise.

Definition 6 *[35] Let* $F(d) \in \mathbf{R}_p^{p\times q}[d]$*,* $p \le q$*. Then* $F(d)$ *is said to be: (i) zero left prime (ZLP) if all the* $p \times p$ *minors of* $F(d)$*,* $a_{p1}, a_{p2}, \ldots, a_{p,\alpha_p}$ *are zero coprime,* i.e.*, they have no common zeros in* $\mathbf{C}^h$*. (ii) minor left prime (MLP) if all the* $p \times p$ *minors of* $F(d)$ *are relatively prime,* i.e.*,* $D_p(F) = 1$*. (iii) factor left prime (FLP) if for any polynomial matrix decomposition* $F(d) = F_1(d)F_2(d)$ *with* $F_1(d) \in \mathbf{R}^{p\times p}[d]$*,* $F_1(d)$ *must be an unimodular matrix.*

Lemma 3 *[35] ZLP* $\Rightarrow$ *MLP* $\Rightarrow$ *FLP.*

Lemma 4 *Let* $F(d) \in \mathbf{R}_p^{p\times q}[d]$*,* $p \le q$ *be right invertible. Then all the reduced* i*-th order minors of* $F(d)$*,* $b_{i1}, b_{i2}, \ldots, b_{i,\alpha_i}$*,* $i = 1, 2, \ldots, p$*, are zero coprime,* i.e.*, there is no* h*-tuple* $(z_1, z_2, \ldots, z_h) \in \mathbf{C}^h$ *such that it is the common zero of* $b_{i1}, b_{i2}, \ldots, b_{i,\alpha_i}$.

Proof. By definition, there exists a matrix $F^+(d) \in \mathbf{R}^{q\times p}[d]$, such that $F(d)F^+(d) = I_p$. It follows from Cauchy-Binet formula that $\forall d = (d_1, d_2, \ldots, d_h) \in \mathbf{C}^h$, $1 = \det\left(F(d)F^+(d)\right) = \sum_{\mathrm{I}\in\mathcal{S}} \det(F_{\mathrm{I}}(d)) \det(F_{\mathrm{I}}^+(d))$, where $\mathcal{S}$ is the family of subsets $\mathrm{I} = \{i_1, i_2, \ldots, i_p\}$ with p elements in $\{1, 2, \ldots, q\}$, $F_{\mathrm{I}}(d) \in \mathbf{R}^{p\times p}[d]$ is a submatrix consisting of all rows and the i_1-th, i_2-th, …, i_p-th columns of $F(d)$, and $F_{\mathrm{I}}^+(d) \in \mathbf{R}^{p\times p}[d]$ is a submatrix consisting of the i_1-th, i_2-th, …, i_p-th rows and all columns of $F^+(d)$. Then by contradiction, we can easily obtain that $F(d)$ is ZLP, namely, all $p \times p$ minors of $F(d)$, $a_{p1}, a_{p2}, \ldots, a_{p,\alpha_p}$, are zero coprime, which, by the Laplace theorem, implies that all $j \times j$ minors of $F(d)$, $a_{j1}, a_{j2}, \ldots, a_{j,\alpha_j}$, $j = 1, 2, \ldots, p-1$, are also zero coprime. In addition, it follows from Lemma 3 that $F(d)$ is MLP, that is, $D_p(F) = 1$, leading to $D_j(F) = 1$, $j = 1, 2, \ldots, p-1$. Hence, all the reduced i-th minors of $F(d)$, i.e., $b_{i1}, b_{i2}, \ldots, b_{i,\alpha_i}$, $i = 1, 2, \ldots, p$, are zero coprime. The proof is complete. ■

Now we are ready to prove Theorem 1.

Proof. According to Lemmas 2 and 4 as well as Lemma 3.1 in [28], $F(d)$ is equivalent to its Smith form, that is, there exist invertible matrices $U(d) \in \mathbf{R}^{p\times p}[d]$ and $V(d) \in \mathbf{R}^{q\times q}[d]$ such that $U(d)F(d)V(d) = [I_p\ 0]$, from which we obtain $F(d)V(d) = \left[U^{-1}(d)\ \ 0\right]$. Partitioning $V(d)$ as $V(d) = [V_1(d)\ V_2(d)]$, where $V_1(d) \in \mathbf{R}^{p\times p}[d]$ and $V_2(d) \in \mathbf{R}^{p\times(q-p)}[d]$. Then we can take $V_2(d)$ as a null-space matrix of $F(d)$, which is clearly left invertible. Thus, the proof is finished. ■

To proceed, for the right invertible matrix $F(d)$ in Theorem 1, we present a procedure to construct a left invertible null-space matrix $N(d)$ step by step. Firstly, by elementary row operations, we transform $F(d)$ into a row-echelon form $F_0(d)$, namely, $U_0(d)F(d) = F_0(d)$, where $U_0(d) \in \mathbf{R}^{p\times p}[d]$ is invertible, and

$$U_0(d)F(d) = \begin{bmatrix} F_{11}(d) & F_{12}(d) & \cdots & F_{1a}(d) \\ & F_{22}(d) & \cdots & F_{2a}(d) \\ & & \ddots & \vdots \\ & & & F_{aa}(d) \end{bmatrix},$$

where $F_{ii}(d) \in \mathbf{R}_{p_i}^{p_i\times q_i}[d]$ is right invertible, $p = p_1 + \cdots + p_a$ and $q = q_1 + \cdots + q_a$. Notice that the procedure can be guaranteed by the equivalence between $F(d)$ and its Smith form, and some column exchanges for $F(d)$ may yield diagonal submatrices with fewer rows, making the following recursive process easier. Next we solve a basis of the solutions of equations $F_0(d)X = 0$, where $X = [X_1^{\mathrm{T}}\ X_2^{\mathrm{T}}\ \cdots\ X_a^{\mathrm{T}}]^{\mathrm{T}} \in \mathbf{R}^q[d]$, $X_i \in \mathbf{R}^{q_i}[d]$. To be specific, by equation $F_{aa}(d)X_a = 0$, we can choose $q_a - p_a$ free variables that are not restricted by other variables and determine the remaining variables; Proceeding this process, we can determine $q_j - p_j$ free variables and the other p_j variables by equation $F_{jj}(d)X_j + \cdots + F_{ja}(d)X_a = 0$,

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$j = 1, 2, \ldots, a-1$. Let the $q-p$ free variables be located in the i_1-th, …, i_{q-p}-th rows of X, denoted as $x^*_{i_1}$, …, $x^*_{i_{q-p}}$. By successively taking $[x^*_{i_1}\ x^*_{i_2}\ \cdots\ x^*_{i_{q-p}}]^{\mathrm{T}} = e_k$, $k = 1, 2, \ldots, q-p$, where e_k is the i-th column of I_{q-p}, we can obtain a basis of the solutions to equations $F_0(d)X = 0$, denoted by $X^*_k \in \mathbf{R}^q[d]$, $k = 1, 2, \ldots, q-p$. Denote $N(d) = [X^*_1\ X^*_2\ \cdots\ X^*_{q-p}] \in \mathbf{R}^{q\times(q-p)}[d]$. Clearly $N(d)$ contains a $(q-p)\times(q-p)$ minor 1, thus it is a left invertible null-space matrix of $F(d)$. A left inverse of $N(d)$ can be constructed by setting the i_k-th columns to be e_k, $k = 1, 2, \ldots, q-p$ and the other columns to be zero vectors. Note that a common left inverse $(N^{\mathrm{T}}N)^{-1}N^{\mathrm{T}}$ in the delay-free case is inapplicable here.

It is important to point out that Theorem 1 can also be obtained from the Quillen-Suslin theorem [25,27,29]. As a main distinguishing feature, our approach leads to a constructive procedure for a null-space matrix and its left inverse, which is intuitive and easily understood.

3 Main Results

3.1 Normal Form

When system (2) has pure relative degree $\{r_1, r_2, \ldots, r_m\}$, a right invertible matrix can be obtained by stacking the coefficient matrices in (3).

Lemma 5 *Assume that system (2) has pure relative degree $\{r_1, r_2, \ldots, r_m\}$. Let $r_1 + r_2 + \cdots + r_m = r$, and $O(d) = \left[O_1^{\mathrm{T}}(d)\ \ O_2^{\mathrm{T}}(d)\ \ \cdots\ \ O_m^{\mathrm{T}}(d)\right]^{\mathrm{T}} \in \mathbf{R}^{r\times n}[d]$, where $O_i(d) = \left[c_i^{\mathrm{T}}(d)\ \ (c_i(d)A(d))^{\mathrm{T}}\ \ \cdots\ \ \left(c_i(d)A^{r_i-1}(d)\right)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbf{R}^{r_i\times n}[d]$. Then matrix $O(d)$ is right invertible, that is, there exists a matrix $Q(d) \in \mathbf{R}^{n\times r}[d]$ such that $O(d)Q(d) = I_r$.*

Proof. Let $Q_0(d) = [Q_{01}(d)\ \cdots\ Q_{0m}(d)] \in \mathbf{R}^{n\times r}[d]$, $Q_{0i}(d) = [A^{r_i-1}(d)\theta_i(d)\ \ A^{r_i-2}(d)\theta_i(d)\ \ \cdots\ \ \theta_i(d)] \in \mathbf{R}^{n\times r_i}[d]$, and $\Theta(d) = B(d)\Gamma^{-1}(d) = [\theta_1(d)\ \cdots\ \theta_m(d)] \in \mathbf{R}^{n\times m}[d]$. From Definition 1 we obtain $c_i(d)A^k(d)\Theta(d) = 0$, $k = 0, 1, \ldots, r_i-2$, $i = 1, 2, \ldots, m$, and if $i = j$, $c_i(d)A^{r_i-1}(d)\theta_j(d) = 1$; otherwise $c_i(d)A^{r_i-1}(d)\theta_j(d) = 0$, $i, j = 1, 2, \ldots, m$. Then it can be calculated that

$$O(d)Q_0(d) = \begin{bmatrix} O_1Q_{01} & \cdots & O_1Q_{0m} \\ \vdots & \ddots & \vdots \\ O_mQ_{01} & \cdots & O_mQ_{0m} \end{bmatrix} \in \mathbf{R}^{r\times r}[d],$$

where the submatrices on the diagonal are all lower-triangular matrices with diagonal element 1, and the non-diagonal submatrices O_iQ_{0j}, $i \neq j$ have lower-triangular form or trapezoidal form, namely,

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ * & 0 & \cdots & 0 & 0 \\ * & * & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & 0 \end{bmatrix} (r_i \geq r_j); \quad \begin{bmatrix} * & \cdots & * & 0 & 0 & \cdots & 0 & 0 \\ * & \cdots & * & * & 0 & \cdots & 0 & 0 \\ * & \cdots & * & * & * & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * & * & \cdots & * & 0 \end{bmatrix} (r_i < r_j),$$

where $* \in \mathbf{R}[d]$ is a possible nonzero element. Note that matrix $O(d)Q_0(d)$ can be transformed into an $r\times r$ identity matrix through elementary operations with the determinant being unchanged, which means that matrix $O(d)Q_0(d)$ is invertible. Thus there exists an invertible matrix $\bar{Q}_0(d) \in \mathbf{R}^{r\times r}[d]$ such that $O(d)Q_0(d)\bar{Q}_0(d) = I_r$. Let $Q(d) = Q_0(d)\bar{Q}_0(d) \in \mathbf{R}^{n\times r}[d]$. Then we get $O(d)Q(d) = I_r$, namely, matrix $O(d)$ is right invertible. The proof is finished. ■

The main result of this subsection is given as follows.

Theorem 2 *Assume that system (2) has pure relative degree $\{r_1, r_2, \ldots, r_m\}$. Then there exists a state transformation $\tilde{x} = T(d)x \triangleq [\eta^{\mathrm{T}}\ \xi^{\mathrm{T}}]^{\mathrm{T}}$, where $T(d) \in \mathbf{R}^{n\times n}[d]$ is an invertible matrix, $\eta = [\eta_1\ \cdots\ \eta_{n-r}]^{\mathrm{T}} \in \mathbf{R}^{n-r}[d]$, $\xi = \left[\xi_1^{\mathrm{T}}\ \cdots\ \xi_m^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbf{R}^r[d]$, $\xi_i = [\xi_{i1}\ \cdots\ \xi_{i,r_i}]^{\mathrm{T}} \in \mathbf{R}^{r_i}[d]$, such that system (2) can be transformed into the following normal form:*

$$\begin{cases} \dot{\eta} = A_{00}(d)\eta + B_0(d)y, \\ \dot{\xi}_{i1} = \xi_{i2}, \\ \quad\vdots \\ \dot{\xi}_{i,r_i-1} = \xi_{i,r_i}, \\ \dot{\xi}_{i,r_i} = C_{0i}(d)\eta + \beta_{0i}(d)\xi + c_iA^{r_i-1}(d)Bu, \\ y_i = \xi_{i1}, \quad i = 1, 2, \ldots, m, \end{cases} \tag{5}$$

where $A_{00}(d) \in \mathbf{R}^{(n-r)\times(n-r)}[d]$, $B_0(d) \in \mathbf{R}^{(n-r)\times m}[d]$, $C_{0i}(d) \in \mathbf{R}^{1\times(n-r)}[d]$ and $\beta_{0i}(d) \in \mathbf{R}^{1\times r}[d]$ are polynomial matrices.

Proof. For the right invertible matrix $O(d)$, by Lemma 5 we have $O(d)Q(d) = I_r$. According to Theorem 1 and the succeeding design procedure, a null-space matrix $N(d) \in \mathbf{R}_{n-r}^{n\times(n-r)}[d]$ and a left inverse matrix of $N(d)$, denoted as $L(d) \in \mathbf{R}_{n-r}^{(n-r)\times n}[d]$, can be constructed, satisfying $O(d)N(d) = 0$ and $L(d)N(d) = I_{n-r}$. In addition, define $\Delta = \Delta_1 \oplus \cdots \oplus \Delta_m \in \mathbf{R}^{m\times r}$, where $\Delta_i = [0_{1,r_i-1} \mid 1] \in \mathbf{R}^{1\times r_i}$. Based on the above preparations, we can construct

$$T_0(d)x = \begin{bmatrix} L - K_0O \\ O \end{bmatrix} x \triangleq \begin{bmatrix} \phi \\ \xi \end{bmatrix}, \ \phi \in \mathbf{R}^{n-r}[d], \tag{6}$$

where $T_0(d) \in \mathbf{R}^{n\times n}[d]$, $K_0 = LB\Gamma^{-1}\Delta \in \mathbf{R}^{(n-r)\times r}[d]$. It can be verified that $T_0(d)$ is invertible and $T_0^{-1}(d) =$

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$[N\ NK_0+Q-NLQ] \triangleq [N\ NK_0+S_0] \in \mathbf{R}^{n\times n}[d]$, where $S_0 \in \mathbf{R}^{n\times r}[d]$. Then we have

$$\begin{aligned}
&\xi_{i1} = c_i x = y_i,\\
&\xi_{i,k+1} = c_i A^k x = \dot{\xi}_{ik},\ k = 1, 2, \ldots, r_i - 1,\\
&\dot{\xi}_{i,r_i} = c_i A^{r_i-1}\dot{x} = c_i A^{r_i} x + c_i A^{r_i-1} B,
\end{aligned} \tag{7}$$

and $\dot{\phi} = (L - K_0O)\dot{x} = (L - K_0O)AN\phi + (LA - K_0OA)(NK_0 + S_0)\xi \triangleq A_{00}\phi + E_0\xi$, which is completely independent of input u, where $A_{00} = (L - K_0O)AN \in \mathbf{R}^{(n-r)\times(n-r)}[d]$, and $E_0 = (LA - K_0OA)(NK_0 + S_0) \in \mathbf{R}^{(n-r)\times r}[d]$.

For the following steps, the derivation for delay-free systems can be closely followed. Partitioning by columns gives $E_0 = [\mu_1\ \cdots\ \mu_m]$, where $\mu_i = [\mu_{i1}\ \cdots\ \mu_{i,r_i}]$. Let $\eta = \phi - \sum_{i=1}^{m}\sum_{j=1}^{r_i-1} h_{ij}\xi_{ij}$, where $\mu_{ij} \in \mathbf{R}^{(n-r)\times 1}[d]$ and $h_{ij} = \mu_{i,j+1} + A_{00}\mu_{i,j+2} + \cdots + A_{00}^{r_i-j-1}\mu_{i,r_i} \in \mathbf{R}^{(n-r)\times 1}[d]$, $j = 0, 1, \ldots, r_i - 1$, $i = 1, 2, \ldots, m$. Then it can be calculated that

$$\begin{aligned}
\dot{\eta} =& A_{00}\eta + \sum_{i=1}^{m}\sum_{j=1}^{r_i-1}\left(\mu_{ij} + A_{00}\mu_{i,j+1} + \cdots + A_{00}^{r_i-j}\mu_{i,r_i}\right)\xi_{ij}\\
&+ \sum_{i=1}^{m}\sum_{j=1}^{r_i}\mu_{ij}\xi_{ij} - \sum_{i=1}^{m}\sum_{j=1}^{r_i-1} h_{ij}\xi_{i,j+1} - \sum_{i=1}^{m}\sum_{j=1}^{r_i-1}\mu_{ij}\xi_{ij}\\
=& A_{00}\eta + \sum_{i=1}^{m}\left(\mu_{i1} + A_{00}\mu_{i2} + \cdots + A_{00}^{r_i-1}\mu_{i,r_i}\right)\xi_{i1}\\
&+ \sum_{i=1}^{m}\sum_{j=1}^{r_i-2} h_{ij}\xi_{i,j+1} + \sum_{i=1}^{m}\mu_{i,r_i}\xi_{i,r_i} - \sum_{i=1}^{m}\sum_{j=1}^{r_i-1} h_{ij}\xi_{i,j+1}\\
=& A_{00}\eta + \sum_{i=1}^{m}\left(\mu_{i1} + A_{00}\mu_{i2} + \cdots + A_{00}^{r_i-1}\mu_{i,r_i}\right)\xi_{i1}\\
\triangleq& A_{00}\eta + B_0 y,
\end{aligned}$$

which is exactly the first equation in (5), where $B_0 = [h_{10}\ h_{20}\ \cdots\ h_{m0}] \in \mathbf{R}^{(n-r)\times m}[d]$ and $h_{i0} = \mu_{i1} + A_{00}\mu_{i2} + \cdots + A_{00}^{r_i-1}\mu_{i,r_i} \in \mathbf{R}^{(n-r)\times 1}[d]$. Denote $H = [H_1\ H_2\ \cdots\ H_m] \in \mathbf{R}^{(n-r)\times r}[d]$, where $H_i = [h_{i1}\ h_{i2}\ \cdots\ h_{i,r_i-1} \mid 0_{n-r,1}] \in \mathbf{R}^{(n-r)\times r_i}[d]$. Then together with equation (6), the state transformation can be expressed as

$$\begin{bmatrix}\eta\\ \xi\end{bmatrix} = \begin{bmatrix}L - (H + K_0)O\\ O\end{bmatrix} x = T(d)x, \tag{8}$$

where $T(d)$ is an invertible state transformation matrix, with the inverse matrix $T^{-1}(d) = [N\ \ G_0] \in \mathbf{R}^{n\times n}[d]$, $G_0 = N(H + K_0) + S_0 \in \mathbf{R}^{n\times r}[d]$. Finally, by using (8), the last equation in (7) becomes $\dot{\xi}_{i,r_i} = C_{0i}\eta + \beta_{0i}\xi + c_i A^{r_i-1}Bu$, where $C_{0i} = c_i A^{r_i}N \in \mathbf{R}^{1\times(n-r)}[d]$ and $\beta_{0i} = c_i A^{r_i}G_0 \in \mathbf{R}^{1\times r}[d]$, which coincides with the normal form (5). The proof is finished. ■

Remark 1 *In the above proof, based on the procedure for constructing the null-space matrix and its left inverse matrix, a novel construction scheme of the state transformation matrix, referred as to the unimodular completion method, is given. Such a construction method is not available in [31], where the Quillen-Suslin theorem was invoked to guarantee its existence.*

Instrumental to the subsequent developments in the paper, we express the normal form (5) as

$$\begin{cases}\dot{\tilde{x}} = \tilde{A}(d)\tilde{x} + \tilde{B}(d)u,\\ y = \tilde{C}\tilde{x},\end{cases} \tag{9}$$

where the coefficient matrices have the structure

$$\begin{aligned}
\tilde{A}(d) &= TAT^{-1} = \left[\begin{array}{c|ccc} A_{00} & A_{01} & \cdots & A_{0m}\\ \hline A_{10} & A_{11} & \cdots & A_{1m}\\ \vdots & \vdots & \ddots & \vdots\\ A_{m0} & A_{m1} & \cdots & A_{mm}\end{array}\right] \in \mathbf{R}^{n\times n}[d],\\
A_{ii}(d) &= \left[\begin{array}{c|c} 0_{r_i-1,1} & I_{r_i-1}\\ \hline * & *\end{array}\right] \in \mathbf{R}^{r_i\times r_i}[d],\\
A_{ij}(d) &= \left[\begin{array}{c} 0_{r_i-1,r_j}\\ \hline *\end{array}\right] \in \mathbf{R}^{r_i\times r_j}[d],\ i, j = 1, 2, \ldots, m,\\
\tilde{B}(d) &= TB = \left[\begin{array}{c} 0_{n-r,m}\\ \hline \tilde{b}\end{array}\right] \Gamma \in \mathbf{R}^{n\times m}[d],\\
\tilde{C} &= CT^{-1} = \left[\begin{array}{c|c} 0_{m,n-r} & \tilde{c}\end{array}\right] \in \mathbf{R}^{m\times n},
\end{aligned} \tag{10}$$

$A_{i0}(d) = \left[0_{r_i-1,n-r}^{\mathrm{T}}\ \ C_{0i}^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbf{R}^{r_i\times(n-r)}[d]$, $A_{0i}(d) = [h_{i0}\ \ 0_{n-r,r_i-1}] \in \mathbf{R}^{(n-r)\times r_i}[d]$, $\tilde{b} = \tilde{b}_{11} \oplus \cdots \oplus \tilde{b}_{mm} \in \mathbf{R}^{r\times m}$, $\tilde{b}_{ii} = \left[0_{r_i-1,1}^{\mathrm{T}}\ 1\right]^{\mathrm{T}} \in \mathbf{R}^{r_i\times 1}$, $\tilde{c} = \tilde{c}_{11} \oplus \cdots \oplus \tilde{c}_{mm} \in \mathbf{R}^{m\times r}$, $\tilde{c}_{ii} = [1\ 0_{1,r_i-1}] \in \mathbf{R}^{1\times r_i}$, and $*$ are possible nonzero matrices whose elements belong to $\mathbf{R}[d]$.

Example 1 *Consider the system* (2) *with coefficient matrices*

$$A(d) = \begin{bmatrix}-1 & 3 & 0 & -1 & 2 & 0\\ 0 & -1 & 1 & 0 & 0 & d_2\\ -1 & 0 & -5 & 1 & -4 & 1\\ 3 & 0 & -2 & -5 & 0 & 0\\ 1 & 0 & 0 & 0 & d_1 & 1\\ 0 & 0 & -1 & 0 & 1 & -2\end{bmatrix},\ B(d) = \begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix},$$

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$$C(d)=\left[\begin{array}{cccccc}0&1&0&0&0&0\\0&0&0&0&1&0\end{array}\right].$$

It can be verified that the system has a pure relative degree $\{r_1,r_2\}=\{2,2\}$ *with invertible coupling matrix* $\Gamma(d)=C(d)A(d)B(d)=\left[\begin{smallmatrix}1&d_2\\0&1\end{smallmatrix}\right]$. *A simple calculation shows that the inverse matrix of* $\Gamma(d)$ *is given by* $\Gamma^{-1}(d)=\left[\begin{smallmatrix}1&-d_2\\0&1\end{smallmatrix}\right]$.

Now we construct an invertible state transformation matrix by following the steps in the proof of Theorem 2, so that the system (2) *can be transformed into the normal form* (5). *It can be calculated that*

$$O(d)=\left[\begin{array}{c}O_1(d)\\O_2(d)\end{array}\right]=\left[\begin{array}{cccccc}0&1&0&0&0&0\\0&-1&1&0&0&d_2\\\hline 0&0&0&0&1&0\\1&0&0&0&d_1&1\end{array}\right].$$

For the right invertible matrix $O(d)$, *the existence of a left invertible null-space matrix can be guaranteed. Applying the constructive procedure following the proof of Theorem 1 results in null-space matrix* $N(d)=\left[\begin{smallmatrix}0&0&0&1&0&0\\-1&0&-d_2&0&0&1\end{smallmatrix}\right]^{\mathrm{T}}$ *and its left inverse* $\left[\begin{smallmatrix}0&0&0&1&0&0\\0&0&0&0&0&1\end{smallmatrix}\right]$. *Note that* $L(d)=\left[\begin{smallmatrix}0&d_1&0&1&-2&0\\0&0&0&0&0&1\end{smallmatrix}\right]$ *is also a left inverse. By calculation, we further have* $K_0=\left[\begin{smallmatrix}0&0&0&0\\0&0&0&1\end{smallmatrix}\right]$ *and* $H(d)=\left[\begin{smallmatrix}d_1-2&0&2d_2-2&0\\0&0&-d_1&0\end{smallmatrix}\right]$. *From equation* (8) *we obtain*

$$T(d)=\left[\begin{array}{cccccc}0&2&0&1&-2d_2&0\\-1&0&0&0&0&0\\\hline 0&1&0&0&0&0\\0&-1&1&0&0&d_2\\0&0&0&0&1&0\\1&0&0&0&d_1&1\end{array}\right].$$

Then the coefficient matrices of the normal form (9) *can be calculated as*

$$\tilde{A}(d)=\left[\begin{array}{cc|cccc}-5&2d_2-3&8&0&\varpi_0(d)&0\\1&-1&-5&0&2d_2-2&0\\\hline 0&0&0&1&0&0\\1&\varpi_1(d)&\varpi_2(d)&\varpi_3(d)&\varpi_4(d)&\varpi_5(d)\\0&0&0&0&0&1\\-1&d_2-1&4&-1&\varpi_6(d)&\varpi_7(d)\end{array}\right],$$

$$\tilde{B}(d)=\left[\begin{array}{cc|ccc}0&0&0&1&0&0\\0&0&0&d_2&0&1\end{array}\right]^{\mathrm{T}},\ \tilde{C}=\left[\begin{array}{cc|cccc}0&0&1&0&0&0\\0&0&0&0&1&0\end{array}\right],\quad(11)$$

where $\varpi_0(d)=-2d_1d_2-10d_2$, $\varpi_1(d)=d_2^2+3d_2+2$, $\varpi_2(d)=-d_2-7$, $\varpi_3(d)=-d_2-6$, $\varpi_4(d)=-d_1d_2^2-3d_1d_2-d_1+3d_2-4$, $\varpi_5(d)=d_2^2+3d_2+1$, $\varpi_6(d)=-d_1d_2+2d_1-2d_2+3$, *and* $\varpi_7(d)=d_1+d_2-2$.

Example 2 *Consider a system* (2) *associated with the linearized model of Mach number dynamics in a cryogenic wind tunnel [26], whose coefficient matrices are described by*

$$A(d)=\left[\begin{array}{ccc}-a&kad_1&0\\0&0&1\\0&-\omega^2&-2\zeta w\end{array}\right],B(d)=\left[\begin{array}{c}0\\0\\\omega^2\end{array}\right],C(d)=\left[\begin{array}{c}d_2\\1\\0\end{array}\right]^{\mathrm{T}},$$

where x_1 *is the Mach number,* x_2 *denotes the actuator position,* x_3 *is the actuator rate,* u *is the input to the actuator servomechanism, and* a, k, ω, ζ *are parameters corresponding to one particular operating point. It can be calculated that* $C(d)B(d)=0$ *and* $C(d)A(d)B(d)=\omega^2$, *which means that the system has a pure relative degree* $r=2$ *with invertible coupling matrix* $\Gamma(d)=\omega^2$.

Then we construct an invertible state transformation matrix by following the steps in the proof of Theorem 2, so that system (2) *can be transformed into the normal form* (5). *It can be calculated that* $O(d)=\left[\begin{smallmatrix}C(d)\\C(d)A(d)\end{smallmatrix}\right]=\left[\begin{smallmatrix}d_2&1&0\\-ad_2&kad_1d_2&1\end{smallmatrix}\right]$, *which, by Lemma 5, is right invertible. Applying the constructive procedure following the proof of Theorem 1 results in null-space matrix* $N(d)=\left[\begin{smallmatrix}1&-d_2&ad_2+kad_1d_2^2\end{smallmatrix}\right]^{\mathrm{T}}$ *and its left inverse* $L(d)=[1\ 0\ 0]$. *By calculation, we further have* $K_0(d)=[0\ 0]$ *and* $H(d)=[0\ 0]^{\mathrm{T}}$. *From equation* (8) *we obtain*

$$T(d)=\left[\begin{array}{ccc}1&0&0\\d_2&1&0\\-ad_2&kad_1d_2&1\end{array}\right],$$

Then the coefficient matrices of the normal form can be calculated as

$$\tilde{A}(d)=\left[\begin{array}{c|cc}\varphi_1(d)&kad_1&0\\\hline 0&0&1\\\varphi_2(d)&\varphi_3(d)&\varphi_4(d)\end{array}\right],\ \tilde{B}(d)=\left[\begin{array}{c}0\\\hline 0\\\omega^2\end{array}\right],\ \tilde{C}=\left[\begin{array}{c}0\\\hline 1\\0\end{array}\right],$$

where $\varphi_1(d)=\varphi_1(d)$, $\varphi_2(d)=(a^2+\omega^2-2\zeta\omega a)d_2+2(ka^2-\zeta\omega ka)d_1d_2^2+k^2a^2d_1^2d_2^3$, $\varphi_3(d)=-\omega^2+(2\zeta\omega ka-ka^2)d_1d_2-k^2a^2d_1^2d_2^2$ *and* $\varphi_4(d)=-2\zeta\omega+kad_1d_2$.

3.2 On the Condition of Pure Relative Degree

In this subsection, we shed light on the condition of pure relative degree, a necessary condition for the existence of the normal form, which is in fact not only necessary

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for constructing the unimodular transformation matrix $T_0(d)$, but also necessary for revealing the zero dynamics. In [19] it is demonstrated that the relative degree of system (1) may be sensitive to infinitesimal delay perturbations. In such a situation, the assumption of pure relative degree is not satisfied. This result is implied by the following proposition, whose proof directly follows from the definition of pure relative degree.

Proposition 1 *If system* (2) *has a pure relative degree, then the relative degree of system* (1) *is independent of the delay values.*

To provide insight in the requirement, we consider system (1) with coefficient matrices

$$A_0 = \begin{bmatrix} 1 & 0 & 2p & p \\ 0 & 0 & 0 & 0 \\ 1 & p & 0 & 0 \\ 0 & -p & 0 & 0 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -2 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \; A_3 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$B_0 = [0\ 1\ 0\ 0]^{\mathrm{T}}$, $C_0 = [1\ 0\ 0\ 0]$, $B_k = [0\ 0\ 0\ 0]^{\mathrm{T}}$, and $C_k = [0\ 0\ 0\ 0]$, $k = 1, 2, 3$, where $p \in \mathbf{R}$ is a parameter. For $p = 0$, its transfer function is given by $G(s) = \frac{\rho_1(s)}{\rho_0(s)}$, where $\rho_0(s) = s^2\left(s^2 - s - 2\mathrm{e}^{-s\tau_2} - \mathrm{e}^{-s\tau_3}\right)$, and $\rho_1(s) = s\left(\mathrm{e}^{-s2\tau_2} - \mathrm{e}^{-s\tau_3}\mathrm{e}^{-s\tau_1}\right) + 4\mathrm{e}^{-s2\tau_1}\mathrm{e}^{-s\tau_2} + 2\mathrm{e}^{-s\tau_3}\mathrm{e}^{-s2\tau_1} + 2\mathrm{e}^{-s\tau_1}\mathrm{e}^{-s2\tau_2} + \mathrm{e}^{-s\tau_3}\mathrm{e}^{-s\tau_1}\mathrm{e}^{-s\tau_2}$. Notice that the relative degree is equal to four whenever $2\tau_2 = \tau_1 + \tau_3$ and three otherwise, revealing the sensitivity to delay changes. If, in addition, the delays are specified as $(\tau_1, \tau_2, \tau_3) = (1, 3, 6)$, then we get $G(s) = \kappa_1(s)/\kappa_0(s)$, where $\kappa_0(s) = s^2\left(s^2 - s - 2\mathrm{e}^{-3s} - \mathrm{e}^{-6s}\right)$, and $\kappa_1(s) = \left(s\left(1 - \mathrm{e}^{-s}\right) + 4\mathrm{e}^{s} + 2\mathrm{e}^{-s} + 2\mathrm{e}^{-2s} + \mathrm{e}^{-4s}\right)\mathrm{e}^{-6s}$. Due to the term $4\mathrm{e}^{s}$ in the numerator (a "negative delay"), the invariant zeros cannot be described as characteristic roots of a dynamical (sub-)system, which also points to the lack of a normal form. See [19] for a plot of the zeros.

Let us now consider the assumption that $\Gamma(0)$ is invertible (inducing a so-called regular relative degree), that is, differentiating the output reveals a direct connection with the input at present time, a.o. The invertibility of $\Gamma(0)$, which excludes the sensitivity issue of the relative degree at any set of nonzero delay values but is weaker than the assumption of pure relative degree, is not sufficient for the existence of a normal form (even if its definition is relaxed to allowing such a coupling matrix). To see this, consider the above example with $p = 1$. Then the transfer function becomes $G(s) = \varrho_1(s)/\varrho_0(s)$, where $\varrho_0(s) = s^2\left(s^2 - s - 2\mathrm{e}^{-s\tau_2} - \mathrm{e}^{-s\tau_3} - 2\right)$, and $\varrho_1(s) = s\left(1 + 2\mathrm{e}^{-s\tau_2} + \mathrm{e}^{-s2\tau_2} - \mathrm{e}^{-s\tau_1}\mathrm{e}^{-s\tau_3}\right) + 2\mathrm{e}^{-s\tau_1} + 4\mathrm{e}^{-s\tau_1}\mathrm{e}^{-s\tau_2} + \mathrm{e}^{-s\tau_1}\mathrm{e}^{-s\tau_3} + 2\mathrm{e}^{-s\tau_1}\mathrm{e}^{-s2\tau_2} + 4\mathrm{e}^{-s2\tau_1}\mathrm{e}^{-s\tau_2} + 2\mathrm{e}^{-s2\tau_1}\mathrm{e}^{-s\tau_3} + 4\mathrm{e}^{-s2\tau_1} + \mathrm{e}^{-s\tau_1}\mathrm{e}^{-s\tau_2}\mathrm{e}^{-s\tau_3}$. In this case, the relative degree is always equal to three and the coupling matrix satisfies $\Gamma(0) = 1$. The invariant zeros can be interpreted as characteristic roots of a time-delay system of neutral type, but not of a (sub-)system of the form $\dot{\eta}(t) = A_{00}(d)\eta(t)$ as in (5), which is of retarded type.

3.3 Properties of the Normal Form

Similar to delay-free systems [38] and commensurate-delay systems [2,12], the invariant zeros of system (2) can also be revealed by normal form (5), and are determined by the η-subsystem, the so-called zero dynamics subsystem, as given by the following theorem. In particular, system (1) is said to be minimum-phase if its invariant zeros all have negative real parts. In this case, we say system (2) is of minimum-phase for $d = (\mathrm{e}^{-s\tau_1}, \ldots, \mathrm{e}^{-s\tau_h})$.

Theorem 3 *Assume that system* (2) *has pure relative degree* $\{r_1, r_2, \ldots, r_m\}$. *Then the invariant zeros of system* (1) *coincide exactly with the eigenvalues of matrix* $A_{00}(d)$, *that is,* $\forall s \in \mathbf{C}$,

$$\det\begin{bmatrix} A(d) - sI_n & B(d) \\ C(d) & 0 \end{bmatrix} = \sigma\det(\Gamma(d))\det(A_{00}(d) - sI_{n-r}),$$

where $d = (\mathrm{e}^{-s\tau_1}, \ldots, \mathrm{e}^{-s\tau_h})$, *and* σ *is equal to* 1 *or* -1.

Proof. By performing the state transformation $\tilde{x} = T(d)x$ and the state feedback transformation $u = \tilde{K}(d)\tilde{x} + \Gamma^{-1}(d)v$ on system (2) successively, where $\tilde{K}(d) = -\Gamma^{-1}(d)\bar{O}(d)T^{-1}(d) \in \mathbf{R}^{m\times n}[d]$, $\bar{O}(d) = \left[(c_1(d)A^{r_1}(d))^{\mathrm{T}} \cdots (c_m(d)A^{r_m}(d))^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbf{R}^{m\times n}[d]$, and $v = [v_1\ v_2\ \cdots\ v_m]^{\mathrm{T}} \in \mathbf{R}^m[d]$ is a new signal, we have

$$\begin{cases} \dot{\tilde{x}} = \left(\tilde{A}(d) + \tilde{B}(d)\tilde{K}(d)\right)\tilde{x} + \tilde{B}(d)\Gamma^{-1}(d)v, \\ y = \tilde{C}\tilde{x}, \end{cases}$$

where

$$\tilde{A}(d) + \tilde{B}(d)\tilde{K}(d) = \left[\begin{array}{c|ccc} A_{00} & A_{01} & \cdots & A_{0m} \\ \hline 0 & \tilde{A}_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{A}_{mm} \end{array}\right] \in \mathbf{R}^{n\times n}[d],$$

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$\tilde{A}_{ii} = \begin{bmatrix} 0_{r_i-1,1} & I_{r_i-1} \\ 0_{1,1} & 0_{1,r_i-1} \end{bmatrix} \in \mathbf{R}^{r_i \times r_i}$. Hence, it follows from the matrix elementary operations that

$$\begin{aligned}
&\det \begin{bmatrix} A(d) - sI_n & B(d) \\ C(d) & 0 \end{bmatrix} \\
&= \det\left(\begin{bmatrix} T^{-1} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} \tilde{A} + \tilde{B}\tilde{K} - sI_n & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ -\tilde{K}T & I_m \end{bmatrix} \right) \\
&= \det \begin{bmatrix} \tilde{A} + \tilde{B}\tilde{K} - sI_n & \tilde{B}\Gamma^{-1} \\ \tilde{C} & 0 \end{bmatrix} \det \begin{bmatrix} I_n & 0 \\ 0 & \Gamma \end{bmatrix} \\
&= \sigma \det(\Gamma) \det(A_{00}(d) - sI_{n-r}) \prod_{i=1}^{m} \det \begin{bmatrix} \tilde{A}_{ii} - sI_{r_i} & \tilde{b}_{ii} \\ \tilde{c}_{ii} & 0 \end{bmatrix} \\
&= \sigma \det(\Gamma) \det(A_{00}(d) - sI_{n-r}), \quad \forall s \in \mathbf{C}.
\end{aligned}$$

The proof is complete. ■

Next, we use the normal form (5) to simplify the criteria of spectral controllability and spectral observability of system (2).

Theorem 4 *Assume that system (2) has a pure relative degree $\{r_1, r_2, \ldots, r_m\}$. Then $(A(d), B(d))$ is spectrally controllable if and only if $(A_{00}(d), B_0(d))$ is spectrally controllable. Similarly, $(A(d), C(d))$ is spectrally observable if and only if $(A_{00}(d), C_0(d))$ is spectrally observable, where $d = (\mathrm{e}^{-s\tau_1}, \ldots, \mathrm{e}^{-s\tau_h})$ and $C_0(d) = \left[C_{01}^{\mathrm{T}}(d)\ C_{02}^{\mathrm{T}}(d)\ \cdots\ C_{0m}^{\mathrm{T}}(d)\right]^{\mathrm{T}} \in \mathbf{R}^{m\times(n-r)}[d]$.*

Proof. It is noted that $[sI_n - A(d)\ B(d)] = T^{-1}(d)\left[sI_n - \tilde{A}(d)\ \tilde{B}(d)\Gamma^{-1}(d)\right](T(d) \oplus \Gamma(d))$. Then by using some elementary column operations of matrices, we obtain $\operatorname{rank}[sI_n - A(d)\ B(d)] = \operatorname{rank}\left[sI_n - \tilde{A}(d)\ \tilde{B}(d)\Gamma^{-1}(d)\right] = \operatorname{rank}[sI_{n-r} - A_{00}(d)\ -h_{10}(d)\ \cdots\ -h_{m0}(d)] + r = \operatorname{rank}[sI_{n-r} - A_{00}(d)\ B_0(d)] + r$, $\forall s \in \mathbf{C}$, which implies that $(A(d), B(d))$ is spectrally controllable if and only if $(A_{00}(d), B_0(d))$ is spectrally controllable. The spectral observability can be proved similarly. Thus the proof is complete. ■

3.4 Inverse System

In light of normal form (5), a low-dimensional inverse system of system (2) can be obtained, as given below.

Theorem 5 *Assume that system (2) has pure relative degree $\{r_1, r_2, \ldots, r_m\}$. Then an inverse system of system (2) is given by*

$$\begin{cases} \dot{\eta} = A_{00}(d)\eta + B_0(d)y, \\ u = -\Gamma^{-1}(d)C_0(d)\eta + \Gamma^{-1}(d)\omega(d), \end{cases} \tag{12}$$

where $\omega(d) = [\omega_1(d)\ \cdots\ \omega_m(d)]^{\mathrm{T}} \in \mathbf{R}^m[d]$, $\omega_j(d) = y_j^{(r_j)} - \sum_{i=1}^{m}\sum_{k=1}^{r_i} \beta_{0j}^{ik}(d) y_i^{(k-1)} \in \mathbf{R}[d]$, and $\beta_{0j}^{ik}(d) \in \mathbf{R}[d]$ satisfies $\beta_{0j}(d) = \left[\beta_{0j}^{1}(d)\ \cdots\ \beta_{0j}^{m}(d)\right] \in \mathbf{R}^{1\times r}[d]$ and $\beta_{0j}^{i}(d) = \left[\beta_{0j}^{i1}(d)\ \cdots\ \beta_{0j}^{i,r_i}(d)\right] \in \mathbf{R}^{1\times r_i}[d]$.

Proof. Obviously, the first equation in (12) holds true. Also, it is derived from equation (3) and normal form (5) that $\Gamma(d)u = -C_0(d)\eta + \omega(d)$, which is exactly the second equation in (12). Hence, the proof is complete. ■

Remark 2 *Under the condition of Theorem 5, another inverse system of system (2) can be obtained by means of the approach in [4], with the form*

$$\begin{cases} \dot{z} = \left(A(d) - B(d)\Gamma^{-1}(d)\bar{O}(d)\right)z + B(d)\Gamma^{-1}(d)Y, \\ u = -\Gamma^{-1}(d)\bar{O}(d)z + \Gamma^{-1}(d)Y, \end{cases}$$

where $Y = \left[y_1^{(r_1)}\ \cdots\ y_m^{(r_m)}\right]^{\mathrm{T}} \in \mathbf{R}^m[d]$. We can easily see that compared with the above inverse system, the inverse system (12) has a lower dimension.

4 Output Feedback Stabilization

The considered output feedback stabilization problem consists of designing a controller based on the direct feedback of outputs and their derivatives for system (2), such that the closed-loop system becomes $\mathcal{H}$-stable. Since the direct measurement of output derivatives may be challenging, an observer can be constructed based on an output differentiator to estimate output derivatives [32]. By using normal form (5), a PD high-gain output feedback controller can be designed to solve this problem, as shown in the following result.

Theorem 6 *Assume that system (2) has pure relative degree $\{r_1, r_2, \ldots, r_m\}$ and is of minimum-phase for $d = (\mathrm{e}^{-s\tau_1}, \ldots, \mathrm{e}^{-s\tau_h})$. Then a PD output feedback controller can be designed as*

$$u = -\Gamma^{-1}(d) \begin{bmatrix} \tilde{\omega}_1(d) + \sum_{j=0}^{r_1-1} f^{r_1-j}\gamma_{1j}y_1^{(j)} \\ \vdots \\ \tilde{\omega}_m(d) + \sum_{j=0}^{r_m-1} f^{r_m-j}\gamma_{mj}y_m^{(j)} \end{bmatrix}, \tag{13}$$

such that the closed-loop system is $\mathcal{H}$-stable, where $\tilde{\omega}_j(d) = \sum_{i=1}^{m}\sum_{k=1}^{r_i} \beta_{0j}^{ik}(d) y_i^{(k-1)}$, and $\beta_{0j}^{ik}(d) \in \mathbf{R}[d]$ is defined in Theorem 5, f is a sufficiently large real number, and γ_{ij}, $j = 0, 1, \ldots, r_i - 1$, $i = 1, 2, \ldots, m$ are real numbers such that $\gamma_i(s) = s^{r_i} + \sum_{j=0}^{r_i-1}\gamma_{ij}s^j$ are Hurwitz polynomials.

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Proof. Let $s_{i1}, \ldots, s_{i,r_i}$ be the roots of the Hurwitz polynomial $\gamma_i(s)$, $i = 1, 2, \ldots, m$, where $\Re(s_{ij}) < 0$ and the complex roots appear in pairs. By applying the method of undetermined coefficients, it can be verified that the polynomial $\hat{\gamma}_i(s) = s^{r_i} + \sum_{j=0}^{r_i-1} f^{r_i-j}\gamma_{ij}s^j$ has roots $fs_{i1}, \ldots, fs_{i,r_i}$, and thus is also Hurwitz.

Since the $\mathcal{H}$-stability of system (2) cannot be changed by the state transformation $\tilde{x} = T(d)x$, next we consider the normal form (5), namely, system (9). It can be obtained that the closed-loop system consisting of system (9) and controller (13) is of the form

$$\left[\begin{array}{c} \dot{\eta} \\ \hline \dot{\xi}_1 \\ \vdots \\ \dot{\xi}_m \end{array}\right] = \left[\begin{array}{c|ccc} A_{00}(d) & A_{01}(d) & \cdots & A_{0m}(d) \\ \hline A_{10}(d) & \hat{A}_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{m0}(d) & 0 & \cdots & \hat{A}_{mm} \end{array}\right] \left[\begin{array}{c} \eta \\ \hline \xi_1 \\ \vdots \\ \xi_m \end{array}\right],$$

where $A_{i0}(d) \in \mathbf{R}^{r_i \times (n-r)}[d]$ is defined in (10), and

$$\hat{A}_{ii} = \left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ \hline -f^{r_i}\gamma_{i0} & \cdots & -f\gamma_{i,r_i-1} & -f^{r_i-1}\gamma_{i1} \end{array}\right] \in \mathbf{R}^{r_i \times r_i}.$$

From matrix theory, the characteristic polynomial of the closed-loop system can be expressed as

$$\det\left(sI_{n-r} - A_{00}(d) - \sum_{i=1}^{m} \frac{h_{i0}(d)C_{0i}(d)}{\hat{\gamma}_i(s)}\right) \prod_{i=1}^{m} \hat{\gamma}_i(s), \quad (14)$$

where $d = (\mathrm{e}^{-s\tau_1}, \ldots, \mathrm{e}^{-s\tau_h})$. We note that when f is sufficiently large, the r characteristic roots of the closed-loop system, corresponding to the second factor, tend to the infinity on the left-half complex plane. Now we investigate the zeros of the first factor of (14). Take a right half plane $\{s \in \mathbf{C} : \Re(s) \geq -\lambda\}$, where $\lambda > 0$ is such that the half plane contains neither invariant zeros nor zeros of $\hat{\gamma}_i$, $i = 1, \ldots, m$, when $f = 1$. From the structure of the first factor in (14), it follows that all its zeros in the half plane must be confined to a compact set Ω, which can be chosen independently from $f \in [1, \infty)$. Furthermore, if $f \to \infty$ the term $\sum_{i=1}^{m} \frac{h_{i0}(d)C_{0i}(d)}{\hat{\gamma}_i(s)}$ uniformly converges to zero on Ω. From Rouché's theorem, we conclude that for sufficiently large f, the first factor in (14) does not have zeros in Ω, and, by extension, in the considered right half plane. Thus when f is sufficiently large, the closed-loop system is $\mathcal{H}$-stable. The proof is complete. ■

Remark 3 *Essentially, the PD high-gain output feedback controller (13) stabilizes the closed-loop system by weakening the coupling between the η-subsystem and ξ-subsystem, and reassigning the poles of the ξ-subsystem.*

Remark 4 *Considering that system (1) is associated with system (2), the controller (13) can be brought back to the delay-differential framework such that when f is sufficiently large, the resulting closed-loop system is exponentially stable.*

Finally, we provide an example to verify the above result.

Example 3 *Consider the system in Example 1 and let $\tau_1 = 1$, $\tau_2 = \sqrt{2}$. Firstly, we check the $\mathcal{H}$-stability of system (2) through the normal form (11). System (2) is unstable due to the positive spectral abscissa 1.2156. However, by Definition 3 and Theorem 3, the system is of minimum-phase for $d = (\mathrm{e}^{-s}, \mathrm{e}^{-\sqrt{2}s})$, inferred from the spectral abscissa of $A_{00}(d)$; -0.6077. Next, according to Theorem 6, a PD high-gain output feedback controller will be designed to to stabilize system (2). Let $\gamma_1(s) = s^2 + 2s + 1$ and $\gamma_2(s) = s^2 + 5s + 6$. Then the controller (13) can be designed as*

$$u = \left[\begin{array}{cccc} -f^2 + 5d_2 + 7 & -2f + 6 & \mu_1(d) & \mu_2(d) \\ -4 & 1 & \mu_3(d) & \mu_4(d) \end{array}\right] \hat{y}, \quad (15)$$

where $\hat{y} = \left[y_1 \ y_1^{(1)} \ y_2 \ y_2^{(1)}\right]^{\mathrm{T}}$, $\mu_1(d) = 6d_2f^2 + 5d_1d_2 - 2d_2^2 + d_1 + 4$, $\mu_2(d) = 5d_2f + d_1d_2 - 5d_2 - 1$, $\mu_3(d) = -6f^2 - \varpi_6(d)$, $\mu_4(d) = -5f - \varpi_7(d)$. Substituting the controller (15) into normal form (11), the characteristic polynomial of the closed-loop system can be obtained as $\hat{\gamma}_1(s)\hat{\gamma}_2(s) \det\left[\begin{smallmatrix} \vartheta_1(s) & \vartheta_2(s) \\ \vartheta_3(s) & \vartheta_4(s) \end{smallmatrix}\right]$, where $d = (\mathrm{e}^{-s}, \mathrm{e}^{-\sqrt{2}s})$, $\hat{\gamma}_1(s) = s^2 + 2fs + f^2$, $\hat{\gamma}_2(s) = s^2 + 5fs + 6f^2$, $\vartheta_1(s) = s + 5 - \frac{8}{\hat{\gamma}_1(s)} - \frac{2d_1d_2+10d_2}{\hat{\gamma}_2(s)}$, $\vartheta_2(s) = -2d_2 + 3 - \frac{8\varpi_1(d)}{\hat{\gamma}_1(s)} + \frac{(2d_1d_2+10d_2)(d_2-1)}{\hat{\gamma}_2(s)}$, $\vartheta_3(s) = -1 + \frac{5}{\hat{\gamma}_1(s)} - \frac{-2d_2+2}{\hat{\gamma}_2(s)}$, and $\vartheta_4(s) = s + 1 + \frac{5\varpi_1(d)}{\hat{\gamma}_1(s)} - \frac{(2d_2-2)(d_2-1)}{\hat{\gamma}_2(s)}$.

By using the controller (15) and considering the gain f as a real variable, the spectral abscissa function is shown in Figure 1, from which we observe that when $f \geq 3$, the closed-loop system is $\mathcal{H}$-stable, inferred from the strictly negative spectral abscissa. Note that for $f \to \infty$, the spectral abscissa of the closed-loop system converges to the one for the zero-dynamics subsystem.

5 Conclusion

This paper was mainly concerned with the normal form of linear systems with non-commensurate state, input and output delays simultaneously, and the application to output feedback stabilization problem. The key to establishing the normal form lies in the invertibility of the coupling matrix and the construction of an invertible state transformation matrix, which direct our research

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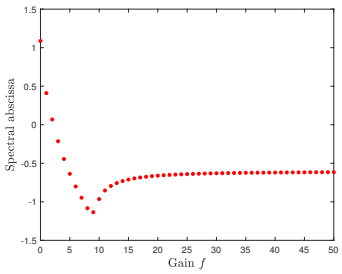

Fig. 1. The spectral abscissa function of the closed-loop system by using controller (15) in Example 3.

toward the ring framework. The invertibility of the coupling matrix was guaranteed by the assumption of the existence of a pure relative degree. An invertible state transformation matrix was constructed by utilizing the proposed unimodular completion method, which is an arduous task, given the immature theory of multivariate polynomial matrices. Finally, by employing the normal form, a novel PD high-gain output feedback controller was designed, so that the output feedback stabilization problem was solved. Considering the restrictive condition of pure relative degree, in future work, the existence conditions and construction methods of an output transformation will be established for system (2) to guarantee the transformed system has a pure relative degree, so that the results of this paper can be applied to a wider range of system categories.

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