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PUNCTURED INTERVALS TILE $\mathbb{Z}^3$

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Abstract. Extending the methods of Metrebian (2018), we prove that punctured intervals tile $\mathbb{Z}^3$. This solves two questions of Metrebian and completely resolves a question of Gruslys, Leader and Tan. We also pose a question that asks whether there is a relation between the genus g (number of holes) in a one-dimensional tile T and a uniform bound d such that T tiles $\mathbb{Z}^d$. An affirmative answer would generalize a conjecture of Gruslys, Leader and Tan (2016).

§1. *Introduction.* Given n, let T be a tile in $\mathbb{Z}^n$, that is, a finite subset of $\mathbb{Z}^n$. The cardinality of T, $|T|$, is the size of T, that is, the number of elements of the subset. Confirming a conjecture of Chalcraft that was posed on MathOverflow, Gruslys *et al.* [2] showed that T tiles $\mathbb{Z}^d$ for some d. This is an existence result and they wondered about better bounds in terms of the dimension n and the size $|T|$. They conjectured the following for the case $n = 1$.

CONJECTURE 1.1 [2]. For any positive integer t, there is a number d such that any tile T in $\mathbb{Z}$ with $|T| = t$ tiles $\mathbb{Z}^d$.

Let us note that Adler and Holroyd [1] had earlier investigated which tiles in $\mathbb{Z}$ can tile $\mathbb{Z}$. When dealing with one-dimensional tiles, we find it convenient to use similar notation as in [1]: a tile T in $\mathbb{Z}$ which is the union of n intervals I_1 up to I_n, such that the length of interval I_i is a_i and the gap between I_i and I_{i+1} is b_i, will be denoted by $[a_1(b_1)a_2(b_2)a_3\dots(b_{n-1})a_n]$. We will call $g = n-1$ the genus of the interval T. Note that for $g = 0$, a tile $[k]$ is indeed a translate of $\{1, 2, \dots k\}$, so this particular case corresponds with classical notation in set theory. The main example we will be working with will be the punctured interval $[k(1)\ell]$ which denotes an interval of $k+\ell+1$ points with one point removed, that is, separating two intervals of length k and l.

Wondering about Conjecture 1.1, one may wonder if the dimension d only depends on the genus of the tile instead of the size. Leading to the following question.

QUESTION 1.2. Does there exist a function $d\colon \mathbb{N} \to \mathbb{N}$ such that any tile $T \subset \mathbb{Z}$ with genus g tiles $\mathbb{Z}^{d(g)}$?

Answering this affirmatively would confirm Conjecture 1.1 since $g \leqslant t-1$. As observed in, for example, [2], for any fixed d, there are one-dimensional tiles with large genus which cannot tile $\mathbb{Z}^d$, see § 4. In particular, we observe that $d(g) \geqslant \frac{g}{2}+1$ by taking $k \to \infty$ in Proposition 4.1. We note that Question 1.2 is false for n-dimensional tiles with $n \geqslant 2$, even for genus 0. This is explained in Proposition 4.3, but it does not imply that the generalization of Conjecture 1.1 cannot hold for $T \subset \mathbb{Z}^n$ where $n \geqslant 2$. We make progress on Question 1.2

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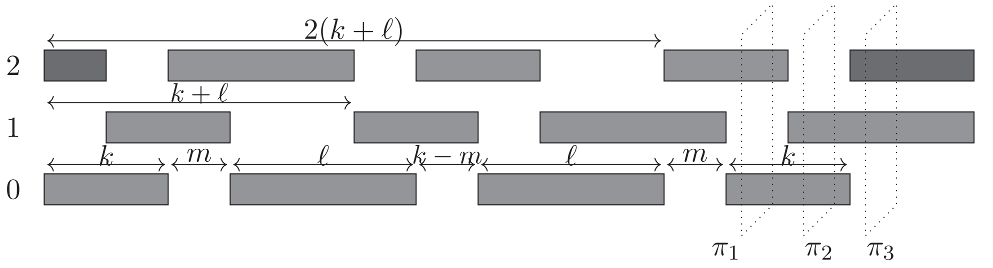

Figure 1: Construction of Y.

for the case $g = 1$ by establishing that one-dimensional tiles $T = [k(m)\ell]$ with $m \leqslant 2$ do tile $\mathbb{Z}^3$, as sketched in the Appendix. In full detail, we prove that punctured intervals do tile $\mathbb{Z}^3$ as our main result.

THEOREM 1.3. *Every punctured interval* $T = [k(1)\ell]$ *does tile* $\mathbb{Z}^3$.

This theorem answers two concrete questions posed by Metrebian [**4**, Question 10,11]. As a corollary, the least d for which $T = [k(1)k]$ tiles $\mathbb{Z}^d$ equals $\min\{k, 3\}$, answering [**2**, Question 21].

In § 2, we prove a lemma implying that it is enough to find some structured partial tilings of $\mathbb{Z}^2$ to prove tiles do tile $\mathbb{Z}^3$. In § 3, we exhibit such partial tilings for punctured intervals. Some divisibility constraints for specific constructions make this a delicate task. For the symmetric tiles $T = [k(1)k]$, the construction depends on $v_2(k)$, the exponent of 2 in the prime factorization of k. So we create infinitely many families of constructions. This is done in Lemma 3.2. Metrebian [**4**] did have such examples already when $v_2(k) \in \{0, 2\}$. An important ingredient to prove the validity of this infinite family of partial tilings is some elementary number theory. In Lemma 3.3, we give constructions for asymmetric tiles $T = [k(1)\ell]$ where $k \neq \ell$. So Lemmas 3.2 and 3.3 together imply Theorem 1.3.

§2. *From partial to complete tilings.* In this section, we will prove that finding certain partial tilings is enough to conclude that a whole tiling does exist. This is done in Lemma 2.1 which is a generalization of [**4**, Lemma 4].

LEMMA 2.1. *Let* T *be the one-dimensional tile* $[k(m)\ell]$. *Suppose there are three disjoint subsets* A, B, C *of* $\mathbb{Z}^d$ *with the same cardinality such that one can tile* $\mathbb{Z}^d \setminus (A \cup B)$, $\mathbb{Z}^d \setminus (A \cup C)$ *and* $\mathbb{Z}^d \setminus (B \cup C)$ *with* T. *Then* T *tiles* $\mathbb{Z}^{d+1}$.

Proof. First assume $m < \min\{k, \ell\}$. We construct a subset $Y \subset \mathbb{Z} \times \{0, 1, 2\}$ such that $|Y \cap (\{z\} \times \{0, 1, 2\})| = 2$ for every $z \in \mathbb{Z}$ and such that T tiles Y. Let $(x, i) \in Y$ for some $x \in \mathbb{Z}$ and $i \in \{0, 1, 2\}$ if and only if

$$x - i(k + l) \equiv 1, 2, \ldots, k; k + m + 1, k + m + 2 \ldots, k + m + \ell \pmod{3k + 3\ell} \text{ or}$$

$$\equiv 2k + \ell + 1, \ldots, 2k + 2\ell;$$

$$2k + 2\ell + m + 1, \ldots, 3k + 2\ell + m \pmod{3k + 3\ell}.$$

The construction has been sketched in Figure 1 for $\{1, 2, \ldots, 3(k + \ell)\} \times \{0, 1, 2\}$. By gluing infinitely many copies of that picture together, one gets the full construction of Y.

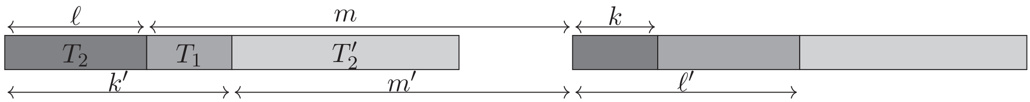

Figure 2: Gluing T_1 and T_2 and copies T'.

Now we explain why this construction meets the conditions we need. Let $S_1 = \{1, 2, \ldots, k\}, S_2 = \{k + m + 1, k + m + 2 \ldots, k + m + \ell\}, S_3 = \{2k + \ell + 1, \ldots, 2k + 2\ell\}$ and $S_4 = \{2k + 2\ell + m + 1, \ldots, 3k + 2\ell + m\}$. Let $S_o = S_1 + S_3$ and $S_e = S_2 + S_4$. Then both $S_o \cup ((k + \ell) + S_o) \cup (2(k + \ell) + S_o)$ and $S_e \cup ((k + \ell) + S_e) \cup (2(k + \ell) + S_e)$ cover all elements in $\frac{\mathbb{Z}}{3(k+\ell)\mathbb{Z}}$ exactly once, from which the result follows.

The elements of $A \cup B \cup C$ can be partitioned into triples $\{a_i, b_i, c_i\}$ since A, B, C have the same cardinality. Every set $\mathbb{Z} \times \{a_i, b_i, c_i\}$ has a subset $Y_i \cong Y$ which can be tiled by T in the same manner, that is, there exists a partition $\{Z_1, Z_2, Z_3\}$ of $\mathbb{Z}$ such that for every i we have $Y_i \cap (\{z\} \times \{a_i, b_i, c_i\}) = \{a_i, b_i\}$ for every $z \in Z_1$, $Y_i \cap (\{z\} \times \{a_i, b_i, c_i\}) = \{a_i, c_i\}$ for every $z \in Z_2$ and $Y_i \cap (\{z\} \times \{a_i, b_i, c_i\}) = \{b_i, c_i\}$ for every $z \in Z_3$. Now $\mathbb{Z}^{d+1} \setminus (\cup_i Y_i)$ can be written as $Z_1 \times (\mathbb{Z}^d \setminus (A \cup B)) \cup Z_2 \times (\mathbb{Z}^d \setminus (A \cup C)) \cup Z_3 \times (\mathbb{Z}^d \setminus (B \cup C))$ and by the assumptions this can be tiled by T as well, so T tiles $\mathbb{Z}^{d+1}$. Looking at Figure 1, every hyperplane π_i will be covered by the intersections with $\cup_i Y_i$ and a partial tiling isomorphic to one of $\mathbb{Z}^d \setminus (A \cup B)$, $\mathbb{Z}^d \setminus (A \cup C)$ or $\mathbb{Z}^d \setminus (B \cup C)$.

When $m \geqslant \min\{k, \ell\}$, where we assume without loss of generality $k = \min\{k, \ell\}$, one can glue two copies T_1, T_2 of T together to a tile T' with $k' = \ell' = k + \ell$ and $m' = m - k$ by taking $T_1 = \{-k, -k + 1, \ldots, -1\} \cup \{m, m + 1, \ldots m + \ell - 1\}$ and $T_2 = \{-k - \ell, -k - \ell + 1, \ldots, -k - 1\} \cup \{m - k, m - k + 1, \ldots, m - 1\}$. See Figure 2 for a depiction. When $m' \geqslant k'$, one can glue $\lfloor m'/k' + 1\rfloor$ copies of T' together, which are translates of T' with initial point at $0, k', \ldots, \lfloor m'/k'\rfloor k'$. Hence we have reduced this to the case which has been proven already. □

§3. *Punctured intervals tile* $\mathbb{Z}^3$. Throughout this section, we let T be a punctured interval tile, which is the union of an interval of length k and an interval of length ℓ with a gap of size 1. So $T = [k(1)\ell]$ equals a translate of $\{-k, -k + 1, \ldots, -1, 1, 2, \ldots, \ell\}$ as a subset of $\mathbb{Z}$. By applying Lemma 2.1, we will prove that T tiles $\mathbb{Z}^3$ for any k, ℓ.

As a warm up and for completeness of presentation, we construct three partial tilings of the plane satisfying the conditions of Lemma 2.1 when T is the symmetric punctured interval $[k(1)k]$ with $k \equiv 1 \pmod 2$. This was also proven in [4, Theorem 3].

LEMMA 3.1 [4]. *If* $2 \nmid k$, *then* $T = [k(1)k]$ *tiles* $\mathbb{Z}^3$.

Proof. Let $A = \{(x, y) \in \mathbb{Z}^2 \mid x \equiv y \pmod{k + 1}, x \equiv 0 \pmod 2\}$, $B = (1, 1) + A$ and $C = (1, 0) + A$. Now one can tile $\mathbb{Z}^2 \setminus (A \cup B)$ horizontally or vertically, as $A \cup B$ consists of infinitely many diagonals with distance k in between.

One can tile $\mathbb{Z}^2 \setminus (A \cup C)$ by placing copies of T vertical and similarly one can tile $\mathbb{Z}^2 \setminus (B \cup C)$ with horizontal copies of T. The sets A, B, C are presented in Figure 3, with the three partial tilings being sketched in Figure 4. Hence the result follows from Lemma 2.1. □

LEMMA 3.2. *Symmetric punctured intervals* $T = [k(1)k]$ *tile* $\mathbb{Z}^3$.

Proof. Let $v_2(k) = n$ and $q = 2^n$. When $n = 0$, the result follows from Lemma 3.1. So from now on, we assume $n \geqslant 1$.

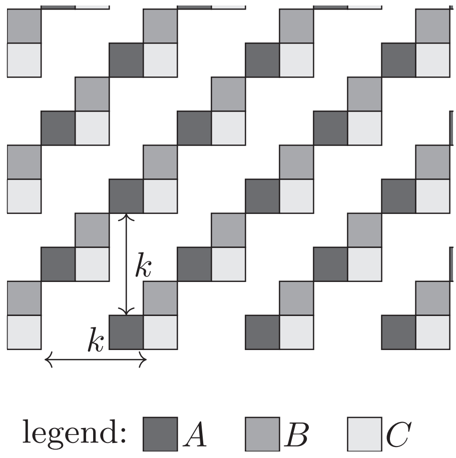

Figure 3: Construction of A, B, C for $T = [k(1)k]$ where $k \equiv 1 \pmod 2$.

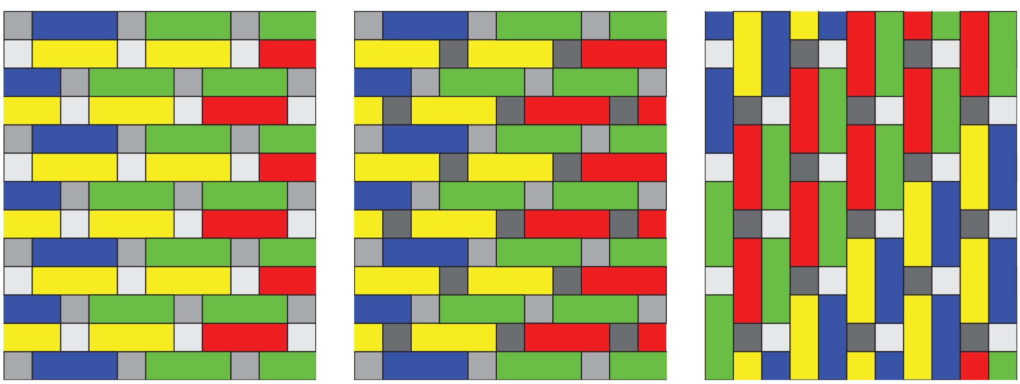

Figure 4 (colour online): Partial tilings for $k \equiv 1 \pmod 2$.

Let $A \subset \mathbb{Z}^2$ be the sets containing the elements (x, y) if and only if

$$x - y\big[2(k+1)(q-1)+1\big] \equiv i(k+1) \pmod{4(k+1)q}$$

for some $0 \leqslant i \leqslant 2q-1$. Let $B = (2q(k+1), 0) + A$ and $C = B + (k, 0)$.

One can see a depiction of this in Figure 5 in the case $q = 2, n = 1$ (for $k = 6$ actually).

Now one can tile $\mathbb{Z}^2 \setminus (A \cup B)$ with T as $A \cup B$ is the union of diagonals which are distance $k+1$ apart. One can tile $\mathbb{Z}^2 \setminus (A \cup C)$ horizontally as well. For this, it is enough to tile one horizontal line as every horizontal line is a translate of that one and due to periodicity in particular the set

$$\big(\mathbb{Z}^2 \setminus (A \cup C)\big) \cap (\{0, 1, \dots, 4q(k+1)-1\} \times \{0\})$$
$$= (\{0, 1, \dots, 4q(k+1)-1\} \times \{0\}) \setminus (A \cup C).$$

For this, use translates of T starting at $(1 + 2i(k+1), 0)$ for $0 \leqslant i \leqslant q-1$ and at $(2i(k+1), 0)$ for $q \leqslant i \leqslant 2q-1$.

To finish, we note that we can tile $\mathbb{Z}^2 \setminus (B \cup C)$ vertically. For this, we only have to check $(\{0\} \times \mathbb{Z}) \setminus (B \cup C)$, since $\gcd\{4(k+1)q, 2(k+1)(q-1)+1\} = 1$ and hence every

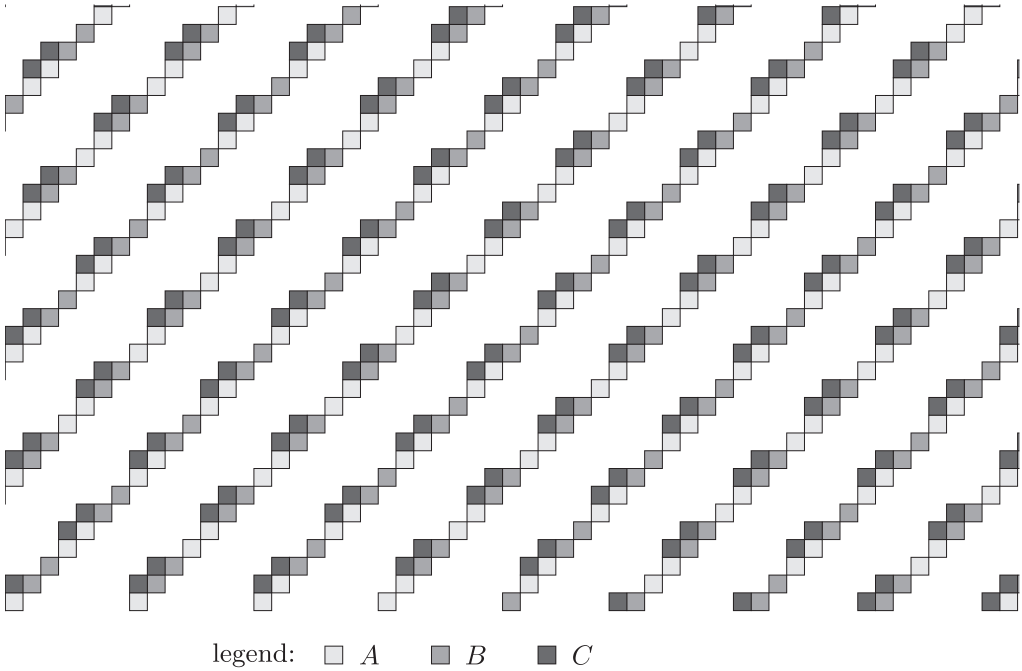

Figure 5: Construction of partial planar tilings where $v_2(k) = 1$.

vertical line is up to some translation identical to every other vertical line. By noting that B and C are subsets of some diagonals on the plane, one checks that

$$(\{0\} \times \mathbb{Z}) \cap B = \{0\} \times \{y \mid y \equiv i(k+1) \pmod{4(k+1)q}, 2q \leqslant i \leqslant 4q-1\}.$$

For this, note that

$$0 - i(k+1) \cdot [2(k+1)(q-1)+1] \equiv i(k+1) \pmod{4q(k+1)}$$

since $k \equiv q \pmod{2q}$. Similarly one has

$$(\{0\} \times \mathbb{Z}) \cap C = \{0\} \times \{y \mid y \equiv i(k+1)+1 \pmod{4(k+1))q}, -1 \leqslant i \leqslant 2(q-1)\}.$$

Hence one can tile $(\{0\} \times \mathbb{Z}) \setminus (B \cup C)$ by putting vertical tiles starting at $(0, i(2k+2) - k + 1)$ for every $i \equiv 0, 1, \ldots, q-1 \pmod{2q}$ and $(0, i(2k+2) - k)$ for every $i \equiv q, q+1, \ldots, 2q-1 \pmod{2q}$. Hence the result follows from Lemma 2.1. □

LEMMA 3.3. *Asymmetric punctured intervals $T = [k(1)\ell]$ with $k > \ell$ tile $\mathbb{Z}^3$.*

Proof. Let $A = \{(x, y) \mid y \equiv x - k \pmod{k+\ell+1}, x \equiv 1, 2, \ldots, k-\ell \pmod{2(k-\ell)}\}$, $B = (k-\ell, k-\ell) + A$ and $C = (k-\ell, 0) + A$.

Note that $A \cup B$ form diagonals distance $k+\ell+1$ apart and so we can tile $\mathbb{Z}^2 \setminus (A \cup B)$ by putting all tiles horizontal or vertical. One can tile $\mathbb{Z}^2 \setminus (A \cup C)$ vertically with copies of T and $\mathbb{Z}^2 \setminus (B \cup C)$ horizontally. A sketch of an example is given in Figure 6. □

§4. *Impossible tilings.* In this section, for the convenience of the reader, we collect two classes of (known) one-dimensional tiles of high genus and give an example of a two-dimensional tile of genus 0 that do not tile $\mathbb{Z}^d$ for a given d.

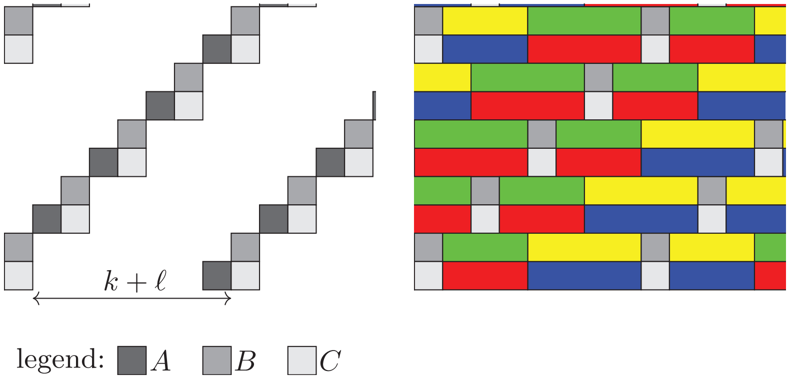

Figure 6 (colour online): Construction of A, B, C for $T = [k(1)\ell]$ where $k = \ell + 1$ with a partial tiling for $\mathbb{Z}^2 \backslash (B \cup C)$.

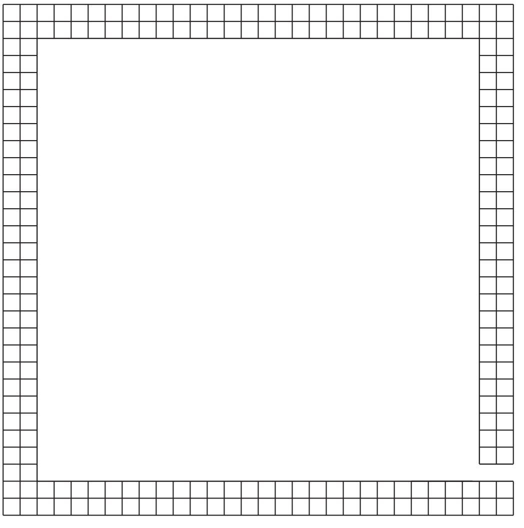

Figure 7: An example of a hook-tile H_{26}.

Let $T_{k,g}$ be the tile $[k\ \underbrace{(k-1)1(k-1)1\ldots(k-1)1(k-1)}_{g \text{ times } (k-1)}\ k]$. Note that $T_{k,k}$ was considered in [2].

Let D_n be the tile $[\underbrace{2(1)2(1)2\ldots(1)2}_{n \text{ times } 2}]$, as considered in [3].

Let H_k be any tile which is formed by taking a $(k+4) \times (k+4)-$square and removing a $k \times k-$square in the center and an additional 1×2 rectangle close to a corner in such a way that the tile has genus 0. An example is presented in Figure 7.

The following proposition shows that for every d, one can find $\{k, g\}$ and n such that neither D_n nor $T_{k,g}$ tiles $\mathbb{Z}^d$. The reason behind this is slightly different for the two tiles. The first uses sparseness of tiles put in one direction. The other considers the intersection of the tiles with subdivisions of $\mathbb{Z}^d$.

PROPOSITION 4.1. *$T_{k,g}$ does not tile $\mathbb{Z}^d$ for $d < \frac{kg+2k-1}{2k+g-1}$ and D_n does not tile $\mathbb{Z}^d$ for $n > 3^{d-1}$.*

Proof. In the case of $T_{k,g}$, one looks to the maximum volume covered by tiles in one of the d orthogonal directions in a hypercube $[N]^d$. When $N \to \infty$, the ratio of the volume covered by these tiles will have a limsup which is at most $\frac{2k+g-1}{kg+2k-1}$ from which the result follows as the sum of the ratios over the d directions should sum to 1.

Next, we consider the D_n. We assume D_n tiles $\mathbb{Z}^d$ and look to the intersection of this fixed tiling with a hypercube $[N]^d$. Look to the 3^d possible (infinite) partitions of $\mathbb{Z}^d$ in hypercubes with side length 3. These correspond with the 3^d possible vertices in $(\frac{\mathbb{Z}}{3\mathbb{Z}})^d$ that represent the corners of all the cubes in the partition. Call a nonempty intersection of $[N]^d$ with a hypercube of side length 3 for a given partition a subregion. We now count the total number $\#D$ of intersections of a subregion of a partition and a D_n which are of size 2 (the intersection has not necessarily to be connected), in two different ways.

For each of the 3^d partitions, there are less than $(\frac{N}{3}+2)^d$ subregions. Each subregion will contain at most $\frac{3^d-1}{2}$ intersections with a D_n of size 2. Hence $\#D < 3^d(\frac{N}{3}+2)^d\frac{3^d-1}{2} = \frac{3^d-1}{2}(N+6)^d$.

On the other hand, there are at least $\frac{(N-6n)^d}{2n}$ copies of D_n completely inside the hypercube. Every D_n of these, intersects n subregions in exactly twp places for each of $2 \cdot 3^{d-1}$ partitions. For 3^{d-1} partitions, these D_n intersects $n-1$ small hypercubes in exactly two places and two small hypercubes in exactly one place. This implies that $\#D \geqslant \frac{(N-6n)^d}{2n} \cdot 3^{d-1}(3n-1)$.

Hence $\frac{3^d-1}{2}(N+6)^d > \frac{(N-6n)^d}{2n} \cdot 3^{d-1}(3n-1)$ for all N, in particular one finds that the leading coefficients satisfy $\frac{3^d-1}{2} \geqslant 3^{d-1} \cdot \frac{3n-1}{2n} \Rightarrow n \leqslant 3^{d-1}$. □

In the case of D_n, this generalizes the "only if" part of [3, Proposition 1].

Let us remark that this also follows from a straightforward generalization of [3, Theorem 1], which concerns "convolutions" of tiles. In case it might be of use to others, we use the notation of [3] to state the generalization (and leave the proof to the reader) of [3, Theorem 1] (where they deal with the case $n = 2$ and $d = 2$).

PROPOSITION 4.2 [3]. *Suppose $T \subset \mathbb{Z}^n$ is a tile. Suppose that $S \subset \mathbb{Z}^d$ is a symmetric tile (i.e., no matter how the tile is oriented, it is a translate of itself). Then if for some $m \in \mathbb{N}$ one has $|1_S \bigstar_m 1_T|_1 < |\overline{1_S}||1_T|$, or if $|1_S \bigstar_m 1_T|_\infty < |1_T|$ and $|\overline{1_S}| \neq 0$, then T does not tile $\mathbb{Z}^d$.*

Next, we prove that H_k does not tile $\mathbb{Z}^d$ for fixed d and k large enough. The idea is essentially similar to the idea used for $T_{k,g}$.

PROPOSITION 4.3. *The tile H_k does not tile $\mathbb{Z}^d$ if $k \geqslant 8\binom{d}{2} - 6$.*

Proof. Note that there are $\binom{d}{2}$ planar directions in $\mathbb{Z}^d$ and two copies of H_k in the same direction cannot overlap. So if H_k would tile $\mathbb{Z}^d$, looking to the hypercube $[N]^d$ for $N \to \infty$, we remark that the limsup of the volume covered by tiles in one direction is bounded by $\frac{8k+14}{(k+4)^2} < \frac{8}{k+6}$. So when $\frac{8}{k+6}\binom{d}{2} \leqslant 1$, a tiling of $\mathbb{Z}^d$ by copies of H_k is impossible. □

Note. This paper is an update of an older version entitled "Symmetric punctured intervals tile $\mathbb{Z}^3$". So we like to give an idea of the changes. In this version, we start our journey

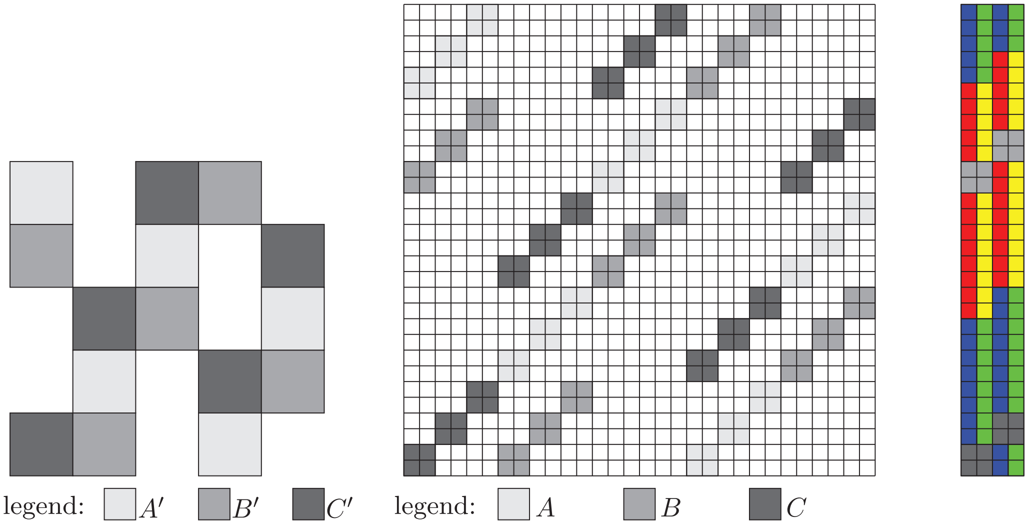

Figure A.1 (colour online): Construction of A, B, C for $T = [8(2)5]$.

more general from Conjecture 1.1 and Question 1.2. By finding a simple construction for the asymmetric punctured intervals in Lemma 3.3, some results in the old versions could be stated more elegantly. Other results could be removed, as they became redundant. The main result now also holds for all punctured intervals and not only symmetric intervals.

A. *Appendix. The tiles* $[k(2)\ell]$ *tile* $\mathbb{Z}^3$. In this section, we briefly sketch the construction for the cases where the gap is of length 2.

If $2 \mid k, \ell$, four copies form a tile that is isomorphic to $[\frac{k}{2}(1)\frac{\ell}{2}]$ and so it is known by Theorem 1.3. When $k \equiv \ell \pmod 4$ are both odd, we have $4 \mid k+\ell+2$ and a construction analogous to Figure 3 does work (k has to be $k+\ell$ instead and every square being a 2×2 square).

The remaining cases where k and ℓ are having different parity, or both are odd and $4 \mid k+\ell$, are handled in the following proposition.

PROPOSITION A.1. *Let* $k, \ell \in \mathbb{Z}^+$ *and* $d = \gcd\{k+1, \ell+1\}$ *where* $k+1 = dx$, $\ell+1 = dy$ *and* $x+y$ *is odd. Then* $[k(2)\ell]$ *does tile* $\mathbb{Z}^3$.

Proof. We start with selecting three transversals in $(\frac{\mathbb{Z}}{(x+y)\mathbb{Z}})^2$. Let $A' =< (-2,1) >$ be the set of elements (additively) generated by $(-2,1)$ in $(\frac{\mathbb{Z}}{(x+y)\mathbb{Z}})^2$. So A' contains exactly the pairs (c_1, c_2) such that $c_1 + 2c_2 \equiv 0 \pmod{x+y}$. Since $\gcd 2x, x+y = 1$, we note that A' is a transversal and that it also can be written as $< (-2x, x) >$. Now let $B' = (0, x) + A'$ and $C' = (-x, 0) + B' = (x, 0) + A'$. Replacing every square here by a $2d \times 2d$ square, where the sets A, B, C are formed by taking the elements on d 2×2 blocks on the diagonals in the corresponding squares of A', B' and C'. So now we have constructed subsets A, B, C in $(\frac{\mathbb{Z}}{2d(x+y)\mathbb{Z}})^2$, where $d(x+y) = k+\ell+2$. We can consider A, B, C as subsets of $\mathbb{Z}^2$ as well, if we consider the coordinates modulo $2(k+\ell+2)$. By applying Lemma 2.1 on these sets A, B, C, we conclude $[k(2)\ell]$ tiles $\mathbb{Z}^3$. For this, note that $\mathbb{Z}^2 \setminus (A \cup B)$ and $\mathbb{Z}^2 \setminus (A \cup C)$ both can be tiled with copies of T which are put horizontally, while $\mathbb{Z}^2 \setminus (B \cup C)$ can be tiled with vertical copies of T.

As an example, the idea is illustrated in Figure A.1 for $k = 8$, $\ell = 5$, so with $d = 3$, $x = 3$ and $y = 2$. The constructions of A', B' and C' in $(\frac{\mathbb{Z}}{5\mathbb{Z}})^2$ is presented on the left, the sets A, B, C in $(\frac{\mathbb{Z}}{30\mathbb{Z}})^2$ are depicted in the middle and part of the tiling of $\mathbb{Z}^2 \setminus (B \cup C)$ is shown on the right side. $\square$

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