INDUCTIVE LIMITS OF NONCOMMUTATIVE CARTAN INCLUSIONS

RALF MEYER, ALI I. RAAD, AND JONATHAN TAYLOR

Abstract. We prove that an inductive limit of aperiodic noncommutative Cartan inclusions is a noncommutative Cartan inclusion whenever the connecting maps are injective, preserve normalisers and entwine conditional expectations. We show that under the additional assumption that the inductive limit Cartan subalgebra is either essentially separable, essentially simple or essentially of Type I we get an aperiodic inclusion in the limit. Consequently, we subsume the case where the building block Cartan subalgebras are commutative and provide a proof of a theorem of Xin Li without passing to twisted étale groupoids.

1. Introduction

The theory of Cartan subalgebras for operator algebras has been prevalent since Murray and von Neumann's construction of the algebras $L^\infty(X,\mu) \rtimes G$ arising from nonsingular group actions on a measurable space $G \curvearrowright (X,\mu)$. Although not termed a Cartan subalgebra then, the distinguished subalgebra $L^\infty(X,\mu) \subseteq L^\infty(X,\mu) \rtimes G$ is indeed a Cartan subalgebra (a regular inclusion of a masa admitting a faithful normal conditional expectation). The abstract definition for a Cartan subalgebra in a von Neumann algebra was later given by Vershik (see [17]). A characterisation of such inclusions was provided by Feldman and Moore in [4]; the Cartan subalgebras are certain subalgebras of von Neumann algebras constructed from measured countable equivalence relations.

The theory of Cartan subalgebras in the setting of C*-algebras was thereafter developed by Kumjian and Renault ([7] and [16], respectively) and characterised as the inclusion of the C_0-functions on the unit space of an étale effective twisted groupoid inside the reduced C*-algebra of the twisted groupoid. Shortly after this characterisation, Exel defined a notion of a noncommutative Cartan subalgebra in [3], where the condition of being maximally commutative was replaced with the condition of having trivial virtual commutants. In the commutative case, this condition is exactly the one of being maximally commutative. Exel also showed that every noncommutative Cartan inclusion was an inclusion inside the reduced cross-sectional C*-algebra of a Fell bundle over an inverse semigroup, where the Cartan subalgebra corresponds to the reduction of the semigroup to the lattice of idempotents. Kwaśniewski and the first author improved upon Exel's theory in [9] by completely characterising the types of actions that yield noncommutative Cartan inclusions.

2010 *Mathematics Subject Classification.* Primary 46L05.

The second author was supported by the Internal KU Leuven BOF project C14/19/088 and project G085020N funded by the Research Foundation Flanders (FWO). The third author was supported by a stipend from the German Academic Exchange Service (DAAD) Funding Program 57450037.

23 Jun 2023 04:46:38 PDT Analysis
220830-fdo64 Version 3 - Submitted to Proc. Amer. Math. Soc.

In recent years, the commutative setting for Cartan subalgebras has attracted a lot of attention. They are related to topological dynamical systems via continuous orbit equivalence, to geometric group theory via quasi-isometry (see [13]), and also to the classification programme for C*-algebras, which aims at classifying a certain class of 'well-behaved' C*-algebras by an invariant consisting of K-theoretic and tracial data. A breakthrough result by Li in [14] shows that every such classifiable C*-algebra has a Cartan subalgebra.

In the same work, Li provides sufficient conditions on connecting maps of an inductive system of Cartan inclusions which guarantee that the inductive limit is a Cartan inclusion. One requires the connecting maps to be injective, map Cartan subalgebra to Cartan subalgebra, normalisers to normalisers and entwine the faithful conditional expectations (see Theorem 1.10 in [14]). In many classes of examples, it is significantly easier to check that such conditions hold rather than attempting to find a Cartan subalgebra in the inductive limit directly (for instance, in AF-algebras, where connecting maps are well-understood). In fact, Li and the second author have used these conditions to construct inductive limit Cartan subalgebras in many classes of AH-algebras, many of which are not classifiable (see [15]). The main result of this article generalises Li's theorem to inductive systems of noncommutative Cartan inclusions:

Theorem (see Theorem 3.9)**.** *Given an inductive system of aperiodic noncommutative Cartan inclusions where the connecting maps are injective and map Cartan subalgebra to Cartan subalgebra, normalisers to normalisers and entwine the conditional expectations, the inductive limit is a noncommutative Cartan inclusion. If the limit Cartan subalgebra has an essential ideal that is separable, simple or of Type I, then the limit inclusion is aperiodic.*

As a consequence, we get a sufficient condition on the level of connecting maps that guarantees that the inductive limit is a canonical noncommutative Cartan inclusion. In the commutative case, it turned out to be much easier to check such conditions rather than work directly with the inductive limit.

Our result also subsumes the result on Cartan subalgebras of inductive limits in Theorem 1.10 in [14] because all commutative Cartan inclusions are aperiodic (see Remark 3.10). Our proof of this theorem does not rely on passing to étale twisted groupoids.

The paper is organised as follows. Section 2 will briefly provide preliminaries on noncommutative and aperiodic Cartan inclusions, and will set up the standing assumptions on our inductive system. Section 3 will present the proof of our main theorem.

2. Preliminaries

Throughout this article, we will consider an inductive system of C*-algebras

$$
\begin{array}{ccccccccc}
A_1 & \xrightarrow{\phi_1} & A_2 & \xrightarrow{\phi_2} & A_3 & \xrightarrow{\phi_3} & \dots & \overset{\mu_n}{\dashrightarrow} & A \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
C_1 & \xrightarrow{\phi_1} & C_2 & \xrightarrow{\phi_2} & C_3 & \xrightarrow{\phi_3} & \dots & \underset{\mu_n}{\dashrightarrow} & C,
\end{array}
$$

23 Jun 2023 04:46:38 PDT Analysis
220830-fdo64 Version 3 - Submitted to Proc. Amer. Math. Soc.

where the vertical arrows are set inclusions. The building block inclusions $C_n \subseteq A_n$ are assumed to be nondegenerate noncommutative Cartan inclusions as in [3, Definition 2.1]. This means the following:

(1) C_n is a C*-subalgebra of A_n that is regular, that is, the set of normalisers $N_{A_n}(C_n) := \{n \in A_n : n^*C_n n \subseteq C_n, nC_n n^* \subseteq C_n\}$ generates A_n as a C*-algebra;
(2) there is a faithful conditional expectation $P_n \colon A_n \twoheadrightarrow C_n$;
(3) C_n contains an approximate unit for A_n;
(4) virtual commutants of C_n in A_n are trivial.

Here a *virtual commutant* of C_n in A_n is a bounded C_n-bimodule map $\varphi\colon J_n \to A_n$ for some closed two-sided ideal $J_n \subseteq C_n$. A virtual commutant of C_n in A_n is trivial if the image of the map lies in C_n. For further details consult [3].

We further assume that the inclusion $C_n \subseteq A_n$ is aperiodic. This means that the Banach C_n-bimodule $X_n = A_n/C_n$ is an aperiodic C_n-bimodule, that is, for each $x \in X_n$ and each nonzero hereditary subalgebra $D \subseteq C_n$ and $\epsilon > 0$, there is a positive element $d \in D$ with $\|d\| = 1$ and $\|dxd\| < \epsilon$ (see [9, Definition 6.1]).

We assume the connecting maps $\{\phi_n\}_{n\in\mathbb{N}}$ to be nondegenerate and injective *-homomorphisms. This gives rise to a nondegenerate inclusion of C*-algebras $C \subseteq A$ with nondegenerate and injective structure *-homomorphisms $\{\mu_n\}_{n\in\mathbb{N}}$. To simplify notation, we may identify building block algebras with their images under connecting maps, so that we may consider the maps ϕ_n and μ_n as inclusions of C*-subalgebras and drop them from our notation.

We further place assumptions on the connecting maps that are analogous to those in Theorem 1.10 in [14], namely:

(1) they map normalisers to normalisers, that is, $\phi_n(N_{A_n}(C_n)) \subseteq N_{A_{n+1}}(C_{n+1})$;
(2) they entwine the conditional expectations, that is, $P_{n+1} \circ \phi_n = \phi_n \circ P_n$.

For an inclusion of C*-algebras $\mathcal{C} \subset \mathcal{A}$ we will call a subset $M \subset \mathcal{A}$ a *slice* for the inclusion if M is a closed linear subspace of $N_{\mathcal{A}}(\mathcal{C})$ that is also a $\mathcal{C}$-bimodule. For $n \in \mathbb{N}$, let S_n be the inverse semigroup of slices for the inclusion $C_n \subseteq A_n$; its multiplication is defined by taking the closure of the linear span of the algebraic multiplication, and the inverse by taking the involution * (see Section 10 in [3]). For subsets A and B of a C*-algebra, we will denote the aforementioned multiplication by $A \cdot B = \overline{\operatorname{span}(AB)}$. For an element $m \in N_{A_n}(C_n)$, $C_n \cdot \{m\} \cdot C_n$ is a slice (see [3, Proposition 10.5]). Every slice is contained in a sum of slices of this form. Indeed, by the Cohen–Hewitt Factorisation Theorem ([6, Theorem 32.22]) we can write every slice M as $C_n M C_n$, which is contained in $\sum_{m\in M} C_n \cdot \{m\} \cdot C_n$.

An inductive system of slices $\mathcal{F} = \{M_n, \phi_n\}_{n\in\mathbb{N}}$ consists of slices $M_n \in S_n$ with $\phi_n(M_n) \subseteq M_{n+1}$. This system of slices gives rise to a limit slice $F_{\mathcal{F}} = \overline{\bigcup_{n\in\mathbb{N}} \mu_n(M_n)}$.

Define $P\colon A \to C$ as the (unique) extension of $P_0\colon \bigcup_n \mu_n(A_n) \to \bigcup_n \mu_n(C_n)$ defined by $P_0(\mu_n(a)) = \mu_n(P_n(a))$, $a \in A_n$. Since the connecting maps entwine conditional expectations and each P_n is contractive, the map P_0 is well-defined and contractive. A conditional expectation $Q\colon B \to D$ is *faithful* if no nonzero positive element of B is mapped to zero. It is *almost faithful* if $Q(x^*b^*bx) = 0$ for all $x \in B$ and some $b \in B$ implies $b = 0$. It is *symmetric* if $Q(b^*b) = 0$ is equivalent to $Q(bb^*) = 0$.

For an inclusion of C*-algebras $\mathcal{C} \subset \mathcal{A}$ a *generalized expectation* is a completely positive contractive map $E : \mathcal{A} \to \tilde{\mathcal{C}}$ such that $E|_{\mathcal{C}} = \mathrm{id}$, where $\mathcal{C} \subset \tilde{\mathcal{C}}$ is an inclusion of C*-algebras. If $\tilde{\mathcal{C}} = I(\mathcal{C})$ (where $I(\mathcal{C})$ is Hamana's injective hull, see [5]) then E is called a *pseudo-expectation*. For details, consult [11, Section 3].

Let $\mathcal{P}$ be a property for C*-algebras (for example, separable). We call a C*-algebra *essentially* $\mathcal{P}$ if it contains an essential ideal with property $\mathcal{P}$. Some results in [11] can only be applied if the C*-algebra A is *essentially separable*, *essentially simple*, or *essentially of Type I*.

3. Inductive limits of noncommutative Cartan inclusions

In this section we prove our main result. Unless otherwise stated, we assume throughout that we are in the setting given in the preliminaries.

Lemma 3.1. *The inclusion $C \subseteq A$ is regular, and P is an almost faithful conditional expectation.*

Proof. We first show that $C \subseteq A$ is a regular inclusion. Let $m \in N_{A_n}(C_n)$. We are going to prove that $\mu_n(m) \in A$ normalises C. Let $c \in C$. There are $c_j \in C_j$ with $\lim c_j = c$. Then

$$\mu_n(m^*)c\mu_n(m) = \lim_{j\to\infty} \mu_n(m^*)\mu_j(c_j)\mu_n(m) = \lim_{j\to\infty} \mu_j\big(\phi_{jn}(m^*)c_j\phi_{jn}(m)\big).$$

Since $\phi_{jn}(m)$ is a normaliser in A_j by assumption, $\phi_{jn}(m^*)c_j\phi_{jn}(m) \in C_j$ and so the limit belong to C. This finishes the proof that $\mu_n(m) \in A$ normalises C. Therefore, the C*-algebra generated by $N_A(C)$ contains the C*-algebra generated by all the $\mu_n(A_n)$, which is all of A.

Next, we show that P is almost faithful. Equivalently, $\mathcal{N} := \{a \in A : P(b^*a^*ab) = 0 \text{ for all } b \in A\}$ vanishes. [8, Proposition 2.2] implies that $\mathcal{N}$ is the largest ideal of A contained in $\ker(P)$. We know that $\mathcal{N}_n = \mathcal{N} \cap \mu_n(A_n)$ is an ideal of $\mu_n(A_n)$. Thus $\mathcal{N} = \overline{\bigcup_n \mathcal{N}_n}$ (see [2, Lemma III.4.1]). Since P vanishes on $\mathcal{N}_n$, it follows that P_n vanishes on $\mu_n^{-1}(\mathcal{N}_n)$. Since P_n is almost faithful, even faithful, this forces $\mathcal{N}_n = \{0\}$. Hence $\mathcal{N} = \{0\}$ as desired. □

Lemma 3.2. *Let $\mathcal{F} = \{M_n, \phi_n\}_{n\in\mathbb{N}}$ be an inductive system of slices with limit F. Then F is a slice for the inclusion $C \subseteq A$.*

Proof. We know that F normalises C because each M_n normalises C_n. It is clear that F is a closed linear subspace. We claim that $FC \subseteq F$. Fix $c_k \in C_k$ and consider $f = \lim_i \mu_i(m_i) \in F$. Then $fc_k = \lim_i \mu_i(m_i\phi_{ik}(c_k))$ which is a limit of elements in $\mu_i(M_i)$ and hence belongs to F. As F is closed it follows that $FC \subseteq F$. A similar proof shows $CF \subseteq F$. □

We now let S be the collection of all limits of inductive systems $\mathcal{F} = \{M_n, \phi_n\}_{n\in\mathbb{N}}$.

Lemma 3.3. *The collection S is an inverse semigroup of slices, under the multiplication $\cdot$ and the inverse *.*

Proof. Let F_1 and F_2 be the limits of inductive systems of slices $\mathcal{F}_1 = \{M_n, \phi_n\}_{n\in\mathbb{N}}$ and $\mathcal{F}_2 = \{N_n, \phi_n\}_{n\in\mathbb{N}}$, respectively. We first show that $F_1 \cdot F_2$, is the limit of the inductive system of slices $\{M_n \cdot N_n, \phi_n\}_{n\in\mathbb{N}}$. First, $\bigcup_n \mu_n(\operatorname{span}(M_nN_n))$ is dense in $\bigcup_n \mu_n(M_n \cdot N_n)$, and $\bigcup_n \mu_n(\operatorname{span}(M_nN_n)) = \operatorname{span}((\bigcup_n \mu_n(M_n))(\bigcup_n \mu_n(N_n)))$.

Then

$$(1)\quad F_1\cdot F_2=\overline{\operatorname{span}((\overline{\bigcup_n \mu_n(M_n))(\bigcup_n \mu_n(N_n)}))}$$
$$=\overline{\bigcup_n \mu_n(\operatorname{span}(M_nN_n))}=\overline{\bigcup_n \mu_n(M_n\cdot N_n)}.$$

Let us now show that the involution * of the C*-algebra A acts as a generalized inverse. Let $F\in S$ be the limit of the system $\{M_n,\phi_n\}$, so that $F=\overline{\cup_{n\in\mathbb{N}}\mu_n(M_n)}$. Now note that by continuity of the involution * we have $F^*=\overline{\cup_{n\in\mathbb{N}}\mu_n(M_n^*)}$. Hence by (1) it follows that

$$F\cdot F^*\cdot F=\overline{\bigcup_n \mu_n(M_n\cdot M_n^*\cdot M_n)}=\overline{\bigcup_n \mu_n(M_n)}=F.$$

Similarly one can show that $F^*\cdot F\cdot F^*=F^*$. To obtain uniqueness of the generalized inverse, note that if $F_1,F_2\in S$ are idempotents, then they are ideals of C and hence commute. Now [12, Theorem 3 in Chapter 1.1] gives uniqueness of the generalized inverse and hence S is an inverse semigroup. □

Lemma 3.4. *The linear span of elements in S is dense in A.*

Proof. Any $m\in N_{A_n}(C_n)$ is contained in the slice $C_n\cdot\{m\}\cdot C_n$. Thus the limit of the inductive system of slices $\{C_k\cdot\{\phi_{kn}(m)\}\cdot C_k,\phi_k\}_{k\geq n}$ contains $\mu_n(m)$. Hence the linear span of elements of S contains the linear span of $\bigcup_n \mu_n(N_{A_n}(C_n))$. The latter is dense in A because each inclusion $C_n\subseteq A_n$ is regular. □

Remark 3.5. Lemma 3.4 implies that S is a saturated grading for A (see [9, Definition 2.1]). Then [9, Remark 2.8] shows that there is a canonical surjective *-homomorphism $U\colon C\rtimes S\to A$.

Lemma 3.6. *Let $U\colon C\rtimes S\to A$ be the universal surjective *-homomorphism and let $EL\colon C\rtimes S\to M_{\mathrm{loc}}(C)$ be the canonical generalized conditional expectation defined in* [9, Proposition 2.9]*. Then $EL=P\circ U$, and so EL takes values in $C\subseteq M_{\mathrm{loc}}(C)$.*

Proof. By Lemma 3.4, elements of S span a dense subset of A. Since conditional expectations are linear and continuous, it suffices to consider restrictions to building blocks of inductive systems of slices. Consider such a building block M_n. It suffices to show that $EL(k)=P_n(k)$ for all $k\in M_n$. On $M_n\cap C_n$ both P_n and EL restrict to the identity map as this is contained in C_n. The expectation EL is zero on the complement $M_n\cdot(M_n\cap C_n)^\perp$ by construction (see [1, Lemma 4.5]). To see that the expectation P_n is zero on $M_n\cdot(M_n\cap C_n)^\perp$, note that P_n preserves slices by [9, Lemma 4.10]. Then $P_n(M_n\cdot(M_n\cap C_n)^\perp)\subseteq P(M_n)\cdot(M_n\cap C_n)^\perp\subseteq(M_n\cap C_n)\cdot(M_n\cap C_n)^\perp=\{0\}$. Since $C_n\subseteq A_n$ is a Cartan inclusion, the slice M_n decomposes as $M_n=M_n\cap C_n\oplus M_n\cdot(M_n\cap C_n)^\perp$ by [9, Proposition 2.17]. So $P_n=EL$ on each slice. Then P and EL agree on the inductive limit slices belonging to S. Thus $EL=P\circ U$. □

Corollary 3.7. *The inductive limit expectation P is symmetric.*

Proof. Lemma 3.6 says that $EL=P\circ U$. Fix $a\in A$ with $P(a^*a)=0$. There is $a_0\in C\rtimes S$ with $U(a_0)=a$. By [10, Theorem 4.11], the expectation EL is symmetric. So $EL(a_0^*a_0)=P(U(a_0)^*U(a_0))=P(a^*a)=0$ implies $EL(a_0a_0^*)=0$. Then $P(aa^*)=P(U(a_0)U(a_0)^*)=EL(a_0a_0^*)=0$. This shows that P is symmetric. □

23 Jun 2023 04:46:38 PDT Analysis
220830-fdo64 Version 3 - Submitted to Proc. Amer. Math. Soc.

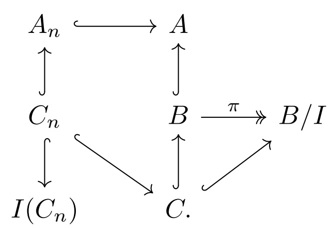
Corollary 3.8. *The inductive limit expectation P is faithful.*

Proof. By Lemma 3.1 and Corollary 3.7, P is almost faithful and symmetric. Therefore, it is faithful by [10, Corollary 3.8]. □

Theorem 3.9. *Let $C_n \subseteq A_n$ be aperiodic noncommutative Cartan inclusions. Let $\phi_n\colon A_n \to A_{n+1}$ be injective and nondegenerate *-homomorphisms that satisfy $\phi_n(C_n) \subseteq C_{n+1}$, $\phi(N_{A_n}(C_n)) \subseteq N_{A_{n+1}}(C_{n+1})$, and $P_{n+1} \circ \phi_n = \phi_n \circ P_n$. Then $C = \varinjlim(C_n, \phi_n) \subseteq A = \varinjlim(A_n, \phi_n)$ is a noncommutative Cartan inclusion. If, in addition, C is essentially separable, essentially simple or essentially of Type I, then the inclusion $C \subseteq A$ is aperiodic.*

Proof. We are going to prove by contradiction that C detects ideals in all intermediate C*-algebras $C \subseteq B \subseteq A$. Together with the results in [11], this will imply our claims. So assume that this fails. Then there is a nonzero ideal $I \subseteq B$ with $I \cap C = \{0\}$. Fix a nonzero positive element $b \in I$. Corollary 3.8 says that the conditional expectation P is faithful. So there is $\epsilon > 0$ with $\|P(b)\| > \epsilon$. Then there are $n \in \mathbb{N}$ and a positive element $b_n \in A_n$ with $\|b_n - b\| < \epsilon/3$.

Let $I(C_n)$ denote Hamana's injective envelope of C_n (see [5]) and let $\pi\colon B \to B/I$ be the quotient map. We have got the following commutative diagram:



Since $I(C_n)$ is injective, the identity homomorphism on C_n extends to a completely positive contractive map $Q\colon B/I \to I(C_n)$. Next, $Q \circ \pi$ extends to a completely positive contraction $R\colon A \to I(C_n)$. By construction, $R(b) = Q(\pi(b)) = 0$ and then $\|R(b_n)\| = \|R(b_n - b)\| \le \|b_n - b\| < \epsilon/3$. The reverse triangle inequality implies $\epsilon/3 > \|P(b_n) - P(b)\| \ge \big|\|P(b_n)\| - \|P(b)\|\big|$ and then $\|P(b_n)\| > 2\epsilon/3 > \epsilon/3 > \|R(b_n)\|$. Hence $P|_{A_n} \ne R|_{A_n}$. However, both P and R are completely positive contractions extending the identity map on C_n, which makes them generalized expectations for the inclusion $C_n \subseteq A_n$. It is well known that $M_{\mathrm{loc}}(C_n) \subseteq I(C_n)$. So P and R produce two different pseudo-expectations for the inclusion $C_n \subseteq A_n$. Then [11, Theorem 3.6] implies that this inclusion is not aperiodic, in contradiction to our assumption. This finishes the proof that C detects ideals in any intermediate C*-algebra B.

The canonical expectation EL on $C \rtimes S$ takes values in C by Lemma 3.6. Therefore, the reduced and essential crossed products agree for the relevant action of S on A. Since $EL = P \circ U$ and P is faithful, it also follows that the canonical *-homomorphism $U\colon C \rtimes S \to A$ descends to an isomorphism $A \cong C \rtimes_{\mathrm{r}} S$. If $T \subseteq S$ is an inverse subsemigroup that contains all idempotents of S, then $C \subseteq C \rtimes_{\mathrm{r}} T \subseteq C \rtimes_{\mathrm{r}} S$ is an intermediate C*-algebra, and we have shown that C detects ideals in it. Now [11, Proposition 6.7] shows that the action of S is purely outer. Then the inclusion $C \subseteq A$ is a noncommutative Cartan inclusion by [9, Theorem 4.3]. If C also contains an essential ideal that is separable, simple, or of Type I, then the inclusion $C \subseteq A$ is even aperiodic by the conditional implications in [11, Figure 1]. □

23 Jun 2023 04:46:38 PDT Analysis
220830-fdo64 Version 3 - Submitted to Proc. Amer. Math. Soc.

Remark 3.10. The properties of being of Type I, separable, or simple each pass to inductive limits of C*-algebras. Thus, if all the noncommutative Cartan subalgebras C_n in the building blocks are of Type I, separable, or simple, then the inclusion $C \subseteq A$ is again aperiodic. This, however, may break down if the building block subalgebras are only essentially of Type I, simple or separable, because essential ideals in C_n need not survive to ideals in C.

In particular, if each C_n is commutative, then it is of Type I and so there is no difference between aperiodic and purely outer actions. Hence Theorem 3.9 subsumes the setting of [14, Theorem 1.10]. Moreover the argument we give does not rely on passing to étale twisted groupoids.

Remark 3.11. In the case where each $C_n = C_0(X_n)$ is commutative, consider the Gelfand dual continuous map $f_n : X_{n+1} \to X_n$ inducing $\phi_n|_{C_n}$. If f_n is an open map, then the inclusion $C_n \subseteq C_{n+1}$ induced by $\phi_n|_{C_n}$ is anti-aperiodic, that is, contains no non-zero aperiodic C_n-bimodules. Indeed, for any non-zero function $g \in C_0(X_{n+1})$ there is an open subset $V \subseteq X_{n+1}$ where $|g(x)| > ||g||/2$ for all $x \in V$. Since f_n is open we see that $C_0(f_n(V))$ is an ideal in $C_0(X_n)$, and for any $h \in C_0(f_n(V))$ with $||h|| = 1$ we have $||hgh|| \geq \sup_{x\in V} |g(x)||h(f_n(x))|^2 > ||g||/2$, so can never satisfy Kishimoto's condition.

In this situation we have by [11, Proposition 3.9] that generalised expectations for the inclusion $C_n \subseteq A_n$ taking values in C_{n+1} are unique, since the inclusion $C_n \subseteq A_n$ is aperiodic, and the inclusion $C_n \subseteq C_{n+1}$ is anti-aperiodic. The maps $\phi_n \circ P_n$ and $P_{n+1} \circ \phi_n$ are both such generalised expectations, so must then be equal. Thus our *-homomorphisms $A_n \to A_{n+1}$ entwine conditional expectations automatically if the Gelfand duals of the restrictions $C_n \to C_{n+1}$ are open.

If the Gelfand dual map is not open then inclusions of commutative C*-algebras may not be anti-aperiodic. For example consider $C[0,1] \subseteq C[0,2]$ induced by the continuous function $f : [0,2] \to [0,1]$, defined by

$$f(t) := \begin{cases} t, & t \leq 1 \\ 1, & t > 1. \end{cases}$$

The $C[0,1]$-subbimodule $C_0(1,2]$ is annihilated by the essential ideal $C_0[0,1) \subseteq C[0,1]$. By [10, Lemma 5.12] the bimodule $C_0(1,2]$ is a non-zero aperiodic $C[0,1]$-subbimodule of $C[0,2]$, hence the inclusion is not anti-aperiodic.

References

[1] Alcides Buss, Ruy Exel, and Ralf Meyer, *Reduced C^*-algebras of Fell bundles over inverse semigroups*, Israel J. Math. **220** (2017), no. 1, 225–274, DOI 10.1007/s11856-017-1516-9. MR3666825

[2] Kenneth R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, vol. 6, Amer. Math. Soc., Providence, RI, 1996. MR1402012

[3] Ruy Exel, *Noncommutative Cartan subalgebras of C^*-algebras*, New York J. Math. **17** (2011), 331–382, available at `http://nyjm.albany.edu/j/2011/17-17.html`. MR2811068

[4] Jacob Feldman and Calvin C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. II*, Trans. Amer. Math. Soc. **234** (1977), no. 2, 325–359, DOI 10.2307/1997925. MR578730

[5] Masamichi Hamana, *Injective envelopes of C^*-algebras*, J. Math. Soc. Japan **31** (1979), no. 1, 181–197, DOI 10.2969/jmsj/03110181. MR519044

[6] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis. Vol. II*, Grundlehren der Mathematischen Wissenschaften, Springer, Berlin, 1975. Structure and analysis for compact groups. Analysis on locally compact abelian groups. MR0396828

[7] Alexander Kumjian, *On C^*-diagonals*, Canad. J. Math. **38** (1986), no. 4, 969–1008, DOI 10.4153/CJM-1986-048-0. MR854149
[8] Bartosz Kosma Kwaśniewski, *Exel's crossed product and crossed products by completely positive maps*, Houston J. Math. **43** (2017), no. 2, 509–567. MR3690127
[9] Bartosz Kosma Kwaśniewski and Ralf Meyer, *Noncommutative Cartan* C^**-subalgebras*, Trans. Amer. Math. Soc. **373** (2020), no. 12, 8697–8724, DOI 10.1090/tran/8174. MR4177273
[10] ———, *Essential crossed products by inverse semigroup actions: Simplicity and pure infiniteness*, Doc. Math. **26** (2021), 271–335, DOI 10.25537/dm.2021v26.271-335. MR4246403
[11] Bartosz Kosma Kwaśniewski and Ralf Meyer, *Aperiodicity: the almost extension property and uniqueness of pseudo-expectations*, Int. Math. Res. Not. IMRN **18** (2022), 14384–14426, DOI 10.1093/imrn/rnab098. MR4485960
[12] Mark V. Lawson, *Inverse semigroups: the theory of partial symmetries*, World Scientific Publishing Co., River Edge, NJ, 1998. MR1694900
[13] Xin Li, *Dynamic characterizations of quasi-isometry and applications to cohomology*, Algebr. Geom. Topol. **18** (2018), no. 6, 3477–3535, DOI 10.2140/agt.2018.18.3477. MR3868227
[14] ———, *Every classifiable simple* C^**-algebra has a Cartan subalgebra*, Invent. Math. **219** (2020), no. 2, 653–699, DOI 10.1007/s00222-019-00914-0. MR4054809
[15] Xin Li and Ali Imad Raad, *Constructing C*-diagonals in AH-algebras*, posted on 2021, DOI 10.48550/ARXIV.2111.11765.
[16] Jean Renault, *Cartan subalgebras in C^*-algebras*, Irish Math. Soc. Bull. **61** (2008), 29–63, available at `http://www.maths.tcd.ie/pub/ims/bull61/S6101.pdf`. MR2460017
[17] A. M. Vershik, *Nonmeasurable decompositions, orbit theory, algebras of operators*, Soviet Math. Dokl. **12** (1971), 1218–1222. MR0287331

Ralf Meyer, Mathematisches Institut, Universität Göttingen, Bunsenstrasse 3-5, 37073 Göttingen, Germany
Email address: `rmeyer2@uni-goettingen.de`

Ali I. Raad, Department of Mathematics, KU Leuven, 200B Celestijnenlaan, 3001 Leuven, Belgium
Email address: `ali.imadraad@kuleuven.be`

Jonathan Taylor, Mathematisches Institut, Universität Göttingen, Bunsenstrasse 3-5, 37073 Göttingen, Germany
Email address: `jonathan.taylor@mathematik.uni-goettingen.de`