

ARITHMETICALLY EQUIVALENT NUMBER FIELDS HAVE APPROXIMATELY THE SAME SUCCESSIVE MINIMA

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ABSTRACT. Let K and K' be arithmetically equivalent number fields, both of degree d . We prove that K and K' have the same successive minima, up to a constant depending only on d . We give examples showing that one cannot expect equality.

1. INTRODUCTION

Two number fields K and K' are said to be *arithmetically equivalent* if their Dedekind zeta functions are equal: $\zeta_K(s) = \zeta_{K'}(s)$. Arithmetically equivalent number fields are not necessarily isomorphic, but they always have the same degree, discriminant, signature, Galois closure and more, by a result of Perlis [13] using Gassmann's theorem [8]. However, for instance their class number and regulator can differ [7]. One of the main problems is to figure out how "isomorphic" arithmetically equivalent fields are, i.e. which arithmetic invariants are equal and which ones can differ. We refer to [17, 18, 16] for a more elaborate discussion.

In this article, we study the successive minima of arithmetically equivalent number fields, which we briefly recall. Let K be a degree d number field with ring of integers \mathcal{O}_K and denote by $\sigma_1, \dots, \sigma_d : K \rightarrow \mathbf{C}$ the embeddings of K into \mathbf{C} . Then the *Minkowski embedding* is the map

$$\iota : K \rightarrow \mathbf{C}^d : v \mapsto (\sigma_1(v), \dots, \sigma_d(v)).$$

This embedding yields a norm $\|\cdot\|$ on K , which for $v \in K$ we normalize as

$$\|v\| = \sqrt{\frac{1}{d}(|\sigma_1(v)|^2 + \dots + |\sigma_d(v)|^2)}.$$

There are different conventions in the literature regarding the Minkowski embedding and the corresponding norm on K , for instance by treating the real and complex embeddings separately. However, the above choice of embedding is the most natural in many settings, including ours (see e.g. [12, 11]). Then the *i -th successive minimum* λ_i of K is the smallest real number such that the set

$$\{v \in \mathcal{O}_K \mid \|v\| \leq \lambda_i\}$$

contains i \mathbf{Q} -linearly independent elements. These successive minima can be seen as an approximate refinement of the discriminant of K . Indeed, by Minkowski's

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second theorem [15, Thm. 16], the successive minima satisfy

$$\lambda_1 \lambda_2 \cdots \lambda_d = \Theta_d \left(\sqrt{|\Delta_K|} \right),$$

where Δ_K is the discriminant of K . Since arithmetically equivalent number fields have the same discriminant, it is natural to wonder whether they have (approximately) the same successive minima as well. This problem seems not to have been studied before, and we give a positive answer to this question.

Theorem 1. *Let K and K' be two arithmetically equivalent number fields of degree d . Let $\lambda_1 \leq \dots \leq \lambda_d$ and $\lambda'_1 \leq \dots \leq \lambda'_d$ be the multiset of successive minima of K and K' . Then there is a constant c_d depending only on d such that for all i*

$$\lambda_i \leq c_d \lambda'_i.$$

The strength of this result lies in the independence of the constant c_d on the specific number fields K and K' . It is natural to normalize the successive minima and instead look at the quantities

$$\mu_1 = \frac{\log \lambda_1}{\log(|\Delta_K|)/2}, \dots, \mu_d = \frac{\log \lambda_d}{\log(|\Delta_K|)/2}.$$

Indeed, Minkowski's second theorem implies that the μ_i sum up to roughly 1, i.e. $\sum_i \mu_i = 1 + O_d(\log(|\Delta_K|)^{-1})$. Now we can rephrase our result as stating that

$$|\mu_i - \mu'_i| \leq \frac{\log c_d}{\log |\Delta_K|}.$$

So if the absolute discriminant $|\Delta_K|$ is large compared to c_d , then the μ_i and μ'_i are very close together. Of course, the strength of the result also depends on the size of the constant c_d . In Section 4 we will prove that one may take $c_d = d$, and additionally provide a more direct comparison between the lattices \mathcal{O}_K and $\mathcal{O}_{K'}$ by constructing an injective linear map $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ of operator norm bounded by d^2 .

We give two proofs of Theorem 1. The first one passes through the common Galois closure L of K and K' and expresses the successive minima of K and K' in terms of the successive minima of lattices in L coming from irreducible representations of $\text{Gal}(L/\mathbf{Q})$. There is an analogue of Theorem 1 in the function field setting, where one has an exact equality of scollar invariants of Gassmann equivalent function fields [5] (see [9, Sec. 7] for this analogy), and this proof is similar in spirit. However, compared to the function field setting this proof does not seem to give good bounds on the required constant c_d . In particular, a rough argument shows that this proof gives $c_d = (d!)^3$.

The second proof is more direct, by explicitly constructing a \mathbf{Z} -linear map $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ using ideas from [2, 17]. Here we obtain much better bounds on c_d , in particular proving Theorem 1 with $c_d = d$. Moreover, this method is also well-suited to explicit computations in low degree. We prove in Section 5 that one may take $c_7 = 3$ and $c_8 = 3$.

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2. SOME REPRESENTATION THEORY

In this section, we recall two representation theoretic lemmas which we will need. Throughout, G will be a finite group. For a representation V , we will denote its character by χ_V . For two representations U, V of G , we denote by $\langle V, U \rangle$ the inner product $\langle \chi_V, \chi_U \rangle$.

Lemma 2. *Let V be an irreducible \mathbf{Q} -representation of G . Then there exists an $f_V \in \mathbf{Z}[G]$ such that f_V induces the zero map on U for every irreducible \mathbf{Q} -representation $U \neq V$ of G and such that f_V is invertible as a \mathbf{Q} -linear map on V .*

Proof. Consider the element

$$f_V = \sum_{g \in G} \chi_V(g^{-1})g \in \mathbf{Z}[G]$$

and note that it induces the zero map on every irreducible \mathbf{Q} -representation $U \neq V$ of G , see e.g. [14, Thm. 8]. On the other hand, we see that on $V \otimes \mathbf{C}$ the map f_V becomes invertible, and hence the same holds over \mathbf{Q} . \square

Lemma 3. *Let V be a \mathbf{Q} -representation of G and let H be a subgroup of G . Then the dimension of V^H is equal to $\langle V, \mathbf{Q}[G/H] \rangle$.*

Proof. Denote by 1 the trivial representation. We have that

$$\begin{aligned} \dim V^H &= \text{mult}(1, \text{Res}_H^G V) = \dim_{\mathbf{Q}} \text{Hom}_H(1, \text{Res}_H^G V) \\ &= \dim_{\mathbf{Q}} \text{Hom}_G(\text{Ind}_H^G 1, V) = \langle V, \mathbf{Q}[G/H] \rangle, \end{aligned}$$

where we have used Frobenius reciprocity. \square

Note that if V is absolutely irreducible, i.e. $\langle V, V \rangle = 1$, then $\langle V, \mathbf{Q}[G/H] \rangle$ is equal to the multiplicity of V in $\mathbf{Q}[G/H]$.

3. SUCCESSIVE MINIMA OF ARITHMETICALLY EQUIVALENT NUMBER FIELDS

In this section we give a first proof of Theorem 1. We first introduce some useful notation.

Let A be some object, e.g. an integer or a group. For two multisets of positive real numbers $\{\lambda_1, \dots, \lambda_m\}$ and $\{\lambda'_1, \dots, \lambda'_m\}$ we define

$$\{\lambda_1, \dots, \lambda_m\} \ll_A \{\lambda'_1, \dots, \lambda'_m\}$$

if there exists a constant $c = c_A > 0$ depending on A such that after reordering both multisets as $\lambda_1 \leq \dots \leq \lambda_m, \lambda'_1 \leq \dots \leq \lambda'_m$, we have for all i that

$$\lambda_i \leq c\lambda'_i.$$

We use the notation $\{\lambda_i\}_i =_A \{\lambda'_i\}_i$ to mean that $\{\lambda_i\}_i \ll_A \{\lambda'_i\}_i$ and $\{\lambda'_i\}_i \ll_A \{\lambda_i\}_i$. For a multiset $\{\lambda_1, \dots, \lambda_m\}$ and an integer k we define $k \cdot \{\lambda_1, \dots, \lambda_m\}$ to be the multiset $\{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_m, \dots, \lambda_m\}$ where every λ_i appears k times.

Let K be a degree d number field with ring of integers \mathcal{O}_K . We follow the notation from Section 1 regarding the Minkowski embedding, the resulting norm and the successive minima. Note that if L/K is a field extension then the resulting norm on L extends the one on K , precisely because of the normalization constant $1/\sqrt{d}$ in the definition of the norm. It is not necessarily true that there exists a basis of \mathcal{O}_K as a \mathbf{Z} -module achieving the successive minima, but we will not need more technical notions such as Minkowski reduced bases. For more background on successive minima we refer to [15].

Let L be a Galois extension of \mathbf{Q} with Galois group G . By the normal basis theorem, L is isomorphic to $\mathbf{Q}[G]$ as a $\mathbf{Q}[G]$ -module. For V an irreducible \mathbf{Q} -representation of G , we denote by L_V the *isotypic component* in L corresponding to V . Recall that this consists of all elements of L which generate V as a representation (together with 0). Since we are in characteristic 0, $\mathbf{Q}[G]$ is semi-simple and is the direct sum of its isotypic components. Define $\mathcal{O}_V = \mathcal{O}_L \cap L_V$, which is a sublattice of \mathcal{O}_L of rank $\dim(V)^2/\langle V, V \rangle$. The successive minima of \mathcal{O}_V will play a central role in the proof of Theorem 1, and the following lemma may be of independent interest.

Lemma 4. *Let L be a Galois extension of \mathbf{Q} with Galois group G , and let V be an irreducible \mathbf{Q} -representation of G . Then the multiset of successive minima of \mathcal{O}_V is of the form*

$$\dim(V) \cdot \{\lambda_{V,1}, \lambda_{V,2}, \dots, \lambda_{V,n_V}\},$$

where $n_V = \dim(V)/\langle V, V \rangle$.

In other words, this lemma states that every successive minimum of \mathcal{O}_V appears with multiplicity $\dim(V)$. We call the $\lambda_{V,i}$ the *successive minima of V in L* .

Proof. The proof proceeds inductively. The main point is that if v is in L and g in G , then $\|v\| = \|gv\|$. Take a non-zero vector v_1 in \mathcal{O}_V for which $\|v_1\|$ is minimal. By acting with G on v_1 , we obtain $\dim(V)$ linearly independent vectors in \mathcal{O}_V with the same length, showing that the first successive minimum appears with multiplicity $\dim(V)$. Now take a vector v_2 of shortest length which is linearly independent from $\{gv_1 \mid g \in G\}$, and act with G on v_2 . Continuing in this manner gives the result. \square

Our main tool is the following lemma.

Lemma 5. *Let L be a Galois extension of \mathbf{Q} with Galois group G and let H be a subgroup of G . Let $\{\lambda_1, \dots, \lambda_m\}$ be the successive minima of L^H . Then*

$$\{\lambda_1, \dots, \lambda_m\} =_G \bigcup_V \langle V, \mathbf{Q}[G/H] \rangle \cdot \{\lambda_{V,1}, \dots, \lambda_{V,n_V}\},$$

where the union is as a multiset, and is over all irreducible \mathbf{Q} -representations V of G .

Proof. Recall that the norm on L extends the norm on L^H . Hence the successive minima of L^H are the same as the successive minima of the lattice \mathcal{O}_{L^H} considered as a sublattice of \mathcal{O}_L . So it is enough to prove the result when the λ_i are the successive minima of \mathcal{O}_{L^H} inside L .

Let v_1, \dots, v_m be \mathbf{Q} -linearly independent elements of \mathcal{O}_{L^H} such that $\|v_i\| = \lambda_i$ under the Minkowski embedding of L . This is not necessarily a \mathbf{Z} -basis of \mathcal{O}_{L^H} but it will suffice for our purpose. For every irreducible \mathbf{Q} -representation V , let $w_{V,1}, \dots, w_{V,n_V}$ be in \mathcal{O}_V such that acting with G on $\{w_{V,i}\}_i$ gives a \mathbf{Q} -basis for the isotypic component L_V , and such that $\|w_{V,i}\| = \lambda_{V,i}$ are the successive minima of V in L . We then prove that

$$\{\|v_1\|, \dots, \|v_m\|\} =_G \bigcup_V \langle V, \mathbf{Q}[G/H] \rangle \cdot \{\|w_{V,1}\|, \dots, \|w_{V,n_V}\|\}.$$

For one inequality, consider the \mathbf{Z} -lattice in \mathcal{O}_{L^H} generated by all elements of the form

$$w_{V,i,g} = \sum_{h \in H} h g w_{V,i},$$

where $g \in G$, V is an irreducible \mathbf{Q} -representation of G and $i = 1, \dots, n_V$. The result is a full rank sublattice of \mathcal{O}_{L^H} . Indeed, if v is in L^H then we can write

$$v = \sum_{V,i,g} a_{V,i,g} g w_{V,i},$$

where the sum is over all irreducible \mathbf{Q} -representations V , $i = 1, \dots, n_V$, $g \in G$ and the $a_{V,i,g}$ are in \mathbf{Q} . Then acting with $\sum_{h \in H} h$ gives that

$$v = \frac{1}{\#H} \sum_{V,i,g} a_{V,i,g} w_{V,i,g}.$$

Now, since $\|w_{V,i,g}\| =_G \|w_{V,i}\|$ Lemma 3 implies that

$$\{\|v_1\|, \dots, \|v_m\|\} \ll_G \bigcup_V \langle V, \mathbf{Q}[G/H] \rangle \cdot \{\|w_{V,1}\|, \dots, \|w_{V,n_V}\|\}.$$

For the other direction we follow an inductive procedure. Using Lemma 2 we fix for every irreducible representation V of G an element $f_V \in \mathbf{Z}[G]$ which is invertible as a \mathbf{Q} -linear map on V and acts as the zero map on U , for every irreducible \mathbf{Q} -representation $U \neq V$. Note that every f_V maps elements of \mathcal{O}_L to \mathcal{O}_V . For every $v \in L$ and every V we have that

$$\|f_V(v)\| \ll_G \|v\|.$$

We now inductively pick an irreducible representation V_i of G for $i = 1, \dots, m$ in the following way. We start by taking an irreducible \mathbf{Q} -representation V_1 of G such that $f_{V_1}(v_1)$ is non-zero. If V_1, \dots, V_i have already been constructed, we take any V_{i+1} such that $f_{V_{i+1}}(v_{i+1})$ is \mathbf{Q} -linearly independent from

$$\{f_{V_{i+1}}(v_j) \mid j = 1, \dots, i, \text{ such that } V_j = V_{i+1}\}.$$

Note that we can indeed always take such a V_{i+1} since $\sum_V f_V$ is an invertible \mathbf{Q} -linear map on L . Having picked all V_i , we will have that for each irreducible \mathbf{Q} -representation V the set

$$\{f_V(v_i) \mid V_i = V\}$$

will generate a sublattice of \mathcal{O}_V whose rank is $n_V \langle V, \mathbf{Q}[G/H] \rangle$. Even more, for a fixed i the intersection of this lattice with the \mathbf{Q} -span of all conjugates of $w_{V,i}$ has rank $\langle V, \mathbf{Q}[G/H] \rangle$. Indeed, this follows from Lemma 3 and the fact that f_V is invertible on L_V . Because $\|f_V(v)\| \ll_G \|v\|$, we therefore have that

$$\langle V, \mathbf{Q}[G/H] \rangle \cdot \{\|w_{V,1}\|, \dots, \|w_{V,n_V}\|\} \ll_G \{\|v_i\| \mid i = 1, \dots, m \text{ for which } V_i = V\}.$$

Taking the union over all V finishes the proof. \square

With this, we can prove that arithmetically equivalent number fields have almost the same successive minima.

Proof of Theorem 1. We have two arithmetically equivalent number fields K and K' of degree d . By Gassmann's theorem [8] K and K' have a common Galois closure L with Galois group G and the subgroups H, H' of G for which $K = L^H, K' = L^{H'}$ are *Gassmann equivalent*. This means that for every conjugacy class \mathcal{C} of G we have $\#(\mathcal{C} \cap H) = \#(\mathcal{C} \cap H')$. Equivalently, this means that the $\mathbf{Q}[G]$ -modules $\mathbf{Q}[G/H]$ and $\mathbf{Q}[G/H']$ are isomorphic, see e.g. [17, Lem. 2.7, Thm. 2.8]. But then Lemma 5 implies that the multisets of successive minima of K and K' are asymptotic to each other, up to a factor depending on G . The dependence on G can be replaced by a dependence on d , since G is isomorphic to a subgroup of S_d . \square

A rough analysis shows that the above proof gives the constant $c_d = (d!)^3$. However, in the next section we will give an alternative proof of Theorem 1 which yields that we may take $c_d = d$.

4. BETTER BOUNDS

In this section we present an alternative proof of Theorem 1, which will in particular allow us to prove the result with $c_d = d$. In fact, the same proof allows us to relate the shape of the lattices of arithmetically equivalent number fields more directly as follows.

Proposition 6. *Let K, K' be arithmetically equivalent number fields of degree d . Then there exists a linear map $\phi : K \rightarrow K'$ such that*

- (1) ϕ is an isomorphism of \mathbf{Q} -vector spaces,
- (2) $\phi(\mathcal{O}_K) \subset \mathcal{O}_{K'}$, and
- (3) for every $v \in \mathcal{O}_K$ we have $\|\phi(v)\| \leq d^2 \|v\|$.

In other words, ϕ has operator norm bounded by d^2 .

In Section 5, we analyse this proof of this proposition further to improve this operator norm for degree 7 and 8.

Since the proof of this proposition and of Theorem 1 with $c_d = d$ is similar, we give both at the same time.

Proof of Theorem 1 and of Proposition 6. Let K and K' be arithmetically equivalent number fields of degree d . Let L be their common Galois closure with Galois group G over \mathbf{Q} . Let $H = \text{Gal}(L/K)$ and $H' = \text{Gal}(L/K')$. We will show how to construct elements $\phi \in \mathbf{Z}[G]$ inducing a morphism $\phi : K \rightarrow K'$ with the desired properties.

To construct ϕ , let g_1, \dots, g_d be a complete set of representatives for the left cosets G/H . We take ϕ of the form

$$\phi = \sum_i c_{g_i H} g_i,$$

where $c_{g_i H} \in \mathbf{Z}$. Then it is already clear that $\phi(\mathcal{O}_K) \subset \mathcal{O}_{K'}$. For ϕ to map K to $K' = L^{H'}$, we need the following for every $h' \in H'$. For every i there is a $\pi(i)$ for which $h'g_i = g_{\pi(i)}h_i$ for some $h_i \in H$. Then we must have for every $v \in K$ that

$$h'\phi(v) = \sum_i c_{g_i H} h'g_i v = \sum_i c_{g_i H} g_{\pi(i)} v = \sum_i c_{h'g_i H} g_i v.$$

Therefore, we must have that $c_{h'g_i H} = c_{g_i H}$ for every $h' \in H'$ and every i . In other words, c is constant on the double cosets $H' \backslash G/H$. By [17, Lem. 4.5] there is a bijective correspondence between \mathbf{Z} -valued functions on double cosets $H' \backslash G/H$ and $\mathbf{Z}[G]$ -morphisms $\mathbf{Z}[G/H] \rightarrow \mathbf{Z}[G/H']$. In more detail, if we let g'_1, \dots, g'_d be a complete set of representatives for G/H' , then a $\mathbf{Z}[G]$ -morphism $\psi : \mathbf{Z}[G/H] \rightarrow \mathbf{Z}[G/H']$ determines a $d \times d$ matrix M over \mathbf{Z} via

$$\psi(g_i H) = \sum_j M_{ij} g'_j H'.$$

Then the function c on double cosets corresponding to ψ is given by $H'g_j'^{-1}g_i H \mapsto M_{ij}$.

We now describe the universal morphism $\mathbf{Z}[G/H] \rightarrow \mathbf{Z}[G/H']$ following [2, Sec. 4]. For this, let $t = \#(H' \backslash G/H)$ be the number of double cosets, and let a_1, \dots, a_t be formal variables corresponding to these double cosets. Take $x_1, \dots, x_t \in G$ a set of representatives for these double cosets. Define the matrix A as $A_{ij} = a_\ell$ if $H'g_j'^{-1}g_i H = H'x_\ell H$. Upon fixing integer values for the a_i , we obtain a $\mathbf{Z}[G]$ -morphism $\mathbf{Z}[G/H] \rightarrow \mathbf{Z}[G/H']$, and conversely, every morphism is obtained from such a matrix.

Let us first focus on the proof of Proposition 6. Let f be the determinant of the matrix A , which is an integer polynomial in the a_i homogeneous of degree d . Since $\mathbf{Q}[G/H]$ and $\mathbf{Q}[G/H']$ are isomorphic, there exists an injection $\mathbf{Z}[G/H] \hookrightarrow \mathbf{Z}[G/H']$. For this injection, the corresponding determinant of A is non-zero, and so f is not the zero polynomial. By the Schwartz–Zippel Lemma (see e.g. [4, Cor. 4.1.2]), there exist $(a_i)_i$ for which $f(a_i) \neq 0$ and for which $|a_i| \leq d$ for all i . We fix these a_i and let ψ be the corresponding morphism $\mathbf{Z}[G/H] \rightarrow \mathbf{Z}[G/H']$. Let ϕ be the map $K \rightarrow K'$ as obtained above. By [17, Cor. 4.7], the invertibility of ψ as a morphism $\mathbf{Q}[G/H] \rightarrow \mathbf{Q}[G/H']$ guarantees that the map ϕ is also invertible as a map of vector spaces. Moreover, we have that $\|\phi(v)\| \leq \sum_i |A_{ij}| \|v\|$ (for any j). Hence by our choice of $(a_i)_i$, we have that $\|\phi(v)\| \leq d^2 \|v\|$.

For the improved version of Theorem 1¹, we instead construct multiple maps $K \rightarrow K'$. For each $i = 1, \dots, t$, let $\phi_i : K \rightarrow K'$ be the morphism corresponding to the choice $a_i = 1$ and $a_j = 0$ if $j \neq i$. Note that then ϕ_i maps \mathcal{O}_K to $\mathcal{O}_{K'}$ and for every $v \in \mathcal{O}_K$ we have $\|\phi_i(v)\| \leq d\|v\|$. None of these maps has to be an isomorphism of vector spaces, but since the construction of ϕ is linear in a , the previous paragraph implies that some linear combination of the ϕ_i is invertible. So if $V \subset K$ is a subspace of dimension i , then the spaces $\phi_1(V), \dots, \phi_t(V)$ together span a vector space of dimension at least i . Now if v_1, \dots, v_d are elements of \mathcal{O}_K achieving the successive minima of K , then applying this with $V = \{v_1, \dots, v_i\}$ for $i = 1, \dots, d$ proves Theorem 1 with $c_d = d$. \square

A more careful analysis of the determinant f of the matrix A appearing in the previous proof might allow one to prove Proposition 6 with the operator norm of ϕ bounded by d . In the next section, we take a closer look at f in low degree.

5. LOW DEGREE EXAMPLES

The author thanks Wouter Castryck for providing the code for computing successive minima of number fields in Magma [1].

Throughout this section, let K and K' be two non-isomorphic arithmetically equivalent number fields of degree d . This implies that $d \geq 7$. We let L be their common Galois closure with Galois group G , and let H, H' be the subgroups of G corresponding to K, K' . Let t be the number of double cosets $\#(H' \backslash G / H)$ and let a_1, \dots, a_t be variables. We denote by A the universal $d \times d$ matrix with entries from the a_i as constructed in the proof from the previous section. We let $\phi : K \rightarrow K'$ be the morphism from Proposition 6. Let us analyse the proof of Theorem 1 and Proposition 6 from Section 4 in more detail.

Degree 7. In this case, $G = \mathrm{GL}_3(\mathbf{F}_2) = \mathrm{PSL}_2(\mathbf{F}_7)$ is a group of order 168 and H and H' are both of index 7 [13], see also [3]. We have shown that one may take $c_7 = 7$ and ϕ of operator norm 49 in the previous section. In fact, one may even take $\|\phi\| \leq 3$ and hence also $c_7 = 3$, as we now argue. There are two double cosets in $H' \backslash G / H$ and the corresponding matrix A in $a = a_1, b = a_2$ is given by

$$A = \begin{pmatrix} a & b & b & b & a & a & a \\ b & a & a & b & b & a & a \\ b & b & a & a & a & b & a \\ b & a & b & a & a & a & b \\ a & b & a & a & b & a & b \\ a & a & b & a & b & b & a \\ a & a & a & b & a & b & b \end{pmatrix}.$$

The determinant of this matrix is $f = -32(a-b)^6(32a+24b)$. Hence we can take $a = 0, b = 1$ to get $f \neq 0$. The corresponding morphism $\phi : K \rightarrow K'$ will satisfy $\|\phi(v)\| \leq 3\|v\|$.

¹The author thanks Aurel Page for explaining this argument.

In general however, one cannot expect that $c_7 = 1$. For an explicit example, by [13] we may take K and K' by adjoining a root to \mathbf{Q} of the respective polynomials

$$f(x) = x^7 - 7x + 3, \text{ and } g(x) = x^7 + 14x^4 - 42x^2 - 21x + 9.$$

Then K and K' are arithmetically equivalent but not isomorphic, and their common Galois closure L has Galois group $\text{GL}_3(\mathbf{F}_2)$. Using Magma, we have computed the successive minima of K and K' to be approximately

$$\begin{aligned} K : & \quad \{1.00, 1.29, 1.78, 2.48, 2.57, 2.62, 2.96\}, \\ K' : & \quad \{1.00, 1.82, 1.83, 2.23, 2.31, 2.52, 2.82\}. \end{aligned}$$

More generally, one can construct an infinite family of arithmetically equivalent number fields whose Galois closures have Galois group $\text{GL}_3(\mathbf{F}_2)$, of arbitrarily large discriminant. Explicitly, for rational numbers s, t define the polynomial

$$\begin{aligned} f_{s,t}(x) = & x^7 + (-6t + 2)x^6 + (8t^2 + 4t - 3)x^5 + (-s - 14t^2 + 6t - 2)x^4 \\ & + (s + 6t^2 - 8t^3 - 4t + 2)x^3 + (8t^3 + 16t^2)x^2 + (8t^3 - 12t^2)x - 8t^3 \in \mathbf{Q}[x]. \end{aligned}$$

By work of LaMacchia [10], this polynomial has Galois group $\text{GL}_3(\mathbf{F}_2)$ over $\mathbf{Q}(s, t)$. Hence by Hilbert irreducibility there are infinitely many specializations $s, t \in \mathbf{Q}$ for which $f_{s,t}(x)$ is irreducible with Galois group $\text{GL}_3(\mathbf{F}_2)$. Bosma and de Smit [3] proved that if $f_{s,t}(x)$ is irreducible with Galois group $\text{GL}_3(\mathbf{F}_2)$ then the degree 7 number fields K and K' defined by $f_{s,t}$ and $f_{-s,t}$ are non-isomorphic and arithmetically equivalent. Experiments in Magma for various values of $s, t \in \mathbf{Q}$ show that the successive minima of K and K' are very close together. In fact, for these computations the value of c , i.e. the ratio between the successive minima of K and K' , never exceeded $3/2$.

Degree 8. In degree 8, by [3, Thm. 3] there are two options for the Galois group G and the corresponding subgroups H, H' . Namely, either $G = \text{AGL}_1(\mathbf{Z}/8\mathbf{Z})$, or $G = \text{GL}_2(\mathbf{F}_3)$.

Assume first that that $G = \text{AGL}_1(\mathbf{Z}/8\mathbf{Z})$, which is a group of order 32. This group is isomorphic to the subgroup of S_8 generated by $(12345678), (13)(57)(26)$ and $(15)(37)$. Under this isomorphism, the index 8 subgroups H and H' which are non-conjugate but Gassmann equivalent are $H = \langle (13)(57)(26), (15)(37) \rangle$ and $H' = \langle (17)(26)(35), (17)(35)(48) \rangle$. There are 4 double cosets of H, H' in G , and the corresponding matrix A in the variables a, b, c, d is

$$A = \begin{pmatrix} a & c & d & b & b & d & a & c \\ c & a & c & d & a & b & b & d \\ a & b & d & c & c & d & a & b \\ d & c & a & c & b & a & d & b \\ c & d & b & a & d & c & b & a \\ b & a & b & d & a & c & c & d \\ b & d & c & a & d & b & c & a \\ d & b & a & b & c & a & d & c \end{pmatrix}.$$

This matrix has determinant $f = 64(b-c)^4(a-d)^2(a-b-c+d)(a+b+c+d)$. The points which give the best estimates for the constant are $(a, b, c, d) = \pm(2, 1, 0, 0)$

or $\pm(1, 1, 0, -1)$. Indeed, this gives that $\|\phi\| \leq 6$, and so the successive minima of K and K' differ by at most a factor 6. In fact, we can do slightly better by noting that each of the maps ϕ_i constructed in Section 4 has norm at most 2. Hence the successive minima of K and K' differ by at most a factor 2. We note that the morphism corresponding to $(2, 1, 0, 0)$ was also considered in [6, p. 145], where it was used to relate the class numbers of certain families of arithmetically equivalent number fields.

Now assume that $G = \mathrm{GL}_2(\mathbf{F}_3)$, which is a group of order 48. This group acts on \mathbf{F}_3^2 , and we let H be the stabilizer of $(1, 0)$ under this action. Define H' to be $\{g^T \mid g \in H\}$. These subgroups are Gassmann equivalent but non-conjugate. There are 3 double cosets, and the matrix A in the variables a, b, c is

$$A = \begin{pmatrix} a & b & b & c & a & c & a & c \\ c & a & c & b & b & c & a & a \\ c & c & a & c & a & b & b & a \\ b & c & a & a & c & c & a & b \\ b & a & c & c & a & a & c & b \\ c & b & b & a & c & a & c & a \\ a & c & a & b & b & a & c & c \\ a & a & c & a & c & b & b & c \end{pmatrix}.$$

The determinant of this matrix is $f = -9(a - c)^4(a - 2b + c)^3(a + \frac{2}{3}b + c)$. Taking $a = 1, b = c = 0$ yields $\|\phi\| \leq 3$ and shows that the successive minima of K and K' differ by at most a factor 3.

In conclusion, we may take $c_8 = 3$. Similar techniques can be used to provide bounds in higher degrees as well.

To end, let us note that in the above examples the determinant f is of a very specific form. Indeed, it always splits as a product of only a few linear factors. A more careful analysis of this determinant might allow one to prove Proposition 6 with a bound on $\|\phi\|$ which is linear in d .

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