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Stringy invariants of singular algebraic varieties

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Contents

Introduction

The research domain of this thesis is *algebraic geometry*, a branch of pure mathematics. Algebraic geometry has a very long history; one can even consider the geometry of the ancient Greeks as a part of it.

For nonsingular projective complex algebraic varieties, the *Hodge numbers* are well known classical invariants. Batyrev has tried to generalize these invariants for a class of singular projective varieties (the allowed singularities are called Gorenstein canonical) as follows. He defined a rational function in two variables, the *stringy E-function*, by using data from a resolution of singularities. He showed that this function does not depend on the chosen resolution, and thus it is an invariant of the variety. When this function is a polynomial, he defined the *stringy Hodge numbers*, essentially as the coefficients of this polynomial. For a nonsingular projective variety, the stringy Hodge numbers coincide with the classical Hodge numbers. Moreover, they have a lot of analogous properties. But there is one problem: classical Hodge numbers are certainly nonnegative, since they are dimensions of certain vector spaces, but for stringy Hodge numbers the nonnegativity is not at all clear. It was conjectured by Batyrev, and it is the subject of this thesis.

We have been able to prove Batyrev's conjecture for varieties with certain mild isolated singularities. A nice corollary is the proof for the conjecture for threefolds in full generality (for surfaces the conjecture is trivially true). Moreover, the proofs suggested that a more general question for the power series development of not necessarily polynomial stringy E-functions is worth further investigation. However, we found an example that gives a negative answer to this question. In our opinion this provides some evidence that Batyrev's conjecture might not be true. In this thesis, we also compute explicit formulae for the contribution of so called A-D-E singularities to the stringy E-function.

We begin with an introductory chapter. It starts with a short overview of classical Hodge theory, Deligne's mixed Hodge structures and the Hodge-Deligne polynomial. Then we are ready for the definition of Batyrev's stringy E-function and a discussion of Veys' generalized stringy invariants. We will use this material throughout the thesis, most of the time without an explicit reference to this introduction. We end the introduction by a sketch of the history of the subject and by a summary of the results of this thesis.

We use the following conventions. By an algebraic variety we mean an integral separated scheme of finite type over an algebraically closed field (in our case always the complex numbers), although we do not use the scheme language (so the notion of variety of Chapter 1 of Hartshorne's book [Ha] suffices). In particular, a variety is always irreducible (unless otherwise stated).

0.1 Pure and mixed Hodge structures and the Hodge-Deligne polynomial

(0.1.1) For a smooth projective algebraic variety X of dimension d, there is a well known direct sum decomposition of the de Rham cohomology (isomorphic to the singular cohomology):

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q},$$

with

$$H^{p,q} = \overline{H^{q,p}}.$$

It is called the *Hodge decomposition* (see for example [GH, Chapter 0]). The cohomology class of a complex valued C^∞-differential form ω belongs to $H^{p,q}$ if ω can be expressed in terms of local complex coordinates $(z_1,\ldots,z_d)$ by means of p dz's and q $d\overline{z}$'s. The dimensions of the spaces $H^{p,q}$ are called the *Hodge numbers* of X and are denoted by $h^{p,q}(X) := \dim H^{p,q}(X)$. They satisfy the following symmetry relations:

$$h^{p,q}(X) = h^{q,p}(X) = h^{d-p,d-q}(X) = h^{d-q,d-p}(X).$$

The following definition is the abstract analogue of the Hodge decomposition. For the rest of this section we follow mainly Srinivas' text [Sr].

Definition. A *pure Hodge structure of weight* $n \in \mathbb{Z}_{\geq 0}$ consists of the data $V = (V_{\mathbb{Q}}, \{V^{p,q}\}_{p+q=n})$, where

1. $V_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}$-vector space,
2. each $V^{p,q}$ is a linear subspace of $V_{\mathbb{C}} := V_{\mathbb{Q}} \otimes \mathbb{C}$, such that
   - $V_{\mathbb{C}} = \oplus_{p+q=n} V^{p,q}$ (this is called the *Hodge decomposition* of $V_{\mathbb{C}}$),
   - $V^{p,q} = \overline{V^{q,p}}$ for all p, q, where for a linear subspace $W \subset V_{\mathbb{C}}$, $\overline{W}$ denotes its complex conjugate (the complex conjugation is induced by complex conjugation on the $\mathbb{C}$-factor).

A morphism of type (r, s) between pure Hodge structures V and V' is a linear map $f : V \to V'$ such that the induced map $f \otimes \mathbb{C} : V_{\mathbb{C}} \to V'_{\mathbb{C}}$ maps $V^{p,q}$ to $V'^{p+r,q+s}$. Another way to describe a pure Hodge structure is by using the *Hodge filtration*. For a pure Hodge structure of weight n, put

$$F^p V_{\mathbb{C}} := \oplus_{p' \geq p} V^{p',n-p'} \subset V_{\mathbb{C}}.$$

Then $\{F^p V_{\mathbb{C}}\}$ is a finite decreasing filtration by linear subspaces, such that

(1) the natural map $F^p V_{\mathbb{C}} \oplus \overline{F^{n-p+1} V_{\mathbb{C}}} \to V_{\mathbb{C}}$ induced by inclusions, is an isomorphism,

(2) $F^p V_{\mathbb{C}} \cap \overline{F^{n-p} V_{\mathbb{C}}} = V^{p,n-p}$.

Conversely, any finite dimensional $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$ with a finite decreasing filtration

$$V_{\mathbb{C}} = F^0 V_{\mathbb{C}} \supset F^1 V_{\mathbb{C}} \supset \cdots F^n V_{\mathbb{C}} \supset F^{n+1} V_{\mathbb{C}} = \{0\}$$

on $V_{\mathbb{C}}$ such that the isomorphisms (1) hold, can be made into a pure Hodge structure of weight n by setting $V^{p,n-p} := F^p V_{\mathbb{C}} \cap \overline{F^{n-p} V_{\mathbb{C}}}$.

(0.1.2) Inspired by his previous work on the Weil conjectures, Deligne was able to generalize the classical Hodge decomposition for smooth projective varieties to a structure theorem for the cohomology of arbitrary complex algebraic varieties (see [De1] and [De2]). The fundamental notion is that of a mixed Hodge structure.

Definition. A *mixed Hodge structure* $V = (V_{\mathbb{Q}}, \{W_n V_{\mathbb{Q}}\}, \{F^p V_{\mathbb{C}}\}$ consists of the following data:

1. a finite dimensional $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$,
2. a finite *increasing* filtration $\{W_n V_{\mathbb{Q}}\}$ on $V_{\mathbb{Q}}$ by linear subspaces, called the *weight filtration*,
3. a finite *decreasing* filtration $\{F^p V_{\mathbb{C}}\}$ on $V_{\mathbb{C}} = V_{\mathbb{Q}} \otimes \mathbb{C}$ by linear subspaces, called the *Hodge filtration*,

such that the Hodge filtration induces a pure Hodge structure of weight n on the n-th graded piece

$$\mathrm{Gr}_n^W V_{\mathbb{Q}} = \frac{W_n V_{\mathbb{Q}}}{W_{n-1} V_{\mathbb{Q}}}.$$

In more detail, the Hodge filtration induces a filtration (also denoted by $F^\bullet$) on

$$\mathrm{Gr}_n^W V_{\mathbb{Q}} \otimes \mathbb{C} = \frac{W_n V_{\mathbb{Q}} \otimes \mathbb{C}}{W_{n-1} V_{\mathbb{Q}} \otimes \mathbb{C}}$$

by

$$\begin{aligned} F^p(\mathrm{Gr}_n^W V_{\mathbb{Q}} \otimes \mathbb{C}) &= \frac{(F^p V_{\mathbb{C}} \cap (W_n V_{\mathbb{Q}} \otimes \mathbb{C})) + W_{n-1} V_{\mathbb{Q}} \otimes \mathbb{C}}{W_{n-1} V_{\mathbb{Q}} \otimes \mathbb{C}} \\ &\cong \frac{F^p V_{\mathbb{C}} \cap (W_n V_{\mathbb{Q}} \otimes \mathbb{C})}{F^p V_{\mathbb{C}} \cap (W_{n-1} V_{\mathbb{Q}} \otimes \mathbb{C})}. \end{aligned}$$

We will often be sloppy and speak of the mixed Hodge structure on $V_{\mathbb{C}}$ instead of on $V_{\mathbb{Q}}$. A morphism between mixed Hodge structures V and V' is a linear map $f : V_{\mathbb{Q}} \to V'_{\mathbb{Q}}$ that is compatible with the weight filtration (i.e. $f(W_n V_{\mathbb{Q}}) \subset W'_n V'_{\mathbb{Q}}$) and such that the induced map $f \otimes \mathbb{C}$ is compatible with the Hodge filtration. It is remarkable that this implies strict compatibility (see [De1, Théorème (2.3.5)]):

$$f(W_n V_{\mathbb{Q}}) = W'_n V'_{\mathbb{Q}} \quad \text{and} \quad (f \otimes \mathbb{C})(F^p V_{\mathbb{C}}) = F'^p V'_{\mathbb{C}}.$$

This strictness implies that the functors W_n and F^p from the category of mixed Hodge structures to the category of $\mathbb{Q}$- or $\mathbb{C}$-vector spaces are exact. We denote by $H^{p,q}$ the functor that maps a mixed Hodge structure V to the (p,q)-component of the pure Hodge structure $\mathrm{Gr}^W_{p+q}V_{\mathbb{Q}}$; this functor is exact as well.

(0.1.3) Now we come to the main theorem of [De1] and [De2].

Theorem. *Let X be a complex algebraic variety (possibly reducible), Y a closed subvariety. The singular cohomology groups $H^n(X,Y,\mathbb{Q})$ can be equipped with a functorial mixed Hodge structure; this means that for a map of pairs $f:(X,Y)\to(X',Y')$, the natural map*

$$f^*: H^n(X',Y',\mathbb{Q}) \to H^n(X,Y,\mathbb{Q})$$

is a morphism of mixed Hodge structures. Moreover,

- *for a smooth projective variety X and for $Y=\emptyset$, this mixed Hodge structure coincides with the pure Hodge structure on $H^n(X,\emptyset,\mathbb{Q}) = H^n(X,\mathbb{Q})$ given by the classical Hodge decomposition,*

- *the boundary map in the long exact sequence for the cohomology of the pair (X,Y) is a morphism of mixed Hodge structures,*

- *for any pair (X,Y) with $\dim X = d$, we have for all n*

$$W_{-1}H^n(X,Y,\mathbb{Q}) = 0, W_{2d}H^n(X,Y,\mathbb{Q}) = H^n(X,Y,\mathbb{Q}),$$

$$F^0H^n(X,Y,\mathbb{C}) = H^n(X,Y,\mathbb{C}), F^{d+1}H^n(X,Y,\mathbb{Q}) = 0.$$

We have given here only a selection of some of the properties satisfied by this mixed Hodge structure. Many more are valid, we mention two others that we will use: for a complete, possibly singular, variety X of dimension d, the weight filtration satisfies $W_dH^n(X,\mathbb{Q}) = H^n(X,\mathbb{Q})$ and for a smooth, possibly non-complete, variety X of dimension d, we have $W_{d-1}H^n(X,\mathbb{Q}) = 0$.

(0.1.4) Let X now be an arbitrary variety (not necessarily irreducible), choose a compactification $\overline{X}$ and define $\partial\overline{X} := \overline{X}\setminus X$. The cohomology group $H^n(\overline{X},\partial\overline{X},A)$ (for any abelian coefficient group A) does not depend on the chosen compactification. It is called the n-th *cohomology group of X with compact support* and denoted $H^n_c(X,A)$ (it can also be defined as a direct limit over the compact subsets K of X of the spaces

$H^n(X, X \setminus K, A)$ as in [GreH]). In particular, for $A = \mathbb{Q}$ it carries a mixed Hodge structure. Of course, when X is compact, cohomology with compact support coincides with singular cohomology. Denote the dimension of $H^{p,q}(H_c^n(X, \mathbb{C}))$ by $h^{p,q}(H_c^n(X, \mathbb{C}))$. The following definition is a nice application of Deligne's mixed Hodge theory. It was first mentioned by Danilov and Khovanskiĭ in [DK].

Definition. Let X be an arbitrary complex algebraic variety (not necessarily irreducible). Define the *Hodge-Deligne polynomial* of X by the formula

$$H(X; u, v) := \sum_{i=0}^{2d} (-1)^i \sum_{p,q} h^{p,q}(H_c^i(X, \mathbb{C})) u^p v^q \ \in \mathbb{Z}[u, v].$$

We remark that some authors insert an extra factor $(-1)^{p+q}$ in this definition. Note that $H(X; 1, 1)$ is equal to the topological Euler characteristic $\chi(X)$ of X. The Hodge-Deligne polynomial is itself a generalized Euler characteristic; it satisfies the following properties:

- if Y is Zariski closed in X, then $H(X) = H(X \setminus Y) + H(Y)$,
- the Hodge-Deligne polynomial $H(X \times Y)$ of a product of two varieties equals $H(X) \cdot H(Y)$.

The first property follows essentially from the long exact sequence for compactly supported cohomology and the second from a Kunneth isomorphism (and the compatibility of these properties with the mixed Hodge structure!). We conclude this section with some easy examples and additional properties.

1. The Hodge-Deligne polynomial of projective space $\mathbb{P}^r$ equals $(uv)^r + (uv)^{r-1} + \cdots + 1$.
2. Thus the Hodge-Deligne polynomial of affine space $\mathbb{A}^r$ must be $(uv)^r$.
3. For a smooth projective curve C of genus g, $H(C) = uv - gu - gv + 1$.
4. If $f : X \to Y$ is a locally trivial fibration (with respect to the Zariski topology) with fibre F, then $H(X) = H(F) \cdot H(Y)$.

5. The symmetry relations of the classical Hodge numbers of a smooth projective variety X of dimension d can be expressed by

$$H(X;u,v) = (uv)^d H(X;u^{-1},v^{-1}).$$

0.2 Batyrev's stringy invariants

(0.2.1) The subject of this thesis is Batyrev's stringy E-function, defined by him in 1997 and published in [Ba2]. Its definition is motivated from string theory in theoretical physics. He used it in the first place to formulate a topological mirror symmetry test for Calabi-Yau varieties with singularities (for more on mirror symmetry and Calabi-Yau varieties, see Section 0.4). For nonsingular d-dimensional Calabi-Yau varieties V and V^* forming a mirror pair this test was formulated as a relation on the Hodge numbers. Expressed with Hodge-Deligne polynomials it becomes

$$H(V;u,v) = (-u)^d H(V^*;u^{-1},v).$$

We will define the singularities that are allowed by Batyrev first. By a log resolution of a variety Y we mean a proper birational morphism $f : X \to Y$ from a nonsingular variety X such that f is an isomorphism, when restricted to the inverse image of the nonsingular part of Y, and such that the inverse image of the singular locus is a divisor on X with smooth irreducible components and only normal crossings. By Hironaka's celebrated work [Hi], a log resolution always exists.

Definition. A normal variety Y is called $\mathbb{Q}$-*Gorenstein* if a multiple rK_Y of its canonical divisor is Cartier for some $r \in \mathbb{Z}_{>0}$ (we call Y *Gorenstein* if K_Y itself is Cartier). For a log resolution $f : X \to Y$ with irreducible exceptional components $D_i, i \in I$, we can then write $rK_X - f^*(rK_Y) = \sum_i b_i D_i$, with the $b_i \in \mathbb{Z}$. This is also formally written as $K_X - f^*(K_Y) = \sum_i a_i D_i$, with $a_i = b_i/r$. The variety Y is called *terminal, canonical, log terminal* or *log canonical* if $a_i > 0, a_i \geq 0, a_i > -1$ or $a_i \geq -1$, respectively, for all i (this does not depend on the chosen log resolution). We say that Y is *strictly log canonical* if Y is log canonical but not log terminal. The number a_i is called the *discrepancy coefficient* of D_i and the difference $K_X - f^*(K_Y)$ is called the *discrepancy.*

This definition plays a key rôle in the Minimal Model Program. Let us illustrate it with an example. Consider a normal affine hypersurface Y in $\mathbb{A}^n$ given by the equation $f(x_1, \ldots, x_n) = 0$. The class K_Y is associated to the sheaf of rational differential forms, regular on the nonsingular part of Y. For our hypersurface, these are generated by

$$\frac{dx_1 \wedge \ldots \wedge dx_{n-1}}{\partial f / \partial x_n} = -\frac{dx_1 \wedge \ldots \wedge dx_{n-2} \wedge dx_n}{\partial f / \partial x_{n-1}} = \cdots = (-1)^{n-1} \frac{dx_2 \wedge \ldots \wedge dx_n}{\partial f / \partial x_1},$$

and thus this sheaf is invertible and K_Y is Cartier (more generally, a normal irreducible divisor on a smooth variety is always Gorenstein). For f of the form $x_1^k + \cdots + x_n^k$ for $k, n \geq 2$, one blow up in the unique singular point suffices to construct a log resolution, and it is easy to compute that Y is terminal for $n \geq k+2$, canonical for $n \geq k+1$, log canonical for $n \geq k$ and strictly log canonical for $n = k$.

(0.2.2) Now we are ready to define the stringy E-function. We discuss its properties and give the additional definitions of the stringy Euler number and the stringy Hodge numbers. All of this is Batyrev's work (see [Ba2]).

Definition. Let Y be a normal irreducible complex variety with at most log terminal singularities and let $f : X \to Y$ be a log resolution. Denote the irreducible components of the exceptional locus by D_i, $i \in I$, and write D_J for $\cap_{i \in J} D_i$ and D_J° for $D_J \setminus \cup_{i \in I \setminus J} D_i$, where J is any subset of I ($D_\emptyset$ is taken to be X). The *stringy E-function* of Y is

$$E_{st}(Y; u, v) := \sum_{J \subset I} H(D_J^\circ; u, v) \prod_{i \in J} \frac{uv - 1}{(uv)^{a_i + 1} - 1},$$

where a_i is the discrepancy coefficient of D_i and where the product $\prod_{i \in J}$ is 1 if $J = \emptyset$.

Batyrev proved that this definition is independent of the chosen log resolution ([Ba2, Theorem 3.4]). His proof uses motivic integration, a theory based on an idea of Kontsevich and mainly developed by Denef and Loeser. An overview of this theory is provided in [Ve1]. Alternatively, one can use the Weak Factorization Theorem by Abramovich, Karu, Matsuki and Włodarczyk from [AKMW]. The 'E' in the name comes from the Hodge-Deligne polynomials that are sometimes called E-polynomials.

(0.2.3) Remark.

(1) Batyrev formulated his mirror symmetry test for singular Calabi-Yau varieties V and V^* as the equality

$$E_{st}(V;u,v) = (-u)^d E_{st}(V^*;u^{-1},v).$$

(2) If Y is smooth, then $E_{st}(Y) = H(Y)$ and if Y admits a crepant resolution $f: X \to Y$ (i.e. such that the discrepancy is 0), then $E_{st}(Y) = H(X)$.

(3) If Y is Gorenstein, then all $a_i \in \mathbb{Z}_{\geq 0}$ and $E_{st}(Y)$ becomes a rational function in u and v. It is then an element of $\mathbb{Z}[[u,v]] \cap \mathbb{Q}(u,v)$.

(4) The *stringy Euler number* of Y is defined as

$$\lim_{u,v \to 1} E_{st}(Y;u,v) = \sum_{J \subset I} \chi(D_J^\circ) \prod_{j \in J} \frac{1}{a_j + 1}.$$

(5) It is easy to deduce the following alternative expression for the stringy E-function:

$$E_{st}(Y) = \sum_{J \subset I} H(D_J;u,v) \prod_{i \in J} \frac{uv - (uv)^{a_i+1}}{(uv)^{a_i+1} - 1}.$$

(0.2.4) Assume moreover that Y is projective of dimension d. Then Batyrev proved the following instance of Poincaré and Serre duality ([Ba2, Theorem 3.7]):

(i) $E_{st}(Y;u,v) = (uv)^d E_{st}(Y;u^{-1},v^{-1})$ (compare this with the last example of (0.1.4)),

(ii) $E_{st}(Y;0,0) = 1$.

If Y has at worst Gorenstein canonical singularities and if $E_{st}(Y;u,v)$ is a polynomial $\sum_{p,q} a_{p,q} u^p v^q$, he defined the *stringy Hodge numbers* of Y as $h_{st}^{p,q}(Y) := (-1)^{p+q} a_{p,q}$. It is easy to see that

(1) they can only be nonzero for $0 \leq p,q \leq d$,

(2) $h_{st}^{0,0}(Y) = h_{st}^{d,d}(Y) = 1$,

(3) $h_{st}^{p,q}(Y) = h_{st}^{q,p}(Y) = h_{st}^{d-p,d-q}(Y) = h_{st}^{d-q,d-p}(Y)$,

(4) if Y is smooth, the stringy Hodge numbers are equal to the usual Hodge numbers.

There is though one desirable property that is not clear at all: nonnegativity!

(0.2.5) Conjecture (Batyrev). *Stringy Hodge numbers are nonnegative.*

This is the problem that we want to investigate in this thesis. Note that it is not clear when to expect that the stringy E-function is a polynomial. In fact we have focused on the following more general question, which is very natural in view of our results from Chapter 1.

Question. *Let Y be a d-dimensional projective variety with at most Gorenstein canonical singularities. Write the stringy E-function of Y as a power series $\sum_{i,j\geq 0} b_{i,j}u^i v^j$. Is it true that $(-1)^{i+j}b_{i,j} \geq 0$ for $i+j \leq d$?*

Thanks to the properties of stringy Hodge numbers, a positive answer to this question implies Batyrev's conjecture. However, we will see in Chapter 4 that the answer to this question is 'no' in general!

Example. The conjecture is true for varieties that admit a crepant resolution. This is the case for all canonical surface singularities, which are exactly the two-dimensional A-D-E singularities [Re3, (4.9),(3)] (see also Theorem (2.4.1) for $m = 3$).

Remark. For a complete surface X with at most log terminal singularities, Veys showed that

$$E_{st}(X) = \sum_{p,q\in\mathbb{Z}} (-1)^{p+q} h_{st}^{p,q} u^p v^q + \sum_{r\notin\mathbb{Z}} h_{st}^{r,r}(uv)^r,$$

with all $h_{st}^{p,q}$ and $h_{st}^{r,r}$ nonnegative [Ve2, Theorem 6.5].

0.3 Veys' ideas for general singularities

(0.3.1) Veys has generalized Batyrev's stringy invariants to the class of all $\mathbb{Q}$-Gorenstein varieties without strictly log canonical singularities. In a first paper ([Ve2]) he did this in a geometric way for surfaces. He studied the minimal log resolution of a normal surface singularity that is not log canonical and noted that an exceptional component with discrepancy -1 is always isomorphic to $\mathbb{P}^1$ and intersects at most two other components, whose discrepancies are not -1. Let S then be a normal surface without strictly log canonical singularities (the $\mathbb{Q}$-Gorenstein condition is not necessary for surfaces) and denote by $D_i, i \in I$, the exceptional components of the minimal log resolution. Let Z be the set $\{i \in I \mid a_i = -1\}$. Then Veys defined the stringy E-function of S as follows:

$$\begin{aligned} E_{st}(S) \quad := \quad & \sum_{J \subset I \setminus Z} H(D_J^\circ; u, v) \prod_{i \in J} \frac{uv - 1}{(uv)^{a_i+1} - 1} \\ & + \sum_{i \in Z} \frac{\kappa_i (uv-1)^2}{((uv)^{a_{i_1}+1} - 1)((uv)^{a_{i_2}+1} - 1)}, \end{aligned}$$

where for $i \in Z$ we denote $\kappa_i = -D_i^2$, and where a_{i_1} and a_{i_2} denote the discrepancies of the components that intersect D_i (if that is only one component, we put $a_{i_2} = 0$). The reason why this formula is 'right', is nicely explained in [Ve2, (3.3)]. The formula gives the same result for any log resolution that has the same property concerning components with discrepancy -1 as the minimal one, and the 'Poincaré duality' formula

$$E_{st}(S; u, v) = (uv)^2 E_{st}(S; u^{-1}, v^{-1})$$

is still valid.

About the sign of the coefficients in the case that Veys' generalized stringy E-function for a surface is a polynomial, nothing can be expected. Indeed, in Chapter 1 we give an example where the coefficients have the 'wrong' sign. We will not use Veys' higher dimensional generalization in this thesis, but we include the interesting construction for completeness.

(0.3.2) In the paper [Ve3] Veys handled the higher-dimensional case, assuming the Minimal Model Program. For a $\mathbb{Q}$-Gorenstein variety Y without strictly log canonical singularities he chooses a (relative) log minimal model. This is a proper birational morphism $p : Y^m \to Y$ from a variety Y^m with the following properties:

- Y^m is $\mathbb{Q}$-factorial (i.e. for every Weil divisor, a positive integer multiple is Cartier),

- Y^m has at most log terminal singularities,
- $K_{Y^m} + F$ is p-nef, where we denote by F the reduced exceptional divisor of p (i.e. the intersection number $(K_{Y^m} + F) \cdot C \geq 0$ for all irreducible curves on Y^m for which $p(C)$ is a point).

The existence of such an object is claimed by the Minimal Model Program and is proved up to dimension 3. Write $F = \sum_i F_i$, with the $F_i, i \in I^m$, its irreducible components. Then one can look at a log resolution $h : X \to Y^m$ of the pair (Y^m, F), that is a log resolution of Y^m such that the union of the exceptional components and the strict transforms of the F_i form a normal crossing divisor with smooth irreducible components. Let I be the index set of all these divisors (so $I^m \subset I$) and denote them by $E_i, i \in I$. Define numbers ν_i and N_i by

$$K_X - h^*(K_{Y^m} + F) = \sum_{i \in I} (\nu_i - 1) E_i,$$

$$h^*(K_{Y^m} + F - p^*(K_Y)) = \sum_{i \in I} N_i E_i.$$

The properties of Y^m and the condition that Y has no strictly log canonical singularities assure that $\nu_i \geq 0 \geq N_i$ and that $(\nu_i, N_i) \neq (0,0)$ for all i. Then Veys defined the following stringy invariant (in fact he defined a more general invariant on the level of the Grothendieck ring):

$$\mathcal{Z}(Y, s) := \sum_{J \subset I} H(E_J^\circ; u, v) \prod_{i \in J} \frac{uv - 1}{(uv)^{\nu_i + sN_i} - 1}.$$

So by adding the variable s, the exceptional divisors with discrepancy -1 are no longer a problem (we have $a_i + 1 = \nu_i + N_i$)! This definition does not depend on the chosen log minimal model, nor on the chosen log resolution. A Poincaré duality formula can be given as well. For a surface S, Veys showed that both generalizations of Batyrev's stringy E-function coincide in the following sense: $E_{st}(S) = \lim_{s \to 1} \mathcal{Z}(Y, s)$ (note that it is not a priori clear that this limit exists). It is an intriguing open question whether this limit exists also in the higher dimensional case and whether a geometric construction of this limit can be given as in the surface case.

0.4 Sketch of the historical context

(0.4.1) The paper [Ba2] that contains Conjecture (0.2.5) is a kind of end product of a long process by Batyrev and others towards a good definition of stringy invariants; namely one that can be applied in a mirror symmetry test (from the mathematical point of view, Veys' generalizations from the previous section show that a lot more can be done). In string theory in theoretical physics, one studies the physics of one-dimensional extended objects (strings). As they propagate, strings sweep out a 2-dimensional 'worldsheet' Σ in a spacetime manifold M. Physicists associate a so called *quantum field theory* to Σ. Explaining what this is, is beyond the scope of this thesis, but an excellent informal introduction for mathematicians to this notion and to mirror symmetry is provided by [GP1]. An introduction that asks more knowledge of physics is [GO]. For physical reasons, the quantum field theory associated to Σ should be conformally invariant and supersymmetric. This leads to severe restrictions on the spacetime M. It must typically be a compact complex Kähler manifold with trivial canonical line bundle (the preferred complex dimension for string theory is 3). Such a manifold is called *Calabi-Yau*.

Sometimes another spacetime $\widetilde{M}$ gives rise to the same quantum field theory (modulo a natural involution). In that case one shows that the Hodge numbers of M and $\widetilde{M}$ satisfy

$$h^{p,q}(M) = h^{d-p,q}(\widetilde{M}), \tag{$*$}$$

where d is the dimension of M and $\widetilde{M}$. So the Hodge diamond of M and $\widetilde{M}$ are related by a reflection about a diagonal axis; for this reason M and $\widetilde{M}$ are called *mirror manifolds*. Relation $(*)$ is called the *mirror symmetry test*. One believes that a mirror dual of M always exists, but at this point one must allow singular Calabi-Yau's (and probably also other generalizations of Calabi-Yau manifolds, see for instance [BB2]). For example, Greene and Plesser gave a mirror construction for Calabi-Yau hypersurfaces M of Fermat type in a weighted projective space of dimension 4. The mirror $\widetilde{M}$ is then given by a quotient of M under the action of a certain group G and is thus in general singular (see [GP1, Section 3] and [GP2]). In this case, a crepant resolution N of $\widetilde{M}$ exists and is still Calabi-Yau; one has to apply the mirror symmetry test to M and N. In general, such a crepant resolution might not exist and for other resolutions of singularities N the mirror symmetry test for M and N fails. This shows the need for a definition of *stringy Hodge numbers* $h_{st}^{p,q}$ for M and $\widetilde{M}$ such that

$$h_{st}^{p,q}(M) = h_{st}^{d-p,q}(\widetilde{M}).$$

An even more demanding task would be to define an underlying *stringy cohomology theory*, equipped with a pure Hodge structure.

(0.4.2) A first attempt to define stringy Hodge numbers was undertaken by Batyrev and Dais in [BD] for varieties with at most Gorenstein toroidal or quotient singularities. Finally, in the paper [Ba2], Batyrev moves the accent from stringy Hodge numbers towards the *stringy E-function.* It allowed him to formulate a general mirror symmetry test for Calabi-Yau varieties with arbitrary Gorenstein canonical singularities. He only defines stringy Hodge numbers for varieties with a polynomial stringy E-function. Borisov and Mavlyutov show that both definitions of stringy Hodge numbers do not always coincide and they give reasons why the definition via the stringy E-function should be considered as the best one (see [BM, Section 2]). In the same paper they give several conjectural constructions for the stringy cohomology for special classes of varieties. Moreover, they show that these constructions coincide with the orbifold cohomology of Chen and Ruan from [CR] under the relevant assumptions.

Related work by Borisov and Libgober can be found in [BL]. In this paper they define an elliptic genus for log terminal complex projective varieties and they show that it is connected with Batyrev's stringy E-function. De Fernex, Lupercio, Nevins and Uribe constructed in [dFLNU] a theory of stringy Chern classes for log terminal varieties, also related to Batyrev's stringy invariants. Aluffi defined these stringy Chern classes independently in [Al]. Finally, we remark that there are generalized versions for most of these notions (including Batyrev's) to pairs (X, D) with D a $\mathbb{Q}$-divisor on X and with a log terminality condition.

Summary of results

This thesis consists of four chapters. In Chapter 1 we obtain a positive answer to Batyrev's conjecture, and even to Question (0.2.5), for a class of mild isolated singularities. The singularities that are allowed depend on the dimension of the variety. For this we use recent results by de Cataldo and Migliorini in a crucial way. A nice corollary is the proof of Batyrev's conjecture for threefolds in full generality. We also give an explicit description of the stringy Hodge numbers in these cases, essentially in terms of the cohomology of our singular variety Y. Moreover, we show that stringy Hodge numbers satisfy the relation $h_{st}^{p,q}(Y) \leq h_{st}^{p+1,q+1}(Y)$, for $p, q \leq d-1$ and $p+q \leq d-2$, where d is the dimension of Y. This relation is well known for classical Hodge numbers of smooth projective varieties.

In Chapter 2 we compute the contribution of a so called A-D-E singularity to the stringy E-function in arbitrary dimension. This was already done by Dais and Roczen in [DR] for 3-dimensional varieties, but their computation of the D and E cases contains an error. We correct this error and simplify their formulae for the A case considerably; in fact all the formulae that we have computed are rather simple. The used method is explicitly computing a log resolution. The main theorem of Chapter 1 implies then a positive answer to Question (0.2.5) for varieties with at most A-D-E singularities. Moreover, the obtained formulae enable us to say in which cases the stringy E-function becomes polynomial for such varieties.

In Chapter 3 we present another way to compute the stringy E-function of canonical affine hypersurface singularities, non-degenerate with respect to their Newton polyhedron. The stringy E-function can in that case be seen as a 'residue' of the Hodge zeta function, a specialization of Denef and Loeser's motivic zeta function. For non-degenerate hypersurfaces there exists an algorithm to compute the motivic zeta function from the Newton polyhedron, published by Artal, Cassou-Noguès, Luengo and Melle and based on work by Denef and Hoornaert. This method can be used to check the formulae for the A-D-E singularities from Chapter 2, and we do this for one example.

In the final chapter, we show by example that the answer to Question

(0.2.5) is 'no' in general. This is somewhat surprising and it gives some evidence that Batyrev's conjecture might not be true. The example consists of a 6-dimensional variety with isolated terminal singularities. The computation with the Newton polyhedron is a lot easier then the computation of an explicit log resolution in this case.

We end the thesis by some conclusory remarks. The most important is the following. If Batyrev's conjecture is true after all, its proof must use the polynomial condition in a crucial way, but it is not clear how this condition can be exploited.

Chapter 1

Nonnegativity of stringy Hodge numbers for a class of mild isolated singularities and for threefolds

Abstract

We give a positive answer to Question (0.2.5) for projective varieties with certain isolated singularities in arbitrary dimension (the allowed singularities depend on the dimension) and for projective threefolds in full generality. Furthermore, we give explicit descriptions of the stringy Hodge numbers in these cases and we prove inequalities for them, analogous to well known inequalities for classical Hodge numbers of smooth projective varieties. For all of this, we use recent results by de Cataldo and Migliorini. About the sign of the coefficients of Veys' generalized stringy E-function, nothing can be expected. Indeed, in the end we show by example that they can have the 'wrong' sign.[1]

1.1 Preliminaries

(1.1.1) Let Y be a projective variety of dimension n with at most isolated singularities. Let $f : X \to Y$ be a resolution of singularities with X projective, such that $f : f^{-1}(Y_{ns}) \to Y_{ns}$ is an isomorphism, where Y_{ns} is the

[1]This chapter corresponds roughly to [SV]; the results are stated more generally and are extended (also in joint work with W. Veys).

nonsingular part of Y, and such that $f^{-1}(y)$ is a divisor for every singular point $y \in Y$. Denote by D the total inverse image of the singular points. We will use the following result of de Cataldo and Migliorini in a crucial way. The proof of this theorem for $n = 3$ and for one singular point is given in [dCM2, Theorem 2.3.4]. From this proof it is clear that the same argument works for any dimension and any number of singular points. For completeness and because we need the same construction later, we include it anyway. By $H^\bullet(\,\cdot\,)$ we always mean singular cohomology with coefficients in $\mathbb{C}$.

Theorem. *The map $H^i(X) \to H^i(D)$, induced by inclusion, is surjective for $i \geq n$.*

Proof. We embed Y in a projective space $\mathbb{P}^r$ and we take a generic hyperplane section Y_s (so this hyperplane section is smooth and does not contain any singular point of Y). Consider the inverse image $X_s := f^{-1}(Y_s)$ and denote $Y \backslash Y_s$ by Y_0 and $X \backslash X_s$ by X_0. Because Y_0 is affine, $H^i(Y_0) = 0$ for $i > n$ (this was probably first proved by Kaup in [Ka]). The sheaves $R^k f_* \mathbb{C}_{X_0}$ are skyscraper sheaves above the singular points for $k > 0$, so they are flasque and $H^j(Y_0, R^k f_* \mathbb{C}_{X_0}) = 0$ for $k, j > 0$. The Leray spectral sequence for $f : X_0 \to Y_0$ (given by $E_2^{p,q} = H^p(Y_0, R^q f_* \mathbb{C}_{X_0})$ and abutting to $H^\bullet(X_0, \mathbb{C})$) gives then an isomorphism $H^i(X_0) \cong H^i(D)$ for $i > n$, because $E_2^{0,i} = E_\infty^{0,i} = H^i(D)$ and because $E_2^{p,q} = E_\infty^{p,q} = 0$ for $p > 0$ and $p + q = i$. There is also a surjection $H^n(X_0) \to H^n(D)$ because $E_2^{0,n} = E_\infty^{0,n} = H^n(D)$. Proposition (8.2.6) from [De2] states that the image of $H^i(X)$ in $H^i(D)$ is equal to the image of $H^i(X_0)$ in $H^i(D)$, whence the result follows. ■

The following corollary is immediate from the theorem; it is stated in [dCM1, Corollary 2.1.11] as a corollary of far more deep results and it was already proved by Steenbrink in [St2, Corollary (1.12)] for the case where D is a divisor with smooth components and normal crossings.

Corollary. *The a priori mixed Hodge structure on $H^i(D)$ is pure for $i \geq n$.*

(1.1.2) For threefolds and fourfolds, de Cataldo and Migliorini also make the following construction in [dCM2], that can again be generalized immediately to any dimension (it also follows from the results of [dCM1], see example 2.4 in that paper). Their starting point is the long exact sequence for cohomology with compact support (notation as in the proof of Theorem (1.1.1), assume $n \geq 3$)

$$H_c^0(X_0) \to H^0(X) \to H^0(X_s) \to H_c^1(X_0) \to \cdots \to H^{n-1}(X)$$

$$\to H^{n-1}(X_s) \to H_c^n(X_0) \to H^n(X) \to H^n(X_s) \to \cdots .$$

Thanks to Poincaré duality, the vector spaces $H_c^i(X_0)$ can be replaced by $H_{2n-i}(X_0)$, which are by the proof above isomorphic to $H_{2n-i}(D)$ for $i < n$. Moreover, the dualized version of the theorem states that $H_{2n-i}(D)$ maps injectively to $H_{2n-i}(X)$, and the latter is isomorphic to $H^i(X)$. These maps are all compatible, so the above long exact sequence splits in short exact sequences (note also that $H_c^0(X_0) \cong H_{2n}(D) = 0$ and $H_c^1(X_0) \cong H_{2n-1}(D) = 0$)

$$0 \to H^0(X) \to H^0(X_s) \to 0,$$

$$0 \to H^1(X) \to H^1(X_s) \to 0,$$

$$0 \to H_{2n-2}(D) \to H^2(X) \to H^2(X_s) \to 0,$$

$$\vdots$$

$$0 \to H_{n+2}(D) \to H^{n-2}(X) \to H^{n-2}(X_s) \to 0,$$

$$0 \to H_{n+1}(D) \to H^{n-1}(X) \to H^{n-1}(X_s) \to H_c^n(X_0) \to \cdots .$$

Let us denote $\ker(H^i(X) \to H^i(D))$ by K^i. We deduce the following isomorphisms from the first sequence (the middle one is the dual statement of the first sequence, the other ones are Poincaré duality)

$$H^{2n-2}(X_s) \cong H_0(X_s) \cong H_0(X) \cong H^{2n}(X).$$

Analogously, we get

$$H^{2n-3}(X_s) \cong H^{2n-1}(X), H^{2n-4}(X_s) \cong K^{2n-2}, \ldots, H^n(X_s) \cong K^{n+2},$$

and a surjection

$$H^{n-1}(X_s) \to K^{n+1}.$$

The point is that the Hard Lefschetz Theorem holds on $H^\bullet(X_s)$, and thus there are isomorphisms $H^i(X_s) \to H^{2n-2-i}(X_s)$ for $i \leq n-1$. This gives rise to the following list of isomorphisms, essentially given by the cup product with the fundamental class $\eta_s \in H^2(X)$ of X_s:

$$(\cup\eta_s)^n : H^0(X) \to H^{2n}(X),$$

$$(\cup\eta_s)^{n-1} : H^1(X) \to H^{2n-1}(X),$$

$$(\cup\eta_s)^{n-2} : \frac{H^2(X)}{H_{2n-2}(D)} \to K^{2n-2},$$

$$\vdots$$

$$(\cup\eta_s)^2 : \frac{H^{n-2}(X)}{H_{n+2}(D)} \to K^{n+2}.$$

The map

$$\cup\eta_s : \frac{H^{n-1}(X)}{H_{n+1}(D)} \to K^{n+1}$$

decomposes via $H^{n-1}(X_s)$ as a composition of a surjection and an injection. De Cataldo and Migliorini show that the Hodge-Riemann bilinear relations on $H^{n-1}(X_s)$ ensure that $\cup\eta_s$ is an isomorphism as well. Their results can be summarized in the following theorem (see also the beginning of Section 2.4 of [dCM2]).

Theorem. *The spaces*

$$H^0(X), H^1(X), \frac{H^2(X)}{H_{2n-2}(D)}, \dots, \frac{H^{n-1}(X)}{H_{n+1}(D)}, H^n(X),$$

$$K^{n+1}, \dots, K^{2n-2}, H^{2n-1}(X), H^{2n}(X)$$

satisfy the Hard Lefschetz Theorem with respect to $\cup\eta_s$.

Remark. The choice of the space $H^n(X)$ in this sequence is somewhat arbitrary, $H^n(X)$ is certainly 'big enough'. It can in fact be replaced by $K^n = \ker(H^n(X) \to H^n(D))$. To prove this, it suffices to show that

$$H^{n-2}(X) \xrightarrow{\cup\eta_s} H^n(X) \to H^n(D)$$

forms a complex. If we dualize, this means that

$$H_n(D) \to H_n(X) \to H_{n-2}(X)$$

should be a complex as well, where $H_n(X) \to H_{n-2}(X)$ corresponds to intersecting with X_s. And this is clear.

From the theorem and this remark we deduce the following corollary.

Corollary. *For $0 \le i \le n-2$, the maps $\cup\eta_s : K^{n+i} \to K^{n+i+2}$ are surjective.*

(1.1.3) We also need the construction of the mixed Hodge structure on the cohomology of an algebraic set D with smooth projective irreducible components and normal crossings. This can be found in [GS, Section 4] and [KK, p.149-156]. The construction is as follows. Denote the irreducible components of D by $D_i, i \in I = \{1, \dots, \alpha\}$ and put for $j \ge 0$

$$D^{(j)} := \coprod_{\substack{J \subset I \\ |J| = j+1}} D_J,$$

where $D_J = \cap_{i \in J} D_i$. So all $D^{(k)}$ are smooth and projective, and we write $A^i(D^{(k)})$ for the C^∞-differential i-forms with values in $\mathbb{C}$ on $D^{(k)}$. The inclusion map $D^{(k)} \hookrightarrow D^{(k-1)}$ defined by mapping $D_{i_1} \cap \ldots \cap D_{i_{k+1}}$ into $D_{i_1} \cap \cdots \cap D_{i_{l-1}} \cap D_{i_{l+1}} \cap \ldots \cap D_{i_{k+1}}$ is denoted by $\delta_l^{(k)}$. There exists a spectral sequence $\{E_r, d_r\}$, abutting to $H^\bullet(D)$ and degenerating at the E_2 level, with $E_0^{p,q} = A^q(D^{(p)})$ and $d_0 : E_0^{p,q} \to E_0^{p,q+1}$ given by differentiation of forms. Then $E_1^{p,q}$ is $H^q(D^{(p)})$; and $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is defined by $\sum_{l=1}^{p+2} (-1)^l (\delta_l^{(p+1)})^*$. There is a pure Hodge structure of weight q on $E_1^{p,q}$, and d_1 is a morphism of Hodge structures, so $E_2^{p,q}$ inherits a pure Hodge structure and this provides $H^\bullet(D)$ with a mixed Hodge structure.

1.2 Main results and corollaries

(1.2.1) Definition. Let Y be a d-dimensional normal variety with $d \ge 2$. We say that Y *satisfies condition* $(*)$ if

- Y has at most isolated Gorenstein singularities,
- for a log resolution $f : X \to Y$ with irreducible exceptional components $D_i, i = 1, \dots, \alpha$, the discrepancy coefficients a_i of D_i are strictly greater than $\lfloor \frac{d-4}{2} \rfloor$ for all i.

Thus this definition includes the canonical surface singularities, isolated Gorenstein canonical threefold singularities and isolated Gorenstein terminal fourfold and fivefold singularities. Note that the second condition does not depend on the chosen log resolution. This can be seen as follows (this is in fact a copy of the proof that the notions 'terminal', 'canonical', etc. do not depend on the chosen resolution). Suppose that we are given two log resolutions $f : X \to Y$ and $f' : X' \to Y$ of Y, such that the discrepancy coefficients of the irreducible components of the exceptional locus of f are $> \lfloor \frac{d-4}{2} \rfloor$. Then we have a birational map $X \dashrightarrow X'$, and we can resolve its indeterminacies by subsequent blow-ups in centers lying inside the exceptional locus of f until we get a smooth variety Z with morphisms $g : Z \to X$ and $g' : Z \to X'$ such that $f \circ g = f' \circ g'$. By performing even more blow-ups, we may assume that $f \circ g$ is still a log resolution of Y. The discrepancy coefficients of the exceptional components of g for the map $f \circ g$ must certainly be $> \lfloor \frac{d-4}{2} \rfloor$, by [GH, p.605]. So whatever components are contracted by g', all discrepancy coefficients of f' remain $> \lfloor \frac{d-4}{2} \rfloor$.

(1.2.2) Theorem. *Assume that Y is projective and satisfies condition $(*)$. Then the answer to Question (0.2.5) is 'yes' for Y.*

Proof. For $d = 2$ there is nothing to prove in view of Example (0.2.5). Let us first handle the case where $d \geq 4$. Take a log resolution $f : X \to Y$; we use the same notations for the irreducible components of the exceptional locus and their discrepancies as in the definition above. Put $I := \{1, \ldots, \alpha\}$. Denote the total exceptional locus by D. Consider the alternative formula (0.2.3) (5) for the stringy E-function:

$$E_{st}(Y) = \sum_{J \subset I} H(D_J; u, v) \prod_{i \in J} \frac{uv - (uv)^{a_i+1}}{(uv)^{a_i+1} - 1},$$

and write this as a power series $\sum_{i,j \geq 0} b_{i,j} u^i v^j$. It suffices to prove that $(-1)^{i+j} b_{i,j} \geq 0$ for $i \geq j$ and $i + j \leq d$. We denote the Hodge-Deligne polynomial of X by $\sum_{i,j} a_{i,j} u^i v^j$ and for $J \subset I, J \neq \emptyset$ we denote the Hodge-Deligne polynomial of D_J by $\sum_{i,j} a^J_{i,j} u^i v^j$. The power series development of $\frac{uv-(uv)^{a_i+1}}{(uv)^{a_i+1}-1}$ for $a_i > 0$ is equal to

$$\begin{aligned}&(uv - (uv)^{a_i+1})(-1 - (uv)^{a_i+1} - (uv)^{2a_i+2} - (uv)^{3a_i+3} - \cdots) \\ &\quad = -uv + (uv)^{a_i+1} - (uv)^{a_i+2} + (uv)^{2a_i+2} - (uv)^{2a_i+3} + \cdots .\end{aligned}$$

Since we assume condition $(*)$, we can write $b_{i,j}$ for $i+j \leq d$ and $i \geq j$ as

$$b_{i,j} = a_{i,j} + \sum_{k=1}^{j} (-1)^k \sum_{\substack{J \subset I \\ |J|=k}} a^J_{i-k,j-k} + R,$$

where

$$R = \begin{cases} \sum_{\substack{l=1,\ldots,\alpha \\ a_l = d/2-1}} a^{\{l\}}_{0,0} & \text{if } d \text{ even}, i=j=d/2, \\ \sum_{\substack{l=1,\ldots,\alpha \\ a_l=(d-3)/2}} a^{\{l\}}_{1,0} & \text{if } d \text{ odd}, i=(d+1)/2, j=(d-1)/2, \\ 0 & \text{otherwise.} \end{cases}$$

In any case, R has got the right sign, so it suffices to study the rest of the formula for $b_{i,j}$. We rewrite it as follows, using the symmetry of usual Hodge numbers:

$$a_{d-i,d-j} + \sum_{k=1}^{j} (-1)^k \sum_{\substack{J \subset I \\ |J|=k}} a^J_{d-i,d-j}.$$

Set $q = 2d-i-j$ and look at the $\{E_1^{p,q}, d_1\}$-term of the spectral sequence from (1.1.3), for all p. This gives a complex:

$$H^q(D^{(0)}) \to H^q(D^{(1)}) \to \cdots \to H^q(D^{(i-1)}) \to 0.$$

Here $H^q(D^{(i)}) = 0$ since $D^{(i)}$ has dimension $d-i-1$. The cohomology of this complex is given by $E_2^{0,q}$ to $E_2^{i-1,q}$, but because of the purity of $H^r(D)$ for $r > q$ (Corollary (1.1.1)), $E_2^{1,q}$ up to $E_2^{i-1,q}$ must be zero! This means that the complex is exact. The purity of $H^q(D)$ itself implies that $H^q(D) = \ker(H^q(D^{(0)}) \to H^q(D^{(1)}))$. So we get an exact sequence where all arrows are morphisms of pure Hodge structures:

$$0 \to H^q(D) \to H^q(D^{(0)}) \to \cdots \to H^q(D^{(i-1)}) \to 0.$$

We apply the exact functor $H^{d-i,d-j}$ and note that $H^{d-i,d-j}(H^q(D^{(s)})) = 0$ for $s = j, \ldots, i-1$ since $D^{(s)}$ has dimension $d-s-1$. Counting the dimensions of the resulting exact sequence of vector spaces gives

$$\dim H^{d-i,d-j}(H^q(D)) = \sum_{k=1}^{j} (-1)^{k+1} \sum_{\substack{J \subset I \\ |J|=k}} (-1)^{2d-i-j} a^J_{d-i,d-j}.$$

The surjectivity of the morphism of Hodge structures $H^q(X) \to H^q(D)$ from Theorem (1.1.1) translates by applying $H^{d-i,d-j}$ to

$$(-1)^{2d-i-j} a_{d-i,d-j} \geq \sum_{k=1}^{j} (-1)^{k+1} \sum_{\substack{J \subset I \\ |J|=k}} (-1)^{2d-i-j} a^J_{d-i,d-j},$$

and thus $(-1)^{2d-i-j} b_{i,j} = (-1)^{i+j} b_{i,j} \geq 0$.

For $d = 3$ it can happen that some a_i are zero. We can express the $b_{i,j}$ as follows:

$$\begin{cases} b_{0,0} = a_{0,0} = 1 \geq 0, \\ b_{1,0} = a_{1,0} \leq 0, \\ b_{2,0} = a_{2,0} \geq 0, \\ b_{3,0} = a_{3,0} \leq 0, \\ b_{1,1} = a_{1,1} - \sum_{\substack{k=1,\ldots,\alpha \\ a_k \neq 0}} a^{\{k\}}_{0,0} = a_{2,2} - \sum_{\substack{k=1,\ldots,\alpha \\ a_k \neq 0}} a^{\{k\}}_{2,2}, \\ b_{2,1} = a_{2,1} - \sum_{\substack{k=1,\ldots,\alpha \\ a_k \neq 0}} a^{\{k\}}_{1,0} = a_{2,1} - \sum_{\substack{k=1,\ldots,\alpha \\ a_k \neq 0}} a^{\{k\}}_{2,1}. \end{cases}$$

An analogous reasoning as above gives here that $H^4(D) = \oplus_{k=1}^{\alpha} H^4(D_k)$ and $H^3(D) = \oplus_{k=1}^{\alpha} H^3(D_k)$. Using the surjectivity of $H^i(X) \to H^i(D)$ for $i = 3, 4$ and applying the functors $H^{2,2}$, respectively $H^{2,1}$, immediately gives that $b_{1,1} \geq 0$ and $b_{2,1} \leq 0$. ■

(1.2.3) Corollary. *The answer to Question (0.2.5) is 'yes' for all threefolds Z with Gorenstein canonical singularities.*

Proof. The main theorem of [Re2] states that there exists a projective variety Y with terminal singularities and a projective birational morphism $g : Y \to Z$ that is crepant. It follows then from [Ba2, Theorem 3.12] that $E_{st}(Y) = E_{st}(Z)$. Since terminal singularities in dimension 3 are isolated (see for example [Ma, Corollary 4-6-6]), the corollary follows immediately from Theorem (1.2.2). ■

(1.2.4) It is not true in general that $(-1)^{i+j} b_{i,j} \geq 0$ for $0 \leq i, j \leq d$, as is shown by the following example (same notation as in the theorem above). Consider the variety $Z' = \{x_1^3 x_5 + x_2^4 + x_3^2 x_5^2 + x_4^2 x_5^2 = 0\} \subset \mathbb{P}^4$. The singular locus of Z' consists of the line $\{x_1 = x_2 = x_5 = 0\}$ and an E_6 singularity at the point $P = \{(0, 0, 0, 0, 1)\}$. We can resolve the singular line by four blow-ups (first in the singular line itself, then in a surface isomorphic to $\mathbb{P}^2$ and afterwards consecutively in two curves isomorphic to $\mathbb{P}^1$). We are then left with a variety Z which has a unique E_6 singularity at P. The stringy E-function of Z can be written as $H(Z\setminus\{P\}; u, v)+$ (contribution of the singular point). For the second term we refer to Theorem (2.4.1) from Chapter 2, it is equal to $1 + \frac{(uv)^2(2(uv)^6-2(uv)^5+(uv)^4-(uv)^2+2uv-2)}{(uv)^7-1}$. The first term can be computed to be $(uv)^3 + 7(uv)^2 + 7uv$. As a power series, $E_{st}(Z) = 1 + 7uv + 9(uv)^2 - (uv)^3 + (uv)^4 - (uv)^6 + \cdots$ and thus $b_{3,3} < 0$.

(1.2.5) Remark. In Theorem (1.2.2) we proved in fact that the numbers $(-1)^{i+j}b_{i,j}$ for $i+j \le d$ and $d \ge 4$ are given by (with the same notations as in the theorem)

$$(-1)^{i+j}b_{i,j} = \dim\ker(H^{d-i,d-j}(H^{2d-i-j}(X)) \to H^{d-i,d-j}(H^{2d-i-j}(D))) + S,$$

where

$$S = \begin{cases} \sum_{\substack{l=1,\dots,\alpha \\ a_l=d/2-1}} \dim H^{d-1,d-1}(H^{2d-2}(D_l)) & \text{if } d \text{ even}, i=j=d/2, \\ \\ \sum_{\substack{l=1,\dots,\alpha \\ a_l=(d-3)/2}} \dim H^{d-2,d-1}(H^{2d-3}(D_l)) & \text{if } d \text{ odd}, \ i=(d+1)/2, \\ & \qquad j=(d-1)/2, \\ 0 & \text{otherwise.} \end{cases}$$

In the following proposition we give a more intrinsic explicit description for these numbers. The 2-dimensional case is trivial (you just find the Hodge numbers of the crepant resolution), the 3-dimensional case is handled in (1.2.6).

Proposition. *Assume that Y is projective and satisfies condition $(*)$, with $d = \dim Y \ge 4$. Take a log resolution $f : X \to Y$ with $D_l, l = 1,\dots,\alpha$, the irreducible components of the exceptional locus and with a_l the discrepancy of D_l. Write $E_{st}(Y)$ as a power series $\sum_{i,j\ge 0} b_{i,j}u^iv^j$. Then for $i+j \le d$ we have*

$$(-1)^{i+j}b_{i,j} = \dim H^{d-i,d-j}(H^{2d-i-j}(Y)) + S,$$

where S is as in the remark.

Proof. The term S is in fact equal to the term R from the proof of Theorem (1.2.2). By the remark above, it is thus sufficient to prove that

$$\dim\ker(H^{d-i,d-j}(H^{2d-i-j}(X)) \to H^{d-i,d-j}(H^{2d-i-j}(D))) \\ = \dim H^{d-i,d-j}(H^{2d-i-j}(Y))$$

for $i \geq j$ and $i+j \leq d$. Let $y_1, \dots, y_s$ be the (isolated) singular points of Y and take disjoint contractible neighbourhoods U_j for the complex topology around the points y_j. From the Leray spectral sequence for $f : V_j = f^{-1}(U_j) \to U_j$ it follows that $H^m(V_j) = H^m(f^{-1}(y_j))$ for all m. Now we consider the long exact sequences for $(X, \cup_j V_j)$ and $(Y, \cup_j U_j)$. We get the following diagram with exact rows (we use that $H^m(\cup_j V_j) = H^m(D)$ and $H^m(\cup_j U_j) = 0$ for $m > 0$, where D is the total exceptional locus):

$$\begin{array}{ccccccccc}
\to H^{d-1}(D) & \to & H^d(X, \cup_j V_j) & \to & H^d(X) & \to \cdots \to & H^{2d}(X) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\to \quad 0 & \to & H^d(Y, \cup_j U_j) & \to & H^d(Y) & \to \cdots \to & H^{2d}(Y) & \to & 0.
\end{array}$$

By excision we have

$$\begin{aligned}
H^m(Y, \cup_j U_j) &\cong H^m(Y \setminus \{y_1, \dots, y_s\}, \cup_j U_j \setminus \{y_1, \dots, y_s\}) \\
&\cong H^m(X \setminus D, \cup_j V_j \setminus D) \\
&\cong H^m(X, \cup_j V_j)
\end{aligned}$$

for all m and then the Mayer-Vietoris construction (see [Ro, Lemma 6.6]) gives us a long exact sequence (note that $H^{2d-1}(D) = H^{2d}(D) = 0$)

$$\to H^{d-1}(D) \to H^d(Y) \to H^d(X) \to H^d(D) \to H^{d+1}(Y) \to \cdots$$

$$\cdots \to H^{2d-1}(X) \to 0 \to H^{2d}(Y) \to H^{2d}(X) \to 0.$$

This sequence is also used in the work of Caibăr, see for instance [Ca, Section 3]. Thanks to Theorem (1.1.1) this sequence splits as follows:

$$\to H^{d-1}(D) \to H^d(Y) \to H^d(X) \to H^d(D) \to 0, \qquad (d)$$

$$0 \to H^{d+1}(Y) \to H^{d+1}(X) \to H^{d+1}(D) \to 0, \qquad (d+1)$$

$$\vdots$$

$$0 \to H^{2d-2}(Y) \to H^{2d-2}(X) \to H^{2d-2}(D) \to 0, \qquad (2d-2)$$

$$0 \to H^{2d-1}(Y) \to H^{2d-1}(X) \to 0, \qquad (2d-1)$$

$$0 \to H^{2d}(Y) \to H^{2d}(X) \to 0. \qquad (2d)$$

Now we apply the exact functor $H^{d-i,d-j}$ to sequence $(2d-i-j)$ and we use the remark above to deduce the result. We must note that for $i+j=d$, $H^{d-i,d-j}(H^{d-1}(D)) = 0$. This is [KK, Corollary 2 p.154], it also follows immediately from the discussion in (1.1.3). ■

Remark. We see from the short exact sequences $(d+1)$ to $(2d)$ that $H^k(Y)$ carries a pure Hodge structure for $k > d$. From the above proof it is clear that this works for any projective Y with at most isolated singularities. Steenbrink proved this even for complete varieties with only isolated singularities [St2, Theorem (1.13)].

(1.2.6) Proposition. *Let Z be a projective threefold with Gorenstein canonical singularities. Write the stringy E-function of Z as a power series $\sum_{i,j\geq 0} b_{i,j}u^i v^j$. Take a partial crepant resolution $g : Y \to Z$ as in the proof of Corollary (1.2.3). Then for $i+j \leq 3$ we have*

$$(-1)^{i+j} b_{i,j} = \dim H^{3-i,3-j}(H^{6-i-j}(Y)).$$

Proof. Recall from (1.2.3) that Y has terminal singularities (which are automatically isolated) and that $E_{st}(Z) = E_{st}(Y)$. For Y the analogue of Remark (1.2.5) (with $S=0$) is valid as well and thus we can proceed exactly as in the proof of the previous proposition. ■

(1.2.7) In view of the previous results it is tempting to define the numbers $(-1)^{i+j}b_{i,j}$ for d-dimensional varieties as in (1.2.5) or (1.2.6) and for $i+j \leq d$ as generalized stringy Hodge numbers $h^{i,j}_{st}$, making it unnecessary to assume that the stringy E-function is a polynomial. One could just put $h^{d-i,d-j}_{st} := h^{i,j}_{st}$. Since the power series of $\frac{uv-(uv)^{a_i+1}}{(uv)^{a_i+1}-1}$ contains only powers of uv, we have that the sum in the power series $\sum_{i,j\geq 0} b_{i,j}u^i v^j$ of the stringy E-function only runs over those (i,j) with $|i-j| \leq d$. With the above definition we can write the stringy E-function as

$$\sum_{i,j=0}^{d} h^{i,j}_{st} u^i v^j + \sum_{k=1}^{\infty} \sum_{\substack{i,j\geq k \\ i+j=d+k}} c_{i,j} u^i v^j,$$

so this means that we take 'the largest possible symmetrical polynomial piece' apart. Note that as a rational function, $f(u,v) := \sum\sum c_{i,j}u^iv^j$ also satisfies the symmetry relation $f(u,v) = (uv)^d f(u^{-1}, v^{-1})$.

(1.2.8) In order to make the analogy between classical Hodge numbers and (generalized) stringy Hodge numbers even bigger, we remark the following. From the Lefschetz decomposition and its compatibility with the Hodge decomposition it follows that classical Hodge numbers of a smooth d-dimensional projective variety satisfy $h^{p,q} \leq h^{p+1,q+1}$ for $0 \leq p, q \leq d-1$ and $p+q \leq d-2$. The well known fact that $h^{p,p} \geq 1$ for $0 \leq p \leq d$ can also be seen as a corollary of these relations, since it is trivial that $h^{0,0} = 1$. We also show these properties for the numbers $(-1)^{i+j}b_{i,j}$. Again the 2-dimensional case is trivial.

Proposition. *Let Y be a projective threefold with Gorenstein canonical singularities or let Y be projective of dimension $d \geq 4$ and satisfy condition $(*)$. Write $E_{st}(Y)$ as a power series $\sum_{i,j\geq 0} b_{i,j}u^iv^j$. Then for $0 \leq i, j \leq d-1$ and $i+j \leq d-2$, we have $(-1)^{i+j}b_{i,j} \leq (-1)^{i+j+2}b_{i+1,j+1}$. In particular, for $i \leq d/2$, one has $b_{i,i} \geq 1$.*

Proof. We give the proof for a d-dimensional variety Y satisfying condition $(*)$. The case of a threefold with Gorenstein canonical singularities can be handled analogously via a partial crepant resolution. From Remark (1.2.5) we get the description

$$(-1)^{i+j}b_{i,j} = \dim\ker(H^{d-i,d-j}(H^{2d-i-j}(X)) \to H^{d-i,d-j}(H^{2d-i-j}(D))) + S,$$

where the term S only appears for $i+j = d$. The corollary from (1.1.2) states that there are surjections

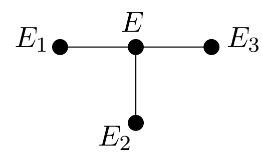
$$\cup\eta_s : \ker(H^{2d-i-j-2}(X) \to H^{2d-i-j-2}(D)) \to \ker(H^{2d-i-j}(X) \to H^{2d-i-j}(D))$$

and these maps are morphisms of Hodge structures of type $(1,1)$. This proves the first statement of the proposition, the second follows since $b_{0,0} = 1$. ■

1.3 Veys' generalizations

(1.3.1) In this section we want to find out if Batyrev's conjecture holds for Veys' generalized stringy E-functions from [Ve2] and [Ve3] (see also

Section (0.3)). From [Ve2, Example 6.8] it follows already that the conjecture is probably not true. In this example, Veys computes the contribution of a triangle singularity to the generalized stringy E-function. A triangle singularity is a surface singularity with dual graph of the minimal resolution equal to

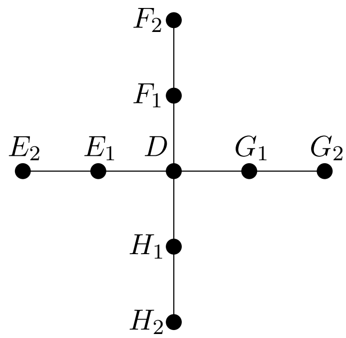


where all the curves are isomorphic to $\mathbb{P}^1$ and where the discrepancy coefficients are -2 for E and -1 for E_1, E_2, E_3. The contribution of this singularity to the generalized stringy E-function is then

$$-(uv)^2 - (n_1 + n_2 + n_3 - 2)uv,$$

where n_i is $-E_i^2$. If we take a projective surface with one such singular point, we already find that the constant term of the generalized stringy E-function must be zero (see also [Ve2, Proposition 4.5]).

(1.3.2) In order to find a counterexample to Batyrev's conjecture for the generalized stringy E-function, we try to find a surface with two triangle singularities (or singularities with a comparable contribution). This will make the constant term of the generalized stringy E-function equal to -1. The easiest example that we found is the following. Our starting point is the variety $X = \{x^3t^5 + y^8 + z^2t^6 = 0\} \subset \mathbb{P}^3$. This variety has a so called E_{14} singularity in the origin of the affine chart with coördinates (x, y, z), and this is a triangle singularity (in [Di, p.63] one can find a list of the hypersurface triangle singularities). But the line $\{y = t = 0\}$ is singular on X. By blowing up with this line as centre, and afterwards performing two more blow-ups in singular curves, we end up with a variety $\widetilde{X}$ with one other singular point besides the E_{14} singularity. This other singularity is in the origin of the chart $\{x^3 + y^3t + yt^3 = 0\} \subset \mathbb{A}^3$. If we blow it up once, we obtain four singular points of type A_2. We have to blow up each of these points once more to get the following dual resolution graph:



Here all curves are rational, and the discrepancies are -3 for D, -2 for E_1, F_1, G_1, H_1 and -1 for the others. The E_i, F_i, G_i, H_i have self-intersection number -2, since they come from an A_2 singularity. The contribution of this singular point to the generalized stringy E-function is then

$$\frac{(uv-3)(uv-1)}{(uv)^{-2}-1}+\frac{4(uv-1)^2}{(uv)^{-1}-1}+\frac{4(uv-1)^2}{((uv)^{-2}-1)((uv)^{-1}-1)}+\frac{8(uv-1)}{(uv)^{-1}-1}$$

and this can be simplified to $-(uv)^2-4uv$. Thus $\widetilde{X}$ is a counterexample to Batyrev's conjecture for Veys' generalized stringy E-function.

We can also compute the total generalized stringy E-function of $\widetilde{X}$. For the nonsingular part of the affine piece $\{x^3+y^8+z^2=0\}\subset\mathbb{A}^3$, we can use Proposition 2.8 from [Da] (see also Section 4.1). This gives $(uv)^2-1$ as result. And the nonsingular part of $\widetilde{X}$ that lies above the singular line of X contributes $3uv$. So the total generalized stringy E-function is

$$-(uv)^2-(n_1+n_2+n_3-1)uv-1,$$

where the n_i are minus the self-intersection numbers of the exceptional components of the triangle singularity.

Chapter 2

Stringy E-functions of varieties with A-D-E singularities

Abstract

In this chapter we compute explicit and fairly simple formulae for the contribution of an A-D-E singularity to the stringy E-function in arbitrary dimension. For 3-dimensional A-D-E singularities this was already done by Dais and Roczen, but their formulae for the D and E cases contain inaccuracies. We correct these errors and simplify their formulae for the A-case considerably. With the aid of these results we can say when the stringy E-function of a variety with A-D-E singularities is a polynomial and we can give a positive answer to Question (0.2.5) for projective varieties with such singularities.[1]

2.1 A-D-E singularities and their desingularization

(2.1.1) First we recall the defining equations of the A-D-E singularities.

Definition. By a d-dimensional ($d \geq 2$) A-D-E singularity we mean a singularity that is analytically isomorphic to the germ at the origin of one of the following hypersurfaces in $\mathbb{A}_{\mathbb{C}}^{d+1}$ (with coordinates $(x_1, \ldots, x_{d+1})$):

[1] This chapter corresponds to [Sch].

Type A_n: $x_1^{n+1} + x_2^2 + x_3^2 + \cdots + x_{d+1}^2 = 0$ for $n \geq 1$,

Type D_n: $x_1^{n-1} + x_1 x_2^2 + x_3^2 + \cdots + x_{d+1}^2 = 0$ for $n \geq 4$,

Type E_6: $x_1^3 + x_2^4 + x_3^2 + \cdots + x_{d+1}^2 = 0$,

Type E_7: $x_1^3 + x_1 x_2^3 + x_3^2 + \cdots + x_{d+1}^2 = 0$,

Type E_8: $x_1^3 + x_2^5 + x_3^2 + \cdots + x_{d+1}^2 = 0$.

Some of their properties are listed in [DR, Remark 1.10].

(2.1.2) We will now construct a log resolution for these singularities by performing successive blow-ups, but we will only do this for $d \geq 4$. The case $d = 2$ is well known and the construction in the 3-dimensional case can be found in detail in [DR, Section 2]; in fact, our procedure is quite analogous. The main differences are:

(1) For $d \geq 4$, every blow-up adds just one component to the exceptional locus, whereas you can get two planes intersecting in a line as new exceptional divisors after a single blow-up in the 3-dimensional case (e.g. after the first blow-up in cases D and E).

(2) In the higher dimensional case, the analogue of this line will be a singular line on the exceptional divisor, thus in order to get an exceptional locus with smooth irreducible components one has to blow up in such lines, which is not necessary for $d = 3$.

An example will make this clear: blow up in the singular point of the defining hypersurface in the E_6 case. For a suitable choice of coordinates one finds $\{z_3^2 + z_4^2 = 0\} \subset \mathbb{P}^3_{\mathbb{C}}$ as equation of the exceptional locus for $d = 3$, and for $d \geq 4$ one finds $\{z_3^2 + z_4^2 + \cdots + z_{d+1}^2 = 0\} \subset \mathbb{P}^d_{\mathbb{C}}$ (this is irreducible, but the line $\{z_3 = \cdots = z_{d+1} = 0\}$ is singular).

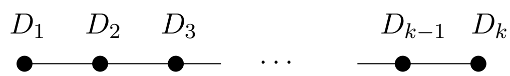
In what follows we use the same name for a divisor D at the moment of its creation as at all later stages (instead of speaking of the strict transform of D). We work out the details for the case of a D_n singularity with even n and we discuss the results briefly in the other cases. We write m for the number of variables ($m \geq 5$) and use coordinates $(x_1, \ldots, x_m)$ on $\mathbb{A}^m$.

(2.1.3) **Case A**

Consider the hypersurface $X = \{x_1^{n+1} + x_2^2 + \cdots + x_m^2 = 0\} \subset \mathbb{A}^m$ for $m \geq 5$.

(1) n odd, $n = 2k - 1$, with $k \geq 1$.

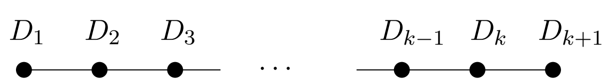
Blowing up an A_n singularity yields an A_{n-2} singularity (that lies on the exceptional locus) and nothing else happens. Thus after k point blow-ups we already have a log resolution. The intersection diagram looks like



where D_i is created after the i-th blow-up. At the moment of its creation, D_i (for $i \in \{1, \ldots, k-1\}$) is isomorphic to the singular quadric $\{x_2^2 + \cdots + x_m^2 = 0\}$ in $\mathbb{P}^{m-1}$, and its singular point is the center of the next blow-up. The last divisor D_k is isomorphic to the nonsingular quadric in $\mathbb{P}^{m-1}$. In the end the intersection of two exceptional divisors is isomorphic to a nonsingular quadric in $\mathbb{P}^{m-2}$.

(2) n even, $n = 2k$, with $k \geq 1$.

This case is almost the same as the previous one. After k point blow-ups the strict transform of X is nonsingular, but the last created divisor D_k still has a singular point, so we have to perform an extra blow-up (with exceptional divisor D_{k+1} isomorphic to $\mathbb{P}^{m-2}$). As intersection diagram we find



with all D_i ($i \in \{1, \ldots, k\}$) isomorphic to the singular quadric $\{x_2^2 + \cdots + x_m^2 = 0\}$ in $\mathbb{P}^{m-1}$ at the moment of their creation. Again, all intersections are isomorphic to the nonsingular quadric in $\mathbb{P}^{m-2}$.

(2.1.4) $\boxed{\textbf{Case D}}$

Now we study $X = \{x_1^{n-1} + x_1 x_2^2 + x_3^2 + \cdots + x_m^2 = 0\} \subset \mathbb{A}^m$ for $m \geq 5$ and $n \geq 4$. Note that you also find singularities for $n = 2$ and $n = 3$, but they are analytically isomorphic to two A_1 and one A_3 singularity, respectively.

(1) n even, $n = 2k$, with $k \geq 2$.

Step 1: We blow up X in the origin. Take $(x_1, \ldots, x_m) \times (z_1, \ldots, z_m)$ as coordinates on $\mathbb{A}^m \times \mathbb{P}^{m-1}$ and consider the reducible variety X' in $\mathbb{A}^m \times \mathbb{P}^{m-1}$ given by the equations

$$\begin{cases} x_1^{2k-1} + x_1 x_2^2 + x_3^2 + \cdots + x_m^2 = 0 & \\ x_i z_j = x_j z_i & \forall\, i, j \in \{1, \ldots, m\}. \end{cases}$$

In the open set $z_1 \neq 0$, X' is isomorphic to $\{x_1^2(x_1^{2k-3} + x_1 x_2^2 + x_3^2 + \cdots + x_m^2) = 0\} \subset \mathbb{A}^m$ by replacing x_j by $x_1 \frac{z_j}{z_1}$ and renaming the affine coordinate $\frac{z_j}{z_1}$ as x_j for $j = 2, \ldots, m$. The equation $x_1 = 0$ describes the exceptional locus, while the other equation gives us the strict transform of X, in which we are interested. Their intersection is the first exceptional divisor, we call it D_1. We can do the same thing for any open set $z_i \neq 0$ and thus we can describe X' by the following set of equations:

$$\begin{cases} x_1^2(x_1^{2k-3} + x_1 x_2^2 + x_3^2 + \cdots + x_m^2) = 0 & (1) \\ x_2^2(x_1^{2k-1} x_2^{2k-3} + x_1 x_2 + x_3^2 + \cdots + x_m^2) = 0 & (2) \\ x_3^2(x_1^{2k-1} x_3^{2k-3} + x_1 x_2^2 x_3 + 1 + x_4^2 + \cdots + x_m^2) = 0 & (3) \\ \qquad\vdots & \vdots \\ x_m^2(x_1^{2k-1} x_m^{2k-3} + x_1 x_2^2 x_m + x_3^2 + \cdots + x_{m-1}^2 + 1) = 0. & (m) \end{cases}$$

One sees from this that globally $D_1 \cong \{x_3^2 + \cdots + x_m^2 = 0\} \subset \mathbb{P}^{m-1}$, which has a singular line $\{x_3 = \cdots = x_m = 0\}$ (located in charts (1) and (2)). Notice that for $k \geq 3$, we have a D_{n-2} singularity in chart (1) and a singularity that is analytically isomorphic to an A_1 in the origin of chart (2). In the other charts both D_1 and the strict transform of X are nonsingular, so we have no problems there. We will assume now that $k \geq 4$ and we will see later what happens if $k = 2, 3$.

Step 2: Let us first get rid of the A_1 singularity. Thus we blow up in the origin of chart (2). Since this blow-up is an isomorphism outside this point, we preserve the other coordinate charts and we replace chart (2) by the following charts:

$$\begin{cases} x_1^4x_2^2(x_1^{4k-6}x_2^{2k-3}+x_2+x_3^2+\cdots+x_m^2)=0 & (2.1)\\ x_2^4(x_1^{2k-1}x_2^{4k-6}+x_1+x_3^2+\cdots+x_m^2)=0 & (2.2)\\ x_2^2x_3^4(x_1^{2k-1}x_2^{2k-3}x_3^{4k-6}+x_1x_2+1+x_4^2+\cdots+x_m^2)=0 & (2.3)\\ \quad\vdots & \vdots\\ x_2^2x_m^4(x_1^{2k-1}x_2^{2k-3}x_m^{4k-6}+x_1x_2+x_3^2+\cdots+x_{m-1}^2+1)=0. & (2.m) \end{cases}$$

Now we see that the strict transform $\widetilde{X}$ of X is nonsingular in this part, but we still have the singular line on D_1 (in charts (1) and (2.1) now). Our new exceptional divisor, we call it E_1, is globally a nonsingular quadric in $\mathbb{P}^{m-1}$.

We check immediately that D_1 and E_1 intersect transversally outside the singular line of D_1: take a point $P=(0,0,\alpha_3,\ldots,\alpha_m)$ on their intersection in chart (2.1) for example (thus $\alpha_3^2+\cdots+\alpha_m^2=0$). We assume that P does not lie on the singular line on D_1 (so at least one of the α_i is nonzero), since we will blow it up later. The local ring $\mathcal{O}_{P,\widetilde{X}}$ is isomorphic to $\left(\frac{\mathbb{C}[x_1,\ldots,x_m]}{I}\right)_{m_P}$ with $I=(x_1^{4k-6}x_2^{2k-3}+x_2+x_3^2+\cdots+x_m^2)$ and $m_P=\frac{(x_1,x_2,x_3-\alpha_3,\ldots,x_m-\alpha_m)}{I}$. As a $\mathbb{C}$-vector space, $\frac{m_P}{m_P^2}$ has dimension $m-1$ and is isomorphic to $\frac{(x_1,x_2,x_3-\alpha_3,\ldots,x_m-\alpha_m)}{(x_1^2,x_1x_2,x_2^2,x_3^2-2\alpha_3x_3+\alpha_3^2,\ldots)+I}$. It is generated by the set $\{x_1,x_2,x_3-\alpha_3,\ldots,x_m-\alpha_m\}$ and the last $m-1$ generators are linearly dependent, since

$$\begin{aligned} &x_2+2\alpha_3(x_3-\alpha_3)+\cdots+2\alpha_m(x_m-\alpha_m)\\ &\quad=\quad x_2+2\alpha_3x_3+\cdots+2\alpha_mx_m\\ &\quad=\quad x_1^{4k-6}x_2^{2k-3}+x_2+x_3^2+\cdots+x_m^2-(x_1^{4k-7}x_2^{2k-4})x_1x_2\\ &\qquad\quad -(x_3^2-2\alpha_3x_3+\alpha_3^2)-\cdots-(x_m^2-2\alpha_mx_m+\alpha_m^2)\\ &\quad=\quad 0, \end{aligned}$$

and thus x_1 and x_2 must be linearly independent. Hence D_1 and E_1 have normal crossings at $(0,0,\alpha_3,\ldots,\alpha_m)$. Later on, we will not check the normal crossings condition any more, it will be satisfied for all divisors in the end.

Step 3: We tackle the D_{n-2} singularity in chart (1) now. We blow up in its origin:

$$\begin{cases} x_1^4(x_1^{2k-5}+x_1x_2^2+x_3^2+\cdots+x_m^2)=0 & (1.1)\\ x_1^2x_2^4(x_1^{2k-3}x_2^{2k-5}+x_1x_2+x_3^2+\cdots+x_m^2)=0 & (1.2)\\ x_1^2x_3^4(x_1^{2k-3}x_3^{2k-5}+x_1x_2^2x_3+1+x_4^2+\cdots+x_m^2)=0 & (1.3)\\ \qquad\vdots & \vdots\\ x_1^2x_m^4(x_1^{2k-3}x_m^{2k-5}+x_1x_2^2x_m+x_3^2+\cdots+x_{m-1}^2+1)=0. & (1.m)\end{cases}$$

It is no surprise that we find a D_{n-4} singularity in the origin of chart (1.1) and an A_1 in the origin of chart (1.2). The newly created divisor, called D_2, intersects D_1 and has a singular line in charts (1.1) and (1.2); the singular line of D_1 from chart (1) is transferred to chart (1.2).

Step 4: We blow up in the origin of chart (1.2). The singularity is resolved and the new divisor E_2 intersects both D_1 and D_2:

$$\begin{cases} x_1^8x_2^4(x_1^{4k-10}x_2^{2k-5}+x_2+x_3^2+\cdots+x_m^2)=0 & (1.2.1)\\ x_1^2x_2^8(x_1^{2k-3}x_2^{4k-10}+x_1+x_3^2+\cdots+x_m^2)=0 & (1.2.2)\\ x_1^2x_2^4x_3^8(x_1^{2k-3}x_2^{2k-5}x_3^{4k-10}+x_1x_2+1+x_4^2+\cdots+x_m^2)=0 & (1.2.3)\\ \qquad\vdots & \vdots\\ x_1^2x_2^4x_m^8(x_1^{2k-3}x_2^{2k-5}x_m^{4k-10}+x_1x_2+x_3^2+\cdots+x_{m-1}^2+1)=0. & (1.2.m)\end{cases}$$

The singular lines on D_1 and D_2 are separated and go to charts (1.2.2) and (1.2.1) respectively.

We continue in this way, performing alternate blow-ups in a D_i and an A_1, until we have to blow up in a D_4 singularity.

Step $n-3$: We blow up in the origin of the chart $x_1^{2k-4}(x_1^3+x_1x_2^2+x_3^2+\cdots+x_m^2)=0$.

$$\begin{cases} x_1^{2k-2}(x_1+x_1x_2^2+x_3^2+\cdots+x_m^2)=0 & (1')\\ x_1^{2k-4}x_2^{2k-2}(x_1^3x_2+x_1x_2+x_3^2+\cdots+x_m^2)=0 & (2')\\ x_1^{2k-4}x_3^{2k-2}(x_1^3x_3+x_1x_2^2x_3+1+x_4^2+\cdots+x_m^2)=0 & (3')\\ \qquad\vdots & \vdots\\ x_1^{2k-4}x_m^{2k-2}(x_1^3x_m+x_1x_2^2x_m+x_3^2+\cdots+x_{m-1}^2+1)=0. & (m')\end{cases}$$

In fact (j') stands here for $(\underbrace{1.1\ldots1}_{k-2 \text{ times}}.j)$. We get three singular points, all analytically isomorphic to an A_1 singularity. Both present divisors (we call them of course D_{k-2} and D_{k-1}) have a singular line and in fact all the singular points lie on the singular line of D_{k-1}. One of the singular points, the origin of chart $(2')$, lies on the intersection of D_{k-2} and D_{k-1}. Note that the singular points $(0, i, 0, \ldots, 0)$ and $(0, -i, 0, \ldots, 0)$ of chart $(1')$ correspond to the points $(-i, 0, \ldots, 0)$ and $(i, 0, \ldots, 0)$ of chart $(2')$ respectively.

Step $n-2$: We deal with the origin of chart $(2')$ first. Blowing it up yields a divisor E_{k-1} that intersects D_{k-1} and D_{k-2}:

$$\begin{cases} x_1^{4k-4}x_2^{2k-2}(x_1^2x_2 + x_2 + x_3^2 + \cdots + x_m^2) = 0 & (2'.1) \\ x_1^{2k-4}x_2^{4k-4}(x_1^3x_2^2 + x_1 + x_3^2 + \cdots + x_m^2) = 0 & (2'.2) \\ x_1^{2k-4}x_2^{2k-2}x_3^{4k-4}(x_1^3x_2x_3^2 + x_1x_2 + 1 + x_4^2 + \cdots + x_m^2) = 0 & (2'.3) \\ \qquad\vdots & \vdots \\ x_1^{2k-4}x_2^{2k-2}x_m^{4k-4}(x_1^3x_2x_m^2 + x_1x_2 + x_3^2 + \cdots + x_{m-1}^2 + 1) = 0. & (2'.m) \end{cases}$$

The other two singularities lie in charts $(1')$ and $(2'.1)$. The singular lines on D_{k-2} and D_{k-1} get separated and go to charts $(2'.2)$ and $(2'.1)$, respectively.

Step $n-1$: After a coordinate transformation the equation of chart $(1')$ becomes $x_1^{2k-2}(x_1x_2(x_2+2i) + x_3^2 + \cdots + x_m^2 = 0$. To put the same point in the origin, we have to change the equation of chart $(2'.1)$ to $(x_1-i)^{4k-4}x_2^{2k-2}(x_1x_2(x_1-2i) + x_3^2 + \cdots + x_m^2) = 0$ for example. In this step we blow up both charts in the origin and we call the new divisor F_1:

$$\begin{cases} x_1^{2k}(x_2(x_1x_2 + 2i) + x_3^2 + \cdots + x_m^2) = 0 & (1'.1) \\ x_1^{2k-2}x_2^{2k}(x_1(x_2 + 2i) + x_3^2 + \cdots + x_m^2) = 0 & (1'.2) \\ x_1^{2k-2}x_3^{2k}(x_1x_2(x_2x_3 + 2i) + 1 + x_4^2 + \cdots + x_m^2) = 0 & (1'.3) \quad \text{and} \\ \qquad\vdots & \vdots \\ x_1^{2k-2}x_m^{2k}(x_1x_2(x_2x_m + 2i) + x_3^2 + \cdots + x_{m-1}^2 + 1) = 0 & (1'.m) \end{cases}$$

$$\begin{cases} x_1^{2k}(x_1-i)^{4k-4}x_2^{2k-2}(x_2(x_1-2i)+x_3^2+\cdots+x_m^2)=0 & (2'.1.1)\\ (x_1x_2-i)^{4k-4}x_2^{2k}(x_1(x_1x_2-2i)+x_3^2+\cdots+x_m^2)=0 & (2'.1.2)\\ (x_1x_3-i)^{4k-4}x_2^{2k-2}x_3^{2k}(x_1x_2(x_1x_3-2i)+1+\cdots+x_m^2)=0 & (2'.1.3)\\ \qquad\vdots & \vdots\\ (x_1x_m-i)^{4k-4}x_2^{2k-2}x_m^{2k}(x_1x_2(x_1x_m-2i)+x_3^2+\cdots+1)=0. & (2'.1.m)\end{cases}$$

The last singular point and the singular line on D_{k-1} are now in charts $(1'.2)$ and $(2'.1.1)$.

Step n: Before blowing up the final singular point, we first do a coordinate transformation in chart $(1'.2)$ to get the equation $x_1^{2k-2}(x_2-2i)^{2k}(x_1x_2+x_3^2+\cdots+x_m^2)=0$ and in chart $(2'.1.1)$ to get $(x_1+2i)^{2k}(x_1+i)^{4k-4}x_2^{2k-2}(x_1x_2+x_3^2+\cdots+x_m^2)=0$. The new exceptional divisor is called F_2.

$$\begin{cases} x_1^{2k}(x_1x_2-2i)^{2k}(x_2+x_3^2+\cdots+x_m^2)=0 & (1'.2.1)\\ x_1^{2k-2}(x_2-2i)^{2k}x_2^{2k}(x_1+x_3^2+\cdots+x_m^2)=0 & (1'.2.2)\\ x_1^{2k-2}(x_2x_3-2i)^{2k}x_3^{2k}(x_1x_2+1+x_4^2+\cdots+x_m^2)=0 & (1'.2.3)\\ \qquad\vdots & \vdots\\ x_1^{2k-2}(x_2x_m-2i)^{2k}x_m^{2k}(x_1x_2+x_3^2+\cdots+x_{m-1}^2+1)=0 & (1'.2.m)\end{cases} \quad \text{and}$$

$$\begin{cases} x_1^{2k}(x_1+2i)^{2k}(x_1+i)^{4k-4}x_2^{2k-2}(x_2+x_3^2+\cdots+x_m^2)=0 & (2'.1.1.1)\\ (x_1x_2+2i)^{2k}(x_1x_2+i)^{4k-4}x_2^{2k}(x_1+x_3^2+\cdots+x_m^2)=0 & (2'.1.1.2)\\ (x_1x_3+2i)^{2k}(x_1x_3+i)^{4k-4}x_2^{2k-2}x_3^{2k}(x_1x_2+1+\cdots+x_m^2)=0 & (2'.1.1.3)\\ \qquad\vdots & \vdots\\ (x_1x_m+2i)^{2k}(x_1x_m+i)^{4k-4}x_2^{2k-2}x_m^{2k}(x_1x_2+x_3^2+\cdots+1)=0. & (2'.1.1.m)\end{cases}$$

The singular line on D_{k-1} is moved to charts $(1'.2.2)$ and $(2'.1.1.1)$.

In the next $k-1$ steps we blow up in the singular lines on the divisors D_i. This gives rise to new exceptional divisors which will be denoted by G_i. After $k-1$ steps we finally have a log resolution; we will perform steps $n+1$ and $n+k-1$ explicitly.

Step n+1: To cover the singular line on D_1 completely, we have to perform the blow-up in charts (2.1) and (1.2.2). In chart (2.1) we have to blow up the reducible variety $Y=\{x_1^4x_2^2(x_1^{4k-6}x_2^{2k-3}+x_2+x_3^2+\cdots+x_m^2)=0\}\subset\mathbb{A}^m$ in the line $\{x_2=\cdots=x_m=0\}$. The strict transform of Y and the exceptional locus form a reducible variety in $\mathbb{A}^m\times\mathbb{P}^{m-2}$, given by the equations

$$\begin{cases} x_1^4 x_2^2 (x_1^{4k-6} x_2^{2k-3} + x_2 + x_3^2 + \cdots + x_m^2) = 0 & \\ x_i z_j = x_j z_i & \forall\, i,j \in \{2,\ldots,m\}, \end{cases}$$

where $(z_2,\ldots,z_m)$ are homogenous coordinates on $\mathbb{P}^{m-2}$. As for a point blow-up, we can replace x_j by $x_i \frac{z_j}{z_i}$ in the open set $z_i \neq 0$ and rename $\frac{z_j}{z_i}$ as x_j. Hence we get the following equations for Y':

$$\begin{cases} x_1^4 x_2^3 (x_1^{4k-6} x_2^{2k-4} + 1 + x_2 x_3^2 + \cdots + x_2 x_m^2) = 0 & (2.1.2) \\ x_1^4 x_2^2 x_3^3 (x_1^{4k-6} x_2^{2k-3} x_3^{2k-4} + x_2 + x_3 + x_3 x_4^2 + \cdots + x_3 x_m^2) = 0 & (2.1.3) \\ \vdots & \vdots \\ x_1^4 x_2^2 x_m^3 (x_1^{4k-6} x_2^{2k-3} x_m^{2k-4} + x_2 + x_3^2 x_m + \cdots + x_{m-1}^2 x_m + x_m) = 0. & (2.1.m) \end{cases}$$

The equations after blowing up in $\{x_1 = x_3 = \cdots = x_m = 0\}$ in chart (1.2.2) are:

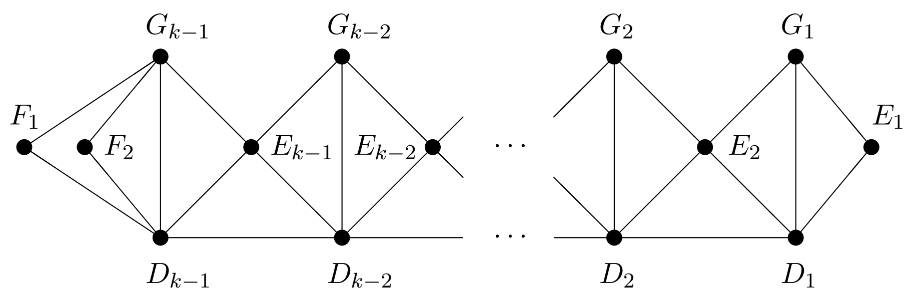
$$\begin{cases} x_1^3 x_2^8 (x_1^{2k-4} x_2^{4k-10} + 1 + x_1 x_3^2 + \cdots + x_1 x_m^2) = 0 & (1.2.2.1) \\ x_1^2 x_2^8 x_3^3 (x_1^{2k-3} x_2^{4k-10} x_3^{2k-4} + x_1 + x_3 + x_3 x_4^2 + \cdots + x_3 x_m^2) = 0 & (1.2.2.3) \\ \vdots & \vdots \\ x_1^2 x_2^8 x_m^3 (x_1^{2k-3} x_2^{4k-10} x_m^{2k-4} + x_1 + x_3^2 x_m + \cdots + x_{m-1}^2 x_m + x_m) = 0. & (1.2.2.m) \end{cases}$$

Step $n+k-1$: Here we have to consider charts $(1'.2.2)$ and $(2'.1.1.1)$ in which D_{k-1} still has a singular line with equations $\{x_1 = x_3 = \cdots = x_m = 0\}$ and $\{x_2 = x_3 = \cdots = x_m = 0\}$, respectively. Blowing it up yields

$$\begin{cases} x_1^{2k-1} (x_2 - 2i)^{2k} x_2^{2k} (1 + x_1 x_3^2 + \cdots + x_1 x_m^2) = 0 & (1'.2.2.1) \\ x_1^{2k-2} (x_2 - 2i)^{2k} x_2^{2k} x_3^{2k-1} (x_1 + x_3 + \cdots + x_3 x_m^2) = 0 & (1'.2.2.3) \\ \vdots & \vdots \\ x_1^{2k-2} (x_2 - 2i)^{2k} x_2^{2k} x_m^{2k-1} (x_1 + x_3^2 x_m + \cdots + x_m) = 0 & (1'.2.2.m) \end{cases} \quad \text{and}$$

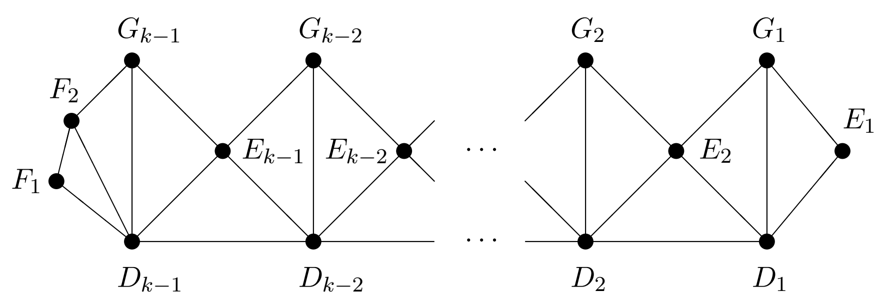
$$\begin{cases} x_1^{2k} (x_1 + 2i)^{2k} (x_1 + i)^{4k-4} x_2^{2k-1} (1 + x_2 x_3^2 + \cdots + x_2 x_m^2) = 0 & (2'.1.1.1.2) \\ x_1^{2k} (x_1 + 2i)^{2k} (x_1 + i)^{4k-4} x_2^{2k-2} x_3^{2k-1} (x_2 + x_3 + \cdots + x_3 x_m^2) = 0 & (2'.1.1.1.3) \\ \vdots & \vdots \\ x_1^{2k} (x_1 + 2i)^{2k} (x_1 + i)^{4k-4} x_2^{2k-2} x_m^{2k-1} (x_2 + x_3^2 x_m + \cdots + x_m) = 0. & (2'.1.1.1.m) \end{cases}$$

From these calculations, we can deduce the intersection diagram. We leave it to the reader to check the details. It can easily be seen that the same diagram is valid for $k = 2, 3$.



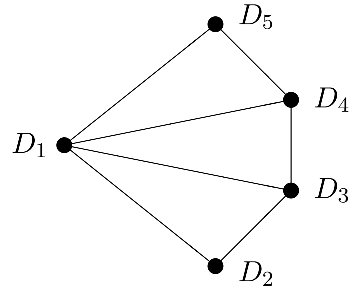
(2) n odd, $n = 2k+1$, with $k \geq 2$.

The first $2k-4$ steps are completely analogous to the case where n is even. Now we end up with the equation $x_1^{2k-4}(x_1^4 + x_1x_2^2 + x_3^2 + \cdots + x_m^2) = 0$ which has a D_5 singularity in the origin. Blowing this up gives one A_3 singularity on the new divisor D_{k-1} (the equation of the first chart is $x_1^{2k-2}(x_1^2 + x_1x_2^2 + x_3^2 + \cdots + x_m^2 = 0)$). We already know that this can be resolved by two consecutive blow-ups, creating divisors F_1 and F_2. Afterwards, the singular lines on the D_i must be blown up. Explicit computations lead to the following intersection diagram:



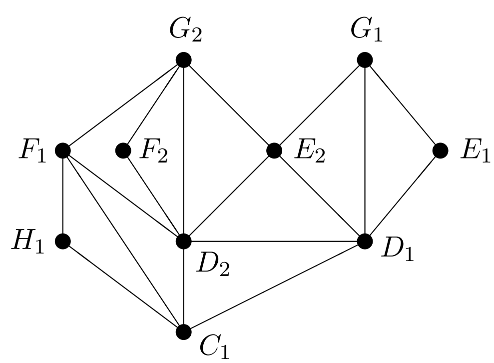
(2.1.5) $\boxed{\textbf{Case } \mathbf{E_6}}$

After blowing up in the origin we get an A_5 singularity and a singular line on the first exceptional divisor D_1. To resolve the A_5 singularity we need three more point blow-ups (creating D_2, D_3 and D_4) and in the end we blow up in the singular line (giving rise to a divisor D_5). We find as intersection graph:

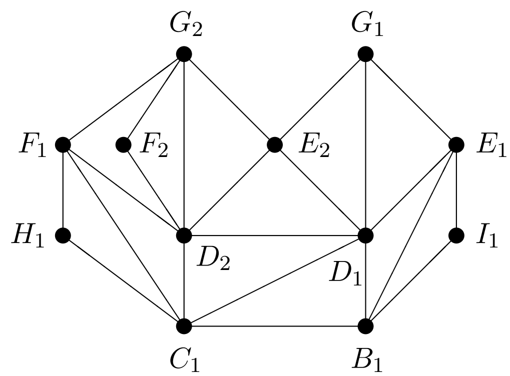


(2.1.6) $\boxed{\textbf{Cases } \mathbf{E_7} \textbf{ and } \mathbf{E_8}}$

An E_7 becomes a D_6 after one step and calculating the intersections gives the following diagram



where C_1 is the very first exceptional divisor and where H_1 arises after blowing up the singular line on C_1. The other divisors come from the D_6 singularity. Notice the difference between F_1 and F_2. It is easy to see that an E_8 singularity passes to an E_7 after one blow-up, with again a singular line on the first exceptional divisor B_1. We denote the divisor that appears after blowing up in this singular line by I_1 and we find the following intersection graph:



2.2 Computation of the Hodge-Deligne polynomials

(2.2.1) Denote by a_r, b_r, c_r $(r \geq 2)$ the Hodge-Deligne polynomials of

- $\{x_1^2 + \cdots + x_r^2 = 0\} \subset \mathbb{P}_{\mathbb{C}}^{r+1}$,
- $\{x_1^2 + \cdots + x_r^2 = 0\} \subset \mathbb{P}_{\mathbb{C}}^{r}$,
- $\{x_1^2 + \cdots + x_r^2 = 0\} \subset \mathbb{P}_{\mathbb{C}}^{r-1}$,

respectively, where $\mathbb{P}^s$ gets coordinates $(x_1, \ldots, x_{s+1})$. We will be able to express all the needed Hodge-Deligne polynomials in terms of a_r, b_r and c_r, and these last expressions are well known. For completeness we include their computation in the following lemma. *From now on, we write w as abbreviation of uv.*

Lemma. *The formulae for a_r, b_r and c_r are given in the following table*:

	r *even*	r *odd*
a_r	$\frac{w^{r+1}-1}{w-1} + w^{\frac{r}{2}+1}$	$\frac{w^{r+1}-1}{w-1}$
b_r	$\frac{w^{r}-1}{w-1} + w^{\frac{r}{2}}$	$\frac{w^{r}-1}{w-1}$
c_r	$\frac{w^{r-1}-1}{w-1} + w^{\frac{r}{2}-1}$	$\frac{w^{r-1}-1}{w-1}$

Proof. Denote by d_r the Hodge-Deligne polynomial of $\{x_1^2+\cdots+x_r^2+1=0\}\subset\mathbb{A}^r$. First we compute d_r by induction on r. Since d_2 is the Hodge-Deligne polynomial of a conic with two points at infinity, it equals $w-1$. The variety $\{x_1^2+x_2^2+x_3^2+1=0\}\subset\mathbb{A}^3$ can be regarded as $\mathbb{P}^1\times\mathbb{P}^1$ minus a conic and thus $d_3=(w+1)^2-(w+1)=w^2+w$. For $r\geq 4$ we use the isomorphism $\{x_1^2+\cdots+x_r^2+1=0\}\cong\{x_1x_2+x_3^2+\cdots+x_r^2+1=0\}$. If $x_1=0$ in this last equation, then the contribution to d_r is wd_{r-2} and if $x_1\neq 0$, then it is $(w-1)w^{r-2}$, so we have the recursion formula $d_r=wd_{r-2}+(w-1)w^{r-2}$. From this it follows that $d_r=w^{r-1}-w^{\frac{r}{2}-1}$ if r is even and $d_r=w^{r-1}+w^{\frac{r-1}{2}}$ if r is odd.

For a_2 we find $2w^2+w+1$ and we have the recursion formula $a_r=a_{r-1}+w^2d_{r-1}$ for $r\geq 3$. The formulae for b_r and c_r can be deduced similarly. ■

(2.2.2) For the remainder of this section, we will calculate the Hodge-Deligne polynomials of the pieces D_J° (see the definition of the stringy E-function). Since we are mainly interested in the contribution of the singular point (by which we mean $E_{st}(X)-H(D_\emptyset^\circ)=E_{st}(X)-H(X\setminus\{0\})$, where X is a defining variety of an A-D-E singularity), we will do this for $J\neq\emptyset$.

We remark here the following. In the defining formula of the stringy E-function we need the Hodge-Deligne polynomials of the D_J° at the end of the resolution process. Note however that we can compute them immediately after they are created, since a blow-up is an isomorphism outside its center. So we just have to subtract contributions of intersections with previously created divisors and already present centers of future blow-ups from the global Hodge-Deligne polynomial in the right way.

The case of an A-D-E surface singularity is well known and for threefold singularities we refer again to [DR], so we consider here the higher dimensional case. Parallel to the previous section, we work out the details for the case D_n, n even, and state the results in the other cases. We use the same notations as in the previous section.

(2.2.3) $\boxed{\textbf{Case A}}$

From the description in (2.1.3), one gets the following:

(1) n odd

$$\begin{aligned}
&H(D_1^\circ) = b_{m-1} - 1 \\
&H(D_i^\circ) = b_{m-1} - c_{m-1} - 1 && (i = 2, \dots, k-1) \\
&H(D_k^\circ) = c_m - c_{m-1} \\
&H(D_i \cap D_{i+1}) = c_{m-1} && (i = 1, \dots, k-1)
\end{aligned}$$

(2) n even

$$\begin{aligned}
&H(D_1^\circ) = b_{m-1} - 1 \\
&H(D_i^\circ) = b_{m-1} - c_{m-1} - 1 && (i = 2, \dots, k) \\
&H(D_{k+1}^\circ) = w^{m-2} + \dots + 1 - c_{m-1} \\
&H(D_i \cap D_{i+1}) = c_{m-1} && (i = 1, \dots, k)
\end{aligned}$$

(2.2.4) $\boxed{\textbf{Case D}}$

(1) n even

All the needed information can be read off from the equations in (2.1.4). We follow the same steps.

Step 1: The first exceptional divisor is globally isomorphic to $\{x_3^2 + \dots + x_m^2 = 0\} \subset \mathbb{P}^{m-1}$, which has a singular line that contains the two singular points of the surrounding variety. Hence $H(D_1^\circ) = a_{m-2} - (w+1)$.

Step 2: One sees that E_1 is a nonsingular quadric in $\mathbb{P}^{m-1}$ that intersects D_1 in $\{x_3^2 + \dots + x_m^2 = 0\} \subset \mathbb{P}^{m-2}$, for a suitable choice of coordinates. Thus $H(E_1^\circ) = c_m - b_{m-2}$. The intersection of D_1 and E_1 contains one point of the singular line on D_1 and hence $H((D_1 \cap E_1)^\circ) = b_{m-2} - 1$.

Step 3: Analogous to step 1 one finds that D_2 is isomorphic to $\{x_3^2 + \dots + x_m^2 = 0\} \subset \mathbb{P}^{m-1}$, with a singular line that contains two singular points of the surrounding variety. Now D_2 intersects D_1 in $\{x_3^2 + \dots + x_m^2 = 0\} \subset \mathbb{P}^{m-2}$. This intersection has exactly one point (the origin of coordinate chart (1.2)) in common with the singular lines on D_2 and D_1. The conclusion is that $H(D_2^\circ) = a_{m-2} - (w+1) - b_{m-2} + 1$ and $H((D_1 \cap D_2)^\circ) = b_{m-2} - 1$.

Step 4: For $H(E_2^\circ)$ we find $c_m - 2b_{m-2} + c_{m-2}$, where $2b_{m-2}$ comes from the intersections with D_1 and D_2 and c_{m-2} from the intersection with $D_1 \cap D_2$. We also have that $H((D_1 \cap E_2)^\circ) = H((D_2 \cap E_2)^\circ) = b_{m-2} - c_{m-2} - 1$, where the -1 comes from a point on the singular lines on the D_i. Finally $H(D_1 \cap D_2 \cap E_2) = c_{m-2}$.

Analogously, for all i from 3 to $k-2$, we have $H(D_i^\circ) = a_{m-2} - (w+1) - b_{m-2} + 1$, $H((D_{i-1} \cap D_i)^\circ) = b_{m-2} - 1$, $H(E_i^\circ) = c_m - 2b_{m-2} + c_{m-2}$, $H((D_{i-1} \cap E_i)^\circ) = H((D_i \cap E_i)^\circ) = b_{m-2} - c_{m-2} - 1$ and $H(D_{i-1} \cap D_i \cap E_i) = c_{m-2}$.

Step $n-3$: In this step three singular points are created, but since they are all on the singular line on D_{k-1}, we still find $H(D_{k-1}^\circ) = a_{m-2} - (w+1) - b_{m-2} + 1$ and $H((D_{k-2} \cap D_{k-1})^\circ) = b_{m-2} - 1$.

Step $n-2$: Again nothing special happens: $H(E_{k-1}^\circ) = c_m - 2b_{m-2} + c_{m-2}$, $H((D_{k-2} \cap E_{k-1})^\circ) = H((D_{k-1} \cap E_{k-1})^\circ) = b_{m-2} - c_{m-2} - 1$ and $H(D_{k-2} \cap D_{k-1} \cap E_{k-1}) = c_{m-2}$.

Step $n-1$ *and step* n: Both F_1 and F_2 are nonsingular quadrics in $\mathbb{P}^{m-1}$ and their intersection with D_{k-1} is $\{x_3^2 + \cdots + x_m^2 = 0\} \subset \mathbb{P}^{m-2}$, which has one point in common with the singular line on D_{k-1}. Thus $H(F_1^\circ) = H(F_2^\circ) = c_m - b_{m-2}$ and $H((D_{k-1} \cap F_1)^\circ) = H((D_{k-1} \cap F_2)^\circ) = b_{m-2} - 1$.

Step $n+1$: The singular line on D_1 is except for the origin of coordinate chart (2.1) covered by chart (1.2.2). But after the blow-up, exactly the intersection of E_1 and G_1 lies above the origin of chart (2.1). Thus to calculate $H(G_1^\circ)$, it suffices to consider only charts (1.2.2.1) to (1.2.2.m). In chart (1.2.2.3) G_1 is just isomorphic to $\mathbb{A}^{m-2}$. The piece of G_1 that is covered by chart (1.2.2.4) but not by (1.2.2.3) is isomorphic to $\mathbb{A}^{m-3}$ and so on, until we add an affine line to G_1 in chart (1.2.2.m). The intersection of G_1 with E_2 is isomorphic to $\mathbb{P}^{m-3}$. It is not so hard to see that $H(D_1 \cap E_2 \cap G_1) = c_{m-2}$ (note that the equations of (the strict transform of) D_1 in chart (1.2.2.3) for instance are $x_1 = 0$ and $1 + x_4^2 + \cdots + x_m^2 = 0$), and from this it follows that $H((D_1 \cap G_1)^\circ) = (w-1)c_{m-2}$ (the w comes from the x_2-coordinate that can be chosen freely in every chart). Now we also have $H((E_2 \cap G_1)^\circ) = w^{m-3} + \cdots + 1 - c_{m-2}$ and $H(G_1^\circ) = w^{m-2} + \cdots + w - (w^{m-3} + \cdots + 1) - wc_{m-2} + c_{m-2} = w^{m-2} - 1 - (w-1)c_{m-2}$. One gets from charts (2.1.2) to (2.1.m) that $H((E_1 \cap G_1)^\circ) = w^{m-3} + \cdots + 1 - c_{m-2}$ and that $H(D_1 \cap E_1 \cap G_1) = c_{m-2}$.

More conceptually, G_1 is a locally trivial $\mathbb{P}^{m-3}$-bundle over the singular line on D_1 and $E_1 \cap G_1$ and $E_2 \cap G_1$ are two fibers. Thus $H(G_1) = (w+1)(w^{m-3} + \cdots + 1)$ and $H(E_i \cap G_1) = w^{m-3} + \cdots + 1$. Furthermore, we can consider the singular line on D_1 as a family of A_1 singularities and thus $D_1 \cap G_1$ is a family of nonsingular quadrics in $\mathbb{P}^{m-3}$. This implies that $H(D_1 \cap G_1) = (w+1)c_{m-2}$ and $H(D_1 \cap E_i \cap G_1) = c_{m-2}$.

In exactly the same way one finds that (for $i \in \{2, \ldots, k-2\}$) $H(G_i^\circ) = w^{m-2} - 1 - (w-1)c_{m-2}$, $H((D_i \cap G_i)^\circ) = (w-1)c_{m-2}$, $H((E_i \cap G_i)^\circ) = H((E_{i+1} \cap G_i)^\circ) = w^{m-3} + \cdots + 1 - c_{m-2}$ and $H(D_i \cap E_i \cap G_i) = H(D_i \cap E_{i+1} \cap G_i) = c_{m-2}$.
Step $n+k-1$: This step looks very much like step $n+1$. It suffices to consider charts $(1'.2.2.1)$ to $(1'.2.2.m)$ to compute $H(G_{k-1}^\circ)$. One checks that $H(D_{k-1} \cap F_1 \cap G_{k-1}) = H(D_{k-1} \cap F_2 \cap G_{k-1}) = c_{m-2}$, $H((F_1 \cap G_{k-1})^\circ) = H((F_2 \cap G_{k-1})^\circ) = w^{m-3} + \cdots + 1 - c_{m-2}$, $H((D_{k-1} \cap G_{k-1})^\circ) = (w-2)c_{m-2}$ and thus $H(G_{k-1}^\circ) = w^{m-2} + \cdots + w - 2(w^{m-3} + \cdots + 1) - (w-2)c_{m-2}$. From charts $(2'.1.1.1.2)$ to $(2'.1.1.1.m)$ we get $H(D_{k-1} \cap E_{k-1} \cap G_{k-1}) = c_{m-2}$ and $H((E_{k-1} \cap G_{k-1})^\circ) = w^{m-3} + \cdots + 1 - c_{m-2}$. A conceptual explanation like in step $n+1$ can be given here too.

(2) n odd

There are only seven changes in comparison with the case where n is even. First remark that $F_1 \cap G_{k-1}$ and $D_{k-1} \cap F_1 \cap G_{k-1}$ are empty, but instead $H((F_1 \cap F_2)^\circ) = c_{m-1} - c_{m-2}$ and $H(D_{k-1} \cap F_1 \cap F_2) = c_{m-2}$. The other five changes are the following:

$$\begin{aligned}
&H(F_1^\circ) = b_{m-1} - b_{m-2}\\
&H(F_2^\circ) = c_m - c_{m-1} - b_{m-2} + c_{m-2}\\
&H(G_{k-1}^\circ) = w^{m-2} - 1 - (w-1)c_{m-2}\\
&H((D_{k-1} \cap F_2)^\circ) = b_{m-2} - c_{m-2} - 1\\
&H((D_{k-1} \cap G_{k-1})^\circ) = (w-1)c_{m-2}
\end{aligned}$$

(2.2.5) $\boxed{\textbf{Case } \mathbf{E_6}}$

We just list the results.

$$
\begin{aligned}
&H(D_1^\circ) = a_{m-2} - w - 1\\
&H(D_2^\circ) = b_{m-1} - b_{m-2}\\
&H(D_3^\circ) = b_{m-1} - b_{m-2} - c_{m-1} + c_{m-2}\\
&H(D_4^\circ) = c_m - b_{m-2} - c_{m-1} + c_{m-2}\\
&H(D_5^\circ) = w^{m-2} + \cdots + w - wc_{m-2}\\
&H((D_1 \cap D_2)^\circ) = b_{m-2} - 1\\
&H((D_1 \cap D_3)^\circ) = H((D_1 \cap D_4)^\circ) = b_{m-2} - c_{m-2} - 1\\
&H((D_1 \cap D_5)^\circ) = wc_{m-2}\\
&H((D_2 \cap D_3)^\circ) = H((D_3 \cap D_4)^\circ) = c_{m-1} - c_{m-2}\\
&H((D_4 \cap D_5)^\circ) = w^{m-3} + \cdots + 1 - c_{m-2}\\
&H(D_1 \cap D_2 \cap D_3) = H(D_1 \cap D_3 \cap D_4) = H(D_1 \cap D_4 \cap D_5) = c_{m-2}
\end{aligned}
$$

(2.2.6) $\boxed{\textbf{Cases } \mathbf{E_7} \textbf{ and } \mathbf{E_8}}$

Let us first treat the E_8 case. From the intersection diagram it follows that we have to compute forty-seven Hodge-Deligne polynomials (there are twelve divisors, twenty-three intersections of two divisors and twelve intersections of three divisors). But there are twenty polynomials coming from the 'D_6 part' of the diagram that are left unchanged here. So we will only write down the other twenty-seven.

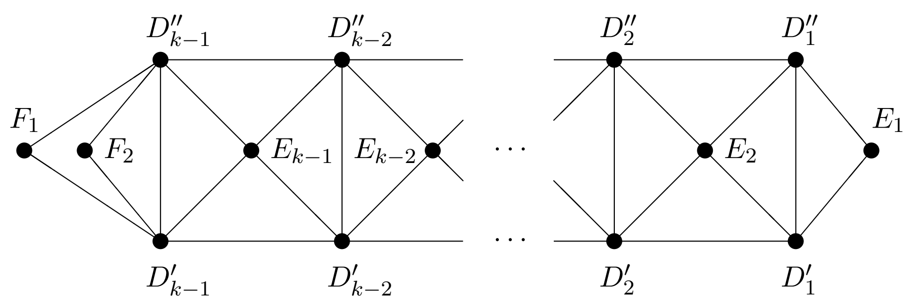
$$
\begin{aligned}
&H(B_1^\circ) = a_{m-2} - w - 1\\
&H(C_1^\circ) = a_{m-2} - b_{m-2} - w\\
&H(D_1^\circ) = H(D_2^\circ) = a_{m-2} - 2b_{m-2} + c_{m-2} - w + 1\\
&H(E_1^\circ) = H(F_1^\circ) = c_m - 2b_{m-2} + c_{m-2}\\
&H(H_1^\circ) = H(I_1^\circ) = w^{m-2} + \cdots + w - wc_{m-2}\\
&H((B_1 \cap C_1)^\circ) = H((B_1 \cap I_1)^\circ) = H((C_1 \cap H_1)^\circ) = wc_{m-2}\\
&H((B_1 \cap D_1)^\circ) = H((B_1 \cap E_1)^\circ) = H((C_1 \cap D_1)^\circ)\\
&\quad = H((C_1 \cap D_2)^\circ) = H((C_1 \cap F_1)^\circ) = H((D_1 \cap D_2)^\circ)\\
&\quad = H((D_1 \cap E_1)^\circ) = H((D_2 \cap F_1)^\circ) = b_{m-2} - c_{m-2} - 1\\
&H((E_1 \cap I_1)^\circ) = H((F_1 \cap H_1)^\circ) = w^{m-3} + \cdots + 1 - c_{m-2}\\
&H(B_1 \cap C_1 \cap D_1) = H(B_1 \cap D_1 \cap E_1) = H(B_1 \cap E_1 \cap I_1)\\
&\quad = H(C_1 \cap D_1 \cap D_2) = H(C_1 \cap D_2 \cap F_1) = H(C_1 \cap F_1 \cap H_1) = c_{m-2}
\end{aligned}
$$

For the E_7 case, we can skip all expressions involving the divisors B_1 and/or I_1. This leaves us with thirty-seven polynomials and apart from the following five, they are all the same as in the E_8 case.

$$
\begin{aligned}
&H(C_1^\circ) = a_{m-2} - w - 1\\
&H(D_1^\circ) = a_{m-2} - b_{m-2} - w\\
&H(E_1^\circ) = c_m - b_{m-2}\\
&H((C_1 \cap D_1)^\circ) = H((D_1 \cap E_1)^\circ) = b_{m-2} - 1
\end{aligned}
$$

2.3 Computation of the discrepancy coefficients

(2.3.1) In this section we compute the last data that we need: the discrepancy coefficients. As already mentioned in Example (0.2.5), all the two dimensional A-D-E's admit a crepant resolution, this means that all the discrepancies are 0. For the three-dimensional case, the computations are done in [DR, Section 3], but the authors are a bit inaccurate. Let us again consider the case D_n, n even, with $k = \frac{n}{2}$. The intersection diagram is as follows:



Compared to the higher dimensional cases, the D_i fall apart into two components D'_i and D''_i, and there are no divisors G_i needed. If we denote by $\varphi : \widetilde{X} \to X$ the log resolution, with X the defining variety of the D_n singularity and $\widetilde{X}$ the strict transform of X, then φ can be decomposed into k birational morphisms

$$
\widetilde{X} = X_k \xrightarrow{\varphi_k} X_{k-1} \longrightarrow \cdots \longrightarrow X_2 \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 = X,
$$

where the exceptional locus of φ_1 is $\{D'_1, D''_1\}$, of φ_i ($2 \leq i \leq k-1$) is $\{D'_i, D''_i, E_{i-1}\}$ and of φ_k is $\{F_1, F_2, E_{k-1}\}$, again using the same name for the divisors at any stage of the decomposition of φ. We can also decompose $K_{\widetilde{X}} - \varphi^*(K_X)$ as

$$\left[\sum_{i=1}^{k-1} \varphi_k^*(\varphi_{k-1}^* \cdots (\varphi_{i+1}^*(K_{X_i} - \varphi_i^*(K_{X_{i-1}}))) \cdots)\right] + K_{X_k} - \varphi_k^*(K_{X_{k-1}}).$$

Dais and Roczen calculated that for instance $\varphi_2^*(D_1') = D_1' + D_2' + E_1$ and $\varphi_2^*(D_1'') = D_1'' + D_2'' + E_1$, but D_1' and D_1'' are not Cartier. Their sum $D_1' + D_1''$ is Cartier and it turns out that $\varphi_2^*(D_1' + D_1'') = D_1' + D_1'' + D_2' + D_2'' + E_1$ instead of $\cdots + 2E_1$. This kind of error occurs also in the following stages for this type of singularity and also for type D_n, n odd, and for types E_6, E_7 and E_8. In the next table, we list the discrepancies. We use notations analogous to our notations from section 2, but they differ from the notations in [DR]. The coefficients that we have corrected are in boldface.

Singularity		Discrepancy
A_n	n even $n = 2k$ $k \geq 1$	$\sum_{i=1}^{k} iD_i + (n+2)D_{k+1}$
	n odd $n = 2k-1$ $k \geq 1$	$\sum_{i=1}^{k} iD_i$
D_n	n even $n = 2k$ $k \geq 2$	$\sum_{i=1}^{k-1} \big(iD_i' + iD_i'' + \mathbf{2i}E_i\big) + \mathbf{k}F_1 + \mathbf{k}F_2$
	n odd $n = 2k+1$ $k \geq 2$	$\sum_{i=1}^{k-1} \big(iD_i' + iD_i'' + \mathbf{2i}E_i\big) + \mathbf{k}F_1 + \mathbf{2k}F_2$

E_6	$D_1' + D_1'' + \mathbf{2}D_2 + \mathbf{4}D_3 + \mathbf{6}D_4$
E_7	$C_1' + C_1'' + 2D_1' + 2D_1'' + 4D_2' + 4D_2''$ $+\mathbf{3}E_1 + \mathbf{7}E_2 + \mathbf{6}F_1 + \mathbf{5}F_2$
E_8	$B_1' + B_1'' + 2C_1' + 2C_1'' + 4D_1' + 4D_1'' + 7D_2' + 7D_2''$ $+\mathbf{6}E_1 + \mathbf{12}E_2 + \mathbf{10}F_1 + \mathbf{8}F_2$

Remark. Dais and Roczen used their results to contradict a conjecture of Batyrev about the range of the string-theoretic index (that is the denominator of the stringy Euler number, see [Ba2, Conjecture 5.9], [DR, Remark 1.9]). Luckily, this follows already from the formulae for the A case, to which we do not correct anything. We will only simplify their formulae in this case.

(2.3.2) Now we consider the higher dimensional case. As an example, we compute the discrepancy coefficient of the divisor E_i for an $(m-1)$-dimensional D_n singularity, where $n = 2k$ is even, $i \in \{1, \ldots, k-1\}$ and $m \geq 5$. Let X be the defining variety $\{x_1^{n-1} + x_1 x_2^2 + x_3^2 + \cdots + x_m^2 = 0\} \subset \mathbb{A}^m$, and let $\varphi : \widetilde{X} \to X$ be the log resolution constructed in Section 2.1. We take a coordinate chart that covers a piece of E_i; in the notation of Section 2.1, this could be for example chart $(\underbrace{1.1\ldots1}_{i-1 \text{ times}}.2.3)$ describing an open affine set $U \subset \widetilde{X}$:

$$y_1^{2k-2i+1} y_2^{2k-2i-1} y_3^{4k-4i-2} + y_1 y_2 + 1 + y_4^2 + \cdots + y_m^2 = 0.$$

In this chart, $y_1 = 0$ gives a local equation for divisor D_{i-1}, $y_2 = 0$ for D_i and $y_3 = 0$ for our divisor E_i. The map $\varphi : U \to X$ can be found from the resolution process. Here it will be

$$\varphi(y_1, \ldots, y_m) = (y_1 y_2 y_3^2, y_1^{i-1} y_2^i y_3^{2i-1}, y_1^{i-1} y_2^i y_3^{2i}, y_1^{i-1} y_2^i y_3^{2i} y_4, \ldots, y_1^{i-1} y_2^i y_3^{2i} y_m).$$

The section $\frac{dx_1 \wedge \ldots \wedge dx_{m-1}}{2x_m}$ is locally a generator of the sheaf $\mathcal{O}_X(K_X)$ ($2x_m = \frac{\partial f}{\partial x_m}$, where f is the equation of X) and we have to compare its pull-back under φ with the generator $\frac{dy_1\wedge\ldots\wedge dy_{m-1}}{2y_m}$ of $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})|_U$. We have

$$\varphi^*(\frac{dx_1\wedge\ldots\wedge dx_{m-1}}{2x_m}) = y_1^{(i-1)(m-3)}y_2^{i(m-3)}y_3^{2i(m-3)}\frac{dy_1\wedge\ldots\wedge dy_{m-1}}{2y_m},$$

which learns us that the discrepancy coefficient of E_i is $2i(m-3)$. And we get the discrepancy coefficient of D_i for free, it is $i(m-3)$. In general, the following can be proved by this kind of calculations.

Proposition. *For all divisors that are created after a point blow-up, except for divisor $D_{\frac{n}{2}+1}$ in the A_n (n even) case, the discrepancy coefficient is $m-3$ times the coefficient of the corresponding divisor(s) in the 3-dimensional case (see the table in (2.3.1)).*

What about the other divisors ? They are all created after blowing up a nonsingular surrounding variety in a point (case A_n, n even) or a line (other cases). We consider again the case of a D_n singularity, with n even. Denote by $X^{(i)}$ the variety obtained after $n+i$ steps in the resolution process of Section 2.1 ($i\in\{0,\ldots,k-2\}$). The log resolution $\varphi:\widetilde{X}\to X$ can be decomposed as follows:

$$\widetilde{X}\ \xrightarrow{\chi^{(i+1)}}\ X^{(i+1)}\ \xrightarrow{\varphi^{(i+1)}}\ X^{(i)}\ \xrightarrow{\psi^{(i)}}\ X,$$

where $\varphi^{(i+1)}$ is the blow-up of the singular line on the divisor $D_{i+1}\subset X^{(i)}$ and where $\chi^{(i+1)}$ and $\psi^{(i)}$ are compositions of other blow-ups. Note that all the singular lines on $X^{(0)}$ are disjoint. Thus, to compute the discrepancy coefficient of G_{i+1}, it suffices to look at its coefficient in $K_{X^{(i+1)}}-(\psi^{(i)}\circ\varphi^{(i+1)})^*(K_X)$. This is equal to

$$K_{X^{(i+1)}}-(\varphi^{(i+1)})^*((\psi^{(i)})^*(K_X)-K_{X^{(i)}})-(\varphi^{(i+1)})^*(K_{X^{(i)}}).$$

It follows from [GH, p.608] that the last term is $-K_{X^{(i+1)}}+(m-3)G_{i+1}$ ($X^{(i)}$ is nonsingular !). And in the second term we only get a nonzero coefficient for G_{i+1} from $-(\varphi^{(i+1)})^*(-(i+1)(m-3)D_{i+1})$ (this follows from [GH, p.605], and the exact coefficient is $2(i+1)(m-3)$ because the multiplicity of a generic point of the singular line on D_{i+1} is 2). This gives us $2(i+1)(m-3)+(m-3)=(2i+3)(m-3)$ as discrepancy coefficient for G_{i+1}. In all other cases where we blow up in a line, the multiplicity of a generic point of the singular line will also be 2 and thus we have the following proposition.

Proposition. *For all divisors that are created after a blow-up in a singular line of another divisor D, the discrepancy coefficient is*

$$2(\textit{discrepancy coefficient of } D) + (m-3).$$

The reader may check that the same arguments give $(n+1)(m-3)+1$ as coefficient for $D_{\frac{n}{2}+1}$ in the case A_n, n even.

These results lead to the observation that any variety with at most A-D-E singularities satisfies condition $(*)$ (see Definition (1.2.1)), and thus we have the following theorem, as a corollary of Theorem (1.2.2).

Theorem. *Let Y be a projective variety with at most A-D-E singularities. Then the answer to Question (0.2.5) is positive for Y.*

In the next section we compute explicit formulae for the contribution of an A-D-E singularity to the stringy E-function and these formulae will enable us to say which varieties with A-D-E singularities have a *polynomial* stringy E-function.

2.4 Contribution of an A-D-E singularity to the stringy E-function

(2.4.1) Let X be a defining variety of an A-D-E singularity; hence X is a hypersurface in $\mathbb{A}^m$ $(m \geq 3)$ with a singular point in the origin. By the contribution of the singular point to the stringy E-function, we mean $E_{st}(X) - H(X \setminus \{0\})$. Before stating the formulae, we first remark that we have to make a distinction between m even and m odd, because the required Hodge-Deligne polynomials depend on the parity of the dimension.

Theorem. *The contributions of the $(m-1)$-dimensional A-D-E singularities $(m \geq 3)$ are given in the following tables (where sums like $\sum_{i=2}^{k}$ must be interpreted as 0 for $k=1$).*

Singularity		*Contribution of singular point for odd m*
A_n	n even, $n=2k$, $k\geq 1$	$1+\frac{(w-1)}{(w^{(2k+1)(m-3)+2}-1)}\left(\sum_{i=2}^{k+1} w^{(k+i)(m-3)+2} + \sum_{i=1}^{k} w^{(k+i)(m-3)+\frac{m+1}{2}} + \sum_{i=1}^{k} w^{i(m-3)+\frac{m-1}{2}} + \sum_{i=1}^{k} w^{i(m-3)+1}\right)$
	n odd, $n=2k-1$, $k\geq 1$	$1+\frac{(w-1)}{(w^{k(m-3)+1}-1)}\left(\sum_{i=1}^{k} w^{i(m-3)+1} + \sum_{i=1}^{k-1} w^{i(m-3)+\frac{m-1}{2}}\right)$
D_n	n even, $n=2k$, $k\geq 2$	$1+\frac{(w-1)}{(w^{(2k-1)(m-3)+1}-1)}\left(\sum_{i=1}^{2k-1} w^{i(m-3)+1} + w^{k(m-3)+1}\right)$
	n odd, $n=2k+1$, $k\geq 2$	$1+\frac{(w-1)}{(w^{2k(m-3)+1}-1)}\left(\sum_{i=1}^{2k} w^{i(m-3)+1} + w^{k(m-3)+\frac{m-1}{2}}\right)$
E_6		$1+\frac{(w-1)}{(w^{6m-17}-1)}\left(w^{6m-17}+w^{4m-11}+w^{3m-8} + w^{m-2}+w^{\frac{9m-25}{2}}+w^{\frac{5m-13}{2}}\right)$
E_7		$1+\frac{(w-1)}{(w^{9m-26}-1)}\left(w^{9m-26}+w^{7m-20}+w^{6m-17}+w^{5m-14} + w^{4m-11}+w^{3m-8}+w^{m-2}\right)$
E_8		$1+\frac{(w-1)}{(w^{15m-44}-1)}\left(w^{15m-44}+w^{12m-35}+w^{10m-29}+w^{9m-26} + w^{7m-20}+w^{6m-17}+w^{4m-11}+w^{m-2}\right)$

Singularity		*Contribution of singular point for even m*
A_n	n even, $n=2k$, $k\geq 1$	$1+\frac{(w-1)}{(w^{(2k+1)(m-3)+2}-1)}\left(\sum_{i=2}^{k+1} w^{(k+i)(m-3)+2} + \sum_{i=1}^{k} w^{i(m-3)+1}\right)$

D_n	n *odd* $n=2k-1$ $k\geq 1$	$1+\frac{(w-1)}{(w^{k(m-3)+1}-1)}\left(\sum_{i=1}^{k} w^{i(m-3)+1}+w^{\frac{m}{2}-1}\right)$
	n *even* $n=2k$ $k\geq 2$	$1+\frac{(w-1)}{(w^{(2k-1)(m-3)+1}-1)}\left(\sum_{i=1}^{2k-1} w^{i(m-3)+1}+w^{k(m-3)+1} + \sum_{i=0}^{k-2} w^{(k+i)(m-3)+\frac{m}{2}}+\sum_{i=0}^{k-1} w^{i(m-3)+\frac{m}{2}-1}+w^{\frac{m}{2}-1}\right)$
	n *odd* $n=2k+1$ $k\geq 2$	$1+\frac{(w-1)}{(w^{2k(m-3)+1}-1)}\left(\sum_{i=1}^{2k} w^{i(m-3)+1}+\sum_{i=1}^{k-1} w^{(k+i)(m-3)+\frac{m}{2}} + \sum_{i=0}^{k-1} w^{i(m-3)+\frac{m}{2}-1}\right)$
E_6		$1+\frac{(w-1)}{(w^{6m-17}-1)}\left(w^{6m-17}+w^{4m-11}+w^{3m-8} + w^{m-2}+w^{\frac{11m-30}{2}}+w^{\frac{3m-8}{2}}\right)$
E_7		$1+\frac{(w-1)}{(w^{9m-26}-1)}\left(w^{9m-26}+w^{7m-20}+w^{6m-17}+w^{5m-14} + w^{4m-11}+w^{3m-8}+w^{m-2}+w^{\frac{17m-48}{2}}+w^{\frac{15m-42}{2}}+w^{\frac{11m-30}{2}} + w^{\frac{9m-26}{2}}+w^{\frac{5m-14}{2}}+w^{\frac{3m-8}{2}}+w^{\frac{m-2}{2}}\right)$
E_8		$1+\frac{(w-1)}{(w^{15m-44}-1)}\left(w^{15m-44}+w^{12m-35}+w^{10m-29} + w^{9m-26}+w^{7m-20}+w^{6m-17}+w^{4m-11}+w^{m-2} + w^{\frac{29m-84}{2}}+w^{\frac{27m-78}{2}}+w^{\frac{23m-66}{2}}+w^{\frac{17m-48}{2}} + w^{\frac{15m-44}{2}}+w^{\frac{9m-26}{2}}+w^{\frac{5m-14}{2}}+w^{\frac{3m-8}{2}}\right)$

Proof. Let us first consider the case where $m\geq 5$. We will focus again on the singularity of type D_n for $n=2k$ and also for even m. All the other cases are completely analogous. We just insert the data from Sections 2.1, 2.2 and 2.3 in the defining formula of the stringy E-function and we find the following formula for the contribution of the singularity:

$$\frac{(w^{m-1}-w^2+w^{\frac{m+2}{2}}-w^{\frac{m}{2}})}{(w^{m-2}-1)}+\sum_{i=2}^{k-1}\frac{(w^{m-2}-w+w^{\frac{m}{2}}-w^{\frac{m-2}{2}})(w-1)}{(w^{i(m-3)+1}-1)}+\frac{w^{m-2}(w-1)}{(w^{2m-5}-1)}$$

$$
\begin{aligned}
&+\sum_{i=2}^{k-1}\frac{(w^{m-2}-w^{m-3}-w^{\frac{m-2}{2}}+w^{\frac{m-4}{2}})(w-1)}{(w^{2i(m-3)+1}-1)}+\frac{2w^{m-2}(w-1)}{(w^{k(m-3)+1}-1)}\\
&+\sum_{i=1}^{k-2}\frac{(w^{m-2}-w^{m-3}-w^{\frac{m-2}{2}}+w^{\frac{m-4}{2}})(w-1)}{(w^{(2i+1)(m-3)+1}-1)}\\
&+\frac{(w^{m-2}-2w^{m-3}-w^{\frac{m-2}{2}}+2w^{\frac{m-4}{2}})(w-1)}{(w^{(2k-1)(m-3)+1}-1)}\\
&+\sum_{i=1}^{k-2}\frac{(w^{m-2}-w+w^{\frac{m}{2}}-w^{\frac{m-2}{2}})(w-1)}{(w^{i(m-3)+1}-1)(w^{(i+1)(m-3)+1}-1)}+\frac{(w^{m-2}-w+w^{\frac{m}{2}}-w^{\frac{m-2}{2}})(w-1)}{(w^{m-2}-1)(w^{2m-5}-1)}\\
&+\sum_{i=2}^{k-1}\frac{(w^{m-3}-1+w^{\frac{m-2}{2}}-w^{\frac{m-4}{2}})(w-1)^2}{(w^{i(m-3)+1}-1)(w^{2i(m-3)+1}-1)}+\sum_{i=1}^{k-2}\frac{(w^{m-3}-1+w^{\frac{m-2}{2}}-w^{\frac{m-4}{2}})(w-1)^2}{(w^{i(m-3)+1}-1)(w^{(2i+2)(m-3)+1}-1)}\\
&+\frac{2(w^{m-2}-w+w^{\frac{m}{2}}-w^{\frac{m-2}{2}})(w-1)}{(w^{(k-1)(m-3)+1}-1)(w^{k(m-3)+1}-1)}+\sum_{i=1}^{k-2}\frac{(w^{m-3}-1+w^{\frac{m-2}{2}}-w^{\frac{m-4}{2}})(w-1)^2}{(w^{i(m-3)+1}-1)(w^{(2i+1)(m-3)+1}-1)}\\
&+\frac{(w^{m-2}-2w^{m-3}-w+2+w^{\frac{m}{2}}-3w^{\frac{m-2}{2}}+2w^{\frac{m-4}{2}})(w-1)}{(w^{(k-1)(m-3)+1}-1)(w^{(2k-1)(m-3)+1}-1)}\\
&+\sum_{i=1}^{k-1}\frac{(w^{m-3}-w^{\frac{m-4}{2}})(w-1)^2}{(w^{2i(m-3)+1}-1)(w^{(2i+1)(m-3)+1}-1)}\\
&+\sum_{i=1}^{k-2}\frac{(w^{m-3}-w^{\frac{m-4}{2}})(w-1)^2}{(w^{(2i+2)(m-3)+1}-1)(w^{(2i+1)(m-3)+1}-1)}+\frac{2(w^{m-3}-w^{\frac{m-4}{2}})(w-1)^2}{(w^{k(m-3)+1}-1)(w^{(2k-1)(m-3)+1}-1)}\\
&+\sum_{i=1}^{k-2}\frac{(w^{m-3}-1+w^{\frac{m-2}{2}}-w^{\frac{m-4}{2}})(w-1)^2}{(w^{i(m-3)+1}-1)(w^{(i+1)(m-3)+1}-1)(w^{(2i+2)(m-3)+1}-1)}\\
&+\sum_{i=1}^{k-1}\frac{(w^{m-3}-1+w^{\frac{m-2}{2}}-w^{\frac{m-4}{2}})(w-1)^2}{(w^{i(m-3)+1}-1)(w^{2i(m-3)+1}-1)(w^{(2i+1)(m-3)+1}-1)}\\
&+\sum_{i=1}^{k-2}\frac{(w^{m-3}-1+w^{\frac{m-2}{2}}-w^{\frac{m-4}{2}})(w-1)^2}{(w^{i(m-3)+1}-1)(w^{(2i+2)(m-3)+1}-1)(w^{(2i+1)(m-3)+1}-1)}\\
&+\frac{2(w^{m-3}-1+w^{\frac{m-2}{2}}-w^{\frac{m-4}{2}})(w-1)^2}{(w^{(k-1)(m-3)+1}-1)(w^{k(m-3)+1}-1)(w^{(2k-1)(m-3)+1}-1)}.
\end{aligned}
$$

The terms correspond to the following pieces of the exceptional locus (in that order):

$$
\begin{aligned}
&D_1^\circ, D_i^\circ, E_1^\circ, E_i^\circ, F_i^\circ, G_i^\circ, G_{k-1}^\circ, (D_i\cap D_{i+1})^\circ, (D_1\cap E_1)^\circ, (D_i\cap E_i)^\circ,\\
&(D_i\cap E_{i+1})^\circ, (D_{k-1}\cap F_i)^\circ, (D_i\cap G_i)^\circ, (D_{k-1}\cap G_{k-1})^\circ, (E_i\cap G_i)^\circ,\\
&(E_{i+1}\cap G_i)^\circ, (F_i\cap G_{k-1})^\circ, D_i\cap D_{i+1}\cap E_{i+1}, D_i\cap E_i\cap G_i,\\
&D_i\cap E_{i+1}\cap G_i, D_{k-1}\cap F_i\cap G_{k-1}.
\end{aligned}
$$

By a very long but easy calculation, it can be proved by induction on k that we indeed get the requested formula. We remark here that we have done the computations for $m\geq 5$, for $m=4$ and for $m=3$ separately, and then noticed that the formulae for $m \geq 5$ are correct in the other cases too.

We can now explain why these formulae are also valid for $m = 4$. For the A_n case, this is not a surprise, since the intersection diagram for $m = 4$ is the same as for $m \geq 5$. For the other cases, consider for example a singularity of type D_n, n even. The blow-ups in the singular lines on the divisors D_i in the higher dimensional case correspond here to blow-ups in the intersections $D_i' \cap D_i''$. Performing these unnecessary extra blow-ups yields just another log resolution, and the formula for the contribution of the singularity for that log resolution will be exactly the evaluation of the formula from the first part of the proof for $m = 4$ (notice for instance that the Hodge-Deligne polynomial for D_i° $(i > 1)$ becomes $2w^2 - 2w$ for $m = 4$ and the Hodge-Deligne polynomials for $(D_i')^\circ$ and $(D_i'')^\circ$ will both be $w^2 - w$).

For $m = 3$ it can be checked easily that the formulae are correct but again we give a more conceptual explanation. Compared with the higher dimensional case, all divisors except the last one split into two (distinct) components in the A_n case, for odd n. This is consistent with the Hodge-Deligne polynomials from (2.2.3), evaluated for $m = 3$. For even n, we must notice that the last blow-up is unnecessary for surfaces; performing it anyway does not yield a crepant resolution any more (the last divisor has discrepancy coefficient 1, as it should be, according to (2.3.2)). This last divisor is irreducible and the first $\frac{n}{2}$ blow-ups each add two components to the exceptional locus (compare this with (2.2.3) again). For the D_n case, the analogue of blowing up in a singular line on a divisor D_i would be to blow up in D_i itself, because it is just a line for $m = 3$. Such a blow-up is an isomorphism, and the result is that the divisors D_i are renamed as G_i. As intersection diagram one finds the same as in the higher dimensional case, but without the divisors D_i. To be able to compare this to (2.2.4), we must notice that it is logical to set $a_1 = w + 1$, $c_1 = 0$ and $b_1 = 1$ in (2.2.1). Then indeed all Hodge-Deligne polynomials that describe a piece of a divisor D_i are 0 in (2.2.4) for $m = 3$. For the E cases the same sort of arguments apply. ■

(2.4.2) In the next proposition we exclude the two-dimensional case, since it is trivial.

Proposition. *The stringy E-function of a projective variety X of dimension ≥ 3 with at most A-D-E singularities is a polynomial if and only if* $\dim X = 3$ *and X has singularities of type A_n (n odd) and/or D_l (l even).*

Proof. It follows from Theorem (2.4.1) that the contributions of the singular points for $m \geq 5$ can be written in the following form:

$$1 + \frac{w^2(w^\alpha + a_{\alpha-1}w^{\alpha-1} + \cdots + a_0)}{w^{\alpha+1} + w^\alpha + \cdots + 1},$$

where $\alpha \in \mathbb{Z}_{>0}$ and all $a_i \in \mathbb{Z}_{\geq 0}$. Such expressions or finite sums of such expressions can never be polynomials. For $m = 4$ the contributions are given in the following table.

Singularity		Contribution of singular point
A_n	n even $n = 2k$	$1 + \frac{w^2(w^{2k+2} - w^{k+2} + w^k - 1)}{w^{2k+3} - 1}$
	n odd $n = 2k - 1$	$w + 1$
D_n	n even $n = 2k$	$2w + 1$
	n odd $n = 2k + 1$	$w + 1 + \frac{w^2(w^{2k} - w^{k+1} + w^{k-1} - 1)}{w^{2k+1} - 1}$
E_6		$1 + \frac{w^2(2w^6 - 2w^5 + w^4 - w^2 + 2w - 2)}{w^7 - 1}$
E_7		$w + 1 + \frac{w^2(w^4 - w^3 + w - 1)}{w^5 - 1}$

E_8	$1+\frac{w^2(2w^7-w^6-w^5+2w^4-2w^3+w^2+w-2)}{w^8-1}$

There are exactly two contributions that are polynomials and one sees again that adding a finite number of the non-polynomial expressions never gives a polynomial. ∎

Remark. It is no surprise that the A_n singularities for odd n and the D_l singularities for even l give a polynomial contribution for $m = 4$. This can be seen by the following well known construction (originally from Atiyah, see [At, Section 3]). An A_n singularity can be described as the origin of the hypersurface $X = \{x_1^{n+1} - x_2^2 = x_3x_4\}$ in $\mathbb{A}^4$. Instead of blowing up in the singular point, one can use the divisor $\{x_1^k - x_2 = x_3 = 0\}$ as center for a blow up, with $k = \frac{n+1}{2}$. The result is a proper birational morphism $f : Y \to X$ from a smooth variety Y such that $f : Y \setminus f^{-1}(\mathbf{0}) \to X \setminus \{\mathbf{0}\}$ is an isomorphism. And $f^{-1}(\mathbf{0})$ is only a (rational) curve! Such a resolution is called 'small'. In particular, f is crepant, and thus $E_{st}(X) = E_{st}(Y) = H(Y)$. In this way we also find $w + 1$ as the contribution of the singular point. For a D_l singularity (l even) you have to perform two of such blow-ups to find a crepant morphism from a nonsingular variety. The inverse image of the singular point consists in that case of two intersecting rational curves, leading to a contribution of $2w + 1$. Note that the above table also shows that the other 3-dimensional A-D-E singularities do not admit a small resolution (this gives for instance another proof of [Re2, Corollary (1.16)]).

Example. Consider the variety $X = \{xyz + t^3 + s^3 = 0\} \subset \mathbb{P}^4$, where we use coordinates (x, y, z, t, s). It is clear that the points $(1, 0, 0, 0, 0)$, $(0, 1, 0, 0, 0)$ and $(0, 0, 1, 0, 0)$ are 3-dimensional D_4 singularities. Thus, their contribution to the stringy E-function of X is $3(2w+1)$. To compute the Hodge-Deligne polynomial of X, we divide X in three locally closed pieces:

$$X = (X \cap \{x \neq 0, y \neq 0\}) \sqcup (X \cap \{x \neq 0, y = 0\}) \sqcup (X \cap \{x = 0\}).$$

The Hodge-Deligne polynomial of the first piece is just $(w - 1)w^2$ since y, z, t, s have become affine coordinates and y, t, s can be chosen freely, with $y \neq 0$. The second piece consists of three planes in $\mathbb{A}^3$, intersecting in a line and has Hodge-Deligne polynomial $3(w^2 - w) + w$ and the third piece are three planes in $\mathbb{P}^3$, intersecting in a line, with contribution $3w^2 + w + 1$. Thus $H(X) = w^3 + 5w^2 - w + 1$ and the Hodge-Deligne polynomial of the nonsingular part is $w^3 + 5w^2 - w - 2$. It follows that the stringy E-function of X is equal to $w^3 + 5w^2 + 5w + 1$ and thus the stringy Hodge numbers of X are indeed nonnegative.

Chapter 3

Stringy E-functions of hypersurfaces

Abstract

The stringy E-function for canonical hypersurfaces can be obtained from the motivic zeta function of Denef and Loeser. This is in fact a nice application of the inversion of adjuction theorem by Ein, Mustaţă and Yasuda. If an affine hypersurface is given by a polynomial that is non-degenerate with respect to its Newton polyhedron, then the motivic zeta function and thus the stringy E-function can be computed from this Newton polyhedron (by work of Artal, Cassou-Noguès, Luengo and Melle based on an algorithm of Denef and Hoornaert). We use this to check the formulae for the A-D-E singularities from Chapter 2.

3.1 The motivic zeta function

(3.1.1) First we recall the definition of the Grothendieck group of complex algebraic varieties, denoted $K_0(Var_{\mathbb{C}})$. It is the abelian group generated by the symbols $[X]$, where X is a complex algebraic variety (not necessarily irreducible), and with the following relations:

- if X is isomorphic to Y, then $[X] = [Y]$,
- if Y is a Zariski closed subset of X, then $[X] = [X \setminus Y] + [Y]$.

There is a product structure making $K_0(Var_{\mathbb{C}})$ into a ring, defined by $[X] \cdot [Y] = [X \times Y]$. Thus the Grothendieck ring is the value ring of the 'universal Euler characteristic' on algebraic varieties. The class of the affine line $\mathbb{A}^1$ is usually denoted by $\mathbb{L}$; the class of a point is the unity 1. The localization of $K_0(Var_{\mathbb{C}})$ with respect to the element $\mathbb{L}$ is denoted by $\mathcal{M}_{\mathbb{C}}$.

(3.1.2) Let X be an algebraic variety over $\mathbb{C}$ (not necessarily irreducible). The space of arcs modulo t^{n+1} or the space of n-jets on X is an algebraic variety $\mathcal{L}_n(X)$ such that

$$\begin{aligned}&\{\text{points of } \mathcal{L}_n(X) \text{ with coordinates in } \mathbb{C}\}\\&\quad = \{\text{points of } X \text{ with coordinates in } \frac{\mathbb{C}[t]}{(t^{n+1})}\}.\end{aligned}$$

If X is an affine variety given by equations $f_i(x_1,\ldots,x_m) = 0$ for $i = 1,\ldots,k$, then $\mathcal{L}_n(X)$ is given by the equations in the variables $a_0^{(1)},\ldots,a_n^{(1)},a_0^{(2)},\ldots,a_n^{(2)},\ldots,a_0^{(m)},\ldots,a_n^{(m)}$ expressing that

$$f_i(a_0^{(1)}+a_1^{(1)}t+\cdots+a_n^{(1)}t^n,\ldots,a_0^{(m)}+a_1^{(m)}t+\cdots+a_n^{(m)}t^n) \equiv 0 \mod t^{n+1},$$

for all i. Note that $\mathcal{L}_0(X) = X$. There are natural truncation maps $\pi_n^m : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$ for $m \geq n$. The image $\pi_0^n(\gamma)$ of an n-jet γ is called the origin of γ. One also considers the space of arcs $\mathcal{L}(X)$ on X, it is the inverse limit of the $\mathcal{L}_n(X)$ (this is not finite dimensional if $\dim X > 0$ and is thus not a 'true' algebraic variety).

(3.1.3) Now we come to the definition of the motivic zeta function by Denef and Loeser. Let X be a smooth complex algebraic variety of dimension d and let $f : X \to \mathbb{A}^1$ be a non-constant morphism. This morphism induces morphisms $f_n : \mathcal{L}_n(X) \to \mathcal{L}_n(\mathbb{A}^1)$ for every $n \geq 1$. A point α of $\mathcal{L}_n(\mathbb{A}^1)$ corresponds to an element $\alpha(t) \in \frac{\mathbb{C}[t]}{(t^{n+1})}$ and thus one can consider the function

$$\mathrm{ord}_t : \mathcal{L}_n(\mathbb{A}^1) \to \{0,1,\ldots,n,\infty\},$$

which maps $\alpha(t)$ to the lowest power of t with nonzero coefficient, with $\mathrm{ord}_t(0) = \infty$. Define the set $\mathcal{X}_n$ as

$$\{\gamma \in \mathcal{L}_n(X) \,|\, \mathrm{ord}_t f_n(\gamma) = n\}.$$

This is a locally closed subvariety of $\mathcal{L}_n(X)$. In [DL3, Definition 3.2.1] Denef and Loeser define the following power series over $\mathcal{M}_{\mathbb{C}}$:

$$Z_f(T) := \sum_{n\geq 1} [\mathcal{X}_n]\mathbb{L}^{-nd}T^n.$$

They call it the *naive motivic zeta function* of f; we will just call it the motivic zeta function of f here. For a point $x \in f^{-1}(0)$ there exists also a local version

$$Z_{loc,x,f}(T) := \sum_{n \geq 1} [\mathcal{X}_n \cap (\pi_0^n)^{-1}(x)] \mathbb{L}^{-nd} T^n,$$

obtained by restricting to jets with origin in x. The definition of the motivic zeta function is in fact inspired by its number theoretic analogue: Igusa's p-adic zeta function (see [DL2] for more information).

Denef and Loeser express these power series in terms of an embedded resolution of $f^{-1}(0)$ as follows. Let $h : Y \to X$ be an embedded resolution of $f^{-1}(0)$. So Y is a nonsingular variety, h is a proper birational morphism, the restriction $h : Y \setminus h^{-1}(f^{-1}(0)) \to X \setminus f^{-1}(0)$ is an isomorphism and $h^{-1}(f^{-1}(0))$ is a divisor with smooth irreducible components and normal crossings on Y. Denote the irreducible components of $h^{-1}(f^{-1}(0))$ by $E_i, i \in I$, and for a subset $J \subset I$, write $E_J := \cap_{i \in J} E_i$ and $E_J^\circ := E_J \setminus (\cup_{i \in I \setminus J} E_i)$. Let N_i be the multiplicity of E_i in the divisor of $f \circ h$ and let $\nu_i - 1$ be the multiplicity of E_i in the divisor of $h^* dx$, where dx is a local generator of the sheaf of differential forms of maximal degree on X. These two numbers are called the *numerical data* of E_i. The motivic zeta function is then equal to ([DL3, Corollary 3.3.2], see also [DL2, Theorem 2.2.1])

$$Z_f(T) = \sum_{\emptyset \neq J \subset I} [E_J^\circ] \prod_{i \in J} \frac{(\mathbb{L} - 1)}{\mathbb{L}^{\nu_i} T^{-N_i} - 1}, \tag{1}$$

where this equality must be interpreted in $\mathcal{M}_{\mathbb{C}}[[T]]$. In particular, this formula does not depend on the chosen embedded resolution. For the local motivic zeta function the formula becomes

$$Z_{loc,x,f}(T) = \sum_{\emptyset \neq J \subset I} [E_J^\circ \cap h^{-1}(x)] \prod_{i \in J} \frac{(\mathbb{L} - 1)}{\mathbb{L}^{\nu_i} T^{-N_i} - 1}. \tag{2}$$

The motivic zeta function contains an enormous amount of information. It specializes for example to the topological zeta function (defined by Denef and Loeser in [DL1]), by 'replacing' T by $\mathbb{L}^{-s}$ and applying the topological Euler characteristic (see [DL3, Section 3.4]). In the next section we will see what the connection with Batyrev's stringy E-function is.

3.2 The Hodge zeta function and the stringy E-function

(3.2.1) Let X be a smooth complex algebraic variety and let $f : X \to \mathbb{A}^1$ be a non-constant morphism. The Hodge-Deligne polynomial $H(\,\cdot\,)$ can be seen as a ring morphism from $K_0(Var_{\mathbb{C}})$ to $\mathbb{Z}[u,v]$, since it is a generalized Euler characteristic. This induces a ring morphism from $\mathcal{M}_{\mathbb{C}}[[T]]$ to $\mathbb{Q}(u,v)[[T]]$, and since $H(\mathbb{L}) = uv$, this morphism applied to (1) from the previous section gives

$$\sum_{\emptyset \neq J \subset I} H(E_J^\circ; u, v) \prod_{i \in J} \frac{(uv-1)}{(uv)^{\nu_i} T^{-N_i} - 1}.$$

We will denote this element of $\mathbb{Q}(u,v)[[T]]$ (even of $\mathbb{Q}(u,v)[[T]] \cap \mathbb{Q}(u,v,T)$) by $\mathcal{H}_f(T)$. It is called the *Hodge zeta function*, see for example [Rod].

(3.2.2) Proposition.[1] *Let X be a smooth algebraic variety of dimension d and let $f : X \to \mathbb{A}^1$ be a non-constant morphism such that $X_0 = f^{-1}(0)$ is irreducible, normal and canonical (recall that a hypersurface is automatically Gorenstein). Then*

$$E_{st}(X_0; u, v) = -\frac{1}{uv(uv-1)} (\mathcal{H}_f(T)(T - uv))|_{T=uv},$$

where the evaluation in $T = uv$ makes sense, since the denominator of $\mathcal{H}_f(T)$ contains the factor $T - uv$ only once.

Remark. The stringy E-function for a hypersurface can thus be seen as a 'residue' of the Hodge zeta function.

Proof. Let $h : Y \to X$ be an embedded resolution of X_0, with $E_i, i \in I$ the irreducible components of $h^{-1}(X_0)$ and such that $h : h^{-1}(X_0) \to X_0$ is an isomorphism outside the singular locus Sing X_0 of X_0. For a component E_i that intersects the strict transform $\widetilde{X_0}$ of X_0 (with $E_i \neq \widetilde{X_0}$) we can look at the numerical data (ν_i, N_i) of the embedded resolution, but also at the discrepancy a_i of $\widetilde{X_0} \cap E_i$ for $h|_{\widetilde{X_0}} : \widetilde{X_0} \to X_0$ (this is actually a log resolution!). Then we claim that $a_i + 1 = \nu_i - N_i$. Denote by $J \subset I$ the index set of the components $E_i \neq \widetilde{X_0}$ having nonempty intersection

[1] This proposition was communicated to me by W. Veys.

with $\widetilde{X_0}$, and by J' the index set of all components different from $\widetilde{X_0}$. Let $f : X_0 \hookrightarrow X$ and $g : \widetilde{X_0} \hookrightarrow Y$ be the inclusions. We have

$$K_Y = h^*(K_X) + \sum_{i\in I}(\nu_i - 1)E_i$$

and

$$K_{\widetilde{X_0}} = h|^*_{\widetilde{X_0}}(K_{X_0}) + \sum_{i\in J} a_i(E_i \cap \widetilde{X_0})$$

By the adjunction formula, this latter is also equal to

$$g^*(K_Y + \widetilde{X_0}) = g^*(h^*(K_X) + \sum_{i\in I}(\nu_i - 1)E_i) + g^*(h^*(X_0) - \sum_{i\in J'} N_iE_i)$$

$$= h|^*_{\widetilde{X_0}}(f^*(K_X + X_0)) + \sum_{i\in J}(\nu_i - 1 - N_i)(E_i \cap \widetilde{X_0}),$$

and then applying the adjunction formula once more proves the claim.

For $\widetilde{X_0}$ itself, the numerical data are $(1, 1)$. So the terms of $\mathcal{H}_f(T)$ containing a piece of the Hodge-Deligne polynomial of $\widetilde{X_0}$ assure that the 'residue' of these terms is indeed the stringy E-function (modulo the correction $-\frac{1}{uv(uv-1)}$), since X_0 is canonical and all the $a_i + 1 = \nu_i - N_i$ are thus ≥ 1. If we can show that no exceptional component E_i with $E_i \cap X_0 = \emptyset$ has $\nu_i - N_i = 0$, then we are done. This follows from the inversion of adjunction theorem for a smooth ambient variety by Ein, Mustaţă and Yasuda (see [EMY, Theorem 1.6], later the first two authors generalized this result to local complete intersection varieties in [EM]; see also [K+, Chapter 17] for more information on inversion of adjunction). In their terminology, the minimum of the $a_i + 1$ for $i \in J$ is called the minimal log discrepancy of the pair $(X_0, \emptyset)$ on Sing X_0. They show that it is equal to the minimal log discrepancy of the pair (X, X_0) on Sing X_0, and this is given by the minimum of the $\nu_i - N_i$, for all i with $E_i \neq \widetilde{X_0}$. This proves that all such $\nu_i - N_i \geq 1$. ■

(3.2.3) Remark. If X_0 has an isolated singularity in a point x and if we want to compute the contribution of this singular point to the stringy E-function, we can use the formula

$$-\frac{1}{uv(uv-1)}(\mathcal{H}_{loc,x,f}(T)(T - uv))|_{T=uv},$$

where $\mathcal{H}_{loc,x,f}(T)$ denotes the element of $\mathbb{Q}(u,v)[[T]]\cap\mathbb{Q}(u,v,T)$ obtained by applying the Hodge-Deligne polynomial to formula (2) from Section 3.1.

3.3 Motivic zeta function of non-degenerate affine hypersurfaces

(3.3.1) In this section we discuss the method of Artal, Cassou-Noguès, Luengo and Melle to compute the motivic zeta function of a polynomial that is non-degenerate with respect to its Newton polyhedron (see [ACLM, Chapter 2]). This method is essentially earlier work by Denef and Hoornaert for Igusa's p-adic zeta function ([DH]). First we need a lot of definitions about Newton polyhedra and polyhedral cones. Let $f : \mathbb{A}^d \to \mathbb{A}^1$ be a morphism with $f(\mathbf{0}) = 0$. (so f is just a polynomial $\sum_{\mathbf{n}\in(\mathbb{Z}_{\geq 0})^d} a_{\mathbf{n}}\mathbf{x}^{\mathbf{n}}$, where $\mathbf{x} = (x_1,\ldots,x_d), \mathbf{n} = (n_1,\ldots,n_d)$ and $\mathbf{x}^{\mathbf{n}} = (x_1^{n_1},\ldots,x_d^{n_d})$. The support of f is the set $\mathrm{supp}(f) = \{\mathbf{n} \in (\mathbb{Z}_{\geq 0})^d \,|\, a_{\mathbf{n}} \neq 0\}$. The Newton polyhedron $\Gamma(f)$ of f is the convex hull in $(\mathbb{R}^+)^d$ of

$$\bigcup_{\mathbf{n}\in\ \mathrm{supp}(f)} \mathbf{n} + (\mathbb{R}^+)^d.$$

For the definition of a face of the Newton polyhedron we refer to [Roc, p.162]. In particular, the Newton polyhedron itself is also considered as a face. A $(d-1)$-dimensional face of the Newton polyhedron is called a facet. For a face τ of $\Gamma(f)$, we denote $\sum_{\mathbf{n}\in\tau} a_{\mathbf{n}}\mathbf{x}^{\mathbf{n}}$ by f_τ. The polynomial f is called *non-degenerate at the origin* with respect to its Newton polyhedron if for every compact face τ the subvariety of $(\mathbb{A}^1 \setminus \{0\})^d$ given by $f_\tau = 0$ is nonsingular. It is called *non-degenerate* if the same is true for every face.

For $\mathbf{k} = (k_1,\ldots,k_d) \in \mathbb{R}^d$ set $m_f(\mathbf{k}) := \inf_{\mathbf{x}\in\Gamma(f)}\{\mathbf{k}\cdot\mathbf{x}\}$, with $\cdot$ the standard inner product. In fact this infimum is attained and is thus a minimum. The first meet locus of $\mathbf{k}$ is the set $F(\mathbf{k}) := \{\mathbf{x}\in\Gamma(f)\,|\,\mathbf{k}\cdot\mathbf{x} = m_f(\mathbf{k})\}$. This is a compact face of $\Gamma(f)$ if and only if $\mathbf{k} \in (\mathbb{R}^+ \setminus \{0\})^d$. For a face τ one defines the associated polyhedral cone $\Delta_\tau := \{\mathbf{k} \in (\mathbb{R}^+)^d\,|\,F(\mathbf{k}) = \tau\}$ in the dual space. It is well known that the cones associated with the compact faces form a partition of $(\mathbb{R}^+\setminus\{0\})^d$. A cone Δ is called a rational simplicial cone (of dimension e) if it is generated by

e linearly independent integer vectors $\beta_1, \dots, \beta_e$; thus

$$\Delta = \{\lambda_1\beta_1 + \cdots + \lambda_e\beta_e \mid \lambda_i \in \mathbb{R}^+ \setminus \{0\}\}.$$

Usually one allows the λ_i to be 0 in this definition, but for our goal it is more appropriate not to do that. We are interested in the set of integer points Δ' of such cones Δ:

$$\Delta' := \{\delta \in (\mathbb{Z}_{>0})^d \mid n\delta = \lambda_1\beta_1 + \cdots + \lambda_e\beta_e \text{ for some } n \in \mathbb{Z}_{>0} \text{ and } \lambda_i \in \mathbb{Z}_{>0}\}.$$

Let $\gamma_1, \dots, \gamma_e$ be obtained from the β_i by dividing by the greatest common divisor of the coordinates of β_i. Then we say that Δ' is strictly generated by $\gamma_1, \dots, \gamma_e$ and

$$G_{\Delta'} := \{\delta \in (\mathbb{Z}_{>0})^d \mid \delta = \lambda_1\gamma_1 + \cdots + \lambda_e\gamma_e, 0 < \lambda_i \leq 1\}$$

is called the fundamental set of Δ'.

Every point $\mathbf{k} \in (\mathbb{Z}_{>0})^d$ belongs to a unique cone Δ_τ associated to a compact face τ. Let $\sigma(\mathbf{k})$ be $k_1 + \cdots + k_d$. Artal, Cassou-Noguès, Luengo and Melle define the following term for a compact face τ of the Newton polyhedron of f (inspired by the work of Denef and Hoornaert):

$$S_{\Delta_\tau}(f, T) := \sum_{\mathbf{k} \in (\mathbb{Z}_{>0})^d \cap \Delta_\tau} \mathbb{L}^{-\sigma(\mathbf{k})} T^{m_f(\mathbf{k})}.$$

Note that this element does not need to belong to $\mathcal{M}_{\mathbb{C}}[[T]]$. Artal, Cassou-Noguès, Luengo and Melle show that it belongs to the ring ([ACLM, Lemma 2.1])

$$\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}, (1 - \mathbb{L}^{-\sigma(\mathbf{a})} T^{m_f(\mathbf{a})})^{-1}][T],$$

with $\mathbf{a}$ in the set of vectors such that $\mathbf{a} \cdot \mathbf{x} = M$ is a reduced integral equation of an affine hyperplane containing τ. In fact they give a more general definition for S_{Δ_τ}, but for our purposes this definition is sufficient. The term can be computed by first computing a partition of Δ_τ into rational simplicial cones $\Delta_i, i = 1, \dots, s$. Then $S_{\Delta_\tau}(f, T) = \sum_{i=1}^{s} S_{\Delta_i}(f, T)$. If Δ_i is the cone strictly generated by $\gamma_1, \dots, \gamma_e$ and G_i is the fundamental set of Δ'_i, then one can prove that

$$S_{\Delta_i}(f, T) = \left(\sum_{\mathbf{g} \in G_i \cap (\mathbb{Z}_{>0})^d} \mathbb{L}^{-\sigma(\mathbf{g})} T^{m_f(\mathbf{g})} \right) \prod_{j=1}^{e} \frac{1}{1 - \mathbb{L}^{-\sigma(\gamma_j)} T^{m_f(\gamma_j)}}.$$

For a compact face τ Artal, Cassou-Noguès, Luengo and Melle also define a term $L_\tau(f,T)$, as follows (in fact both this definition of $L_\tau(f,T)$ and our definition of the motivic zeta function differ by a factor $\mathbb{L}^{-d}$ from theirs). Let N_τ be the subvariety of $(\mathbb{A}^1 \setminus \{0\})^d$ defined by $\{f_\tau = 0\}$ and let $[N_\tau]$ be its class in the Grothendieck ring $K_0(Var_\mathbb{C})$. Set

$$L_\tau(f,T) := (\mathbb{L}-1)^d - [N_\tau] + (\mathbb{L}-1)[N_\tau]\frac{\mathbb{L}^{-1}T}{1-\mathbb{L}^{-1}T} \in \mathcal{M}_\mathbb{C}[[T]].$$

Then we can finally state the following theorem ([ACLM, Theorem 2.4]; compare with the second remark after Theorem 4.2 in [DH] by 'replacing' $\mathbb{L}$ by p and T by p^{-s}).

Theorem. *Let f be a polynomial in d variables over the complex numbers with $f(\mathbf{0}) = 0$. Assume that is non-degenerate at the origin with respect to its Newton polyhedron $\Gamma(f)$. Then*

$$Z_{loc,0,f}(T) = \sum_{\substack{\text{compact faces}\\ \tau \text{ of } \Gamma(f)}} L_\tau(f,T) S_{\Delta_\tau}(f,T).$$

When f is non-degenerate with respect to $\Gamma(f)$, an analogous formula for $Z_f(T)$ can be given, by summing over *all* faces of $\Gamma(f)$ (see Theorem 4.2 in [DH]).

(3.3.2) In the next section and in Chapter 4 we will combine this theorem with Remark (3.2.3) to compute the contribution of an isolated singular point of non-degenerate affine hypersurfaces to the stringy E-function. The major advantage of this approach is that there are in general very few denominators compared to the computation via a log resolution. Consider for example an $(m-1)$-dimensional A_n singularity, with n odd, $m \geq 4$. In Chapter 2 we found $k = \frac{n+1}{2}$ different discrepancy coefficients a_i and thus a priori k different denominators $(uv)^{a_i+1} - 1$. Let us now apply the above theorem. In this easy case all dual cones of the compact faces are already simplicial. There is one compact face of dimension $d-1$ and its dual cone is strictly generated by $\mathbf{k} := (1, k, \ldots, k)$. This gives a denominator of the form $((uv)^{(k(m-3)+1)}-1)$, since $\sigma(\mathbf{k}) = (m-1)k+1$ and $m_f(\mathbf{k}) = 2k$ (compare with Theorem (2.4.1)!). The additional generators of dual cones of other compact faces τ are just a number of standard basis vectors. They give a denominator of the form $(uv-1)$, but it will cancel since the Hodge-Deligne polynomial of N_τ contains this factor as many times.

3.4 Contribution of an A-D-E singularity to the stringy E-function revisited

(3.4.1) We will use the results of the previous sections to compute the contribution of an A-D-E singularity to the stringy E-function in another way. We only handle the D_n case (n even), since this is the hardest one. Let $m \geq 3$ be the dimension of the surrounding affine space. We explain how one can find all the necessary data to compute the formula of Theorem (3.3.1). We have put these data in a computer program that allowed us to check the formula from Theorem (2.4.1) for concrete values of m and n. We use the notations of the previous section.

The D_n singularity ($n \geq 4$) was given by the origin of the hypersurface

$$\{f(\mathbf{x}) := x_1^{n-1} + x_1x_2^2 + x_3^2 + \cdots + x_m^2 = 0\} \subset \mathbb{A}^m.$$

So the 0-dimensional faces of the Newton polyhedron are given by the m points

$$(n-1, 0, \ldots, 0), (1, 2, 0, \ldots, 0), (0, 0, 2, 0, \ldots, 0), \ldots (0, \ldots, 0, 2).$$

A compact face is then the convex hull of a number of these points. It is easy to check that f is non-degenerate at the origin with respect to its Newton polyhedron. For a compact face τ we compute first the class of $[N_\tau]$ in the Grothendieck ring, secondly the generators of the dual cones and finally the decomposition in simplicial cones and the integer points in the fundamental sets of these simplicial cones.

(3.4.2) *Computation of the classes* $[N_\tau]$. First note that if τ is 0-dimensional, then the class $[N_\tau]$ is zero. Essentially there are four kinds of classes that we have to compute:

- $W_l :=$ the class of $\{x_1^{n-1} + x_1x_2^2 + x_3^2 + \cdots + x_l^2 = 0\} \subset (\mathbb{A}^1 \setminus \{0\})^m$ for $2 \leq l \leq m$ (for $l = 2$ we mean the class of $\{x_1^{n-1} + x_1x_2^2 = 0\}$),
- $X_l :=$ the class of $\{x_1x_2^2 + x_3^2 + \cdots + x_l^2 = 0\} \subset (\mathbb{A}^1 \setminus \{0\})^m$ for $3 \leq l \leq m$,

- $Y_l :=$ the class of $\{x_1^{n-1} + x_2^2 + \cdots + x_l^2 = 0\} \subset (\mathbb{A}^1 \setminus \{0\})^m$ for $2 \leq l \leq m-1$,
- $Z_l :=$ the class of $\{x_1^2 + \cdots + x_l^2 = 0\} \subset (\mathbb{A}^1 \setminus \{0\})^m$ for $2 \leq l \leq m-2$.

For all of these cases we derive recursion formulae. Let us start with the last one. We easily find $Z_2 = (\mathbb{L}-1)^{m-2}(2\mathbb{L}-2)$. For $l = 3$ we use the isomorphism

$$\{x_1^2 + x_2^2 + x_3^2 = 0\} \subset (\mathbb{A}^1 \setminus \{0\})^m$$
$$\cong \{x_1^2 + x_2 x_3 = 0\} \subset (\{x_2^2 \neq x_3^2\}) \times (\mathbb{A}^1 \setminus \{0\})^{m-2}.$$

For $x_3 = 0$ we find no contribution, since $x_1 \neq 0$. For $x_3 \neq 0$, we can write $x_2 = -\frac{x_1^2}{x_3}$. The condition that $x_2^2 \neq x_3^2$ leads to $x_1^4 \neq x_3^4$ in the (x_1, x_3)-plane. Together with $x_1 \neq 0$ and $x_3 \neq 0$ we find thus $\mathbb{L}^2 - 6(\mathbb{L}-1) - 1 = (\mathbb{L}-1)(\mathbb{L}-5)$ for the contribution of coordinates x_1 and x_3. We can choose $m-3$ coordinates freely and x_2 is determined by x_1 and x_3. Thus

$$Z_3 = (\mathbb{L}-1)^{m-2}(\mathbb{L}-5).$$

For $l \geq 4$ we use the analogous isomorphism

$$\{x_1^2 + \cdots + x_l^2 = 0\} \subset (\mathbb{A}^1 \setminus \{0\})^m$$
$$\cong \{x_1^2 + \cdots + x_{l-2}^2 + x_{l-1}x_l = 0\} \subset (\{x_{l-1}^2 \neq x_l^2\}) \times (\mathbb{A}^1 \setminus \{0\})^{m-2}.$$

Now we find the contribution $\frac{Z_{l-2}}{\mathbb{L}-1}$ for $x_l = 0$ and $(\mathbb{L}-1)^{m-1} - 2\frac{Z_{l-1}}{\mathbb{L}-1}$ for $x_l \neq 0$. Indeed, for both $x_l = x_{l-1}$ and $x_l = -x_{l-1}$ we have to subtract $\frac{Z_{l-1}}{\mathbb{L}-1}$. So we have the recursion formula

$$Z_l = \frac{Z_{l-2}}{\mathbb{L}-1} + (\mathbb{L}-1)^{m-1} - 2\frac{Z_{l-1}}{\mathbb{L}-1}.$$

For Y_2 we find $(\mathbb{L}-1)^{m-1}$ and for $l \geq 3$ we proceed as in the Z-case: $Y_3 = (\mathbb{L}-1)^{m-2}(\mathbb{L}-3)$ and for $l \geq 4$ we have the same recursion formula

$$Y_l = \frac{Y_{l-2}}{\mathbb{L}-1} + (\mathbb{L}-1)^{m-1} - 2\frac{Y_{l-1}}{\mathbb{L}-1}.$$

The X_l are also easy: since x_2 cannot be zero we immediately find $(\mathbb{L}-1)^{m-1} - \frac{Z_{l-2}}{\mathbb{L}-1}$, with $X_3 = (\mathbb{L}-1)^{m-1}$.

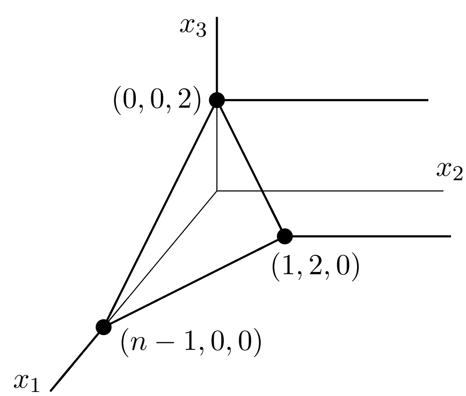
The class W_2 is equal to the class of $\{x_1^{n-2}+x_2^2=0\}\subset(\mathbb{A}^1\setminus\{0\})^m$ and this is $(2\mathbb{L}-2)(\mathbb{L}-1)^{m-2}$. For W_3 we use the isomorphism of varieties in $(\mathbb{A}^1\setminus\{0\})^m$ induced by blowing up:

$$\begin{array}{rcl} W:=\{x_1+x_1x_2^2+x_3^2=0\} & \cong & \{x_1^{n-1}+x_1x_2^2+x_3^2=0\} \\ (x_1,\dots,x_m) & \mapsto & (x_1,x_1^{(n-2)/2}x_2,x_1^{(n-2)/2}x_3,x_4,\dots,x_m). \end{array}$$

It is easy to compute the class of W. If $x_2\neq\pm i$, then the contribution is $(\mathbb{L}-1)^{m-2}(\mathbb{L}-3)$, and for $x_2=\pm i$ we find nothing. For $l\geq 4$, we again have the recursion formula

$$W_l=\frac{W_{l-2}}{\mathbb{L}-1}+(\mathbb{L}-1)^{m-1}-2\frac{W_{l-1}}{\mathbb{L}-1}.$$

(3.4.3) *Computation of the generators of the dual cones.* These generators correspond to the cones that are dual to the facets of the Newton polyhedron (not only to the compact facets). There are $m+2$ such facets. The following picture for $m=3$ might be helpful.



The hyperplane through all the m 0-dimensional faces of $\Gamma(f)$ has equation

$$2x_1+(n-2)x_2+(n-1)x_3+\cdots(n-1)x_m=2(n-1),$$

and thus its dual cone is generated by $\alpha:=(2,n-2,n-1,\dots,n-1)$. If we consider the noncompact facet that contains all points except $(n-1,0,\dots,0)$, we get a dual cone generated by $\beta:=(2,0,1,\dots,1)$. All the standard basis vectors generate dual cones as well. Note that the first one $(1,0,\dots,0)$ is different from the others, since it is dual to a facet that contains only $m-2$ 0-dimensional faces. We call this vector δ, the other standard basis vectors are denoted $\gamma_2 := (0,1,0,\dots,0),\dots,\gamma_m := (0,\dots,0,1)$. For all of these vectors, the essential data are the sum of the coefficients σ and the number m_f (see previous section). We denote these data for a vector $\mathbf{k}$ by $[\sigma(\mathbf{k}), m_f(\mathbf{k})]$. Thus we get the following table

generator	α	β	γ_2	$\dots$	γ_m	δ
$[\sigma, m_f]$	$[(n-1)(m-1)+1, 2(n-1)]$	$[m,2]$	$[1,0]$	$\dots$	$[1,0]$	$[1,0]$

(3.4.4) *Computation of the decomposition in simplicial cones and the integer points in the fundamental sets of these cones.* There is one compact face of dimension $m-1$, with one integer point in the fundamental set of the dual cone, namely α, with data $[(n-1)(m-1)+1, 2(n-1)]$. There are two types of compact faces of dimension $m-2$. The first type does not contain the point $(1,2,0,\dots,0)$ (dual cone generated by α and γ_2, since this compact face is the intersection of the facets dual to α and γ_2) or a point $(0,0,0,\dots,0,2,0,\dots,0)$ with the 2 on position $i \geq 3$, its dual cone is then strictly generated by α and γ_i. The fundamental set has only the integer point $(2, n-1, \dots, n-1)$ for the first cone, and $(2, n-2, n-1, \dots, n-1, n, n-1, \dots, n-1)$ in the second case. The data are $[(n-1)(m-1)+2, 2(n-1)]$. The second type is the compact face that does not contain $(n-1,0,\dots,0)$, with dual cone strictly generated by α and β. This cone has $n-2$ integer vectors in its fundamental set:

$$\{\frac{j}{n-2}\alpha + \frac{n-2-j}{n-2}\beta \,|\, j = 1,\dots,n-3\} \cup \{\alpha+\beta\},$$

with data

$$\{[(j+1)(m-1)+1, 2(j+1)] \,|\, j = 1,\dots,n-3\} \cup \{[n(m-1)+2, 2n]\}.$$

Note that the cone generated by α and δ does not correspond to an $(m-2)$-dimensional compact face of the Newton polyhedron.

For the 3-dimensional cones the situation becomes more complicated. The reason is that the intersection of the facets dual to α, β, γ_2 and δ is also an $(m-3)$-dimensional compact face τ. This means that the cone dual to τ is not simplicial, since it is generated by 4 elements. We decompose it in three pieces; the first piece Δ_1^3 generated by α, β, δ, the second Δ_2^3 generated by α, γ_2, δ and then a 2-dimensional cone $\Delta_{1,2}^3$ generated by α, δ. There are $n-2$ integer vectors in Δ_1^3, namely

$$\{\frac{j}{n-2}\alpha + \frac{n-2-j}{n-2}\beta + \delta \,|\, j = 1, \ldots, n-3\} \cup \{\alpha + \beta + \gamma\},$$

with data

$$\{[(j+1)(m-1)+2, 2(j+1)] \,|\, j = 1, \ldots, n-3\} \cup \{[n(m-1)+3, 2n]\}.$$

For Δ_2^3 we find the integer vectors

$$\{\frac{j}{n-1}\alpha + \frac{j}{n-1}\gamma_2 + \frac{n-1-2j}{n-1}\delta \,|\, j = 1, \ldots, (n-2)/2\}$$

$$\cup\{\frac{j}{n-1}\alpha + \frac{j}{n-1}\gamma_2 + \frac{2n-2-2j}{n-1}\delta \,|\, j = n/2, \ldots, n-2\} \cup \{\alpha + \gamma_2 + \delta\},$$

with data

$$\{[j(m-1)+1, 2j] \,|\, j = 1, \ldots, (n-2)/2\}$$

$$\cup\{[j(m-1)+2, 2j] \,|\, j = n/2, \ldots, n-2\} \cup \{[(n-1)(m-1)+3, 2(n-1)]\}.$$

And for $\Delta_{1,2}^3$ we only find $\alpha + \delta$ with data $[(n-1)(m-1)+2, 2(n-1)]$.

The other 3-dimensional cones are easier. There are $m-2$ cones generated by α, β and a γ_i ($i \geq 3$), with data

$$\{[(j+1)(m-1)+2, 2(j+1)] \,|\, j = 1, \ldots, n-3\} \cup \{[n(m-1)+3, 2n]\}.$$

And then we have $\binom{m-1}{2}$ cones generated by α, a γ_i and a γ_j. These have only one integer vector in their fundamental set, with data $[(n-1)(m-1)+3, 2(n-1)]$.

We can continue in this way for the higher dimensional cones. However, we must be careful: there are three cones that need a different treatment (one of dimension $m-1$ and two of dimension m). First we summarize the results for general k-dimensional cones. We also give the corresponding class in the Grothendieck ring (recall that these classes are 0 for m-dimensional cones). There are $\binom{m-1}{k-1}$ cones generated by α and $k-1$ γ_i's. These have one integer vector in their fundamental set, with data $[(n-1)(m-1)+k, 2(n-1)]$. If γ_2 belongs to the set of generators, then the corresponding class in the Grothendieck ring is of type Y_{m-k+1}, else it is of type W_{m-k+1}. There are $\binom{m-2}{k-2}$ cones generated by α, β and $k-2$ γ_i's ($i \geq 3$) with data

$$\{[(j+1)(m-1)+k-1, 2(j+1)] \mid j = 1, \ldots, n-3\} \cup \{[n(m-1)+k, 2n]\}$$

and with class in the Grothendieck ring of type X_{m-k+2}. And then there are $\binom{m-2}{k-3}$ non-simplicial cones, generated by $\alpha, \beta, \gamma_2, \delta$ and $k-3$ γ_i's ($i \geq 3$). The class in the Grothendieck ring corresponding to these cones is of type Z_{m-k+1}. They can be divided in three simplicial cones; first Δ_1^k generated by α, β, δ and the $k-3$ γ_i's with data

$$\{[(j+1)(m-1)+k-1, 2(j+1)] \mid j = 1, \ldots, n-3\} \cup \{[n(m-1)+k, 2n]\}.$$

The second piece is Δ_2^k generated by α, γ_2, δ and the γ_i's, with data

$$\{[j(m-1)+k-2, 2j] \mid j = 1, \ldots, (n-2)/2\}$$

$$\cup\{[j(m-1)+k-1, 2j] \mid j = n/2, \ldots, n-2\} \cup \{[(n-1)(m-1)+k, 2(n-1)]\}.$$

The third piece is generated by α, δ and the γ_i's and has data $[(n-1)(m-1)+k-1, 2(n-1)]$.

Finally, we handle the three special cases. The $(m-1)$-dimensional cone generated by $\alpha, \gamma_3, \ldots, \gamma_m$ has two integer vectors in its fundamental set, namely

$$(1, \frac{n-2}{2}, \frac{n}{2}, \ldots, \frac{n}{2}) \text{ and } (2, n-2, n, \ldots, n),$$

with data

$$[(m-1)\frac{n}{2}, n-1] \text{ and } [(m-1)n, 2(n-1)].$$

An analogous phenomenon occurs for the m-dimensional cone generated by $\alpha, \gamma_2, \gamma_3, \ldots, \gamma_m$. The data are now

$$[(m-1)\frac{n}{2}+1, n-1] \text{ and } [(m-1)n+1, 2(n-1)].$$

And there is also the m-dimensional cone generated by $\alpha, \beta, \gamma_3, \ldots, \gamma_m$. The integer vectors of the fundamental set of this cone are

$$\{\frac{j}{n-2}\alpha + \frac{n-2-j}{n-2}\beta + \gamma_3 + \cdots + \gamma_m \mid j = 1, \ldots, n-3\}$$

$$\cup\{\alpha + \beta + \gamma_3 + \cdots + \gamma_m\}$$

$$\cup\{\frac{j}{n-2}\alpha + \frac{n-2-2j}{2(n-2)}\beta + \frac{1}{2}\gamma_3 + \cdots + \frac{1}{2}\gamma_m \,|\, j = 1, \ldots, \frac{n-4}{2}\}$$

$$\cup\{\frac{j}{n-2}\alpha + \frac{3(n-2)-2j}{2(n-2)}\beta + \frac{1}{2}\gamma_3 + \cdots + \frac{1}{2}\gamma_m \,|\, j = \frac{n-2}{2}, \ldots, n-2\},$$

with data

$$\{[(j+2)(m-1), 2(j+1)] \,|\, j = 1, \ldots, n-3\} \cup \{[(m-1)(n+1)+1, 2n]\}$$
$$\cup\{[(m-1)(j+1), 2j+1] \,|\, j = 1, \ldots, \frac{n-4}{2}\}$$
$$\cup\{[(m-1)(j+2)+1, 2j+3] \,|\, j = \frac{n-2}{2}, \ldots, n-2\}.$$

(3.4.5) To conclude this section and this chapter, we give an easy explicit example of the above computations. For a 5-dimensional ($m = 6$) D_8 singularity ($n = 8$) our computer program gives the following local motivic zeta function:

$$\begin{aligned} Z_{loc,0,f} = & \frac{T^2}{(\mathbb{L}-T)(\mathbb{L}^{18}-T^7)}\Big((1-\mathbb{L}-2\mathbb{L}^3+2\mathbb{L}^4)T^6 \\ & + (2\mathbb{L}^3-2\mathbb{L}^4-\mathbb{L}^6+\mathbb{L}^7)T^5 + (\mathbb{L}^6-\mathbb{L}^7-\mathbb{L}^8+\mathbb{L}^9)T^4 \\ & + (\mathbb{L}^8-\mathbb{L}^9-\mathbb{L}^{11}+\mathbb{L}^{12})T^3 + (\mathbb{L}^{11}-\mathbb{L}^{12}-\mathbb{L}^{13}+\mathbb{L}^{14})T^2 \\ & + (\mathbb{L}^{13}-\mathbb{L}^{14}-\mathbb{L}^{16}+\mathbb{L}^{17})T + \mathbb{L}^{16}-\mathbb{L}^{17}-\mathbb{L}^{18}+\mathbb{L}^{19}\Big). \end{aligned}$$

Thus by the procedure of Section 3.2 the contribution of such a singularity to the stringy E-function becomes (with $w = uv$)

$$\frac{w^{12}+w^{11}-w^{10}+w^9-w^7+w^6-w^4+2w^3-2w^2-1}{w^{11}-1},$$

and by a short computation this can be seen to be equal to the formula from Theorem (2.4.1).

Chapter 4

An interesting example

Abstract

We show by example that the answer to Question (0.2.5) is 'no' in general. Our example consists of a 6-dimensional projective variety with isolated terminal singularities. We have computed a log resolution that has some components with discrepancy 1. In particular, this shows that Theorem (1.2.2) cannot be extended, since in that theorem we already allow discrepancy coefficients ≥ 2.

4.1 Hodge-Deligne polynomials of quasi-homogeneous hypersurfaces and Fermat hypersurfaces

(4.1.1) In this section we discuss some results obtained by Dais in [Da]. He used these results to compute the contribution of a so called $A_{n,l}^{(r)}$ singularity to the stringy E-function (for the cases $l \mid n$ and $l \mid n+1$). This kind of singularity is given by the origin of

$$\{x_1^{n+1} + x_2^l + \cdots + x_{r+1}^l = 0\} \subset \mathbb{A}^{r+1},$$

for $r \geq l \geq 2$ and $n+1 \geq l$. In particular, this includes the A_n singularities from Chapter 2. We will need Dais' computation of the Hodge-Deligne polynomial of a quasi-homogeneous affine hypersurface with an isolated singularity in the origin and of the Hodge-Deligne polynomial of a Fermat hypersurface.

(4.1.2) Let $f \in \mathbb{C}[x_1, \ldots, x_{r+1}]$ be a quasi-homogeneous polynomial of degree d with respect to the weights $w_1, \ldots, w_{r+1} \in \mathbb{Z}_{>0}$. This means that

$$f(\lambda^{w_1} x_1, \ldots, \lambda^{w_{r+1}} x_{r+1}) = \lambda^d f(x_1, \ldots, x_{r+1}) \text{ for all } \lambda \in \mathbb{C}^*.$$

In particular, $f(\mathbf{0}) = 0$. Assume moreover that $\mathbf{0}$ is an isolated singularity of the hypersurface $Y := \{f = 0\} \subset \mathbb{A}^{r+1} = \mathbb{C}^{r+1}$. The following well known construction is due to Milnor ([Mi]). For small enough $\varepsilon > 0$, the closed ball $B_\varepsilon \subset \mathbb{C}^{r+1}$ of radius ε and centered in the origin intersects $f^{-1}(0)$ transversely. For $0 < \eta \ll \varepsilon$ and for t in the disk $D_\eta \subset \mathbb{C}$ with radius η around the origin, the fibre $f^{-1}(t)$ intersects B_ε transversely as well. Set $X := f^{-1}(D_\eta) \cap B_\varepsilon, X_0 := X \cap f^{-1}(0), X^* = X \setminus X_0$ and $D_\eta^* = D_\eta \setminus \{0\}$. The mapping

$$f|_{X^*} : X^* \to D_\eta^*$$

is a locally trivial C^∞-differentiable fibre bundle. Its fibre X_t over a point $t \in D_\eta^*$ is called the Milnor fibre. The link L is the intersection of $\{f = 0\}$ with the boundary of B_ε. Let F_t be the interior of X_t. It is well known that the cohomology of L and F_t carries a natural mixed Hodge structure. Dais shows that ([Da, Proposition 2.8])

$$H(Y; u, v) = (uv)^r + (-1)^{r-1}(uv - 1) \sum_{p=0}^{r-1} h^{p,r-1-p}(H^{r-1}(L, \mathbb{C}))u^p v^{r-1-p},$$

where $h^{p,r-1-p}(H^{r-1}(L, \mathbb{C}))$ denotes the dimension of the $H^{p,r-1-p}$ component of the mixed Hodge structure on the singular cohomology group $H^{r-1}(L, \mathbb{C})$. These numbers can be computed in terms of $w_1, \ldots, w_{r+1}$, as explained in Theorem 2.6 and Lemma 2.7 of [Da]. Consider the Milnor algebra

$$M(f) := \frac{\mathbb{C}[x_1, \ldots, x_{r+1}]}{\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{r+1}}\right)}.$$

This becomes a finitely generated graded $\mathbb{C}$-algebra if we give x_i degree w_i. The Poincaré series of such an algebra is defined by

$$P_{M(f)}(t) := \sum_{k \geq 0} \dim_{\mathbb{C}}(M(f)_k) t^k,$$

where $M(f)_k$ is the piece of degree k. This series can be calculated by the formula

$$P_{M(f)}(t) = \frac{(1-t^{d-w_1})\cdots(1-t^{d-w_{r+1}})}{(1-t^{w_1})\cdots(1-t^{w_{r+1}})}.$$

Dais shows, referring to work of Griffiths and Steenbrink ([Gr] and [St1]), that the numbers $h^{p,r-1-p}(H^{r-1}(L,\mathbb{C}))$ equal

$$\dim_{\mathbb{C}}(M(f))_{(p+1)d-(w_1+\cdots+w_{r+1})},$$

and thus they can be computed from the Poincaré series.

(4.1.3) We denote the d-dimensional Fermat hypersurface of degree l by $Y_l^{(d)}$. So $Y_l^{(d)}$ is given by

$$\{x_0^l + \cdots + x_{d+1}^l = 0\} \subset \mathbb{P}^{d+1}.$$

To write down the Hodge-Deligne polynomial of $Y_l^{(d)}$ we need an auxiliary definition. Dais considers the numbers

$$\mathcal{G}(\kappa,\lambda\,|\,\nu,\xi) := \sum_{j=0}^{\lambda}(-1)^j\binom{\kappa+1}{j}\binom{\nu(\lambda-j)+\xi}{\kappa}$$

for $(\kappa,\lambda,\nu,\xi) \in \mathbb{Z}^4_{\geq 0}$ and $\kappa \geq \lambda$ (if $m > n$, the binomial coefficient $\binom{n}{m}$ must be interpreted as 0). Then the Hodge-Deligne polynomial of $Y_l^{(d)}$ is given by ([Da, Lemma 3.3])

$$H(Y_l^{(d)};u,v) := \sum_{p=0}^{d} u^p\left(v^p + (-1)^d\mathcal{G}(d+1,p+1\,|\,l-1,p)v^{d-p}\right).$$

4.2 A negative answer to Question (0.2.5)

(4.2.1) Suppose that we would like to prove Theorem (1.2.2) in some other cases. One of the first cases not covered by this theorem, is that of a 6-dimensional projective variety Y with Gorenstein terminal isolated singularities (thus compared with condition $(*)$ we also allow discrepancy coefficients equal to 1). If we develop the stringy E-function of Y in a power series $\sum_{i,j\geq 0} b_{i,j}u^iv^j$, then the problem to prove Theorem (1.2.2) is situated in the coefficient $b_{3,3}$ (the other coefficients $b_{i,j}$ for $i+j \leq 6$ always have the right sign, this can be proved like in Theorem (1.2.2)). We recall some of the notations of the proof of Theorem (1.2.2). Take a log resolution $f : X \to Y$ of Y with irreducible exceptional components $D_i, i \in I$. Let a_i be the discrepancy coefficient of D_i. Set $D_J := \cap_{i \in J} D_i$ for a subset $J \subset I$. Denote the Hodge-Deligne polynomial of X by $\sum_{i,j} a_{i,j} u^i v^j$ and of $D_J, J \neq \emptyset$, by $\sum_{i,j} a^J_{i,j} u^i v^j$. Then we can write $b_{3,3}$ as

$$\begin{aligned} b_{3,3} = a_{3,3} - \sum_{i \in I} a^{\{i\}}_{2,2} + \sum_{\substack{J \subset I \\ |J|=2}} a^J_{1,1} - \sum_{\substack{J \subset I \\ |J|=3}} a^J_{0,0} \\ + \sum_{\substack{i \in I \\ a_i=1}} a^{\{i\}}_{1,1} - \sum_{\substack{i \in I \\ a_i=1}} a^{\{i\}}_{0,0} - \sum_{\substack{\{i,j\} \in I \\ a_i=1 \text{ or } a_j=1}} \delta^{\{i,j\}} a^{\{i,j\}}_{0,0} + \sum_{\substack{i \in I \\ a_i=2}} a^{\{i\}}_{0,0}, \end{aligned} \tag{1}$$

where $\delta^{\{i,j\}} \in \{1,2\}$ is the number of components in $\{i,j\}$ with discrepancy 1. For the terms on the first line, we can proceed exactly as in the proof of Theorem (1.2.2) to prove that their alternating sum is nonnegative. For the terms on the second line this is not clear; and it is actually not true. They can even be 'negative enough' to make $b_{3,3}$ negative. So in order to find an example where this occurs, we need some exceptional components with discrepancy 1 and a lot of intersections with them. At least the first of these needs is fulfilled by the hypersurface singularity $\{x_1^5 + x_2^5 + x_3^6 + x_4^6 + x_5^6 + x_6^6 + x_7^6 = 0\} \subset \mathbb{A}^7$. If we blow up once in the origin, the exceptional locus consists of five irreducible components, all having discrepancy coefficient 1 (thanks to the fifth powers in the equation). To get this singularity on a projective variety, we just take the projective closure:

$$Y := \{x_1^5 z + x_2^5 z + x_3^6 + x_4^6 + x_5^6 + x_6^6 + x_7^6 = 0\} \subset \mathbb{P}^7,$$

where we consider $z = 0$ as the hyperplane at infinity. This operation gives us five additional isolated singularities at infinity. The rest of this section is devoted to computing the stringy E-function of Y. This leads to the following result.

Proposition. *The answer to Question (0.2.5) is 'no' for Y.*

(4.2.2) *Contribution of the singularities to the stringy E-function via a log resolution.* Let us consider the singularities at infinity first. These are all analytically isomorphic to the origin of

$$Y' := \{x_1^2 + x_2^2 + x_3^6 + \cdots + x_7^6 = 0\} \subset \mathbb{A}^7.$$

To resolve them, we have to blow up in the singular point first. This gives two exceptional components, denoted D_1^∞ and D_2^∞, and their intersection becomes the new singular locus. Blowing it up, gives again two exceptional components E_1^∞ and E_2^∞ whose intersection is singular for the strict transform of Y'. Moreover, the components D_1^∞ and D_2^∞ are separated (like in Chapter 2, we use the same name for an exceptional divisor at every moment of the resolution process). Then we only have to perform the blow up in the intersection of E_1^∞ and E_2^∞. This gives one new exceptional component F^∞ and the following intersection diagram:

$$\begin{tikzcd} \overset{D_1^\infty}{\bullet} \arrow[r, dash] & \overset{E_1^\infty}{\bullet} \arrow[r, dash] & \overset{F^\infty}{\bullet} \arrow[r, dash] & \overset{E_2^\infty}{\bullet} \arrow[r, dash] & \overset{D_2^\infty}{\bullet} \end{tikzcd}$$

The discrepancy coefficient of all these components is 4. The components D_1^∞ and D_2^∞ are isomorphic to $\mathbb{P}^5$, E_1^∞ and E_2^∞ are $\mathbb{P}^1$-bundles over $\mathbb{P}^4$ and all intersections are isomorphic to $\mathbb{P}^4$. It is not so easy to compute the Hodge-Deligne polynomial of F^∞. In one of the charts, F^∞ is given by the equations

$$\{x_3 = 0, x_1^2 + x_2^2 + 1 + x_4^6 + x_5^6 + x_6^6 + x_7^6 = 0\} \subset \mathbb{A}^7.$$

This variety is isomorphic to

$$\{x_1 x_2 + 1 + x_4^6 + \cdots + x_7^6 = 0\} \subset \mathbb{A}^6.$$

For $x_1 \neq 0$, one finds a contribution of $(uv-1)(uv)^4$ to $H(F^\infty; u, v)$. For $x_1 = 0$, one finds uv times the Hodge-Deligne polynomial of an affine piece of the 3-dimensional Fermat hypersurface $Y_6^{(3)}$ of degree 6 (in fact $Y_6^{(3)} \setminus Y_6^{(2)}$), and this Hodge-Deligne polynomial can be calculated by (4.1.3). Taking into account the contributions of all other relevant coordinate charts (that can be calculated analogously), one finds

$$H(F^\infty; u, v) = (uv)^5 + 2(uv)^4 + 2(uv)^3 + 2(uv)^2 + 2(uv) + 1 + (uv)H(Y_6^{(3)}; u, v),$$

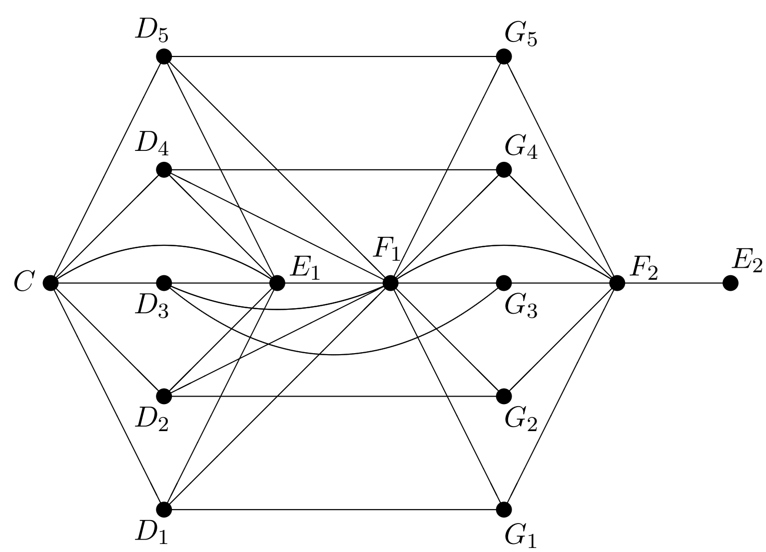
with

$$H(Y_6^{(3)}; u, v) = (uv)^3 + (uv)^2 + uv + 1 - 5u^3 - 5v^3 - 255u^2v - 255uv^2.$$

With these data, one can compute that the contribution of such a singular point to the stringy E-function is

$$A := \frac{(uv-1)}{((uv)^5 - 1)} \Big(5(uv)^5 + (uv)^4 + (uv)^3 + (uv)^2 + uv + 1 \\ - 5u^4v - 5uv^4 - 255u^3v^2 - 255u^2v^3 \Big).$$

The computation of the contribution of the singularity in $(0,\ldots,0,1)$ is a bit complicated, though the results are nice. First we blow up in the singular point itself. As already noticed, this produces five exceptional components $D_1,\ldots,D_5$. After this blow up, they all go through one $\mathbb{P}^4$, and thus they have nowhere normal crossings. The new singular locus is a Fermat hypersurface of degree 6 on this $\mathbb{P}^4$. Blowing up in this singular locus gives two new exceptional divisors E_1 and E_2. They both intersect the D_i, but the new singular locus is the intersection of E_1 and E_2. This singular locus also contains a piece of the intersection of the D_i, and this piece is exactly the intersection of the D_i with E_2 (which is surprisingly only 3-dimensional). Blowing up in this new singular locus splits of E_2 from E_1 and all the D_i. Two new exceptional components F_1 and F_2 appear. They intersect each other, and apart from that, the first intersects E_1 and all the D_i, and the second intersects E_2 and the D_i. The new singular locus consists of five separate pieces; one piece on each D_i. It is exactly the intersection of F_2 with the D_i. Blowing it up gives us five new components $G_1,\ldots G_5$, all intersecting F_1, F_2, and every G_i intersects one D_i (of course we take a compatible numbering). Finally we have a nonsingular strict transform, but still the D_i have no normal crossings. Blowing up in their intersection (which isomorphic to $\mathbb{P}^4$), gives one new exceptional component C (a $\mathbb{P}^1$-bundle over this intersection), intersecting each D_i and also intersecting E_1. We find the following intersection diagram:



We mentioned already that the discrepancy coefficients of the D_i are 1. One finds 6 for C, 5 for E_1 and the G_i, 4 for F_1, 3 for F_2 and 2 for E_2. There are twenty threefold intersections, namely $C \cap E_1 \cap D_i$, $E_1 \cap F_1 \cap D_i$, $F_1 \cap D_i \cap G_i$ and $F_1 \cap F_2 \cap G_i$, where i runs of course from 1 to 5. They are all isomorphic to the Fermat hypersurface $Y_6^{(3)}$. One can count from the diagram that there are thirty-four twofold intersections, all having Hodge-Deligne polynomial $(uv+1)H(E_6^{(3)};u,v)$, except for the $C \cap D_i$, they are isomorphic to $\mathbb{P}^4$. The Hodge-Deligne polynomials of the components itself are

$$\begin{aligned}
&H(C) = (uv+1)((uv)^4 + (uv)^3 + (uv)^2 + uv + 1),\\
&H(D_i) = (uv)^5 + (uv)^4 + (uv)^3 + (uv)^2 + uv + 1 + 3uvH(Y_6^{(3)};u,v),\\
&H(E_1) = H(G_i) = ((uv)^2 + 2uv + 1)H(Y_6^{(3)};u,v),\\
&H(E_2) = ((uv)^2 + uv + 1)H(Y_6^{(3)};u,v),\\
&H(F_1) = H(F_2) = ((uv)^2 + 7uv + 1)H(Y_6^{(3)};u,v).
\end{aligned}$$

By a rather lengthy calculation one can simplify the contribution of this singular point to

$$\begin{aligned}
B :=& \frac{(uv-1)}{((uv)^7-1)}\Big(5(uv)^{10} + (uv)^9 + 7(uv)^8 + 3(uv)^7 + 9(uv)^6 + 4(uv)^5\\
&+ 8(uv)^4 + 2(uv)^3 + 6(uv)^2 + uv + 1 - 5u^9v^6 - 5u^6v^9 - 255u^8v^7\\
&- 255v^7v^8 - 5u^8v^5 - 5u^5v^8 - 255u^7v^6 - 255u^6v^7 - 5u^7v^4 - 5u^4v^7\\
&- 255u^6v^5 - 255u^5v^6 - 5u^6v^3 - 5u^3v^6 - 255u^5v^4 - 255u^4v^5\\
&- 20u^4v - 20uv^4 - 1020u^3v^2 - 1020u^2v^3\Big).
\end{aligned}$$

(4.2.3) *Contribution of the singular points to the stringy E-function via Newton polyhedra.* It is interesting to explain this method too, since in this particular case, it is easy to find the dual cones, and so the contributions of the singular points can easily be calculated by a computer. We will handle the singularity in the origin; the singularities at infinity are even easier. The Newton polyhedron of $f = x_1^5 + x_2^5 + x_3^6 + x_4^6 + x_5^6 + x_6^6 + x_7^6$ has one compact facet with dual 1-dimensional cone generated by $\alpha := (6,6,5,5,5,5,5)$. The other facets are non-compact and simply have the standard basis vectors $\beta_1, \ldots, \beta_7$ as generators of their dual cone. For a j-dimensional cone we want to distinguish in general three types.

- *Type 1*: There are $\binom{5}{j-1}$ cones generated by α and $j-1$ vectors from $\{\beta_3, \ldots, \beta_7\}$ (for $j \geq 1$). So this type does not occur for $j = 7$.

There is one integer vector in the fundamental set, namely the sum of the generators, with data $[37+j-1, 30]$. To this rule, there is one exception: for $j=6$ there are 6 integer vectors:

$$\{(1,\dots,1),\dots,(6,\dots,6)\}$$

with data

$$\{[7,5],[14,10],\dots,[42,30]\}.$$

- *Type 2*: There are $2\binom{5}{j-2}$ cones generated by α, β_1 or β_2 (not both), and $j-2$ other β_i's. There is no 1-dimensional cone of this type. These cones have always one integer vector in their fundamental set with data $[37+j-1, 30]$, except for $j=7$. In that case there are 6 integer vectors with data

$$\{[8,5],[15,10],[22,15],[29,20],[36,25],[43,30]\}.$$

- *Type 3*: Now we choose α, β_1, β_2 and $j-3$ other β_i's as generators. This type does not occur for $j=1,2$ and there are $\binom{5}{j-3}$ cones of this type. There are always 5 integer vectors in the fundamental domain with data

$$\{[6+j,6],[13+j,12],[20+j,18],[27+j,24],[36+j,30]\}.$$

These types correspond to three types of varieties whose classes in the Grothendieck ring must be computed, namely

- *Type 1*: $X_j := \{x_1^5 + x_2^5 + x_3^6 + \cdots + x_{8-j}^6 = 0\} \subset (\mathbb{A}^1 \setminus \{0\})^7$ for $1 \le j \le 6$ (for $j=6$ we mean $X_6 = \{x_1^5 + x_2^5 = 0\}$),
- *Type 2*: $Y_j := \{x_1^5 + x_2^6 + \cdots + x_{8-j}^6 = 0\} \subset (\mathbb{A}^1 \setminus \{0\})^7$ for $2 \le j \le 7$ (with $Y_7 = \{x_1^5 = 0\}$),
- *Type 3*: $Z_j := \{x_1^6 + \cdots + x_{8-j}^6 = 0\} \subset (\mathbb{A}^1 \setminus \{0\})^7$ for $3 \le j \le 7$.

In fact we will not compute the classes, but just the Hodge-Deligne polynomial of these varieties; that is sufficient for the stringy E-function. We will just present one example: Type 1 for $j=4$. Let X'_j, Y'_j and Z'_j be the varieties given by the same equations as X_j, Y_j and Z_j, but considered as a subvariety of $\mathbb{A}^{8-j}$. We can express the Hodge-Deligne polynomial of

X_j as follows (by subtracting from $H(X'_j)$ the intersections with the coordinate hyperplanes, adding again twofold intersections and so on, and multiplying by the right power of $(uv-1)$):

$$\begin{aligned}H(X_4) = (uv-1)^3\Big(&H(X'_4) - 2H(Y'_5) - 2H(X'_5) + H(Z'_6) + 4H(Y'_6) \\ &+ H(X'_6) - 2H(Z'_7) - 2H(Y'_7) + 1\Big).\end{aligned}$$

The Hodge-Deligne polynomials of X'_j, Y'_j and Z'_j can be calculated by the method of Dais from (4.1.2). We get (here the last five are elementary)

$$\begin{aligned}H(X'_4) &= (uv)^3 + 20uv(uv-1) \\ H(Y'_5) &= (uv)^2 \\ H(X'_5) &= (uv)^2 \\ H(Z'_6) &= 6uv - 5 \\ H(Y'_6) &= uv \\ H(X'_6) &= 5uv - 4 \\ H(Z'_7) &= 1 \\ H(Y'_7) &= 1,\end{aligned}$$

and thus we find $H(X_4) = (uv-1)^3((uv)^3 + 16(uv)^2 - 5uv - 12)$. For smaller j, the computation is more complicated, we have written a short computer program for that. In the end, the result is the same as expression B above.

(4.2.4) *Computation of the Hodge-Deligne polynomial of the nonsingular locus.* Let us first compute the contribution of the nonsingular part of Y at infinity. The total part at infinity is given by

$$Y^\infty := \{x_3^6 + x_4^6 + x_5^6 + x_6^6 + x_7^6 = 0\} \subset \mathbb{P}^6.$$

To find the nonsingular part, we just have to remove 5 points. In fact Y^∞ can be found from $Y_6^{(3)}$ by taking twice the projective cone. On the level of the Hodge-Deligne polynomial, one such operation multiplies the original Hodge-Deligne polynomial by uv and adds 1. The contribution at infinity becomes thus (do not forget to subtract the 5 singular points)

$$C := (uv)^5 + (uv)^4 + (uv)^3 + (uv)^2 + uv - 4 - 5u^5v^2 - 5u^2v^5 - 255u^4v^3 - 255u^3v^4,$$

since $H(Y_6^{(3)}; u, v)$ was

$$(uv)^3 + (uv)^2 + uv + 1 - 5u^3 - 5v^3 - 255u^2v - 255uv^2.$$

Finally, we only have to compute the contribution of $\{x_1^5+x_2^5+x_3^6+\cdots+x_7^6=0\}\subset\mathbb{A}^7$ minus the singular point. This can again be done by the method of (4.1.2) and the result is (subtracting also the singular point)

$$D:=(uv)^6-1-(uv-1)(20u^4v+20uv^4+1020u^3v^2+1020u^2v^3).$$

(4.2.5) *The total stringy E-function.* We just add the terms $5A, B, C$ and D and simplify. The result is

$$\begin{aligned}
E_{st}(Y;u,v) &= \frac{1}{((uv)^5-1)((uv)^7-1)}\Big((uv)^{18}+(uv)^{17}+6(uv)^{16}-3(uv)^{15}\\
&+7(uv)^{14}+21(uv)^{13}-20(uv)^{12}-12(uv)^{11}+6(uv)^{10}-14(uv)^9+6(uv)^8\\
&-12(uv)^7-20(uv)^6+21(uv)^5+7(uv)^4-3(uv)^3+6(uv)^2+uv+1\\
&-25(u^{17}v^{14}+u^{14}v^{17}+u^4v+uv^4)-1275(u^{16}v^{15}+u^{15}v^{16}+u^3v^2+u^2v^3)\\
&+20(u^{16}v^{13}+u^{13}v^{16}+u^5v^2+u^2v^5)+1020(u^{15}v^{14}+u^{14}v^{15}+u^4v^3+u^3v^4)\\
&-5(u^{15}v^{12}+u^{12}v^{15}+u^6v^3+u^3v^6)-255(u^{14}v^{13}+u^{13}v^{14}+u^5v^4+u^4v^5)\\
&+10(u^{11}v^8+u^8v^{11}+u^{10}v^7+u^7v^{10})+510(u^{10}v^9+u^9v^{10}+u^9v^8+u^8v^9)\Big).
\end{aligned}$$

So if we develop this in power series, we get a term $-3(uv)^3$, and this gives indeed a negative answer to Question (0.2.5). It is interesting to see what formula (1) from (4.2.1) becomes in this case. Let D^∞ be the total exceptional locus of all singular points at infinity, and let D be the total exceptional locus of the other singular point. There are exact sequences

$$\begin{aligned}
0\to H^{3,3}(H^6(D))\to H^{3,3}(H^6(D^{(0)}))\to H^{3,3}(H^6(D^{(1)}))\\
\to H^{3,3}(H^6(D^{(2)}))\to 0
\end{aligned}$$

and

$$0\to H^{3,3}(H^6(D^\infty))\to H^{3,3}(H^6((D^\infty)^{(0)}))\to H^{3,3}(H^6((D^\infty)^{(1)}))\to 0,$$

as in the proof of Theorem (1.2.2). The dimensions of the spaces in the first sequence are (in the right order) $24, 67, 63, 20$ and of the second sequence $25, 45, 20$ (this can be seen from the explanation in (4.2.2)). And the first part of the formula for $b_{3,3}$ from (4.2.1) becomes

$$a_{3,3}-\sum_{i\in I}a_{2,2}^{\{i\}}+\sum_{\substack{J\subset I\\ |J|=2}}a_{1,1}^J-\sum_{\substack{J\subset I\\ |J|=3}}a_{0,0}^J$$

$$= (1 + 24 + 25) - (67 + 45) + (63 + 20) - 20 = 1,$$

where the 1 comes from the coefficient of $(uv)^3$ in $C + D$. The second part of that formula is

$$\sum_{\substack{i\in I\\ a_i=1}} a_{1,1}^{\{i\}} - \sum_{\substack{i\in I\\ a_i=1}} a_{0,0}^{\{i\}} - \sum_{\substack{\{i,j\}\in I\\ a_i=1 \text{ or } a_j=1}} \delta^{\{i,j\}} a_{0,0}^{\{i,j\}} + \sum_{\substack{i\in I\\ a_i=2}} a_{0,0}^{\{i\}} = 20-5-20+1 = -4,$$

and in that way we find -3 for $b_{3,3}$.

Some ideas for the future

(1) *Polynomial stringy E-functions.* The example of Chapter 4 provides some evidence that Batyrev's conjecture (0.2.5) might not be true. But in his conjecture, he assumes that the stringy E-function is a polynomial. If we compare this to a power series development $\sum_{i,j\geq 0} b_{i,j}u^iv^j$, like in Chapter 1, this means that all $b_{i,j}$ are 0 for $i,j \geq d$, where d is the dimension of the variety Y. Moreover, $b_{d,d} = 1$, and there are symmetry relations $b_{i,j} = b_{d-i,d-j}$ for $i,j \leq d$. If we choose a log resolution $f : X \to Y$, all these relations can be translated to relations between Hodge numbers of X and Hodge numbers of the exceptional components D_i, Hodge numbers of intersections of these components, etc. (the components with the smallest discrepancy coefficients have the most contributions). However, these relations are rather complicated and it is not immediately clear how they can help to prove the conjecture.

(2) *Connection with intersection cohomology.* Let us have another look at Proposition (1.2.5). If we forget the term S, then the numbers $b_{i,j}$ for $i+j \leq d$ come from $H^{d-i,d-j}(H^{2d-i-j}(Y))$, where Y is our singular projective variety of dimension d, satisfying condition $(*)$. By the remark after the proof of Proposition (1.2.5), $H^{2d-i-j}(Y)$ has a pure Hodge structure for $i+j < d$. For $i+j = d$, only the quotient $\frac{H^d(Y)}{W_{d-1}H^d(Y)}$, where $W_\bullet$ denotes the weight filtration, contributes to the numbers $b_{i,j}$. These groups are in fact isomorphic to the intersection cohomology groups (with respect to the middle perversity). Since Y has isolated singularities, it follows immediately from [KW, Proposition 4.4.1] that $IH^{2d-i-j}(Y) \cong H^{2d-i-j}(Y)$ for $i+j < d$. This same proposition says that

$$IH^d(Y) \cong \frac{H^d(Y)}{\ker(H^d(Y) \to H^d(Y \setminus \text{Sing } Y))},$$

where Sing Y is of course the set of singular points of Y. The natural map $H^d(Y) \to IH^d(Y)$ is thus certainly surjective, and the combination of Theorem 1.7 (D') and (E') from [We] gives then

$$IH^d(Y) \cong \frac{H^d(Y)}{\ker(H^d(Y) \to H^d(Y \setminus \text{Sing } Y))} = \frac{H^d(Y)}{W_{d-1}H^d(Y)}.$$

This same result can also be deduced by using Durfee's construction of a mixed Hodge structure on a punctured neighbourhood. Let $T_i, i = 1, \ldots, s$ be disjoint contractible closed neighbourhoods around the singular points y_i. Let T_i^* be $T_i \setminus \{y_i\}$. Then according to [Du, Theorem 3.2], there exists a mixed Hodge structure on the cohomology of the T_i^* such that the Mayer-Vietoris sequence (written down for $m > 0$; then the cohomology of the T_i is trivial)

$$\to H^m(Y) \to H^m(Y \setminus \text{Sing } Y) \to H^m(\cup_i T_i^*) \to H^{m+1}(Y) \to$$

is a sequence of morphisms of mixed Hodge structures. Moreover, by Proposition 3.8 from the same paper, $W_m H^m(\cup_i T_i^*) = 0$ for $m \geq d$ and $W_m H^m(\cup_i T_i^*) = H^m(\cup_i T_i^*)$ for $m \leq d-1$. So if we write down the pieces of weight $d-1$ and d of the sequence above, we find exact sequences

$$\begin{array}{cccccccc}
\to & H^{d-1}(\cup_i T_i^*) & \to & W_{d-1}H^d(Y) & \to & 0 & & \\
 & \| & & \cap & & \cap & & \\
\to & H^{d-1}(\cup_i T_i^*) & \to & H^d(Y) & \to & W_d H^d(Y \setminus \text{Sing } Y) & \to & 0,
\end{array}$$

where we use that $W_m H^m(Y) = H^m(Y)$ for a complete variety and that $W_{m-1}H^m(Z) = 0$ for a smooth variety ([De2, Théorème 8.2.4]). So here we find too that

$$\begin{aligned}
W_{d-1}H^d(Y) &= \ker(H^d(Y) \to W_d H^d(Y \setminus \text{Sing } Y)) \\
&= \ker(H^d(Y) \to H^d(Y \setminus \text{Sing } Y)).
\end{aligned}$$

A natural question is whether intersection cohomology also corresponds to the first terms of the expression for the numbers $b_{i,j}$ for non-isolated singularities.

(3) *Reduction from the canonical to the terminal case.* Let Y be a projective variety with Gorenstein canonical singularities. Assuming the Minimal Model Program, there exists a projective birational morphism f from a terminal variety X, such that $K_X = f^*(K_Y)$, i.e. a crepant morphism (the existence of such crepant morphisms has been proved for threefolds;

we used this result in Chapter 1, but it is an open problem in higher dimensions). In particular, by [Ba2, Theorem 3.12], the stringy E-functions of X and Y are equal. Thus under these assumptions, it is enough to prove Batyrev's conjecture for terminal varieties. However, even if one starts with isolated canonical singularities, the terminal singularities may be non-isolated. An example of this phenomenon is the hypersurface singularity $\{x_1^4 + x_2^4 + x_3^5 + x_4^5 + x_5^5 = 0\} \subset \mathbb{A}^5$. The first blow-up gives a crepant morphism from a terminal variety, but the new singular locus is 1-dimensional. So we cannot apply our results from Chapter 1.

In this context, we can prove the following partial result, for 4-dimensional varieties. Let Z be a 4-dimensional variety with at most Gorenstein canonical isolated singularities. Suppose that $f : Y \to Z$ is a crepant morphism from a terminal projective variety. Take a log resolution $g : X \to Y$ that is also a log resolution of Z. Let D be the total exceptional locus of $f \circ g$, and D' the total exceptional locus of g. Denote the irreducible exceptional components with discrepancy coefficient 0 by $D_i, i = 1, \ldots, \beta$ and the others by $D_i, i = \beta + 1, \ldots, \alpha$. Put

$$D' := \bigcup_{i=\beta+1}^{\alpha} D_i, \quad D^{(j)} := \coprod_{\substack{J \subset \{1,\ldots,\alpha\} \\ |J|=j+1}} D_i, \quad D'^{(j)} := \coprod_{\substack{J \subset \{\beta+1,\ldots,\alpha\} \\ |J|=j+1}} D_i.$$

From Theorem (1.1.1) we know that there are surjections $H^i(X) \to H^i(D)$ for $i \geq 4$, making the mixed Hodge structure on $H^i(D)$ pure.

Proposition.[1] *The cohomology groups $H^4(D'), H^5(D')$ and $H^6(D')$ have a pure Hodge structure as well.*

Proof. Since D' is a normal crossing divisor, we can describe its cohomology by the spectral sequence of (1.1.3). Let us write down the E_1-term explicitly:

[1] For this proposition I am indebted to J. van Hamel.

$$
\begin{array}{c|ccccccccc}
q & & & & & & & & & \\
7 & 0 & & & & & & & & \\
6 & H^6(D'^{(0)}) & \rightarrow & 0 & & & & & & \\
5 & H^5(D'^{(0)}) & \rightarrow & 0 & & & & & & \\
4 & H^4(D'^{(0)}) & \dashrightarrow & H^4(D'^{(1)}) & \rightarrow & 0 & & & & \\
3 & H^3(D'^{(0)}) & \dashrightarrow & H^3(D'^{(1)}) & \rightarrow & 0 & & & & \\
2 & H^2(D'^{(0)}) & \rightarrow & H^2(D'^{(1)}) & \dashrightarrow & H^2(D'^{(2)}) & \rightarrow & 0 & & \\
1 & H^1(D'^{(0)}) & \rightarrow & H^1(D'^{(1)}) & \rightarrow & H^1(D'^{(2)}) & \rightarrow & 0 & & \\
0 & H^0(D'^{(0)}) & \rightarrow & H^0(D'^{(1)}) & \rightarrow & H^0(D'^{(2)}) & \rightarrow & H^0(D'^{(3)}) & \rightarrow & 0 \\
\hline
 & 0 & & 1 & & 2 & & 3 & & 4 \quad p
\end{array}
$$

We claim that the dashed arrows describe surjective maps. Recall that this is the case for the corresponding maps in the E_1-term of the cohomology of D, because of the purity of $H^4(D)$ and $H^5(D)$. Consider for example the following commutative diagram induced by these two spectral sequences (the vertical arrows are just restrictions to direct sums with less summands):

$$
\begin{array}{ccc}
H^4(D^{(0)}) & \xrightarrow{d_1} & H^4(D^{(1)}) \\
\downarrow & & \downarrow \\
H^4(D'^{(0)}) & \xrightarrow{d'_1} & H^4(D'^{(1)})
\end{array}
$$

The surjectivity of d_1 and the restrictions implies then the surjectivity of d'_1. In the same way, one can prove the surjectivity for the other maps from the claim. This means that $E_2^{1,4}, E_2^{1,3}$ and $E_2^{2,2}$ are zero, and this proves the proposition. ∎

However, for a proof of Batyrev's conjecture in this setting, we really need the surjectivity of the map

$$H^4(X) \to H^4(D') = \ker(H^4(D'^{(0)}) \to H^4(D'^{(1)})),$$

and this does not follow from the surjectivity of

$$H^4(X) \to H^4(D) = \ker(H^4(D^{(0)}) \to H^4(D^{(1)})).$$

So the question here is how to prove this surjectivity.

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Nederlandse Samenvatting

N.1 Inleiding en basisbegrippen

(N.1.1) Het uitgevoerde onderzoek behoort tot het domein van de *algebraïsche meetkunde*, een deelgebied van de zuivere wiskunde. Algebraïsche meetkunde kent reeds een zeer lange geschiedenis; zelfs de meetkunde van de oude Grieken kan men hiertoe rekenen.

Meer specifiek hebben wij de *stringy Hodge-getallen* bestudeerd (letterlijk 'snarige' Hodge-getallen), zoals gedefinieerd door Batyrev in [Ba2]. Zoals reeds blijkt uit de naamgeving, liet hij zich bij deze definitie inspireren vanuit de snaartheorie in theoretische natuurkunde. Vooraleer we de definitie geven, voeren we eerst een aantal andere begrippen in.

(N.1.2) Zij X een algebraïsche variëteit (we werken altijd over de complexe getallen, X is hier niet noodzakelijk irreducibel) en Y een gesloten deelvariëteit. Deligne heeft aangetoond dat er een natuurlijke *gemengde Hodge-structuur* ligt op de cohomologiegroepen $H^\bullet(X,Y,\mathbb{C})$. In het bijzonder is ook de cohomologie met compacte drager $H_c^\bullet(X,\mathbb{C})$ van X uitgerust met zo'n gemengde Hodge-structuur. Noteer de dimensie van de $H^{p,q}$-component van $H_c^i(X,\mathbb{C})$ met $h^{p,q}(H_c^i(X,\mathbb{C}))$. Men definieert dan de Hodge-Deligne-polynoom van X door middel van de formule

$$H(X;u,v) := \sum_{i=0}^{2d}(-1)^i \sum_{p,q} h^{p,q}(H_c^i(X,\mathbb{C}))u^p v^q \quad \in \mathbb{Z}[u,v].$$

(N.1.3) Vanaf nu bedoelen we met een variëteit steeds een irreducibele algebraïsche verzameling. Zij Y een normale variëteit. We noemen Y *Gorenstein* indien K_Y Cartier is. Zij $f : X \to Y$ dan een logresolutie van Y. Dit betekent dat X niet-singulier is, dat f een proper birationaal morfisme is dat, beperkt tot het inverse beeld van het niet-singuliere deel van Y, een isomorfisme is, en dat bovendien de exceptionele locus van f een divisor is met gladde componenten en normale snijdingen. Noteer de irreducibele exceptionele componenten met $D_i, i \in I$. We kunnen de canonieke divisor op X dan schrijven als

$$K_X = f^*(K_Y) + \sum_{i \in I} a_i D_i.$$

De gehele getallen a_i worden de *discrepantiecoëfficiënten* genoemd. Men noemt X *terminaal* indien alle $a_i \geq 1$ en canoniek indien alle $a_i \geq 0$ (dit hangt niet af van de gekozen logresolutie).

(N.1.4) Zij Y een Gorenstein canonieke variëteit. Kies een logresolutie $f : X \to Y$. Noteer de irreducibele exceptionele componenten opnieuw met $D_i, i \in I$, en hun discrepanties met a_i. Voor een deelverzameling $J \subset I$ noteren we $D_J := \cap_{i \in J} D_i$ en $D_J^\circ := D_J \setminus \cup_{i \in I \setminus J} D_i$. Batyrev definieerde dan de *stringy E-functie* van Y door

$$E_{st}(Y; u, v) := \sum_{J \subset I} H(D_J^\circ; u, v) \prod_{i \in J} \frac{uv - 1}{(uv)^{a_i+1} - 1} \qquad \in \mathbb{Z}[[u, v]] \cap \mathbb{Q}(u, v).$$

Voor $J = \emptyset$ moet men D_J beschouwen als X en $\prod_{i \in J}$ als 1. Batyrev toonde met behulp van motivische integratie aan dat deze definitie niet afhangt van de gekozen logresolutie. Bovendien voldoet de stringy E-functie aan volgende eigenschappen.

1. Als Y glad is, dan is $E_{st}(Y; u, v) = H(Y; u, v)$. Als Y een crepante logresolutie $f : X \to Y$ toelaat (d.w.z. $K_X = f^*(K_Y)$), dan is $E_{st}(Y; u, v) = H(X; u, v)$.

2. Zij Y projectief van dimensie d. Dan is

$$E_{st}(Y; u, v) = (uv)^d E_{st}(Y; u^{-1}, v^{-1})$$

en $E_{st}(Y; 0, 0) = 1$.

3. Een alternatieve formule voor de stringy E-functie is

$$E_{st}(Y; u, v) = \sum_{J \subset I} H(D_J; u, v) \prod_{i \in J} \frac{uv - (uv)^{a_i+1}}{(uv)^{a_i+1} - 1}.$$

(N.1.5) Veronderstel nu dat Y ten hoogste Gorenstein canonieke singulariteiten heeft en projectief is van dimensie d. Veronderstel bovendien dat de stringy E-functie een polynoom $\sum_{p,q} a_{p,q} u^p v^q$ is. Dan definieert Batyrev de *stringy Hodge-getallen* van Y door $h_{st}^{p,q}(Y) := (-1)^{p+q} a_{p,q}$. Dan kan men gemakkelijk volgende eigenschappen nagaan (merk de analogie op met de klassieke Hodge-getallen van gladde projectieve variëteiten !).

1. Stringy Hodge-getallen kunnen enkel verschillen van 0 voor $0 \leq p, q \leq d$,
2. $h_{st}^{0,0}(Y) = h_{st}^{d,d}(Y) = 1$,
3. $h_{st}^{p,q}(Y) = h_{st}^{q,p}(Y) = h_{st}^{d-p,d-q}(Y) = h_{st}^{d-q,d-p}(Y)$,
4. voor gladde Y zijn de stringy Hodge-getallen gelijk aan de klassieke Hodge-getallen.

Er is echter één gewenste eigenschap die helemaal niet duidelijk is uit de definitie: positiviteit !

Conjectuur (Batyrev)**.** *Stringy Hodge-getallen zijn positief.*

Dit probleem is het onderwerp van deze thesis. Merk op dat het niet duidelijk is wanneer je mag verwachten dat de stringy E-functie een polynoom is. In feite hebben we ons geconcentreerd op de volgende iets algemenere vraag (die op natuurlijke wijze voortvloeide uit onze bekomen resultaten).

Vraag. *Zij Y een d-dimensionale projectieve variëteit met ten hoogste Gorenstein canonieke singulariteiten. Schrijf de stringy E-functie van Y als een machtreeks $\sum_{i,j \geq 0} b_{i,j} u^i v^j$. Is $(-1)^{i+j} b_{i,j} \geq 0$ voor $i + j \leq d$?*

Voorbeeld. Voor variëteiten die een crepante resolutie toelaten is de conjectuur waar. In het bijzonder is dit het geval voor alle Gorenstein canonieke oppervlakken (de enige toegelaten singulariteiten zijn in dat geval A-D-E singulariteiten).

N.2 Overzicht van de belangrijkste resultaten

(N.2.1) In Hoofdstuk 1 bewijzen we volgend positief antwoord op Vraag (N.1.5).

Stelling. *Zij Y een d-dimensionale Gorenstein projectieve variëteit met ten hoogste geïsoleerde singulariteiten ($d \geq 3$). Zij $f : X \to Y$ een logresolutie. Veronderstel dat de discrepantiecoëfficiënten van de exceptionele componenten strikt groter zijn dan $\lfloor \frac{d-4}{2} \rfloor$ (deze voorwaarde hangt niet af van de gekozen logresolutie). Dan is het antwoord op Vraag (N.1.5) 'ja' voor Y.*

Zij D de totale exceptionele locus van f. Om bovenstaande stelling te bewijzen combineren we een recent resultaat van de Cataldo en Migliorini ($H^i(X,\mathbb{C}) \to H^i(D,\mathbb{C})$ is surjectief voor $i \geq d$) met de welbekende constructie van de gemengde Hodge-structuur op de cohomologie van D. Voor het 3-dimensionale geval kunnen we nog meer aantonen. Reid heeft immers bewezen dat er voor een Gorenstein canonieke variëteit Z steeds een partiële crepante resolutie bestaat vanaf een terminale variëteit Y. In dat geval is $E_{st}(Z) = E_{st}(Y)$ en op Y kunnen we bovenstaande stelling toepassen. We besluiten dus dat het antwoord op Vraag (N.1.5) voor het 3-dimensionale geval steeds 'ja' is!

Bovendien konden we onder de voorwaarden van de stelling een expliciete beschrijven geven van de getallen $(-1)^{i+j} b_{i,j}$ (voor $i + j \leq d$) uit de machtreeksontwikkeling $E_{st}(Y; u, v) = \sum_{i,j} b_{i,j} u^i v^j$, essentieel in termen van dimensies van componenten van de gemengde Hodge-structuur op de cohomologie van Y. Voor een 3-dimensionale Gorenstein canonieke variëteit Z is deze beschrijving dan gegeven in termen van de cohomologie van een partiële crepante resolutie Y.

(N.2.2) In Hoofdstuk 2 berekenen we de bijdrage van een A-D-E singulariteit tot de stringy E-functie in willekeurige dimensie, door expliciet een logresolutie te construeren. Voor 3-dimensionale A-D-E singulariteiten was dit reeds gedaan door Dais en Roczen, maar hun formules voor gevallen D en E bevatten een onnauwkeurigheid. We hebben ze verbeterd en bovendien hebben we hun formules voor geval A kunnen vereenvoudigen. In elke dimensie bekomen we overigens relatief eenvoudige formules.

Als gevolg van deze berekeningen stellen we vast dat variëteiten met ten hoogste A-D-E singulariteiten voldoen aan de voorwaarden van Stelling (N.2.1) en dus is het antwoord op Vraag (N.1.5) 'ja' voor zulke variëteiten. We kunnen onze formules ook gebruiken om in te zien dat variëteiten met

A-D-E singulariteiten enkel een polynomiale stringy E-functie hebben indien ze 3-dimensionaal zijn en singulariteiten hebben van type A_n (n oneven) en/of D_l (l even).

(N.2.3) In Hoofdstuk 3 presenteren we een alternatieve methode om de stringy E-functie te berekenen voor affiene hyperoppervlakken die niet-gedegenereerd zijn ten opzichte van hun Newtonpolyeder. Voor nulpuntsverzamelingen van reguliere functies op een gladde variëteit (dus in het bijzonder voor affiene hyperoppervlakken) hebben Denef en Loeser de *motivische zetafunctie* gedefinieerd. Via de *Hodge-zetafunctie* kan de stringy E-functie (voor canonieke nulpuntsverzamelingen) hier ook uit afgeleid worden. Dit is een mooie toepassing van de 'inversie van adjunctie'-stelling van Ein, Mustaţă en Yasuda.

Voor niet-gedegenereerde affiene hyperoppervlakken bestaat er een algoritme om de motivische zetafunctie te berekenen uit de Newtonpolyeder (van Artal, Cassou-Noguès, Luengo en Melle; in essentie werk van Denef en Hoornaert). Dit algoritme hebben we gebruikt om de formules voor de A-D-E singulariteiten uit Hoofdstuk 2 te controleren.

(N.2.4) Laten we Stelling (N.2.1) nog eens nader bekijken. Eén van de gevallen die net niet onder de voorwaarden van deze stelling vallen is dat van een 6-dimensionale variëteit met terminale singulariteiten (waarbij er dus discrepantiecoëfficiënten gelijk aan 1 mogen zijn). In Hoofdstuk 4 tonen we met een voorbeeld aan dat deze stelling ook niet kan uitgebreid worden tot dit geval. In het algemeen is het antwoord op Vraag (N.1.5) dus 'neen'! Dit is ietwat verrassend en het zou er op kunnen wijzen dat de conjectuur van Batyrev in het algemeen misschien niet waar is. De singulariteit die leidde tot dit tegenvoorbeeld is gegeven door de oorsprong van

$$\{x_1^5 + x_2^5 + x_3^6 + x_4^6 + x_5^6 + x_6^6 + x_7^6 = 0\} \subset \mathbb{A}^7.$$

In het bijzonder is de methode van Hoofdstuk 3 om de bijdrage tot de stringy E-functie te berekenen voor deze singulariteit veel gemakkelijker dan het berekenen van een logresolutie.

(N.2.5) We besluiten de thesis met enkele opmerkingen. De belangrijkste is de volgende. Als de conjectuur van Batyrev toch waar is, moet de voorwaarde dat de stringy E-functie een polynoom is op essentiële wijze gebruikt worden in het bewijs. Het is echter niet duidelijk hoe deze conditie uitgebuit kan worden.