KATHOLIEKE UNIVERSITEIT LEUVEN
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The quantum $E(2)$ group

Promotor:
Prof. dr. **A. Van Daele**

Proefschrift voorgedragen tot
het behalen van het doctoraat
in de wetenschappen

door

Arnoud Jacobs

November 2005

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It's Amazing.
With the blink of an eye.
We finally see the light.

Aerosmith

Acknowledgements

Once upon a time, I started a journey through the world of quantum groups. This Ph.D.-thesis is the travel report of the last years of my trip. Since I was not alone on this trip, there are several people to say *'thank you'* to. For this reason, we start the thesis with some acknowledgements.

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Abstract

The subject of this Ph.D.-thesis is to be situated in the theory of *locally compact quantum groups* which is a non-commutative generalization of the well-known theory of locally compact groups. The locally compact groups constitute the basic examples of locally compact quantum groups. For each property about groups, the corresponding property for quantum groups is studied.

The theory of locally compact quantum groups is quite recent. The nowadays formulation of the theory is known since 1999. Already earlier (since about 1961), a lot of researchers made a contribution which is used in the present theory.

J. Kustermans and S. Vaes were the ones that finally came up with a satisfactory definition of a *'locally compact quantum group'*. They gave their main definition in 1999 and built up a whole theory, relying on the results of several other researchers (Van Daele, Woronowicz, Baaj, Skandalis,...)

By now, the Kustermans-Vaes theory is widely spread and further developed. As such, we now have a fully equipped theory at our disposal.

It is of course necessary to have interesting examples. There is no theory that can survive without examples. It turns out not to be an easy task to find a new quantum group and then elaborate its typical features. Throughout the years, there has already been done a lot of effort in order to construct and work out several new examples. It is especially S.L. Woronowicz who has made very valuable contributions hereabouts.

In the thesis, we study the *'quantum $E(2)$ group'*. This is an important and very refined example of a non-compact quantum group. The fundamental work has already been done by Woronowicz in 1991. Also S. Baaj made a very important contribution by proving the existence and uniqueness of the Haar weights. The work of Baaj was published in 1992.

The first chapter of the thesis is devoted to the general theory. In particular, we point our attention to the world of (modular) multiplicative unitaries. A new technique is introduced to study invariant weights within this framework.

In Chapter 2, we show that this new technique provides an elegant way to prove that the quantum $E(2)$ group fits into the Kustermans-Vaes theory. We also seize the opportunity to renew and simplify the known results.

Further, we also study some general features of the quantum $E(2)$ group. We will have a closer look at the amenability properties (Chapter 3) and the representation theory (Chapter 4). In this way, we obtain a fully elaborated example of a non-compact quantum group.

Korte samenvatting

Het onderwerp van deze doctoraatsthesis situeert zich in de theorie van *lokaal compacte kwantumgroepen.* Dit is een niet-commutatieve veralgemening van de welbekende theorie van lokaal compacte groepen. De lokaal compacte groepen vormen dan ook de basisvoorbeelden van lokaal compacte kwantumgroepen. Elke gekende eigenschap over lokaal compacte groepen geeft aanleiding tot een studie van de overeenkomstige eigenschap voor kwantumgroepen.

De theorie van kwantumgroepen is vrij recent. In zijn huidige vorm kennen we deze theorie sinds 1999. Maar reeds eerder (ongeveer vanaf 1961) hebben tal van onderzoekers bijdragen geleverd die tot de huidige theorie geleid hebben.

Het waren uiteindelijk J. Kustermans en S. Vaes die er als eersten in gelukt zijn om een bevredigende definitie te geven van wat een *'lokaal compacte kwantumgroep'* nu precies is. Ze formuleerden hun definitie in 1999 en bouwden daarna een hele theorie uit. Zij steunden hierbij op heel wat resultaten van andere onderzoekers (Van Daele, Woronowicz, Baaj, Skandalis,...).

Ondertussen is de theorie van Kustermans en Vaes wijd verspreid en verder ontwikkeld. We beschikken momenteel over een zeer uitgewerkte theorie.

Het is uiteraard nodig om interessante voorbeelden te hebben. Er is immers geen enkele theorie die kan overleven zonder voorbeelden. Het blijkt echter niet eenvoudig om een nieuwe kwantumgroep te vinden en om dan zijn typische kenmerken te beschrijven. Er is dan ook al heel wat tijd ingestoken om nieuwe voorbeelden te construeren en uit te werken. Het is vooral S.L. Woronowicz die hieromtrent belangrijke bijdragen heeft geleverd.

In de thesis bestuderen we de *'kwantum $E(2)$ groep'*. Dit is een belangrijk en zeer verfijnd voorbeeld van een niet-compacte kwantumgroep. De nodige fundamenten voor de constructie van dit voorbeeld zijn reeds in 1991 gelegd door Woronowicz. Ook S. Baaj leverde een heel belangrijke bijdrage door bestaan en uniciteit van de Haargewichten te bewijzen. Dit gebeurde in 1992.

Het eerste hoofdstuk van de thesis behandelt de algemene theorie. In het bijzonder richten we hierin onze aandacht op de wereld van (modulaire) multiplicatieve unitairen. We introduceren een nieuwe techniek voor het bestuderen van invariante gewichten binnen deze structuur.

In Hoofdstuk 2 tonen we dat deze nieuwe techniek een elegante manier levert om te bewijzen dat de kwantum $E(2)$ groep past in de Kustermans-Vaes theorie. We maken bovendien van de gelegenheid gebruik om de gekende resultaten te hernieuwen en te vereenvoudigen.

We bestuderen verder ook enkele algemene kenmerken van de kwantum $E(2)$ groep. Hierbij passeren vooral de amenability eigenschappen (Hoofdstuk 3) en de representatietheorie (Hoofdstuk 4) de revue. Op deze manier krijgen we een volledig uitgewerkt voorbeeld van een niet-compacte kwantumgroep.

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Introduction

In this thesis, we study the *quantum* $E(2)$ *group* of Woronowicz [158, 159, 149] as a locally compact quantum group in the sense of Kustermans and Vaes [65, 66]. We also consider the (opposite) dual quantum group $\hat{E}(2)$. We will give a (very) detailed treatment of both of these two non-compact quantum groups.

The nowadays theory of locally compact quantum groups is built up starting from the basic *definition of Kustermans and Vaes* (cf. Definition 1.1.6). In its present form, this study of quantum groups in the operator algebra setting is quite elaborated. Due to the work of many researchers, several aspects of the general theory are now well-understood and applications can be found in different domains. In Section 1.1, we will state the main definition and give, in short, an overview of the most important results of the general theory.

Of course, it is important to have enough (non-trivial) examples of locally compact quantum groups illustrating the general theory. With this thesis, we make a contribution to this by giving a detailed study of one such example. We will focus on the quantum $E(2)$ group. The construction and study of examples is really an apart branch in the quantum group theory. In any non-trivial example, a rather high level of complexity of techniques is involved. This is the reason why a complete thesis is needed in order to fully elaborate a single example.

The purpose of this Ph.D.-thesis is to describe all the aspects of the quantum $E(2)$ group in an accurate way. From the main Chapter 2 onwards, we will give the complete construction of this non-compact quantum group and its related objects. While doing this, a lot of our attention will go to the study of the specific properties of both the quantum $E(2)$ group and its dual.

Before we concentrate on the quantum $E(2)$ group, we first explore the world of (modular) multiplicative unitaries. The main focus is on the appearance of *Haar weights* within this framework; see Sections 1.3 and 1.4.

This study of multiplicative unitaries is useful on its own. It eventually leads up to a new technique to construct a family of invariant weights.

Preliminaries

The two quantum groups $E(2)$ and $\hat{E}(2)$ were first introduced and studied by S.L. Woronowicz in [158, 159, 149]. He constructed two pairs (A, Φ) and $(\hat{A}, \hat{\Phi})$

of a C*-algebra together with a comultiplication. The starting point of his construction was a quantum deformation of the group of motions on the (Euclidean) plane. This deformation is explained in [157]. The two considered bi-C*-algebras had enough properties to be called *'locally compact quantum groups'*.

The example of the quantum $E(2)$ group was further explored by S. Baaj in [1, 2]. He was the first to obtain a formula for the Haar weights. Using advanced techniques, he was able to prove the existence of left and right invariant weights on both the bi-C*-algebras (A, Φ) and $(\hat{A}, \hat{\Phi})$. Moreover, he could also prove a uniqueness property. These results completed (in a sense) the quantum group structure of the quantum $E(2)$ group.

However, these results of Woronowicz and Baaj were obtained already in 1991 and 1992. In that time, there was not yet developed a satisfactory general theory of locally compact quantum groups. This was only done in 1999, when J. Kustermans and S. Vaes started to publish their series of papers [65, 66, 67, 68]. So, in a sense, the quantum $E(2)$ group was an example before there was a theory. As such, it played (together with other examples) an important role in the development of a general quantum group theory. All the experts were convinced that any such theory, not including the quantum $E(2)$ group as a basic example, was not a good theory. In this way, the structure of the quantum $E(2)$ group was an indication for finding the adequate axioms in the search for a satisfactory definition of a locally compact quantum group.

Thus, in between the original construction of the quantum $E(2)$ group and the development of the general theory, there lies a period of almost 10 years. In the meanwhile, a lot of important research was done in the quantum group theory. For instance, Baaj and Skandalis published their fundamental paper on *multiplicative unitaries* [4]. Also, there was the proposal of Kirchberg [52] to introduce a polar decomposition for the antipode. Both these innovations had a major effect on the development of the nowadays (operator algebra) approach to quantum groups. Further, several new concepts were introduced and studied in the area of locally compact quantum groups. An example of this is that one began to study *regularity* and *amenability* of quantum groups.

The part of quantum group theory specifically related to the study of examples has also changed a lot through the years. Woronowicz introduced several new concepts which are very useful in the construction of examples of locally compact quantum groups. The two most important ones are the *manageability theory* for multiplicative unitaries [162, 114] and the machinery around the notion of elements *affiliated* to a C*-algebra [157, 161]. We will have a closer look at these concepts in Sections 1.3 to 1.5. It should further be mentioned that Woronowicz also provided us with new typical techniques; see e.g. his study of the quantum $az+b$-group [165] and the quantum $ax+b$-group [169].

Because of all the new features that appeared in the theory of locally compact quantum groups since Woronowicz' construction of the quantum $E(2)$ group, we believe that it is certainly worthwhile to revise and update the treatment of this example. In the thesis, we present a nowadays proof of the fact that the quantum $E(2)$ group is indeed a locally compact quantum group in the sense of Kustermans and Vaes. This basic result is stated in Theorem 2.4.3. It should certainly be remarked that a proof can also be deduced from the earlier results proved by S. Baaj in [1]. We assess our approach as more streamlined.

Actually, the reason for writing this thesis is threefold. As explained above, the first reason is to give a thoroughly nowadays approach to the quantum $E(2)$ group. Secondly, we have a closer look at the technique of Van Daele [148] to prove the invariance of the Haar weights. In Section 1.4, we formulate (and prove) a generalization of this technique in the general framework of modular multiplicative unitaries. In this way, we show that the Van Daele technique is quite general applicable to construct Haar weights in specific examples of quantum groups. We will apply the generalized technique in Chapter 2 when studying the quantum $E(2)$ group. Finally, as a third reason, we also seize the opportunity to study all the known features of locally compact quantum groups in the case of the quantum $E(2)$ group and its dual. We give a detailed study of the *amenability properties* and we unravel the *representation theory*.

The approach followed in the thesis is fully adapted to the nowadays theory of locally compact quantum groups. This is clearly seen as our work is strongly influenced by the recent fundamental work of Kustermans and Vaes [65, 66] and Woronowicz [162, 161] on the topic. As a consequence, our new treatment of the quantum $E(2)$ group fits completely into the framework of Kustermans and Vaes. The thesis is written in the spirit of the recent work of Van Daele [148] and Woronowicz [165, 169] in which the quantum $az+b$-group and the quantum $ax+b$-group are studied. We are convinced that the approach in these articles is a good one to deal with examples of locally compact quantum groups.

Of course, the earlier work of Woronowicz and Baaj is an important source of inspiration. Nevertheless, we will look at the quantum $E(2)$ group from our own point of view. We think that our approach is more instructive than the previous ones, mainly because we can now rely on the recent developments in quantum group theory. Also the fact that we give a complete working-out of the algebraic motivation helps to improve the readability of the text. This algebraic part is included to increase the intuition for the operator algebra results.

We like to point out that our main contribution to the quantum $E(2)$ group is not that we prove revolutionary new facts, but that we give an easy to read and complete study of this example. We also prove some new properties and simplify proofs of known facts. Another achievement is that we show that the Haar weights fit into the general framework built up in Section 1.4.

At some points, our approach to the quantum $E(2)$ group differs from both of the approaches of Woronowicz and Baaj, although there is certainly no fundamental difference. Especially in the main formulas, there will only be minor changes (if any). Throughout the thesis, we will explain some differences and similarities between the different approaches. We further mention that also A. Pal studied the quantum $E(2)$ group [88, 89]. In his work, he mainly focussed on the contraction procedure of Woronowicz [160] relating the quantum $E(2)$ group and the quantum $SU(2)$ group. However, this approach is completely different.

As the central object in this thesis is the quantum $E(2)$ group, it is appropriate to mention a few words about that part of the quantum group theory that is concerned with examples. In Section 1.2, we will discuss some generalities about the usual treatment of examples. Here in the Introduction, we already shortly elucidate some aspects of how to construct (atomic) examples.

Studying a concrete example can be quite different from working in the general theory. It requires special techniques which are not always easy. In his recent work, Woronowicz provided us with several new, typical techniques which are very useful (also see above). Nevertheless, it still takes a lot of work to construct a non-trivial example of a quantum group. Especially the 'atomic examples' (examples built up using Method 1 in Section 1.2) can be quite hard to work with. A lot easier to handle are for instance the examples constructed by using the bicrossed product construction of Vaes and Vainerman [131, 132].

Many examples of bi-C*-algebras arise from a quantum deformation of a classical group (also the quantum $E(2)$ group; see above). These deformations are first obtained on the Hopf *-algebra level and then lifted to the C*-algebra context. In between, there is a Hilbert space step. The lifting procedure is usually rather complicated, but interesting analysis is involved. Especially in the non-compact case, there can appear some non-trivial technical difficulties.

As mentioned above, it is Woronowicz who has done some remarkable work in producing such quantum deformations. However, having a bi-C*-algebra, there remains the problem of constructing Haar weights. In his very recent work [166], Woronowicz displayed a formula for right invariant weights that fits into the framework of manageable multiplicative unitaries. Unfortunately, his technique does not always produce a Haar weight; see Remark 2.7.20.

Preceding the results of Woronowicz, there was already made some progress in the area. The work of Van Daele [148] probably was the first systematic study of Haar weights on quantum group examples. In Section 1.4, we base on the technique of Van Daele to prove a formula for the Haar weights that is more general than the one of Woronowicz. This new formula is strongly motivated by the results in [134]. As we will illustrate in Chapter 2, it yields a quite powerful technique to construct Haar weights in concrete examples.

Examples of (atomic) quantum groups are thus usually constructed on three levels. First, there is the *Hopf *-algebra level*. This level contains the idea behind the deformation and consists of the construction of a dual pair of Hopf *-algebras. Apart from the deformation idea (which is non-trivial), this level mostly does not contain many advanced results. It is mainly used as an algebraic intuition for later results. The next step is the *Hilbert space level*. In this step, the generators of the considered Hopf *-algebras are represented as (possibly unbounded) operators on a Hilbert space. This level contains all the technical details. The last step in the construction procedure is the *C*-algebra level*. Here, the example is studied as a locally compact quantum group. It is also the level where the Haar weights come into the picture.

In Chapter 2, we construct the quantum $E(2)$ group by using the above method. Besides the quantum $E(2)$ group, there are several other examples of locally compact quantum groups built up in this way. The most interesting ones are the quantum $az+b$-group, the quantum $ax+b$-group and the quantum $\widetilde{SU}_q(1,1)$ group. We will give references for these (and other) examples in Section 1.2.

Thus, in this thesis, we explore the example of the quantum $E(2)$ group. We will give a complete description of all the aspects of this non-compact quantum group. On the technical level, we are very precise. We however try to write things in a not too detailed way in order not to let technical details prevent a better understanding of what is really going on. Nevertheless, it should take the reader little effort to make all the arguments completely rigorous.

As will be explained in Section 1.1, locally compact quantum groups are studied in different settings. The main distinction is in the choice of applying either C*-algebra theory or von Neumann algebra theory. The C*-algebra approach is historically the most important one. Especially the last few years, it is clear that the von Neumann algebra approach is coming on. Due to its richness in properties, it can sometimes be very convenient to use the von Neumann algebra theory. This is certainly the case when weight theory is involved, because we then have the Tomita-Takesaki theory at our disposal. Further, the von Neumann algebra setting is more appropriate to deal with technicalities.

For this Ph.D.-thesis, we have opted to mainly work in the (reduced) C*-algebra setting. The reason for this choice is that this setting is the most natural one to study the quantum $E(2)$ group in; e.g. because we can use the theory of elements affiliated to a C*-algebra [157, 161]. We will however not neglect to also consider some results relating to the von Neumann algebra setting.

Structure of the thesis

Before we really jump into the quantum world, we first give a short overview of the *structure* and the *content* of this thesis.

The thesis contains 5 chapters.

Chapter 1 - Locally compact quantum groups. We start, in Chapter 1, by giving a short introduction to the general theory of *locally compact quantum groups*. This Chapter 1 is divided into 5 sections. In the first Section 1.1, we formulate the definition of Kustermans and Vaes (cf. Definition 1.1.6) and we give some related remarks. We also say a few words about the history behind this definition. Further, we mention the most important basic properties of quantum groups. The second Section 1.2 is devoted to *examples* of quantum groups. Here, we discuss a few different methods for constructing examples. We also describe the standard examples and include a list of further known examples.

The next three sections are used to study some important concepts which are particularly useful when studying a concrete quantum group example. In Section 1.3, we initiate the reader to the world of *multiplicative unitaries*. Most of our attention will go to *modular* multiplicative unitaries. We describe the main results of Woronowicz [162] and prove some extra properties.

Also in Section 1.4, the notion of modularity plays an important role. Combining the modularity results with the technique of Van Daele [148], we create a quite general framework wherein one can study invariance of weights. We display a concrete description of a family of weights ψ_q that are strongly right invariant. Each of these weights is a good candidate for being a Haar weight.

Finally, in the fifth and last Section 1.5, we gather some information about the *affiliation relation*. We include these results for sake of completeness.

Chapter 2 - The quantum $E(2)$ group and its dual. This is the main chapter. The complete Chapter 2 is devoted to the construction of the quantum $E(2)$ group. We will handle this in a (very) detailed way. The principal result is that the quantum $E(2)$ group is a locally compact quantum group in the sense of Kustermans and Vaes; see Theorem 2.4.3. Moreover, it is non-compact and non-discrete. Simultaneously with the construction of the quantum $E(2)$ group, we also construct the (opposite) dual quantum $\hat{E}(2)$ group. Of course, we will not only expound the constructions, but also have a close look at the specific properties of both of these two examples.

The constructions of the quantum $E(2)$ group and its dual are performed in Section 2.4 in a short and standard way. The Sections 2.5 to 2.8 contain the details about the typical features of these examples.

The main actor in our construction is the *multiplicative unitary* W. We present a new, direct proof of the fact that this unitary operator is *manageable* in the sense of Woronowicz [162]. This gives us the possibility to simplify the construction of the quantum $E(2)$ group (A, Φ). From the manageability theory, we can directly deduce the basic properties of the C*-algebra A and the comultiplication Φ. For the dual quantum group $(\hat{A}, \hat{\Phi})$, we automatically get similar results.

The next step is the construction of the *Haar weights*. This will be an easy application of the general results proved in Section 1.4. The quantum $E(2)$ group is unimodular; the dual quantum $\hat{E}(2)$ group is non-unimodular.

By using all these results, we will find that both the quantum $E(2)$ group and its dual are locally compact quantum groups according to Definition 1.1.6. After establishing this main result, we display more traits of the bi-C*-algebras (A, Φ) and $(\hat{A}, \hat{\Phi})$. We also construct the (unbounded) antipodes S and $\hat{S}$ through their polar decomposition. We then focus on some distinctive characteristics such as the modular elements and the left and right regular representations. Further, we consider regularity properties and we give some duality theory.

Chapter 3 - Amenability properties. Then, in Chapter 3, we look at some interesting amenability results. We first give a short introduction to the general theory of amenability in the quantum group case. After this, we study the specific amenability properties of the quantum groups $E(2)$ and $\hat{E}(2)$. It turns out that both these quantum groups are strongly amenable. In the two cases, we are able to construct infinitely many linearly independent (left) invariant means. It is the first appearance of this foreseen property.

Chapter 4 - Representation theory. The Chapter 4 is devoted to the representation theory. First, we describe the C*-algebras A and $\hat{A}$ in terms of crossed products and we prove some related universality properties. Using these results, we then completely unravel the representation theory of the quantum $E(2)$ group and its dual. In both cases, we display a detailed description of all the irreducible *-representations and all the corepresentations.

Chapter 5 - Conclusions and open problems. In Chapter 5, we conclude the thesis. Besides the conclusions, we also list some open problems. These may be of interest for future research on the quantum $E(2)$ group.

There are also two appendices.

Appendix A - The function F_μ and its Fourier transform. In the first Appendix A, we study the quantum exponential function F_μ together with its Fourier coefficients. The formulated results are very important in calculations throughout the thesis. They were obtained before by Woronowicz and Baaj. For clearness, we have chosen to rewrite the proofs in our setting.

Appendix B - Technical results. In Appendix B, we collect some technical results. We look at basic features in weight theory and in the theory of one-parameter representations. Further, we discuss the functional calculus.

Notations and conventions

To end the Introduction, we collect some basic notions. We fix the used notations and settle some conventions. Further, we also include standard references.

Throughout the thesis, we make extensive use of operator algebra theory. Basic references for this are [81, 92, 119]. Basic references for the theory of (unbounded) operators on a Hilbert space are [19, 105, 116]. A text about Hilbert space operators which is very well suited for our purposes is [59].

We only include some basic notations here. The more advanced notations (e.g. notations concerning weights or one-parameter representations) are explained in Appendix B. In this Appendix B, the reader can also find some remarks about the functional calculus for normal operators.

First, we fix a number $\mu \in \mathbb{R}$ with $0 < \mu < 1$.

The unit circle in $\mathbb{C}$ is denoted by S^1. Further, we use the following notations:

$$\mathbb{C}^\mu = \{\mu^k z \mid k \in \mathbb{Z}, z \in S^1\}, \qquad \mathbb{C}(\mu^{\frac{1}{2}}) = \{\mu^{\frac{1}{2}k} z \mid k \in \mathbb{Z}, z \in S^1\},$$
$$\mathbb{R}^\mu = \{\mu^k \mid k \in \mathbb{Z}\}, \qquad \mathbb{R}(\mu^{\frac{1}{2}}) = \{\mu^{\frac{1}{2}k} \mid k \in \mathbb{Z}\}.$$

Notations like $\bar{\mathbb{C}}^\mu$ mean that we take the closure; hence $\bar{\mathbb{C}}^\mu = \mathbb{C}^\mu \cup \{0\}$.

For any subset X in a Banach space E, we denote the linear span of X by $\operatorname{span} X$. We use the notation $[X]$ to denote the norm closed linear span and we hereby shorten a notation like $[\{x \mid x \in X\}]$ to $[x \mid x \in X]$. For two linear subsets $A, B \subseteq E$, we always write AB to denote the linear set $\operatorname{span}\{xy \mid x \in A, y \in B\}$ while $A \odot B$ denotes the algebraic tensor product.

The norm on a C*-algebra A is always denoted by $\|\cdot\|$. For a Hilbert space H, we use $\langle \cdot, \cdot \rangle$ to denote the scalar product.

Let X be a locally compact space. Then, we use the notations

$$\begin{aligned} \mathrm{C}(X) &= \{f : X \to \mathbb{C} \mid f \text{ is continuous}\}, \\ \mathrm{C}_b(X) &= \{f \in \mathrm{C}(X) \mid f \text{ is bounded}\}, \\ \mathrm{C}_0(X) &= \{f \in \mathrm{C}(X) \mid f \text{ vanishes at infinity}\}. \end{aligned}$$

If $X_0 \subseteq X$ and $f \in \mathrm{C}(X)$, we use $f|_{X_0}$ to denote the restriction of f to X_0.

For any subset $Y \subseteq X$, we use χ_Y to denote the characteristic function of Y, i.e., we have $\chi_Y(x) = 1$ if $x \in Y$ and $\chi_Y(x) = 0$ if $x \notin Y$.

Further, with A a C*-algebra, we also use the notations

$$\begin{aligned} \mathrm{K}(X, A) &= \{f : X \to A \mid f \text{ is continuous and has compact support}\}, \\ \mathrm{K}(X) &= \{f \in \mathrm{C}(X) \mid f \text{ has compact support}\}. \end{aligned}$$

Let ν be a Borel measure on X. The Hilbert space $\mathrm{L}^2(X, \nu)$ is then standard defined as the set of equivalence classes in

$$\mathfrak{L}^2(X, \nu) = \{f : X \to \mathbb{C} \mid f \text{ is square integrable with respect to } \nu\}$$

The regarding equivalence relation $\sim$ is defined on $\mathfrak{L}^2(X, \nu)$ by setting $f \sim g$ if and only if $f(s) = g(s)$ for almost all $s \in X$.

We remark that we set $\ell^2(X) = L^2(X, \nu)$ when the measure ν is discrete.

In the case that G is a locally compact group with left Haar measure ν, we use the short notation $\mathrm{L}^2(G)$ to denote $\mathrm{L}^2(G, \nu)$.

For any locally compact group G, we use $\mathrm{C}^*_r(G)$ to denote the reduced group C*-algebra. This C*-algebra is built up in Example 1.2.4.

For a C*-algebra A, we denote by $M(A)$ the *multiplier algebra* and by A^η the set of *affiliated elements*. More information about these sets (and topologies on these sets) can be found in Section 1.5. We always use the notation $a \,\eta\, A$ to denote that $a \in A^\eta$. The cone of positive elements in A is denoted by A^+.

Further, A^* is used to denote the set of continuous linear functionals. The subset of positive linear functionals is denoted by A^*_+. For a von Neumann algebra M, we also use the notation M_* for the set of normal functionals. We use M_*^+ to denote the subset of positive normal functionals.

We remark that there is a notion of a *strict linear map* on A. These linear maps can be extended to $M(A)$ in a unique manner. Below, we already make use of these extensions. We refer to Section 1.5 for more details.

For a concrete C*-algebra $A \subseteq B(H)$, we use the notation A' to denote the *commutant* of A. Similarly, A'' denotes the *bicommutant* of A. We have that A' and A'' are von Neumann algebras. Moreover, it is a standard result that the bicommutant A'' is equal to the weak (or strong) closure of A.

Let A and B be two C*-algebras. The notation $A \cong B$ then means that A and B are *-isomorphic, i.e., there exists a *-isomorphism $j : A \to B$. The set of *-automorphisms on A is denoted by $\mathrm{Aut}(A)$. If M is a von Neumann algebra, then $\mathrm{Aut}(M)$ always denotes the set of normal *-automorphisms.

If H is a Hilbert space, we denote by $B(H)$, respectively $B_0(H)$, the C*-algebras of bounded and compact operators on H. As $B(H)$ is a von Neumann algebra, the notation $B(H)_*$ is well-defined. We notice that (under identification) we have $M(B_0(H)) = B(H)$ and $B(H)_* = B_0(H)^*$. If $\xi, \eta \in H$, we define the vector functional $\omega_{\xi,\eta} \in B(H)_*$ by setting $\omega_{\xi,\eta}(x) = \langle x\xi, \eta \rangle$. We also make use of the short notation ω_ξ to denote the positive vector functional $\omega_{\xi,\xi}$.

For an (unbounded) linear map T, we use $D(T)$ to denote its domain. The linear set $\{(\xi, T\xi) \mid \xi \in D(T)\}$ is called the *graph* of T. We say that T is *closed* when the graph of T is closed. A subset $D_0 \subseteq D(T)$ is called a *core* for T when the graph of the restriction of T to D_0 is dense in the graph of T. For any subset $D \subseteq D(T)$, we use $T|_D$ to denote the restriction of T to D.

Let T be a linear operator on a Hilbert space H. The *spectrum* of T is denoted by $\sigma(T)$. Now suppose that T is *densely defined*, i.e., that $D(T)$ is dense in H. We then use T^* to denote the adjoint operator. We say that T is *normal* when $T^*T = TT^*$ and *self-adjoint* when $T = T^*$. We call T *non-singular* when T is injective and *positive* when $\langle T\xi, \xi \rangle \geq 0$ for all $\xi \in H$. We call T *strictly positive* if it is non-singular, positive and self-adjoint. We say that $u \in B(H)$ is a *unitary* when $u^*u = uu^* = 1$. We call $p \in B(H)$ a *projection* when $p = p^* = p^2$.

Let S and T be normal operators on a Hilbert space H. Suppose that S and T commute. We then use $\sigma(S,T)$ to denote their *joint spectrum*.

Also for an anti-linear operator J, we use J^* to denote the adjoint operator. We say that J is an *anti-unitary* operator when $J^*J = JJ^* = 1$. If J is an anti-unitary operator such that $J = J^*$, then J is called a *conjugation*.

Let R and S be two closed operators. In the unbounded case, we have to be very careful when we work with the sum or the product of R and S. Throughout the thesis, we only use the notation $R \dotplus S$ when the sum of R and S is densely defined and closable. Then, $R \dotplus S$ is defined as the closure of $R + S$. Similarly, we only use the notation RS when the product of R and S is densely defined and closable. In this case, RS is defined as the closure of this product. It is important not to forget these two conventions.

We use the symbol $\otimes$ to denote different tensor products. The precise meaning of the symbol depends on its ingredients. If H and K are Hilbert spaces, then $H \otimes K$ is the tensor product Hilbert space. For (possibly unbounded) closed operators R and S on Hilbert spaces H, respectively K, we denote by $R \otimes S$ the (closed) tensor product operator defined on the Hilbert space $H \otimes K$.

In Section 2.1, the symbol $\otimes$ is used to denote the algebraic tensor product between two algebras. If A and B are C*-algebras, then $A \otimes B$ always denotes the minimal C*-algebra tensor product. For von Neumann algebras M and N, we use $M \bar{\otimes} N$ to denote the von Neumann algebra tensor product.

The following standard examples are well-known. For locally compact groups F and G, we have $\mathrm{C}_0(F) \otimes \mathrm{C}_0(G) = \mathrm{C}_0(F \times G)$. For Hilbert spaces H and K, we have $B_0(H) \otimes B_0(K) = B_0(H \otimes K)$ and $B(H) \bar{\otimes} B(K) = B(H \otimes K)$.

Let $A \subseteq B(H)$ and $B \subseteq B(K)$ be two C*-algebras. The flip operator on $H \otimes K$ is denoted by Σ. The flip map on $A \otimes B$ is denoted by $\dot{\sigma}$. Hence, by definition, we have that $\dot{\sigma}(x) = \Sigma x \Sigma$ for every $x \in A \otimes B$.

We will make extensive use of the *leg-numbering notation*. Let A, B and C be three C*-algebras. There exist unique non-degenerate *-homomorphisms

$$\theta_{12} : A \otimes B \to A \otimes B \otimes C, \qquad \theta_{13} : A \otimes B \to A \otimes C \otimes B,$$
$$\theta_{23} : A \otimes B \to C \otimes A \otimes B.$$

such that, for all $a \in A$ and $b \in B$, we have

$$\theta_{12}(a \otimes b) = a \otimes b \otimes 1, \quad \theta_{13}(a \otimes b) = a \otimes 1 \otimes b \quad \text{and} \quad \theta_{23}(a \otimes b) = 1 \otimes a \otimes b.$$

For all $x \in M(A \otimes B)$, we then use the notations

$$x_{12} = \theta_{12}(x), \qquad x_{13} = \theta_{13}(x) \qquad \text{and} \qquad x_{23} = \theta_{23}(x).$$

For any set G, we always use ι (or ι_G) to denote the identity map:

$$\iota : G \to G : x \mapsto x.$$

Let A and B be two C*-algebras and let $\omega \in A^*$. Then, we define $\bar{\omega} \in A^*$ by $\bar{\omega}(x) = \overline{\omega(x^*)}$. For all $a \in M(A)$, we define $a\omega, \omega a \in A^*$ by $a\omega(x) = \omega(xa)$ and $\omega a(x) = \omega(ax)$. We also consider the slice maps

$$\omega \otimes \iota : M(A \otimes B) \to M(B) \qquad \text{and} \qquad \iota \otimes \omega : M(B \otimes A) \to M(B).$$

If moreover $\rho \in B^*$, we consider the tensor product functional

$$\omega \otimes \rho : M(A \otimes B) \to \mathbb{C}.$$

In the von Neumann algebra case, we again make use of the symbol $\bar{\otimes}$ to denote slice maps and tensor product functionals.

Chapter 1

Locally compact quantum groups

As explained in the Introduction, this thesis is completely devoted to the study of the quantum $E(2)$ group in the framework of Kustermans and Vaes [65, 66]. Before we go into the details of the construction of this example, we use this first chapter to give a short introduction to the general theory of *locally compact quantum groups*. The main emphasis lies on the principal results in the theory of Kustermans and Vaes, but we also unfold some aspects of the general concepts introduced by Woronowicz in his study of quantum group examples.

The Sections 1.3 and 1.4 constitute the most important part of Chapter 1. They contain new results concerning the manageability theory which we believe to be interesting. We prove some extra properties of modular multiplicative unitaries and we introduce a new technique to construct Haar weights. In Chapter 2, we will apply these new results in our study of the quantum $E(2)$ group.

We are very brief in our discussion of the general theory. For a complete outline of the theory, we refer to the series of papers [65, 66, 68]. We also mention the paper [67] for those who are looking for a more leisurely approach to locally compact quantum groups. The most extended text about the general theory can be found in the Ph.D.-thesis of S. Vaes [125]. This probably gives the best notes for people who are not yet experts in the subject. For a first encounter with quantum groups, we can also recommend the notes [63] and [69].

We use [125] only as a general reference. Concrete references in this thesis are always made to one of the articles [65, 66, 68]. However, the text [125] is of great importance. Our work is mainly influenced by this fundamental book. We will formulate the results in such a way that they give a concrete illustration of the theory outlined in the first chapter of [125]. In a sense, a shortened version of this thesis could fit into [125] as a (possible) next chapter.

This chapter is split up into 5 sections.

Section 1.1 - The definition of a locally compact quantum group. The first section is devoted to the general Kustermans-Vaes theory. We will give their definition and its main consequences. Further, we introduce the most important objects related to a quantum group together with their basic properties.

Section 1.2 - Examples of quantum groups. Here, we focus on some generalities that concern the study of specific examples of locally compact quantum groups. We describe the basic examples (i.e. the commutative quantum groups and their duals) and we look at some possible methods to construct examples. We also include a list of further known examples.

Section 1.3 - Multiplicative unitaries. In Section 1.3, we explicate the theory of multiplicative unitaries. In particular, we outline Woronowicz' results on manageable (and modular) multiplicative unitaries. These results will be very important in later calculations as the major tool in our construction of the quantum $E(2)$ group is a manageable multiplicative unitary W.

By making ahead use of the results in Section 1.4, it is possible to apply typical quantum group techniques in the setting of modular multiplicative unitaries. This gives us the opportunity to prove some extra properties.

Section 1.4 - A formula for the Haar weights. Also in Section 1.4, we work in the framework of modular multiplicative unitaries. Building on the technique of Van Daele [148], we create a new method to prove invariance of Haar weights. This method is quite general and will provide us with a powerful tool in our study of the Haar weights on the quantum $E(2)$ group.

For a modular multiplicative unitary W, there is a natural way to define an associated bi-C*-algebra (A, Φ) (cf. Section 1.3). The main result of Section 1.4 is stated in Theorem 1.4.26. It describes a concrete family of faithful, lower semi-continuous weights ψ_q that are strongly right invariant on (A, Φ).

Section 1.5 - The affiliation relation. In this last section, we consider the notion of elements affiliated to a C*-algebra. We discuss the main properties of the affiliation relation as they appear in [157, 161]. We also give some remarks about C*-algebras that are generated by affiliated elements.

1.1 The definition of a locally compact quantum group

In this first section, we give a condensed overview of the main results from the general theory of locally compact quantum groups in its present form. As already mentioned above, the nowadays quantum group theory is built up starting from the *definition of Kustermans and Vaes* (cf. Definition 1.1.6). This definition was proposed in 1999 by J. Kustermans and S. Vaes as a possible definition for a (reduced) C*-algebraic quantum group [65, 66]. They thereby settled a research program that was started several decennia ago with the attempt to generalize the beautiful duality theory of abelian locally compact groups; see below.

By now, the definition of Kustermans and Vaes is widely spread and accepted as a 'good' definition. With their definition, J. Kustermans and S. Vaes gave the quantum group theory a very solid base which is of great importance for the further exploration of the theory and the search for new applications. The road to this definition has been quite long. Below, we will very briefly sketch the way that the nowadays theory of quantum groups arose. We refer to the introduction of [125] for a complete description of the historical background. The role of examples is explained in a more detailed way in the next Section 1.2.

The search for a good notion of a locally compact quantum group was originated in order to find a non-commutative analogue of the famous *Pontryagin duality theorem*. After a lot of pioneering work on some special cases, this eventually resulted in the theory of *Kac algebras* which solved the problem of finding a category that includes all locally compact groups and their duals. The theory of Kac algebras was found in the seventies by Kac and Vainerman [136, 137] and, independently, by Enock and Schwarz [26, 27]. We refer to [29] for a complete overview of the Kac algebra theory. For a reformulation of the Kac algebra theory in the C*-algebra language, we can refer to [31, 140].

The theory of Kac algebras defines in an elegant way a category with duality containing all locally compact groups. However, amongst others, Drinfel'd and Woronowicz found (in the eighties) new objects which had enough properties to be called 'quantum group' but did not satisfy the axioms of Kac algebra theory. This indicated that the axioms defining a Kac algebra were too narrow to include all the would-be quantum groups. One began to look for a larger category including all the Kac algebras and the known examples.

The first success in this direction was obtained by S.L. Woronowicz who defined *compact quantum groups* in 1992; published later in [163]. One of the most important features of his theory is the proof of the existence and uniqueness of a Haar measure; also see [145]. As the theory classically agrees with the theory of compact groups, it could not be the final step. Nevertheless, Woronowicz' theory was an enormous breakthrough in the axiomatization of quantum groups. A nice overview of the theory of compact quantum groups can be found in [76]. There is presented a slightly different approach in [82].

Also *discrete quantum groups* were defined and investigated. First, they were studied as the duals of compact quantum groups; see [94]. Effros and Ruan [25] and, independently, Van Daele [146] both gave an intrinsic approach.

Another crucial step towards the nowadays theory of quantum groups was the full development of the theory of *multiplicative unitaries*, i.e., unitary operators satisfying the so-called *pentagonal equation*. W.F. Stinespring was the first to emphasize, in the framework of group duality, the important role played by the fundamental unitary of a locally compact group [117]. Also in the theory of Kac algebras, this fundamental unitary plays a key role; see e.g. [29, 139].

The first systematic study of multiplicative unitaries was made by S. Baaj and G. Skandalis in 1993. In their fundamental paper [4], Baaj and Skandalis also introduced the notions of a *regular and irreducible multiplicative unitary* and of a *Kac system*. Their theory unifies (the basics of) the theories of Kac algebras and compact quantum groups and allows a perfectly working duality theory. Nevertheless, S. Baaj discovered in [1, 2] that the multiplicative unitary of the quantum $E(2)$ group constructed by Woronowicz [159] is not regular and hence does not fit in the Baaj-Skandalis theory; also see Corollary 2.8.24.

In 1996, S.L. Woronowicz proposed the alternative axiom of *manageability* for multiplicative unitaries [162]. Apparently, this seems to be a notion that is very well adapted to quantum groups. The fundamental unitary of any locally compact quantum group is manageable (cf. Proposition 1.1.16). We will see in Sections 1.3 and 1.4 that the manageability theory is very powerful.

Besides the above analytic developments, there was also important work done on the algebraic level. In 1996, A. Van Daele [147] defined in an algebraic way a special class of quantum groups, called *algebraic quantum groups*. Together with J. Kustermans, he also developed a way to construct a C*-algebraic quantum group out of an algebraic quantum group [57, 70]. This leads to a self-dual category including all compact and discrete quantum groups. Of course, this theory is too algebraic to include all locally compact quantum groups.

Other important contributions to the quantum group theory are the work of S. Vaes and A. Van Daele on *Hopf C*-algebras* [124, 133], the work of Masuda and Nakagami on *Woronowicz algebras* [79] and the joint work of Masuda, Nakagami and Woronowicz on *weighted Hopf C*-algebras* [80].

Notwithstanding the beauty and importance of the above developments, none of these theories completely satisfies all the needs for a general theory of locally compact quantum groups. They all suffer from one or more shortcomings. So, one was still looking for a final theory unifying all of the above theories.

The problem was thus settled in 1999 when Kustermans and Vaes gave their (relatively simple) definition of a locally compact quantum group [65, 66]. One year later, they also gave a von Neumann algebraic version [68]. The basic definition of Kustermans and Vaes is both very natural and very powerful. The only (possible) drawback for this definition is the fact that the existence of Haar weights is postulated. However, it seems to be the best definition one can hope for at the moment. A theory that has the existence of the Haar weights as a theorem rather than an axiom still seems far away.

Starting from their basic definition, Kustermans and Vaes have built up a very powerful theory. In their work, they are of course inspired by the aforesaid earlier theories. Throughout the years, many researchers have made a contribution to the further development of the theory of locally compact quantum groups. Several aspects of the theory of locally compact groups are now generalized to the case of quantum groups. Due to all this work, most features of quantum group theory are now understood quite well. This has also lead to interesting applications in other domains; e.g. in the theory of subfactors [127, 129] and in the cohomology theory of locally compact groups [6].

The definition of Kustermans and Vaes

After the above brief history sketch, we now really start overviewing the general theory. In our survey, we will only mention results which will pop up later in the thesis. For proofs, we refer to [66, 68, 125]. Motivational background for the definitions and results can be found in [125, 63, 69].

The first and most important thing to do is of course to formulate the basic definition of Kustermans and Vaes (cf. Definition 1.1.6). Before we do so, it is necessary first to introduce some preliminary standard definitions.

Because a usual group is defined by a space and a multiplication, it is natural to define quantum groups by a *quantum space* and a *comultiplication*. In more mathematical terms, one wants to define locally compact quantum groups by an operator algebra (either a C*-algebra or a von Neumann algebra) and a comultiplication. An initiation to these concepts is thus a good beginning.

First, we give the basic Definition 1.1.1.

Definition 1.1.1 *Let A be a C*-algebra and let $\Phi : A \to M(A \otimes A)$ be a non-degenerate *-homomorphism. We then call (A, Φ) a* bi-C*-algebra *if Φ satisfies* coassociativity*, i.e., if we have $(\Phi \otimes \iota)\Phi = (\iota \otimes \Phi)\Phi$.*

In this case, Φ is called a comultiplication *on the C*-algebra A.*

Remark 1.1.2 *One should observe that Φ has its image in $M(A \otimes A)$ rather than in $A \otimes A$. At first, $\Phi \otimes \iota$ and $\iota \otimes \Phi$ are only defined on $A \otimes A$. Because they are non-degenerate, we can extend these two maps in a unique way to unital *-homomorphisms on $M(A \otimes A)$; see Proposition 1.5.18.*

The following notation is used at several places in this thesis.

Notation 1.1.3 *Let (A, Φ) be a bi-C*-algebra and $\omega, \rho \in A^*$. Then, we define $\omega\rho \in A^*$ by $\omega\rho = (\omega \otimes \rho)\Phi$. If $\omega, \rho \in A^*_+$, then we also have $\omega\rho \in A^*_+$.*

We will define in the Definition 1.1.6 below a *locally compact quantum group* as a bi-C*-algebra satisfying a density condition and possessing a left and a right invariant weight. So, we thus first have to introduce the notion of left (and right) invariance for weights on a bi-C*-algebra.

In the thesis, two different forms of invariance for weights will appear. It is clear that strong invariance is indeed stronger than invariance.

Definition 1.1.4 *Consider a bi-C*-algebra (A, Φ) and a weight φ on A^+.*

- *The weight φ is called* left invariant *if $\varphi\big((\omega \otimes \iota)\Phi(a)\big) = \omega(1)\varphi(a)$ for all $a \in \mathfrak{M}^+_\varphi$ and $\omega \in A^*_+$. It is called* strongly *left invariant if this equality can be extended to hold for all $a \in A^+$ and $\omega \in A^*_+$;*

- *The weight φ is called* right invariant *if $\varphi\big((\iota\otimes\omega)\Phi(a)\big) = \omega(1)\varphi(a)$ for all $a \in \mathfrak{M}_\varphi^+$ and $\omega \in A_+^*$. It is called* strongly *right invariant if this equality can be extended to hold for all $a \in A^+$ and $\omega \in A_+^*$.*

Remark 1.1.5 *Notice that in Definition 1.1.4, we implicitly use the extension of the weight φ to $M(A)^+$ (see Section B.1). For a general bi-C*-algebra (A,Φ), we only know that $(\omega\otimes\iota)\Phi(a) \in M(A)^+$ for $a \in A^+$.*

We are now ready to state the main definition. This powerful definition was introduced by J. Kustermans and S. Vaes in their fundamental papers [65, 66]. It is the starting point of the nowadays theory of quantum groups.

Definition 1.1.6 (Kustermans and Vaes) *Consider a C*-algebra A and a non-degenerate *-homomorphism $\Phi : A \to M(A\otimes A)$ such that*

- $(\Phi\otimes\iota)\Phi = (\iota\otimes\Phi)\Phi$*;*
- $A = [\,(\omega\otimes\iota)\Phi(a) \mid \omega \in A^*, a \in A] = [\,(\iota\otimes\omega)\Phi(a) \mid \omega \in A^*, a \in A]$.

Assume, moreover, the existence of

- *a faithful left invariant approximate KMS-weight φ on (A,Φ);*
- *a right invariant approximate KMS-weight ψ on (A,Φ).*

Then we call (A,Φ) a locally compact quantum group*.*

The next Terminology 1.1.7 is justified by Theorem 1.1.10 below.

Terminology 1.1.7 *Let (A,Φ) be a locally compact quantum group and let φ and ψ be the weights appearing in Definition 1.1.6.*

We call φ the left Haar weight *of (A,Φ) and ψ the* right Haar weight*.*

The next Remark 1.1.8 is quite important. We remark that we use this property in the thesis without explicitly referring to here.

Remark 1.1.8 *It follows from Proposition 1.5.21 that the density conditions in Definition 1.1.6 are a weakening of the* cancellation law*, i.e., the property that the linear sets $\Phi(A)(1\otimes A)$ and $\Phi(A)(A\otimes 1)$ are dense in $A\otimes A$.*

Nevertheless, the cancellation law will hold in every locally compact quantum group. This property is stated in Theorem 1.1.11.

The Remark 1.1.9 below also goes along with the Definition 1.1.6.

Remark 1.1.9 *In the papers [65, 66], the object defined in Definition 1.1.6 is called a* reduced C*-algebraic quantum group*.*

The reason for this terminology is that there exist other, equivalent, approaches to quantum groups. One can work with universal quantum groups *[61] or in the* von Neumann algebraic setting *[68]. See also the Definition 1.1.40.*

The main results of the general theory

We now have a look at the main consequences of Definition 1.1.6. We list the basic results from the general theory. These results are all due to Kustermans and Vaes. As mentioned above, proofs can be found in [66, 68, 125].

Although the Theorems 1.1.10 and 1.1.11 below are very natural, they are far from being trivial. In fact, these two theorems contain some of the most advanced results of the general theory. The full force of the work of Kustermans and Vaes is needed to prove them in a rigorous way.

We state in Theorem 1.1.10 the strongest known uniqueness result concerning the Haar weights. This *uniqueness theorem* is a very crucial property.

Theorem 1.1.10 *Let (A, Φ) be a locally compact quantum group. Let φ and ψ be the weights appearing in Definition 1.1.6.*

Further, assume that η is a non-zero, densely defined, lower semi-continuous weight on A^+. The properties below hold:

- *If η is left invariant, there exists a number $r > 0$ such that $\eta = r\varphi$;*
- *If η is right invariant, there exists a number $r > 0$ such that $\eta = r\psi$.*

The next Theorem 1.1.11 gathers some very deep, non-trivial results. In particular, it gives that the invariance of the Haar weights holds in the strongest form (although only a weak form is required in Definition 1.1.6).

We further mention that the density condition in the Theorem 1.1.11 below is known as the *cancellation law*.

It is clear that Theorem 1.1.11 provides some very useful information.

Theorem 1.1.11 *Let (A, Φ) be a locally compact quantum group. Let φ be the left Haar weight and ψ the right Haar weight.*

We have the following properties:

- *The linear sets $\Phi(A)(A \otimes 1)$ and $\Phi(A)(1 \otimes A)$ are dense in $A \otimes A$;*
- *The weight φ is a faithful, strongly left invariant KMS-weight on (A, Φ);*
- *The weight ψ is a faithful, strongly right invariant KMS-weight on (A, Φ).*

The next Corollary 1.1.12 is an (almost) immediate consequence

Corollary 1.1.12 *Let (A, Φ) be a locally compact quantum group. Let φ be the left Haar weight and ψ the right Haar weight. Then $(A, \dot{\sigma}\Phi)$ is a locally compact quantum group with left Haar weight ψ and right Haar weight φ.*

We call $(A, \dot{\sigma}\Phi)$ the opposite *quantum group of (A, Φ).*

Notation 1.1.13 *We also use $(A, \Phi)^{\mathrm{op}}$ to denote $(A, \dot{\sigma}\Phi)$.*

In order to get more insight on how the general theory works, we discuss the most important objects related to a (fixed) quantum group (A, Φ). We define these objects and describe their standard features.

First, we fix the basic notations. For notational convenience, we keep these notations fixed for the remainder of this Section 1.1.

Notation 1.1.14 *Let (A, Φ) be a locally compact quantum group with left Haar weight φ and right Haar weight ψ. The GNS-construction of the weight φ is denoted by $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$. Because π_φ is faithful, we can (and will) identify the C^*-algebras A and $\pi_\varphi(A)$ and assume that A acts on H_φ.*

Because φ and ψ are faithful KMS-weights (cf. Theorem 1.1.11), we can introduce the following Notation 1.1.15.

Notation 1.1.15 *We denote the unique modular automorphism group of φ by σ. We use σ' to denote the unique modular automorphism group of ψ. We have that σ and σ' are norm continuous one-parameter groups on A.*

The first object that Kustermans and Vaes build up starting from Definition 1.1.6 is the (left) *regular representation* W. This multiplicative unitary is irrefutably one of the main actors in the general theory.

The importance of the operator W can hardly be overestimated. We remark that manageability for multiplicative unitaries is defined in Section 1.3.

We have the following crucial Proposition 1.1.16.

Proposition 1.1.16 *There exists a unique unitary operator $W \in B(H_\varphi \otimes H_\varphi)$ such that*

$$W^* \big(\Lambda_\varphi(x) \otimes \Lambda_\varphi(y)\big) = (\Lambda_\varphi \otimes \Lambda_\varphi)\big(\Phi(y)(x \otimes 1)\big) \tag{1.1}$$

for all $x, y \in \mathfrak{N}_\varphi$.

The unitary W is multiplicative, i.e., it satisfies the pentagonal equation

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

Moreover, we have that W is manageable.

We call W the left regular representation *of (A, Φ).*

The next Remark 1.1.17 is about the proof of Proposition 1.1.16.

Remark 1.1.17 *Using the invariance of the left Haar weight φ, it is not so difficult to construct an isometry W^* satisfying Equation* (1.1)*. However, it requires a lot of hard work to prove that W^* is indeed a unitary operator. For this, one also needs the right Haar weight ψ.*

Thus, we here see one of the reasons for requiring the existence of both a left and a right Haar weight in Definition 1.1.6. The proof of Proposition 1.1.16 is one of the major achievements of Kustermans and Vaes.

The great importance of W is already clear from the following result. It says that we can describe the complete structure of the bi-C*-algebra (A, Φ) in terms of W. We recall that we identify the C*-algebras A and $\pi_\varphi(A)$.

It is lucid that the formulas in Proposition 1.1.18 below are very useful.

Proposition 1.1.18 *We have that*

$$A = [\,(\iota \otimes \omega)W \mid \omega \in B(H_\varphi)_*].$$

For all $x \in A$, we have

$$\Phi(x) = W^*(1 \otimes x)W.$$

The next step is the construction of the *antipode* S. In contrast with Kac algebra theory, the antipode can be unbounded in general. For instance, we will see in the Chapter 2 that this is the case for the quantum $E(2)$ group.

Because of its unboundedness, it sometimes can be quite delicate to work with the antipode. Luckily, we have the polar decomposition which helps us (a lot) to handle the problems caused by the unboundedness; see below.

The next Proposition 1.1.19 describes the main features of the antipode.

Proposition 1.1.19 *There exists a unique closed linear map $S : D(S) \subseteq A \to A$ such that, for every $\omega \in B(H_\varphi)_*$, we have $(\iota \otimes \omega)W \in D(S)$ and*

$$S\big((\iota \otimes \omega)W\big) = (\iota \otimes \omega)W^*$$

and where the linear set $\{\,(\iota \otimes \omega)W \mid \omega \in B(H_\varphi)_*\}$ is a core for S.

This linear map S is called the antipode *of (A, Φ).*

Furthermore, the antipode S has a unique polar decomposition *$S = R\tau_{-\frac{i}{2}}$ where (τ_t) is a norm continuous one-parameter group on A and where R is an involutive *-anti-automorphism of A such that R and τ_t commute for all $t \in \mathbb{R}$.*

We call R the unitary antipode *of (A, Φ) and (τ_t) the* scaling group.

The antipode S has the following properties:

1. *We have that $S = R\tau_{-\frac{i}{2}} = \tau_{-\frac{i}{2}}R$,*
2. *S is densely defined and has dense range,*
3. *S is injective and $S^{-1} = R\tau_{\frac{i}{2}} = \tau_{\frac{i}{2}}R$,*
4. *S is anti-multiplicative: for $x, y \in D(S)$, we have that $xy \in D(S)$ and $S(xy) = S(y)S(x)$,*
5. *For all $x \in D(S)$, we have $S(x)^* \in D(S)$ and $S(S(x)^*)^* = x$,*
6. *We have that $S^2 = \tau_{-i}$.*

Moreover, the commutation relations below hold:

1. $R\tau_z = \tau_z R$ *for all* $z \in \mathbb{C}$,
2. $\Phi\tau_t = (\tau_t \otimes \tau_t)\Phi$ *for all* $t \in \mathbb{R}$,
3. $\Phi R = \dot{\sigma}(R \otimes R)\Phi$,
4. $RS = SR$,
5. $\tau_t S = S\tau_t$ *for all* $t \in \mathbb{R}$.

It is important to notice the two Remarks 1.1.20 and 1.1.21.

Remark 1.1.20 *We can extend the scaling group* (τ_t) *in a unique way to a strictly continuous one-parameter group* $(\bar{\tau}_t)$ *on* $M(A)$*; see Section B.2.*

The antipode S *can therefore be extended to a strictly closed linear map* $\bar{S}$ *on* $M(A)$*. This is done by putting*

$$\bar{S} = R\bar{\tau}_{-\frac{i}{2}} = \bar{\tau}_{-\frac{i}{2}}R.$$

We have that $\bar{S}$ *is the strict closure of* S*.*

Every result of Proposition 1.1.19 has a strict counterpart involving the extension $\bar{S}$*. Moreover, if* $x \in D(S)$ *and* $y \in D(\bar{S})$*, we have that* xy*,* $yx \in D(S)$*.*

We remark that S *itself is strictly closed as a linear map on* A*.*

Remark 1.1.21 *Due to the property* $\Phi R = \dot{\sigma}(R \otimes R)\Phi$ *(cf. Proposition 1.1.19) and the uniqueness of the Haar weights (cf. Theorem 1.1.10), we see that we can take* $\psi = \varphi R$*. From this, we also get that* $\sigma'_t = R\sigma_{-t}R$ *for all* $t \in \mathbb{R}$*.*

The Proposition 1.1.22 below collects some more commutation relations.

Proposition 1.1.22

1. *The three automorphism groups* σ*,* σ' *and* τ *commute pairwise.*
2. *For all* $t \in \mathbb{R}$*, we have the following equations:*

$$\Phi\sigma_t = (\tau_t \otimes \sigma_t)\Phi, \qquad \Phi\sigma'_t = (\sigma'_t \otimes \tau_{-t})\Phi,$$
$$\Phi\tau_t = (\tau_t \otimes \tau_t)\Phi, \qquad \Phi\tau_t = (\sigma_t \otimes \sigma'_{-t})\Phi.$$

By definition of a modular automorphism group, we have $\varphi\sigma_t = \varphi$ and $\psi\sigma'_t = \psi$ for all $t \in \mathbb{R}$. The next Proposition 1.1.23 deals with the remaining invariance properties. It is the appropriate place to introduce the *scaling constant.*

Proposition 1.1.23 *There exists a unique number* $\nu > 0$ *such that, for all* $t \in \mathbb{R}$*, we have*

$$\varphi\sigma'_t = \nu^t\varphi, \qquad \psi\sigma_t = \nu^{-t}\psi,$$
$$\psi\tau_t = \nu^{-t}\psi, \qquad \varphi\tau_t = \nu^{-t}\varphi.$$

We call ν *the* scaling constant *of* (A, Φ)*.*

Remark 1.1.24 *The scaling constant ν is a quite peculiar object. Only very recently, it was discovered that it can happen that $\nu \neq 1$.*

It is proved by A. Van Daele in [148] that in the case of the quantum az+b-group (with complex deformation parameter) and the quantum ax+b-group, we have that the scaling constant is not equal to 1.

The left Haar weight φ and the right Haar weight ψ are related to each other through the *modular element*. We define this element δ below by using the theory of *absolutely continuous KMS-weights* (see Section B.1).

We recall that, on an informal level, we have $\varphi_\delta(x) = \varphi(\delta^{\frac{1}{2}} x \delta^{\frac{1}{2}})$ for $x \in A^+$.

Proposition 1.1.25 *There exists a unique strictly positive element $\delta \, \eta \, A$ such that $\sigma_t(\delta) = \nu^t \delta$ for all $t \in \mathbb{R}$ (where ν is the scaling constant) and $\psi = \varphi_\delta$. We call δ the* modular element *of (A, Φ).*

We have the following basic properties:

1. $\psi = \varphi_\delta$ *and* $\varphi = \psi_{\delta^{-1}}$,
2. $\Phi(\delta) = \delta \otimes \delta$,
3. $R(\delta) = \delta^{-1}$ *and* $\tau_t(\delta) = \delta$ *for all* $t \in \mathbb{R}$,
4. *If* $t \in \mathbb{R}$, *then* $\delta^{it} \in D(\bar{S})$ *and* $\bar{S}(\delta^{it}) = \delta^{-it}$,
5. $\sigma_t(\delta) = \sigma'_t(\delta) = \nu^t \delta$ *for all* $t \in \mathbb{R}$,
6. $\sigma'_t(a) = \delta^{it} \sigma_t(a) \delta^{-it}$ *for all* $t \in \mathbb{R}$ *and* $a \in A$.

The modular element δ indicates how much the left Haar weight φ is right invariant and how much the right Haar weight ψ is left invariant. This loose statement is made precise in the Proposition 1.1.26 below.

Proposition 1.1.26 *Let $x \in M^+$. Take $\xi \in D(\delta^{\frac{1}{2}})$ and $\eta \in D(\delta^{-\frac{1}{2}})$. Then, we have that*

1. $\varphi\big((\iota \otimes \omega_\xi)\Phi(x)\big) = \varphi(x)\langle \delta^{\frac{1}{2}}\xi, \delta^{\frac{1}{2}}\xi \rangle$,
2. $\psi\big((\omega_\eta \otimes \iota)\Phi(x)\big) = \psi(x)\langle \delta^{-\frac{1}{2}}\eta, \delta^{-\frac{1}{2}}\eta \rangle$.

We distinguish between the cases $\delta = 1$ and $\delta \neq 1$.

Definition 1.1.27 *If we have $\delta = 1$, then we call (A, Φ)* unimodular. *In the case $\delta \neq 1$, we call (A, Φ)* non-unimodular.

Thus, (A, Φ) is unimodular if and only if $\varphi = \psi$.

As the final theme in our survey of the general theory, we discuss the duality theory as studied within the Kustermans-Vaes framework. For a locally compact quantum group, there is a natural way to construct the *dual quantum group*. This (not so difficult) construction mainly uses ideas and techniques that are known from the theory of multiplicative unitaries [4, 162].

We are again concise with the details. We only briefly indicate how to construct the dual quantum group. The motivation behind the given definitions can be found in [125]. We also refer to [125] to learn more about the close relation between a quantum group and its dual.

We recall that having a nice duality theory was one of the first motivations to search for a general theory of locally compact quantum groups.

Before constructing the dual of (A, Φ), we look back at Proposition 1.1.18. This proposition says that (A, Φ) lives in the first leg of W. Analogously, the dual quantum group $(\hat{A}, \hat{\Phi})$ will live in the second leg of W. In both cases, the comultiplication can be written in terms of the multiplicative unitary W.

We have the following results.

Proposition 1.1.28 *The linear set $\hat{A}$ defined by*

$$\hat{A} = [\,(\omega \otimes \iota)W \mid \omega \in B(H_\varphi)_*]$$

is a C^*-algebra.

We can define a non-degenerate *-homomorphism $\hat{\Phi} : \hat{A} \to M(\hat{A} \otimes \hat{A})$ by the formula

$$\hat{\Phi}(x) = \Sigma W(x \otimes 1)W^*\Sigma.$$

Moreover, we have the existence of

- *a faithful, strongly left invariant KMS-weight $\hat{\varphi}$ on $(\hat{A}, \hat{\Phi})$;*
- *a faithful, strongly right invariant KMS-weight $\hat{\psi}$ on $(\hat{A}, \hat{\Phi})$.*

Thus, $(\hat{A}, \hat{\Phi})$ is a locally compact quantum group. We call it the dual *of (A, Φ).*

Proposition 1.1.29 *The unitary operator $\Sigma W^* \Sigma$ is the left regular representation of $(\hat{A}, \hat{\Phi})$. We use $\hat{W}$ to denote this multiplicative unitary.*

Notation 1.1.30 *We also use $\widehat{(A, \Phi)}$ to denote the dual of (A, Φ).*

Remark 1.1.31 *In the general theory, it is a common practise to flip the comultiplication when constructing the dual quantum group.*

This choice is not always made. We e.g. leave out the flip when we define the comultiplication on the quantum $\hat{E}(2)$ group. In this way, we get that it actually is the opposite dual of the quantum $E(2)$ group (cf. Proposition 2.8.21).

It is explained in Remark 2.1.2 why we have made this particular choice.

The following Proposition 1.1.32 is the basis for the *corepresentation theory*. This is further explained in Chapter 4.

Proposition 1.1.32 *We have that $W \in M(A \otimes \hat{A})$. Moreover, we have*

$$(\Phi \otimes \iota)W = W_{13}W_{23} \qquad \textit{and} \qquad (\iota \otimes \hat{\Phi})W = W_{13}W_{12}.$$

The next *biduality* theorem is of major importance. It generalizes the well-known *Pontryagin duality theorem* for commutative locally compact groups.

Theorem 1.1.33 *Let $(\hat{\hat{A}}, \hat{\hat{\Phi}})$ be the* bidual *quantum group of (A, Φ) (i.e. the dual of $(\hat{A}, \hat{\Phi})$). Then, we have $(\hat{\hat{A}}, \hat{\hat{\Phi}}) = (A, \Phi)$.*

Remark 1.1.34 *We can write $(\hat{\hat{A}}, \hat{\hat{\Phi}}) = (A, \Phi)$ because we identify $\pi_\varphi(A)$ and A. In general, we have that the bidual quantum group $(\hat{\hat{A}}, \hat{\hat{\Phi}})$ of a locally compact quantum group (A, Φ) is isomorphic to (A, Φ).*
More specific, $\pi_\varphi : A \to \hat{\hat{A}}$ is a *-isomorphism such that $(\pi_\varphi \otimes \pi_\varphi)\Phi = \hat{\hat{\Phi}}\pi_\varphi$.
We have moreover that $\hat{\hat{\varphi}}\pi_\varphi = \varphi$ and $\hat{\hat{\psi}}\pi_\varphi = \psi$.

Special types of quantum groups

It is natural to distinguish *special types* of quantum groups. These special types are easier to handle because we are in a less general case and we thus have more known properties. Historically, the special cases below were treated before there was the general framework of Kustermans and Vaes; see above.

First, we have the *commutative* and the *cocommutative* quantum groups. These are the basic examples of locally compact quantum groups. In Section 1.2, we will describe all the quantum groups which are of one of these two special types. This will be done in Examples 1.2.1 and 1.2.4.

Definition 1.1.35 *Let (A, Φ) be a locally compact quantum group. Then, we call (A, Φ)*commutative *if the C^*-algebra A is commutative. We call (A, Φ)* cocommutative *if we have that $\Phi = \dot{\sigma}\Phi$.*

Other special types of locally compact quantum groups are the *compact* and the *discrete* quantum groups. These classes are more interesting than the commutative and cocommutatives ones because they are less clear-cut.

We refer to [163, 76] for studies of compact quantum groups. The papers [146, 25] contain treatments of discrete quantum groups.

Definition 1.1.36 *Let (A, Φ) be a locally compact quantum group. We then call (A, Φ)* compact *if the C^*-algebra A has a unit* 1.

Further, we call (A, Φ) discrete *if the C^*-algebra $\hat{A}$ has a unit* 1.

The next Lemma 1.1.37 is a basic result.

Lemma 1.1.37 *Let (A,Φ) be a discrete quantum group. Then, there exists a family of natural numbers $(n_i)_{i\in I}$ such that $A = \bigoplus_{i\in I} M_{n_i}(\mathbb{C})$.*

It follows from [68, Proposition 4.2] that commutativity and cocommutativity are dual to each other. It is a direct consequence of Theorem 1.1.33 that also compactness and discreteness are dual notions.

Proposition 1.1.38 *Let (A,Φ) be a locally compact quantum group.*

- *We have that (A,Φ) is commutative if and only if $(\hat{A},\hat{\Phi})$ is cocommutative,*
- *We have that (A,Φ) is compact if and only if $(\hat{A},\hat{\Phi})$ is discrete.*

The next Remark 1.1.39 is about compact quantum groups.

Remark 1.1.39 *The compactness property defined in Definition 1.1.36 above does not coincide with the notion of a* compact quantum group *as introduced by Woronowicz in [163]. For a bi-C^*-algebra (A,Φ) with A unital, we have that (A,Φ) is a compact quantum group according to Definitions 1.1.6 and 1.1.36 if and only if (A,Φ) is a compact quantum group in the sense of Woronowicz such that the Haar state h on (A,Φ) is faithful.*

Let (A,Φ) be a compact quantum group in the sense of Woronowicz. Let h be the Haar state on (A,Φ). It is not so difficult to translate the faithfulness of h in terms of A and Φ. The following conditions are equivalent:

- *The Haar state h is faithful;*
- *For all $x \in A^+$ and $\omega \in A^*_+$, we have that $(\omega\otimes\iota)\Phi(x) = 0$ if and only if $\omega = 0$ and $x = 0$;*
- *For all $x \in A^+$ and $\omega \in A^*_+$, we have that $(\iota\otimes\omega)\Phi(x) = 0$ if and only if $\omega = 0$ and $x = 0$.*

This equivalence is proved in [39].

The von Neumann algebra setting

We now also look at the Kustermans-Vaes definition of a *von Neumann algebraic quantum group*. The von Neumann algebra approach to quantum groups is developed similar to the C*-algebra case. We are concise with the details and refer to [68, 125] for a complete outline of the theory.

It is important to mention that there exists a natural bijective correspondence between (reduced) C*-algebraic quantum groups and von Neumann algebraic quantum groups. This is explained in [68, 125].

We have the Definition 1.1.40 below. Notice that there are no density conditions. These will follow automatically from the axioms in the definition.

We recall that the symbol $\bar{\otimes}$ is used to denote von Neumann algebra tensor products. This was already mentioned in the Introduction.

Definition 1.1.40 ([68]) *Consider a von Neumann algebra M and a normal and unital *-homomorphism $\Phi : M \to M \bar{\otimes} M$ such that*

$$(\Phi \bar{\otimes} \iota)\Phi = (\iota \bar{\otimes} \Phi)\Phi.$$

Assume the existence of two n.s.f. weights φ and ψ on M such that

- $\varphi\big((\omega \bar{\otimes} \iota)\Phi(x)\big) = \omega(1)\varphi(x)$ *for all* $x \in \mathfrak{M}_\varphi^+$ *and* $\omega \in M_*^+$*;*
- $\psi\big((\iota \bar{\otimes} \omega)\Phi(x)\big) = \omega(1)\psi(x)$ *for all* $x \in \mathfrak{M}_\psi^+$ *and* $\omega \in M_*^+$*.*

Then we call (M, Φ) a von Neumann algebraic quantum group*.*

We have the following uniqueness result.

Proposition 1.1.41 *Let (M, Φ) be a von Neumann algebraic quantum group. Let φ and ψ be the weights appearing in Definition 1.1.40.*

Assume that η is an n.s.f. weight on M^+. The properties below hold:

- *If η is left invariant, there exists a number $r > 0$ such that $\eta = r\varphi$;*
- *If η is right invariant, there exists a number $r > 0$ such that $\eta = r\psi$.*

It is therefore justified to use the next Terminlogy 1.1.42.

Terminology 1.1.42 *Let (M, Φ) be a von Neumann algebraic quantum group. Let φ and ψ be the weights appearing in Definition 1.1.6.*

We call φ the left Haar weight *of (M, Φ) and ψ the* right Haar weight*.*

Similar to the C*-algebra case, there is also a full development of the theory of von Neumann algebraic quantum groups. Although we will make use of this theory, we do not include its main results. We only make a small remark about the used terminology. We refer to [68, 125] for a full treatment.

Remark 1.1.43 *Let (M, Φ) be a von Neumann algebraic quantum group with left Haar weight φ and right Haar weight ψ.*

Similar to the C-algebra setting, we can define an antipode S on M with polar decomposition $S = R\tau_{-\frac{i}{2}}$. Because φ and ψ are n.s.f. weights, they also have a modular automorphism group.*

Further, also the notions of modular element and scaling constant have a von Neumann algebra counterpart.

We adopt all notations and terminology from the C-algebra case.*

The *Kac algebras* are special types of von Neumann algebraic quantum groups. From the results in [69], we get the next Lemma 1.1.44.

Lemma 1.1.44 *Let (M, Φ) be a von Neumann algebraic quantum group.*

We then have that (M, Φ) is a Kac algebra *if and only if the scaling group τ is trivial (i.e., we have $\tau_t = \iota$ for all $t \in \mathbb{R}$) and the Haar weights φ and ψ have the same modular automorphism group.*

In this case, the antipode S is thus bounded and satisfies $S^2 = \iota$.

The von Neumann algebra setting is the most suited framework to study *closed quantum subgroups*. We adopt the following Definition 1.1.45.

Definition 1.1.45 *Let (M_1, Φ_1) and (M_2, Φ_2) be two von Neumann algebraic quantum groups. Suppose that there exists a normal, faithful *-homomorphism $\hat{\alpha} : \hat{M}_1 \to \hat{M}_2$ such that $\hat{\Phi}_2\big(\hat{\alpha}(x)\big) = (\hat{\alpha} \,\bar{\otimes}\, \hat{\alpha})\hat{\Phi}_1(x)$ for all $x \in \hat{M}_1$.*

Then, we say that (M_1, Φ_1) is a closed quantum subgroup *of (M_2, Φ_2).*

We also define the concept of a *left coaction* of a von Neumann algebraic quantum group. There is of course an equivalent notion of a right coaction.

Definition 1.1.46 *Let (M, Φ) be a von Neumann algebraic quantum group and let N be a von Neumann algebra.*

Then, let $\alpha : N \to M \,\bar{\otimes}\, N$ be an injective, unital and normal *-homomorphism. We call α a left coaction *of (M, Φ) on N if we have*

$$(\iota \otimes \alpha)\alpha = (\Phi \otimes \iota)\alpha.$$

1.2 Examples of quantum groups

We now point our attention to *examples* of quantum groups. It is of course important to have enough (non-trivial) examples of locally compact quantum groups. Every theory needs some refined examples in order to survive.

This Section 1.2 deals with some general aspects about the study of quantum group examples. We give a few history details and describe possible construction methods. While doing this, we also give references to lots of examples. As it would be almost impossible to give a complete overview of all of the theory concerning examples, we certainly do not claim to have done so.

The two standard examples of locally compact quantum groups are given by the locally compact groups and their duals. We will describe them in Examples 1.2.1 and 1.2.4. However, most of our attention will go to the construction of quantum group examples which do not come from a group.

As explained above (and below), the construction of examples of quantum groups played a crucial role in the search for Definition 1.1.6. Similar to the general quantum group theory, also the study of examples has been developed in different steps. In this process, it happened more than once that a new constructed example leaded to some drastic changes concerning the general theory.

In this context, we should certainly mention the joint work of G.I. Kac and V.G. Paljutkin done in the sixties (thus even before there was the theory of Kac algebras). In [43], they built up a dual pair of quantum groups. These (probably) were the first examples of infinite quantum groups not coming from a group, i.e., which are not commutative or cocommutative. Further, they constructed in [44] their famous dimension 8 example; also see [45, 87].

After the development of Kac algebra theory in the seventies, people of course tried to construct as many examples as possible. Since then, quite a number of Kac algebra examples have been found and investigated. Several different examples (either in the C*-algebra or the von Neumann algebra framework) can be found in [21, 141, 118, 78, 74, 106, 108, 30, 135, 46, 47, 48, 49, 50].

However, in the quest for finding new Kac algebra examples, also new objects were found which had enough properties to be called ‘quantum group’ but did not satisfy the axioms of a Kac algebra. This showed that the theory of Kac algebras was too narrow. The main problem was the assumption that the antipode (which is the quantum analogue of the inverse group operation) has square 1.

The first indication that the category of Kac algebras was probably not the endpoint of quantum group theory was made on the algebraic level. The Hopf algebra examples constructed by V.G. Drinfel’d [24] and M. Jimbo [40] in 1985 were the first examples (known to operator algebra people) of Hopf algebras where the square of the antipode was not equal to 1. The construction of these, by now famous, examples was an important motivation to search for a further development of the analytic theory of quantum groups.

Two years later, there were found new analytical results which were comparable with the above situation. S.L. Woronowicz constructed in 1987 his famous and revolutionary example of the quantum $SU(2)$ group [153]. This gave an example of his theory of *compact matrix pseudo-groups* [154, 155] which was the starting point of the theory of compact quantum groups. It turned out that, also here, the antipode did not have square 1. Hence, the quantum $SU(2)$ group is not a Kac algebra. However, technically the most frightening thing was that the example of the quantum $SU(2)$ group showed that the antipode did not need to be a bounded map (as in the Kac algebra case) but could be a densely defined, unbounded linear map. As mentioned in Section 1.1, it was mainly for this reason that people wanted to develop a larger category of quantum groups on an analytic level, including the Kac algebras and all the known examples.

The example of the quantum $SU(2)$ group was generalized by Woronowicz one year later in 1988. For every $n \in \mathbb{N}\setminus\{0,1\}$, he constructed the compact quantum $SU(n)$ group [156]. Besides these typical examples, also several other compact quantum groups were constructed. We can refer e.g. to [107, 8, 73, 87]. We give a description of the quantum $SU(2)$ group in Example 1.2.10.

Examples of (non-commutative) discrete quantum groups are less common than one could expect. The dual of the quantum $SU(2)$ group is described in [94]. Other discrete examples can e.g. be found in [147].

Later on, in the nineties, there appeared examples of locally compact quantum groups that were neither compact nor discrete. For instance, P. Podleś and S.L. Woronowicz constructed in 1990 a *quantum Lorentz group* which gave the first non-discrete example of what should be a non-compact quantum group [94]. The quantum $E(2)$ group of Woronowicz [158, 159, 149] was the second example of a non-compact and non-discrete locally compact quantum group.

Of course, the existence of these non-compact examples motivated further the necessity of a new theory of *locally compact quantum groups*. As said above, a satisfactory theory was eventually found in 1999 by Kustermans and Vaes [65, 66]. Starting from their main Definition 1.1.6, they built up a theory including both the Kac algebras and all the known examples of quantum groups.

Not so surprisingly, the study of examples gained even more attention after the full development of the Kustermans-Vaes theory. The fact that there finally was a solid framework to study examples within was of course an impetus to search for new examples. Moreover, as e.g. this thesis proves, there also came a renewed interest for earlier constructed examples.

The two standard examples

First, we describe the commutative and the cocommutative examples. These are the two basic types of quantum group examples. We can construct all of them by using the well-known theory of locally compact groups.

We start with the commutative case. Classically, this agrees with the theory of locally compact groups. The commutative quantum groups are the standard examples and as such are a great source of inspiration. Every studied feature in quantum group theory is actuated by a property of locally compact groups.

It can be shown that any commutative quantum group is of the form discussed in the next Example 1.2.1. This is proved e.g. in [133, Proposition 2.5].

Example 1.2.1 *Let G be a locally compact group (where the group operation is denoted by juxtaposition). We use dx to denote the left Haar measure on G.*

We define a commutative C^*-algebra A by $A = \mathrm{C}_0(G)$. We already mentioned in the Introduction that $A \otimes A = \mathrm{C}_0(G \times G)$. Further, we then get from the results in Section 1.5 that $M(A) = \mathrm{C}_b(G)$ and $M(A \otimes A) = \mathrm{C}_b(G \times G)$.

We can define a comultiplication Φ on $A \otimes A$ by the formula

$$\Phi(f)(s,t) = f(st) \tag{1.2}$$

when $f \in \mathrm{C}_0(G)$ and $s, t \in G$.

The coassociativity of Φ *follows immediately from the associativity of the group operation on* G*. For all* $f \in \mathrm{C}_0(G)$ *and* $r, s, t \in G$*, we have*

$$(\Phi \otimes \iota)\Phi(f)(r,s,t) = f\big((rs)t\big) \quad \textit{and} \quad (\iota \otimes \Phi)\Phi(f)(r,s,t) = \big(r(st)\big)$$

The cancellation law in G *is equivalent with the property that the linear spaces*

$$\Phi(A)(A \otimes 1) \quad \textit{and} \quad \Phi(A)(1 \otimes A)$$

are dense in $A \otimes A$*. This is proved e.g. in [76, Proposition 3.3].*

We can define a faithful KMS-weight φ *on* A^+ *by integrating with respect to the left Haar measure. Hence, for all* $f \in \mathrm{C}_0(G)$*, we have*

$$\varphi(f) = \int_G f(x)\, dx.$$

Suppose that $\omega \in A^*_+$*. Then there is a (unique) finite Borel measure* ν *on* G *such that the action of* ω *is given by*

$$\omega(f) = \int_G f(r)\, d\nu(r).$$

Using the Fubini theorem, it is not so difficult to prove that the weight φ *is (strongly) left invariant. For all* $f \in \mathrm{C}_0(G)$*, we have*

$$\begin{aligned}\varphi\big((\omega \otimes \iota)\Phi(f)\big) &= \int \big((\omega \otimes \iota)\Phi(f)\big)\, ds = \iint \Phi(f)(r,s)\, d\nu(r)\, ds \\ &= \iint f(rs) d\nu(r) = \iint f(rs)\, ds\, d\nu(r) = \omega(1)\varphi(f).\end{aligned}$$

Hence, we get that φ *is a left Haar weight on* (A, Φ)*.*

Completely similar, a right Haar weight ψ *can be found by integrating with respect to the right Haar measure.*

We thus get that (A, Φ) *is a locally compact quantum group.*

Further, it is here easy to describe the antipode and the counit. These two maps behave very well in the commutative case.

The antipode S *is a* **-automorphism on* A *defined by*

$$S(f)(t) = f(t^{-1})$$

when $f \in \mathrm{C}_0(G)$ *and* $t \in G$*.*

The (bounded) counit *is a* **-homomorphism* $\varepsilon : A \to \mathbb{C}$ *defined by*

$$\varepsilon(f) = f(e)$$

when $f \in \mathrm{C}_0(G)$ *and where* e *is the unit in* G*.*

We accompany the Example 1.2.1 with two remarks.

Remark 1.2.2 *In Example 1.2.1 above, we also find some motivational remarks explaining (some of) the terminology introduced in Section 1.1. For instance, we see that* coassociaticity *is the quantum analogue of* associaticity.

The notion of a counit *is introduced in Definition 3.1.4.*

Remark 1.2.3 *The commutative case in Example 1.2.1 also explains why we generally let Φ take values in $M(A \otimes A)$ rather than in $A \otimes A$. For a non-compact locally compact group G, we have that*

$$\Phi\big(\mathrm{C}_0(G)\big) \cap \big(\mathrm{C}_0(G) \otimes \mathrm{C}_0(G)\big) = \{0\}.$$

This follows immediately from Equation (1.2).

In the next Example 1.2.4, we construct the *group duals.* In this way, we describe all the *cocommutative* quantum groups; see Remark 1.2.7.

Example 1.2.4 *Let G be a locally compact group.. We use $\hat{A}$ to denote the (reduced) group C*-algebra $\mathrm{C}^*_r(G)$.*

We shortly recall how $\mathrm{C}^*_r(G)$ is constructed.

For every $p \in G$, we denote the corresponding left translation by λ_p, i.e., we have that λ_p is the unitary operator on $\mathrm{L}^2(G)$ defined by

$$(\lambda_p \xi)(t) = \xi(s^{-1}t) \quad \textit{for all} \;\; \xi \in \mathrm{L}^2(G),\, t \in G.$$

For all $f \in K(G)$, we use $\lambda(f) \in B(\mathrm{L}^2(G))$ to denote the left convolution with f. The operator $\lambda(f)$ is defined by putting

$$\lambda(f) = \int f(s)\lambda_s \, ds$$

where ds is the left Haar measure on G.

The group C-algebra $\mathrm{C}^*_r(G)$ is then defined as*

$$\mathrm{C}^*_r(G) = [\, \lambda(f) \mid f \in \mathrm{K}(G)\,].$$

There exists a unique non-degenerate *-homomorphism $\hat{\Phi} : \hat{A} \to M(\hat{A} \otimes \hat{A})$ such that, for all $p \in G$, we have

$$\hat{\Phi}(\lambda_p) = \lambda_p \otimes \lambda_p.$$

We moreover have that the map $\hat{\Phi}$ is a comultiplication, i.e., it satisfies the coassociativity property:

$$(\hat{\Phi} \otimes \iota)\hat{\Phi} = (\iota \otimes \hat{\Phi})\hat{\Phi}.$$

Further, we also have the density properties

$$\hat{A} \otimes \hat{A} = [\hat{\Phi}(\hat{A})(\hat{A} \otimes 1)] = [\hat{\Phi}(\hat{A})(1 \otimes \hat{A})].$$

There exists a faithful KMS-weight $\hat{\varphi}$ on $(\hat{A}, \hat{\Phi})$ which is both left and right invariant. Loosely speaking, $\hat{\varphi}$ is given by the formula

$$\hat{\varphi}\big(\lambda(f)\big) = f(e)$$

We thus get that $(\hat{A}, \hat{\Phi})$ is a locally compact quantum group.

Further, we give a short description of the antipode and the counit. These maps are in most cases quite well-behaved.

The antipode $\hat{S}$ is a *-anti-automorphism on $\hat{A}$ such that, for all $f \in L^1(G)$, we have the formula

$$\hat{S}(\lambda(f)) = \lambda(\tilde{f})$$

Here, the function $\tilde{f} \in L^1(G)$ is defined by $\tilde{f}(p) = \Delta(p)^{-1} f(p^{-1})$ when $p \in G$ and where Δ is the modular function of G.

The counit $\hat{\varepsilon}$ is a linear map such that, for all $f \in L^1(G)$, we have

$$\hat{\varepsilon}(\lambda(f)) = \int_G f(p)\, dp$$

This linear map is bounded if and only if G is amenable.

For the ease of notation, we introduce the following convention.

Notation 1.2.5 *Whenever we use the notations $(\mathrm{C}_0(G), \Phi)$ and $(\mathrm{C}^*_r(G), \hat{\Phi})$, we always assume to be working with Examples 1.2.1 and 1.2.4.*

When this notation may cause any confusion, we will use a more precise notation with a subscript; e.g. we denote $(\mathrm{C}_0(G), \Phi_G)$.

It is said above that the quantum groups $(\mathrm{C}_0(G), \Phi)$ and $(\mathrm{C}^*_r(G), \hat{\Phi})$ are dual to each other. We give a short description of this duality.

Lemma 1.2.6 *Let G be a locally compact group. We can define a multiplicative unitary $W \in L^2(G \times G)$ by the formula*

$$W(\xi)(p, q) = \xi(p, p^{-1}q)$$

when $\xi \in L^2(G \times G)$ and $p, q \in G$.

The unitary operator W is the left regular representation of $\big(\mathrm{C}_0(G), \Phi\big)$.

Further, the two equalities below hold:

- $\mathrm{C}_0(G) = [\,(\iota \otimes \omega)W \mid \omega \in B(L^2(G))_*]$,
- $\mathrm{C}^*_r(G) = [\,(\omega \otimes \iota)W \mid \omega \in B(L^2(G))_*]$.

We also have that

$$\Phi(x) = W^*(1 \otimes x)W \quad \textit{and} \quad \hat{\Phi}(y) = \Sigma W(y \otimes 1)W^*\Sigma$$

when $x \in \mathrm{C}_0(G)$ *and* $y \in \mathrm{C}_r^*(G)$.

These formulas precisely mean that $(\widehat{\mathrm{C}_0(G), \Phi}) = (\mathrm{C}_r^*(G), \hat{\Phi})$.

The duality results in Lemma 1.2.6 have some nice consequences. We formulate two of them in the Remarks 1.2.7 and 1.2.8.

Remark 1.2.7 *We already mentioned that every commutative quantum group is of the form* $(\mathrm{C}_0(G), \Phi)$ *with* G *a locally compact group.*

From Lemma 1.2.6 and Proposition 1.1.38, we then get that every cocommutative quantum group is of the form $(\mathrm{C}_r^*(G), \hat{\Phi})$ *with* G *a locally compact group.*

Remark 1.2.8 *Let now* G *be an* abelian *locally compact group. Then, we have that* $\mathrm{C}_r^*(G) \cong \mathrm{C}_0(\hat{G})$ *where* $\hat{G}$ *is the dual group of* G.

The duality in Lemma 1.2.6 here takes the form $(\widehat{\mathrm{C}_0(G), \Phi_G}) = (\mathrm{C}_0(\hat{G}), \Phi_{\hat{G}})$.

In this special case, we can thus rewrite Theorem 1.1.33 as

$$(\mathrm{C}_0(\hat{\hat{G}}), \Phi_{\hat{\hat{G}}}) = (\mathrm{C}_0(G), \Phi_G).$$

This is precisely the famous Pontryagin duality theorem.

The next Proposition 1.2.9 sums up the main parts of the above results.

Proposition 1.2.9 *Let* G *be a locally compact group.*

We have that $(\mathrm{C}_0(G), \Phi)$ *and* $(\mathrm{C}_r^*(G), \hat{\Phi})$ *are locally compact quantum groups.*

Furthermore, we have $(\widehat{\mathrm{C}_0(G), \Phi}) = (\mathrm{C}_r^*(G), \hat{\Phi})$.

The examples $(\mathrm{C}_0(G), \Phi)$ *describe all the commutative quantum groups while the examples* $(\mathrm{C}_r^*(G), \hat{\Phi})$ *describe all the cocommutative quantum groups.*

The construction of examples

We mentioned above that, similar to the development of the general theory, the construction of examples of quantum groups (especially non-compact ones) has been a slow process. However, due to the work of several researchers, the list of examples is getting quite respectable by now. For the remainder of this section, we discuss possible methods for constructing examples.

We divide the construction methods in two categories. This will be done in terms of the examples that are produced. We will distinguish between *atomic* and *non-atomic* examples. This terminology is certainly not standard, but it is also not new. It appeared before in [54].

There first are the *atomic examples.* By 'atomic', we mean that these examples (up till now) cannot be constructed out of simpler quantum groups through one of the different existing theoretical construction procedures such as the bicrossed product construction or the double crossed product construction. Except for the two standard examples described in Examples 1.2.1 and 1.2.4, these examples are usually built up using the 'three level approach' introduced by Woronowicz. All the other examples (thus built up using a construction procedure with other quantum groups as ingredients) are called *non-atomic.*

Not so unexpectedly, the two introduced categories both have their advantages and disadvantages. Below, we will have a look at this and other aspects of the different types of construction methods.

Method 1: Atomic examples

From the above description, it is obvious that the notion of an example being 'atomic' is not clear-cut. More or less, we called the category of atomic examples into existence to have a short collective noun to adumbrate those examples which are constructed along the lines set up by Woronowicz.

Constructing new atomic examples requires a strong intuition in combination with non-trivial mathematics. The work of Woronowicz in this branch of quantum group theory is very important. From the end of the eighties onwards, he provided us with many new examples and he also introduced several typical techniques [162, 161] which are useful in the study of an example. As it is said above, these techniques are best usable in the C*-algebra framework.

Atomic examples are built up 'from nowhere' only using a 'good idea'. All the necessary ingredients are constructed from the bottom up. When constructing the example, the work is always done in three stages:

1. The Hopf *-algebra level;
2. The Hilbert space level;
3. The C*-algebra level (or von Neumann algebra level).

The Hopf *-algebra level is mainly used as an intuition. It gives the algebraic motivation for the later results on the operator algebra level. The Hilbert space level deals with the technical details. The main part here is the construction of a *multiplicative unitary.* Finally, we arrive at the C*-algebra level. This gives the example as a locally compact quantum group.

For the third level, one can thus also choose to work in the von Neumann algebra framework. Depending on the considered example, the best choice can either be to work with C*-algebras or with von Neumann algebras.

The aforesaid 'good idea' for the construction of an atomic example is mostly found as follows. We start from a classical group G consisting of matrices. Then, we work along the lines of the following recipe:

1. We look at the Hopf *-algebra $\mathcal{A}$ of polynomial functions on the group G and we try to find some natural generators and relations for $\mathcal{A}$.
2. We then deform these relations by some complex number q and consider the Hopf *-algebra $\mathcal{A}_q$ generated by the generators together with the deformed relations. After this and inspired by the usual comultiplication on $\mathcal{A}$, we try to define a comultiplication Δ from $\mathcal{A}_q$ to the algebraic tensor product $\mathcal{A}_q \odot \mathcal{A}_q$ in such a way that $(\mathcal{A}_q, \Delta)$ is again a Hopf *-algebra.
3. The next step is to represent the deformed generators of $\mathcal{A}_q$ by (possibly unbounded) closed operators on a Hilbert space H.
4. Now, we look for a C*-algebra $A \subseteq B(H)$ that is 'generated' by these Hilbert space operators (cf. [161]). This is done by taking appropriate functions of the generators. Further, we try to define a comultiplication $\Phi : A \to M(A \otimes A)$ that agrees with Δ on the generators.
5. The construction of the example is ended by defining left and right Haar weights and proving their invariance.

Although the above five-step recipe is a useful guideline, it certainly does not guarantee that it will be easy to construct new examples. First of all, it is already non-trivial to find a possible deformation on the Hopf *-algebra level.

It also requires a lot of hard work to perform the lifting procedure. Technically the most difficult part lies in the search for a good comultiplication Φ on the C*-algebra A and in the proof of its coassociativity. In comparison with finding the correct formula for Φ, the construction of the Haar weights is not that difficult. For this, the technique presented in Section 1.4 can be helpful.

In the compact case, the above construction method works really well. There, it can be used in a rather straightforward way. Historically the first quantum group built up using the above recipe was famous example of the non-commutative quantum $SU_\lambda(2)$ group. We construct this compact quantum group in Example 1.2.10 below. More general, there are the compact quantum $SU_\lambda(n)$ groups with $n \in \mathbb{N} \setminus \{0, 1\}$ [156]. However, we only consider the case $n = 2$.

The quantum $SU_\lambda(2)$ group was first constructed by Woronowicz in 1987 [153]. It is probably the most studied example of a (compact) quantum group.

Example 1.2.10 *Take* $\lambda \in \mathbb{R}$ *with* $0 < |\lambda| < 1$.

First, recall that $SU(2)$ *is the group of unitary complex matrices in* $M_2(\mathbb{C})$ *with determinant* 1. *We can describe* $SU(2)$ *in a more concrete way as*

$$SU(2) = \left\{ \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \in M_2(\mathbb{C}) \;\middle|\; a, c \in \mathbb{C} \textit{ such that } |a|^2 + |c|^2 = 1 \right\}.$$

We consider the functions $\alpha, \gamma \in \mathrm{C}(SU(2))$ *defined by*

$$\alpha \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} = a \qquad \text{and} \qquad \gamma \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} = c$$

when $a, c \in \mathbb{C}$ *such that* $|a|^2 + |c|^2 = 1$.

Now, we let $\mathcal{A}_0$ *be the dense unital* **-subalgebra of* $\mathrm{C}(SU(2))$ *generated by* α *and* γ. *We can define a* **-homomorphism* $\Delta : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{A}_0$ *by the formula* $\Delta(f)(s,t) = f(st)$ *when* $f \in \mathcal{A}_0$ *and* $s, t \in SU(2)$.

It is easy to check that $(\mathcal{A}_0, \Delta)$ *is a Hopf* **-algebra. We have that*

$$\Delta(\alpha) = \alpha \otimes \alpha - \gamma^* \otimes \gamma \qquad \text{and} \qquad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Now, we let $\mathcal{A}_0$ *be the dense unital* **-subalgebra of* $\mathrm{C}(SU(2))$ *generated by* α *and* γ. *We can define a* **-homomorphism* $\Delta : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{A}_0$ *by the formula* $\Delta(f)(s,t) = f(st)$ *when* $f \in \mathcal{A}_0$ *and* $s, t \in SU(2)$.

It is easy to check that $(\mathcal{A}_0, \Delta)$ *is a Hopf* **-algebra. We have*

$$\Delta(\alpha) = \alpha \otimes \alpha - \gamma^* \otimes \gamma \qquad \text{and} \qquad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

As said above, the idea is now to deform the Hopf **-algebra* $(\mathcal{A}_0, \Delta)$ *by using the parameter* λ. *We stick to* $(\mathcal{A}_0, \Delta)$ *for denoting the deformation.*

We denote by $\mathcal{A}_0$ *the universal* **-algebra with unit* 1 *generated by two elements* α *and* γ *satisfying the relations*

$$\alpha\gamma = \lambda\gamma\alpha \ , \qquad \alpha\gamma^* = \lambda\gamma^*\alpha \ , \qquad \gamma^*\gamma = \gamma\gamma^*,$$
$$\alpha^*\alpha + \gamma^*\gamma = 1 \ , \qquad \alpha\alpha^* + \lambda^2\gamma\gamma^* = 1.$$

The universality property of the **-algebra* $\mathcal{A}_0$ *implies the existence of a unique* **-homomorphism* $\Delta : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{A}_0$ *such that*

$$\Delta(\alpha) = \alpha \otimes \alpha - \lambda\, \gamma^* \otimes \gamma \qquad \text{and} \qquad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Again, it is easy to check that $(\mathcal{A}_0, \Delta)$ *is a Hopf* **-algebra. It is straightforward to produce a co-unit and an antipode.*

From the results in [144], we then get that

$$\mathcal{A}_0 \otimes \mathcal{A}_0 = \operatorname{span}\{\Delta(a)(b \otimes 1) \mid a, b \in \mathcal{A}_0\} = \operatorname{span}\{(a \otimes 1)\Delta(b) \mid a, b \in \mathcal{A}_0\}.$$

The next step is to represent the generators α *and* γ *as bounded operators on a Hilbert space. We give the concrete choice that is typically used for this.*

Consider the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z})$. *Let* $(e_n)_{n \in \mathbb{N}}$ *be the standard orthonormal basis of* $\ell^2(\mathbb{N})$ *and let* $(f_k)_{k \in \mathbb{Z}}$ *be the standard orthonormal basis of* $\ell^2(\mathbb{Z})$. *We also set* $e_{-1} = 0$.

There are unique operators $S, T \in B(\mathcal{H})$ *such that*

$$S(e_n \otimes f_k) = \sqrt{1 - \lambda^{2n}}\, e_{n-1} \otimes f_k \qquad \text{and} \qquad T(e_n \otimes f_k) = \lambda^n\, e_n \otimes f_{k+1}$$

for all $n \in \mathbb{N}$ *and* $k \in \mathbb{Z}$.

Some easy calculations show that the operators S and T satisfy the same relations as α and γ do. The universality of $\mathcal{A}_0$ therefore implies the existence of a unique *-representation $\pi : \mathcal{A}_0 \to B(\mathcal{H})$ such that $\pi(\alpha) = S$ and $\pi(\gamma) = T$.

It is not too arduous to prove that the *-representation π is injective. From the defining relations, we immediately get that $\mathcal{A}_0$ is the linear span of elements of the form $\alpha^k(\gamma^l)^k\gamma^m$ and $(\alpha^*)^{k'}(\gamma^*)^{l'}\gamma^{m'}$ where $k, k', l, l', m, m' \in \mathbb{N}$. It is a little bit tedious but not too hard to check that the operators $S^k(T^*)^l T^m$ and $(S^*)^{k'}(T^*)^{l'}T^{m'}$, where $k, k', l, l', m, m' \in \mathbb{N}$ and $k' \neq 0$, are linearly independent. The injectivity of π clearly follows from this result.

We can now use a standard procedure to construct the enveloping C-algebra of $\mathcal{A}_0$. Because of the existence of the unital *-representation π, we can define a map $\|\cdot\|_0 : \mathcal{A}_0 \to [0, +\infty]$ by setting*

$$\|a\|_0 = \sup\big\{\|\theta(a)\| \mid \theta \textit{ is a unital }^*\textit{-representation of the }^*\textit{-algebra } \mathcal{A}_0 \textit{ on a Hilbert space}\big\}.$$

If K is a Hilbert space and $\theta : \mathcal{A}_0 \to B(K)$ is a unital *-representation, then we of course have that $\theta(\alpha)^*\theta(\alpha) + \theta(\gamma)^*\theta(\gamma) = 1$ which gives that $\|\theta(\alpha)\| \leq 1$ and $\|\theta(\gamma)\| \leq 1$. This implies that the number $\|x\|_0$ is finite for any $x \in \mathcal{A}_0$.

It is now easy to check that $\|\cdot\|_0$ satisfies all the needed relations for being a C-seminorm. The injectivity of π gives that $\|\cdot\|_0$ is actually a C*-norm.*

At this point, we can define a C-algebra A as the closure of $\mathcal{A}_0$ with respect to the above defined C*-norm $\|\cdot\|_0$.*

By the definition of the norm $\|\cdot\|_0$, there exists a unique unital *-homomorphism $\Phi : A \to A \otimes A$ that extends the comultiplication Δ. In other words, we have a unique unital *-homomorphism $\Phi : A \to A \otimes A$ such that

$$\Phi(\alpha) = \alpha \otimes \alpha - \lambda\gamma^* \otimes \gamma \qquad \textit{and} \qquad \Phi(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

The properties of $(\mathcal{A}_0, \Delta)$ directly imply that this comultiplication Φ makes the pair (A, Φ) into a compact quantum group according to Woronowicz.

We call (A, Φ) the quantum $SU_\lambda(2)$ group.

From the results in [163, 145], it ensues that there exists a unique Haar state h on (A, Φ) that is both left and right invariant.

Let $\pi_r : A \to B(K)$ be the unique unital *-homomorphism extending π. Then, an explicit formula for h is given by

$$h(a) = (1 - \mu^2)\sum_{n=0}^{\infty} \mu^{2n}\big\langle \pi_r(a)(e_n \otimes f_0), e_n \otimes f_0\big\rangle$$

when $a \in A$. We refer to [154] where the invariance of h is proved.

Furthermore, it is shown in [83] that the Haar state h is faithful on A.

The above properties together yield that the quantum $SU_\lambda(2)$ group (A,Φ) is a compact quantum group according to Definitions 1.1.6 and 1.1.36.

There exists a bounded counit $\varepsilon : A \to \mathbb{C}$ which is determined by the formulas

$$\varepsilon(\alpha) = 1 \quad \textit{and} \quad \varepsilon(\gamma) = 0.$$

For the antipode $S : D(S) \subseteq A \to A$, we have $\alpha, \alpha^*, \gamma, \gamma^* \in D(S)$ and

$$S(\alpha) = \alpha^*, \quad S(\alpha^*) = \alpha \quad \textit{and} \quad S(\gamma) = -\lambda\gamma, \quad S(\gamma^*) = -\lambda^{-1}\gamma^*.$$

This antipode S is unbounded on A.

In order to construct non-compact examples, it is a lot more difficult to apply in a straightforward way the construction scheme displayed on page 43. One of the crucial differences with the compact case is that the represented generators can now be unbounded. In this case, there is no unequivocal procedure to find the C*-algebra which they generate. It might even happen that no such C*-algebra exists (cf. Section 1.5). However, this is not the main problem.

There are more serious problems with concern to the comultiplication. First of all, there typically appear some *spectral conditions* to ensure the existence of a comultiplication on the C*-algebra level (cf. Remark 4.2.6).

When people started to study examples of non-compact, non-discrete, quantum groups in the operator algebra framework, they also ran up against a much more fundamental problem concerning the comultiplication. While some of the known candidates were very simple on the Hopf *-algebra level, it turned out to be impossible to lift these examples to the operator algebra level. Probably the most famous examples of this type are the quantum $ax{+}b$-group and the quantum $SU_q(1,1)$-group (cf. [157]). The technical problem is related to a known problem in Hilbert space operator theory, namely the existence of symmetric operators with no self-adjoint extensions. Woronowicz and Zakrzewski were able to overcome this problem in several cases and in particular, were able to obtain (a form of) the quantum $ax{+}b$ group on the operator algebra level [169]. The needed analysis is quite advanced, but very interesting. Koelink and Kustermans succeeded in constructing the quantum $\widetilde{SU}_q(1,1)$-group [54].

To overcome the problems that are encountered, one has to concentrate on one example at a time. In each specific case, it is inevitable to elaborate typical techniques to find a solution. Several people made a contribution to this, but the most credit undoubtedly should have to go to S.L. Woronowicz for all of his important work on various highly sophisticated examples.

When dealing with non-compact examples, the problem of constructing the bi-C*-algebra (A,Φ) is most of the times solved by defining a 'good' multiplicative unitary W that contains all the information about A and Φ. However, it is a highly non-trivial problem to find such a multiplicative unitary. Although the results on the Hopf *-algebra level will give us some intuition, it takes a lot of work to define W in a correct way and to prove its multiplicativity.

Luckily, the hard work will pay off. Once we have found a formula for W, we can find a very good companion in the basic results known from the theory of multiplicative unitaries. The results in Sections 1.3 and 1.4 will provide us then with a possible scheme for the construction.

Also in our treatment of the quantum $E(2)$ group, we will adopt this way of working. We apply in Chapter 2 the following construction scheme:

1. Find a 'good' multiplicative unitary $W \in B(H \otimes H)$.
2. Define a bi-C*-algebra (A, Φ) by the following formulas:
   - $A = [\,(\omega \otimes \iota)W \mid \omega \in B(H)_*]$,
   - $\Phi(x) = W(x \otimes 1)W^*$ when $x \in A$.
3. Use the method in Section 1.4 to construct the Haar weights.

This programme can also be followed to construct all the other non-compact examples mentioned in the list below.

The following ambivalence occurs in the study of quantum groups.

Remark 1.2.11 *In the general theory, we start with the Haar weights and then we define the regular representation and the antipodes.*

In concrete examples, we usually work in the opposite direction.

First, there is the built-up of the multiplicative unitary and its related antipodes. The construction of the Haar weights follows as a later step.

We end the discussion of atomic examples by giving a list of some of the most important examples that are constructed by using the three level construction procedure described above. We give references to articles where the example is constructed or where some basic properties are discussed:

– quantum $SU(2)$ group [153, 154, 94, 71, 11, 138],

– quantum $SU(n)$ group [156, 83, 76, 143, 151, 13, 14],

– quantum $E(2)$ group [157, 158, 159, 149, 2, 1, 88, 89, 103]; see Chapter 2,

– quantum $az{+}b$-group [165, 101, 112, 113, 148, 166, 97, 110, 75, 77],

– quantum $ax{+}b$-group [169, 102, 148, 166, 164, 109],

– quantum $\widetilde{SU}_q(1,1)$-group [54, 55, 56].

We listed these examples in historical order of the first appearance in a published paper. One should notice that the quantum $E(2)$ group was the first example of a non-compact quantum group built up as an atomic example.

Method 2: Non-atomic examples

The ‘atomic’ approach is of course not the only possible method to construct quantum group examples. There is a multitude of other possible construction procedures. We give a few details about some of them.

It is obvious that we cannot include all known methods, so we restrict ourselves to a few interesting cases. For simplicity, we use the general term ‘non-atomic example’ to denote any example not built up via the above Method 1.

As it is said above, it can require a lot of work to give the complete construction of one atomic example. The difficulty is that there is no general approach to find a solution to the problems one encounters while constructing such an example. It is therefore needed to consider separately each specific case.

For this reason, it is clear that it would be very much desirable to also have some general constructions of locally compact quantum groups which allow to construct many concrete examples in a unified way.

We here mention three such construction procedures: *twisting*, the *double crossed product construction* and the *cocycle bicrossed product construction*. These three methods are quite regularly used and can be applied to construct both compact and non-compact examples. The basic idea is to use known examples to build up more difficult ones. In all three cases, it turns out to be most advisable to work in the von Neumann algebraic setting. The corresponding C*-algebraic versions are usually more involved and thus require some extra effort.

The three above mentioned methods all work via the same concept. All the generated examples are built up by means of a general theoretical construction procedure using certain fairly general mathematical structures (like locally compact groups or quantum groups) as ingredients.

Because of this, the hard part in the construction of a non-atomic example lies in the search for a set of suitable ingredients needed to perform one of the known construction procedures. Once these are found, one can rely on the general properties to build up the example and study its features. This makes it more standard to study a non-atomic example than an atomic example. Of course, it also requires the knowledge of the used construction method.

The main advantage of the ‘non-atomic’ methods is the fact that one can generate a multitude of examples by varying the ingredients. However, these methods sometimes preserve too much of the properties of the original ingredients one starts from. For instance, up till now, all known non-atomic examples have trivial scaling constant. As it is mentioned before, there are constructed atomic examples with scaling constant $\nu \neq 1$; see Remark 1.1.24.

Nonetheless, this certainly does not mean that the above mentioned non-atomic construction procedures did not provide us with some new insights. They have generated examples that are very useful in the study of some important features. The (cocycle) bicrossed product construction is used, for instance, to construct examples of non-semi-regular quantum groups [5]. It is also used to produce non-trivial quantum group examples that are not amenable [23].

There is a quite efficient interplay between the two introduced categories of construction methods. Every time there is constructed a new atomic example, people try to find enough related material in order that the example can serve as a building ingredient for (one of) the known non-atomic procedures.

Remark 1.2.12 *It sometimes happens that a quantum group example can be constructed using either Method 1 or Method 2.*

For instance, this is the case with the quantum $\mathrm{GL}(2, \mathbb{C})$*-group.*

This example is constructed in [100] by using the ideas of Method 1. However, it is shown in [96] that it can also be obtained as a double crossed product of the quantum $az+b$*-group and its dual.*

We classify the quantum $\mathrm{GL}(2, \mathbb{C})$*-group to be a non-atomic example.*

We end this discussion by giving some basic references to articles where more details can be found. For the reader who is looking for more information about the general construction procedures or for concrete constructions of non-atomic examples, we can refer to the following papers:

1. Twisting: [30, 135, 69, 74, 86, 106, 107],
2. Double crossed products: [7, 4, 80, 1, 50],
   - quantum Lorentz group [94, 95, 98, 99] (with quantum $SU(2)$ group),
   - quantum Lorentz group [168] (with quantum $E(2)$ group),
   - quantum $\mathrm{GL}(2, \mathbb{C})$-group [100, 96] (with quantum $az+b$ group),
3. Bicrossed products: [131, 132, 128, 5, 42, 69, 4, 78, 21, 23, 51, 118, 38],
   - quantum Heisenberg group [141, 75, 34, 35].

1.3 Multiplicative unitaries

As it is said in the Introduction and Sections 1.1 and 1.2, multiplicative unitaries play a major role in the quantum group theory. We now give a systematic study of these important unitary operators. In the next Section 1.4, we will look at their relation with Haar weights and prove some useful new results.

Multiplicative unitaries are now present in purely theoretical considerations in the general theory of Kustermans and Vaes [66, 68]. They are also particularly useful when considering concrete examples of quantum groups. It is no surprise that the first appearance of these unitary operators already dates back to the very early stages of the quantum group theory.

The first who put forward the great importance of multiplicative unitaries was W.F. Stinespring in 1959 when he studied duality for unimodular locally compact groups [117]. He showed that the left regular representation W of a unimodular locally compact group G (cf. Lemma 1.2.6) contains complete information about the algebraic and topological structure of the group G. The non-unimodular case was considered by M. Takesaki [121]. Also the work of L. Vanheeswijck [139] concerning group duality is to be situated in this scheme.

By definition, a unitary operator is called a *multiplicative unitary* if it satisfies the so-called *pentagonal equation*. This equation was written down for the first time by G.I. Kac in [41]. Kac defined multiplicative unitaries in an abstract way and showed that the structure of two unimodular Kac algebras in duality can be described in terms of the associated regular representation.

The concept of a *multiplicative unitary* was further explored by S. Baaj and G. Skandalis. They published in 1993 their fundamental paper [4] in which they give an axiomatic approach to the Kac-Takesaki operator. The basic results in this comprehensive study can certainly be considered as the foundation of the nowadays abstract theory of multiplicative unitaries.

S.L. Woronowicz in 1996 gave the theory of multiplicative unitaries a new impulse by his *manageability theory* [162, 114]. In the thesis, we will mainly follow this approach to study multiplicative unitary operators.

The definition of a multiplicative unitary

First, we recall the Definition 1.3.1 of a multiplicative unitary. We remark that we use the *leg-numbering notation* as it is introduced in the Introduction.

Definition 1.3.1 *Let $W \in B(H \otimes H)$ be a unitary operator. We call W a* multiplicative unitary *if W satisfies the* pentagonal equation

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

The next Proposition 1.3.2 is mentioned in [4] and can easily be checked.

Proposition 1.3.2 *Let $W \in B(H \otimes H)$ be a multiplicative unitary. It then also holds that $\Sigma W^* \Sigma$ is a multiplicative unitary.*

We call $\Sigma W^* \Sigma$ the dual *multiplicative unitary of W.*

Not all multiplicative unitaries are useful. The pentagonal equation on its own is not powerful enough to prove much interesting results. In order to work properly with a multiplicative unitary, we need to impose extra conditions.

Baaj and Skandalis defined the notions of *regularity* [4] and *semi-regularity* [1] as possible extra conditions. In both cases, there is elaborated a quite satisfactory theory. The Baaj-Skandalis theory however does not apply to all multiplicative unitaries related to quantum groups. It is proved in [5] that the left regular representation of a quantum group need not be semi-regular.

We state the Definition 1.3.3 of regularity and semi-regularity without going into any detail. For the basic results concerning these concepts, we refer to [4, 1]. The papers [5, 77] contain more recent research about the subject.

Definition 1.3.3 ([4, 1]) *Let $W \in B(H \otimes H)$ be a multiplicative unitary. We then consider the algebra $\mathcal{C}(W)$ defined by*

$$\mathcal{C}(W) = [\,(\omega \otimes \iota)(\Sigma W) \mid \omega \in B(H)_*].$$

We call W regular *if $\mathcal{C}(W) = B_0(H)$ and* semi-regular *if $\mathcal{C}(W) \supseteq B_0(H)$.*

The notions of regularity and semi-regularity for multiplicative unitaries are nowadays mostly used to classify locally compact quantum groups.

Looking at the left regular representation, we can use them to define special types of quantum groups. This classification is for instance important with concern to the continuity of coactions as is explained in [5, 129].

Definition 1.3.4 *Let (A, Φ) be a locally compact quantum group and W its left regular representation. We then call (A, Φ)* regular *if and only if W is regular. We call (A, Φ)* semi-regular *if and only if W is semi-regular.*

It is not always easy to check (semi-)regularity in concrete cases. There are various examples for which this property is still unknown.

Luckily, it is known for several categories of quantum groups to be regular. We list the most important ones in Remark 1.3.5 below.

Remark 1.3.5 *All Kac algebras are regular quantum groups; see [4, 27]. We get from [130, Remark 2.12] that this is also the case for all C^*-algebraic quantum groups arising from algebraic quantum groups in the sense of [57, 70].*

We have in particular that commutative and cocommutative quantum groups are regular and that compact and discrete quantum groups are regular.

The quantum $E(2)$ group is semi-regular, but not regular (cf. Corollary 2.8.25). Examples of non-semi-regular quantum groups can be found in [5].

Manageability theory for multiplicative unitaries

S.L. Woronowicz defined in [162] the notion of *manageability* for multiplicative unitaries. This condition was later weakened to *modularity* [114]. There is by now developed a full-blown manageability theory; see [162, 114, 166].

The concept of a modular multiplicative unitary is the central subject in this Section 1.3. First, we formulate the basic results from [162, 114]. We then further elaborate the manageability theory. We prove interesting new results that are common for locally compact quantum groups, but were not yet known in the framework of modular multiplicative unitaries. Further, we introduce a stronger form of modularity in which case calculations can be diminished.

In Section 1.4, we give a profound study of the relation between the notion of modularity for multiplicative unitaries and (right) Haar weights. In this way, we narrow the chasm with the Kustermans-Vaes theory.

There is an interplay between the results in Sections 1.3 and 1.4. Some results of Section 1.4 are used in proofs in Section 1.3 and vice versa. Throughout the two sections, we indicate the mathematical dependency in a clear way.

We like to stress that the manageability theory is a strong tool. It is used both in the general theory and in the construction of examples.

From the general theory, we know that the left (and right) regular representation of a locally compact quantum group is always a manageable multiplicative unitary. Conversely, a manageable (modular) multiplicative unitary can be very advantageous when constructing examples of quantum groups. This is already said in Section 1.2 and is corroborated by the results in Section 1.4.

The usefulness of the manageability theory in the construction of examples is illustrated by our approach to the quantum $E(2)$ group; see Chapter 2.

As said above, we adhere to the Woronowicz approach to multiplicative unitaries throughout the complete thesis. For this reason, we give a full and detailed discussion of the theory of modular multiplicative unitaries. We however only include proofs for new results not appearing in [162, 114].

It's now about time to start with the mathematics.

The Notation 1.3.6 below concerns the complex conjugate Hilbert space $\overline{H}$ of a Hilbert space H and is only used in this Section 1.3.

Notation 1.3.6 *For a Hilbert space H, we denote by $\overline{H}$ the complex conjugate Hilbert space. For any element $x \in H$, the corresponding element in $\overline{H}$ is denoted by $\overline{x}$. Then $L : H \to \overline{H} : x \mapsto \overline{x}$ is an anti-unitary map.*

In particular, we have that $\langle \overline{x}, \overline{y} \rangle = \langle y, x \rangle$ for all $x, y \in H$.

For a closed operator m on H, the transpose of m is denoted by $m^\top$. This means by definition that $D(m^\top) = \overline{D(m^*)}$ and $m^\top \overline{x} = \overline{m^* x}$ for $x \in D(m^*)$. In other words, we have that $m^\top = L m^* L^*$. If $m \in B(H)$, then $m^\top$ is the unique bounded operator on $\overline{H}$ such that $\langle m^\top \overline{x}, \overline{y} \rangle = \langle m y, x \rangle$ for all $x, y \in H$. Clearly, the map $B(H) \to B(\overline{H}) : m \mapsto m^\top$ is an *-anti-isomorphism of C-algebras.*

Setting $\overline{\overline{x}} = x$, we identify the Hilbert spaces $\overline{\overline{H}}$ and H. By this identification, we have $m^{\top\top} = m$ for any closed operator m on H.

For every $\omega \in B(H)^*$, we define $\omega^\top \in B(\overline{H})^*$ by the formula $\omega^\top(m^\top) = \omega(m)$ when $m \in B(H)$. If $\omega \in B(H)_*$, we have that $\omega^\top \in B(\overline{H})_*$. Further, it is clear that $\omega^{\top\top} = \omega$ for every $\omega \in B(H)^*$.

The next Definition 1.3.7 defines the notions of *manageability* and *modularity* for multiplicative unitaries. It is taken from [114].

Definition 1.3.7 ([162, 114]) *Let $W \in B(H \otimes H)$ be a multiplicative unitary. We say that W is* modular *if there exist two strictly positive operators Q and $\hat{Q}$ on H and a unitary operator $\breve{W} \in B(\overline{H} \otimes H)$ such that:*

1. *We have that*
$$W(\hat{Q} \otimes Q)W^* = \hat{Q} \otimes Q,$$
2. *For all $x, z \in H$ and $u \in D(Q)$, $y \in D(Q^{-1})$, we have*
$$\big\langle W(z \otimes y), x \otimes u \big\rangle = \big\langle \breve{W}(\overline{x} \otimes Q^{-1}y), \overline{z} \otimes Qu \big\rangle$$

In the case $\hat{Q} = Q$, we say that W is manageable.

We sum up the most important results concerning modularity. The two main Theorems 1.3.8 and 1.3.9 show that a modular multiplicative unitary is a very powerful tool. This is endorsed by the new results proved further below.

The next Theorem 1.3.8 gives that the modularity property is preserved if we consider the dual multiplicative unitary.

Theorem 1.3.8 ([162, 114]) *Let W be a multiplicative unitary. In the case that W is modular, also $\Sigma W^* \Sigma$ is modular. The operators Q and $\hat{Q}$ appearing in Definition 1.3.7 then exchange their positions.*

If W is manageable, then also $\Sigma W^* \Sigma$ is manageable.

The power of the manageability theory is illustrated by the advanced results in the next Theorem 1.3.9. This main theorem gathers together the basic properties of modular multiplicative unitaries. In particular, we see that a modular multiplicative unitary gives rise to many related objects.

The Theorem 1.3.9 below was first proved by P. Soltan and S.L. Woronowicz. We refer to [162, 114] for a full elaboration.

Theorem 1.3.9 ([162, 114]) *Let $W \in B(H \otimes H)$ be a modular multiplicative unitary. We define*
$$A = [\,(\omega \otimes \iota)W \mid \omega \in B(H)_*] \quad \textit{and} \quad \hat{A} = [\,(\iota \otimes \omega)W \mid \omega \in B(H)_*].$$
The following properties hold:

1. *We have that A and $\hat{A}$ are non-degenerate C^*-algebras in $B(H)$,*
2. *We have $W \in M(B_0(H) \otimes A)$ and $W \in M(\hat{A} \otimes B_0(H))$,*
3. *We have $W \in M(\hat{A} \otimes A)$,*

4. *There exists a non-degenerate* **-homomorphism* $\Phi : A \to M(A \otimes A)$ *that is uniquely determined by the formula*

$$(\iota \otimes \Phi)W = W_{12}W_{13}.$$

Moreover, the three properties below hold:

   (a) *We have* $\Phi(x) = W(x \otimes 1)W^*$ *for all* $x \in A$,
   (b) *We have that* Φ *is* coassociative, *i.e.,* $(\Phi \otimes \iota)\Phi = (\iota \otimes \Phi)\Phi$,
   (c) *The linear spaces* $\Phi(A)(1 \otimes A)$ *and* $\Phi(A)(A \otimes 1)$ *are dense in* $A \otimes A$.

5. *There exists a unique closed linear operator* $S : D(S) \subseteq A \to A$ *such that* $\{(\omega \otimes \iota)W \mid \omega \in B(H)_*\}$ *is a core for* S *and*

$$S\big((\omega \otimes \iota)W\big) = (\omega \otimes \iota)W^*$$

for every $\omega \in B(H)_*$.

6. *The linear map* S *has a unique* polar decomposition

$$S = R\tau_{-\frac{i}{2}}$$

where

   (a) (τ_t) *is a norm continuous one-parameter group on* A *such that* τ_t *is a* **-automorphism of* A *for every* $t \in \mathbb{R}$,
   (b) R *is an* **-anti-automorphism of* A *such that* $R^2 = \iota$,
   (c) $R\tau_t = \tau_t R$ *for all* $t \in \mathbb{R}$.

7. *Moreover, we have that*

   (a) S *is densely defined and has dense range,*
   (b) S *is injective and* $S^{-1} = R\tau_{\frac{i}{2}} = \tau_{\frac{i}{2}}R$,
   (c) S *is anti-multiplicative: for* $x, y \in D(S)$, *we have that* $xy \in D(S)$ *and* $S(xy) = S(y)S(x)$,
   (d) *For all* $x \in D(S)$, *we have* $S(x)^* \in D(S)$ *and* $S(S(x)^*)^* = x$,
   (e) $S^2 = \tau_{-i}$.

8. *We have the following commutation relations:*

   (a) $R\tau_z = \tau_z R$ *for all* $z \in \mathbb{C}$,
   (b) $\Phi\tau_t = (\tau_t \otimes \tau_t)\Phi$ *for all* $t \in \mathbb{R}$,
   (c) $\Phi R = \dot{\sigma}(R \otimes R)\Phi$,
   (d) $RS = SR$,
   (e) $\tau_t S = S\tau_t$ *for all* $t \in \mathbb{R}$.

The next 5 remarks all allude to Theorem 1.3.9 above.

Remark 1.3.10 *We use the polar decomposition $S = R\tau_{-\frac{i}{2}}$ where Woronowicz in [162] uses $S = R\tau_{\frac{i}{2}}$. This is of course just a minor change. It is made to have a conformity with the conventions in the Kustermans-Vaes theory.*

We have that S is strictly closed as a linear map on A.

The Theorem 1.3.9 is proved in [162, 114] only in the case that H is separable. It is easy to adapt the proofs in order to hold in the general case.

If the Hilbert space H is separable, we have that A and $\hat{A}$ are separable.

Remark 1.3.11 *The coassociativity of Φ is a direct consequence of the fact that W is a multiplicative unitary.*

Let $x \in A$. We then have

$$(\Phi \otimes \iota)\big(\Phi(x)\big) = W_{12}\Phi(x)_{13}W_{12}^* = W_{12}W_{13}(x \otimes 1 \otimes 1)W_{13}^*W_{12}^*$$

and

$$(\iota \otimes \Phi)\big(\Phi(x)\big) = W_{23}\Phi(x)_{12}W_{23}^* = W_{23}W_{12}(x \otimes 1 \otimes 1)W_{12}^*W_{23}^*.$$

Using these results, we see that

$$\begin{aligned}(\Phi \otimes \iota)\big(\Phi(x)\big) &= W_{12}W_{13}W_{23}(x \otimes 1 \otimes 1)W_{23}^*W_{13}^*W_{12}^* \\ &= W_{23}W_{12}(x \otimes 1 \otimes 1)W_{12}^*W_{23}^* = (\iota \otimes \Phi)\big(\Phi(x)\big).\end{aligned}$$

The second equality follows from the pentagonal equation.

Remark 1.3.12 *The observant reader has probably noticed that the approach in Theorem 1.3.9 above is somewhat different from the Kustermans-Vaes theory expounded in Section 1.1. We here take A to be in the 'second leg' of W while we have the converse result in Proposition 1.1.18.*

This difference is not so important. In order to have clear references, we twice have chosen to adopt the definitions from the original articles.

Remark 1.3.13 *Similar to Remark 1.1.20, we can use the polar decomposition $S = R\tau_{-\frac{i}{2}}$ to extend S to a strictly closed linear map $\bar{S}$ on $M(A)$ satisfying strict versions of all the properties of S.*

In what follows, we will use S to also denote the extension $\bar{S}$.

Remark 1.3.14 *Theorem 1.3.9 also holds in a von Neumann algebraic version. For this, we consider the von Neumann algebras $M = A''$ and $\hat{M} = \hat{A}''$.*

From Theorem 1.3.9, we get that $W \in \hat{M} \bar{\otimes} M$.

The formula $\Phi(x) = W(x \otimes 1)W^*$ for all $x \in A$ gives that we can extend Φ to a normal and unital *-homomorphism

$$\Phi : M \to M \bar{\otimes} M.$$

The results in [162, 114] imply that we can extend τ to a strongly continuous one-parameter group of normal *-automorphisms on M and that R can be extended to a normal *-anti-automorphism on M. The polar decomposition $S = R\tau_{-\frac{i}{2}}$ can then be used to find a strongly closed extension of the linear map S.

It is clear that we can similarly consider $\hat{\Phi} : \hat{M} \to \hat{M} \bar{\otimes} \hat{M}$.

Using these extensions, it is standard to prove that all the properties described in Theorem 1.3.9 have a von Neumann algebraic counterpart.

Throughout the text, the above considered extensions are regularly used. They are denoted by the same notations as in the C-algebra framework.*

The next Notation 1.3.15 is introduced for notational convenience.

Notation 1.3.15 *For the remainder of Sections 1.3 and 1.4, we fix a modular multiplicative unitary W together with all its related objects. Hence, all used regular notations refer to the objects related to W as introduced above.*

In particular, the notations (A, Φ) and $S = R\tau_{-\frac{i}{2}}$ respectively designate the bi-C-algebra and the antipode related to W as defined in Theorem 1.3.9.*

All dual objects (related to $\Sigma W^* \Sigma$) are denoted with a ˆ-symbol. This e.g. gives us a bi-C-algebra $(\hat{A}, \hat{\Phi})$ and an antipode $\hat{S} = \hat{R}\hat{\tau}_{-\frac{i}{2}}$.*

The Terminology 1.3.16 is as customary.

Terminology 1.3.16 *We call the pair (A, Φ) the* bi-C*-algebra associated to W*. It is standard to call Φ the* comultiplication *and S the* antipode*. We call τ the* scaling group *and R the* unitary antipode*.*

In the rest of the Section 1.3, we describe (a lot) further features of the modular multiplicative unitary W. We will see that W satisfies several useful properties. This endorses that the modularity theory constitutes a spruce whole.

We recall that all notations refer to objects related to the modular multiplicative unitary W, i.e., introduced in Definition 1.3.7 or Theorem 1.3.9.

It is easy to compute the *-algebra structure on the elements $(\omega \otimes \iota)W$. The following results are standard; see [4].

Lemma 1.3.17 *Let $\omega, \omega_1, \omega_2 \in B(H)_*$. Then, we have that*

1. $(\omega_1 \otimes \iota)W + (\omega_2 \otimes \iota)W = \big((\omega_1 + \omega_2) \otimes \iota\big)W$,
2. $\big((\omega_1 \otimes \iota)W\big)\big((\omega_2 \otimes \iota)W\big) = (\omega_1\omega_2 \otimes \iota)W$,
3. $\big((\omega \otimes \iota)W\big)^* = (\bar{\omega} \otimes \iota)W^*$.

The next Proposition 1.3.18 describes how Φ acts on the basic elements.

Proposition 1.3.18 *Let $\omega \in B(H)_*$. We then have*

$$\Phi\big((\omega \otimes \iota)W\big) = (\omega \otimes \iota \otimes \iota)(W_{12}W_{13}).$$

Proof. Let $\omega \in B(H)_*$. We have that

$$\begin{aligned}\Phi\big((\omega \otimes \iota)W\big) &= W\big((\omega \otimes \iota)W \otimes 1\big)W^* \\ &= (\omega \otimes \iota \otimes \iota)(W_{23}W_{12}W_{23}^*) = (\omega \otimes \iota \otimes \iota)(W_{12}W_{13}).\end{aligned}$$

The last equality follows from the pentagonal equation. ∎

The following propositions are useful in further calculations. They describe the action of the scaling group τ and the unitary antipode R.

We first give some basic results.

Proposition 1.3.19 ([162, 114]) *The properties below hold:*

1. *For all $a \in A$ and $t \in \mathbb{R}$, we have*
$$\tau_t(a) = Q^{-2it}aQ^{2it},$$
2. *If $\omega \in B(H)_*$, then $(\omega \otimes \iota)W \in D(\tau_{-\frac{i}{2}})$ and $\tau_{-\frac{i}{2}}\big((\omega \otimes \iota)W\big) = (\omega^\top \otimes \iota)\check{W}$,*
3. *For $\omega \in B(H)_*$, we have $R\big((\omega \otimes \iota)W\big) = (\omega^\top \otimes \iota)\check{W}^*$.*

Proposition 1.3.20 *Let $t \in \mathbb{R}$. For all $\omega \in B(H)_*$, we have that*

$$\tau_t\big((\omega \otimes \iota)W\big) = (\omega_t \otimes \iota)W$$

where $\omega_t \in B(H)_*$ is defined by $\omega_t(x) = \omega(\hat{Q}^{2it}x\hat{Q}^{-2it})$.

Proof. Let $t \in \mathbb{R}$ and $\omega \in B(H)_*$. Using Proposition 1.3.19, we get

$$\begin{aligned}\tau_t\big((\omega \otimes \iota)W\big) &= Q^{-2it}\big((\omega \otimes \iota)W\big)Q^{2it} \\ &= (\omega_t \otimes \iota)\big((\hat{Q}^{-2it} \otimes Q^{-2it})W(\hat{Q}^{2it} \otimes Q^{2it})\big) = (\omega_t \otimes \iota)W.\end{aligned}$$

The last equality holds because W and $\hat{Q} \otimes Q$ strongly commute. ∎

For the analytic continuations τ_z, we have the next property.

Proposition 1.3.21 *Let $z \in \mathbb{C}$. Take $\xi \in D(\hat{Q}^{-2\,\mathrm{Im}z})$ and $\eta \in D(\hat{Q}^{\,2\,\mathrm{Im}z})$. Then, we have that $(\omega_{\xi,\eta} \otimes \iota)W \in D(\tau_z)$. Further, we have*

$$\tau_z\big((\omega_{\xi,\eta} \otimes \iota)W\big) = (\omega_{\hat{Q}^{-2iz}\xi,\hat{Q}^{-2i\bar{z}}\eta} \otimes \iota)W.$$

Proof. From Proposition B.2.21, we at once get that it is sufficient to prove that $(\omega_{\xi,\eta}\otimes\iota)W$ is a middle multiplier of Q^{2iz}, Q^{-2iz} and that (the closure of) $Q^{2iz}\big((\omega_{\xi,\eta}\otimes\iota)W\big)Q^{-2iz}$ is equal to $(\omega_{\hat{Q}^{-2iz}\xi,\hat{Q}^{-2i\overline{z}}\eta}\otimes\iota)W$.

For all $p\in D(Q^{2\,\mathrm{Im}z})$ and $q\in D(Q^{-2\,\mathrm{Im}z})$, we have

$$\begin{aligned}\big\langle(\omega_{\xi,\eta}\otimes\iota)WQ^{-2iz}p,Q^{-2i\overline{z}}q\big\rangle &= \big\langle W(\xi\otimes Q^{-2iz}p),\eta\otimes Q^{-2i\overline{z}}q)\big\rangle\\ &= \big\langle W(\hat{Q}\otimes Q)^{-2iz}(\hat{Q}^{2iz}\xi\otimes p),\eta\otimes Q^{-2i\overline{z}}q\big\rangle\\ &= \big\langle(\hat{Q}\otimes Q)^{-2iz}W(\hat{Q}^{2iz}\xi\otimes p),\eta\otimes Q^{-2i\overline{z}}q\big\rangle\\ &= \big\langle W(\hat{Q}^{2iz}\xi\otimes p),\hat{Q}^{2i\overline{z}}\eta\otimes q)\big\rangle = \big\langle(\omega_{\hat{Q}^{2iz}\xi,\hat{Q}^{-2i\overline{z}}\eta}\otimes\iota)Wp,q\big\rangle.\end{aligned}$$

In the third step, we use that W and $\hat{Q}\otimes Q$ strongly commute.

From the above calculation, we get that $(\omega_{\xi,\eta}\otimes\iota)WQ^{-2iz}p\in D(Q^{2iz})$ and

$$Q^{2iz}\big((\omega_{\xi,\eta}\otimes\iota)W\big)Q^{-2iz}p = \big((\omega_{\hat{Q}^{-2iz}\xi,\hat{Q}^{-2i\overline{z}}\eta}\otimes\iota)W\big)p.$$

This ends the proof of the proposition. ∎

By using Proposition B.2.10, we can deduce the following result.

Corollary 1.3.22 *Let $H_0=\{\xi\in H\mid \xi\in D(\hat{Q}^y)$ for all $y\in\mathbb{C}\}$. For $\xi,\eta\in H_0$, we have that $(\omega_{\xi,\eta}\otimes\iota)W$ is analytic with respect to τ. For every $z\in\mathbb{C}$, we have that* $\mathrm{span}\{(\omega_{\xi,\eta}\otimes\iota)W\mid\xi,\eta\in H_0\}$ *is a core for τ_z.*

We also mention the next formula.

Proposition 1.3.23 *Let $\xi\in D(\hat{Q})$ and $\eta\in D(\hat{Q}^{-1})$. Then, we have*

$$R\big((\omega_{\xi,\eta}\otimes\iota)W\big) = (\omega_{\hat{Q}\xi,\hat{Q}^{-1}\eta}\otimes\iota)W^*.$$

Proof. From Theorem 1.3.9, we know that $R=\tau_{\frac{i}{2}}S$. We get that

$$\begin{aligned}R\big((\omega_{\xi,\eta}\otimes\iota)W\big) &= \tau_{\frac{i}{2}}S\big((\omega_{\xi,\eta}\otimes\iota)W\big) = \tau_{\frac{i}{2}}\big((\omega_{\xi,\eta}\otimes\iota)W^*\big)\\ &= \big(\tau_{-\frac{i}{2}}(\omega_{\eta,\xi}\otimes\iota)W\big)^* = ((\omega_{\hat{Q}^{-1}\eta,\hat{Q}\xi}\otimes\iota)W)^* = (\omega_{\hat{Q}\xi,\hat{Q}^{-1}\eta}\otimes\iota)W^*.\end{aligned}$$

The fourth equality follows from Proposition 1.3.21. ∎

We now prove two properties which are well-known in the Kustermans-Vaes theory, but were not yet formulated in the context of modular multiplicative unitaries. We will use them in later calculations.

The first result immediately follows from Definition 1.3.7.

Proposition 1.3.24 *For all $t\in\mathbb{R}$, we have*

$$(\hat{\tau}_t\otimes\tau_t)(W) = W.$$

It is somewhat harder to prove the second one.

Proposition 1.3.25 *We have* $(\hat{R} \otimes R)(W) = W$.

Proof. Take $x, z \in H$ and $w \in D(Q^{-1})$, $v \in D(Q)$. From Definition 1.3.7, it follows that

$$\big\langle W(z \otimes Qv), x \otimes Q^{-1}w \big\rangle = \big\langle \breve{W}(\overline{x} \otimes v), \overline{z} \otimes w \big\rangle.$$

This equality precisely means that

$$(\iota \otimes \omega_{Qv,Q^{-1}w})W = ((\iota \otimes \omega_{v,w})\breve{W})^{\top}. \tag{1.3}$$

Notation 1.3.6 says that $T : B(H) \to B(\overline{H}) : m \mapsto m^{\top}$ is a *-anti-isomorphism. From Theorem 2.3 in [114], we know that $\breve{W} = (T \otimes R)(W^*)$. Using this formula, it is not difficult to check that

$$((\iota \otimes \omega_{v,w})\breve{W})^{\top} = (\iota \otimes \omega_{v,w} R)W^*. \tag{1.4}$$

Equations (1.3) and (1.4) together give that

$$(\iota \otimes \omega_{Qv,Q^{-1}w})W = (\iota \otimes \omega_{v,w} R)W^*.$$

Now, we apply $\hat{R}$ to this equation. Using the dual version of Proposition 1.3.23, we get that

$$(\iota \otimes \omega_{v,w})W^* = (\iota \otimes \omega_{v,w})\big((\hat{R} \otimes R)(W^*)\big).$$

We can conclude that $(\hat{R} \otimes R)(W^*) = W^*$. ∎

This result also implies the following.

Proposition 1.3.26 *Let* $\omega \in B(H)_*$. *Then, we can take* $\omega_{\hat{R}} \in B(H)_*$ *such that* $\omega_{\hat{R}}(x) = \omega(\hat{R}(x))$ *for all* $x \in \hat{A}$. *We have that*

$$R\big((\omega \otimes \iota)W\big) = (\omega_{\hat{R}} \otimes \iota)W.$$

Proof. Let $\omega \in B(H)_*$. From Proposition 1.3.25, we get

$$R\big((\omega \otimes \iota)W\big) = R\big((\omega \otimes \iota)(\hat{R} \otimes R)(W)\big) = (\omega_{\hat{R}} \otimes \iota)W.$$

In this calculation, we also use that $R^2 = \iota$. ∎

We now prove the small Lemma 1.3.27 that contains a simple but useful property concerning the modular multiplicative unitary W.

Lemma 1.3.27 *Let* $\xi, \eta \in H$. *Assume that* $(\omega_{\xi,\eta} \otimes \iota)W = 0$ *or* $(\iota \otimes \omega_{\xi,\eta})W = 0$. *We then have* $\langle \xi, \eta \rangle = 0$.

Proof. Take $\xi, \eta \in H$ such that $(\omega_{\xi,\eta} \otimes \iota)W = 0$. For all $\rho \in B(H)_*$, we then get that $\omega_{\xi,\eta}\big((\iota \otimes \rho)W\big) = 0$ and hence $\omega_{\xi,\eta}(x) = 0$ for all $x \in \hat{M}$. This implies that $\langle \xi, \eta \rangle = 0$. The case $(\iota \otimes \omega_{\xi,\eta})W = 0$ is proved in a similar way. ∎

The next Proposition 1.3.28 is of a similar kind as Lemma 1.3.27.

Proposition 1.3.28 *Let $\omega \in B(H)^+_*$. If $(\omega \otimes \iota)W = 0$ or $(\iota \otimes \omega)W = 0$, then we have that $\omega = 0$.*

Proof. This property can easily be proved by applying the same technique as in the proof of Lemma 1.3.27. ∎

In the remainder of the Section 1.3, we will (quite heavily) rely on one of the most important propositions proven in Section 1.4. On the other hand, some of the properties formulated below will also be used to strengthen the main results of Section 1.4. We can thus speak of a beneficial interplay.

We now postulate a special case of Proposition 1.4.13. It gives the existence of an n.s.f. weight ψ_1 on $B(H)^+$ satisfying some right invariance property. As it is said above, we postpone the proof until Section 1.4.

We remark that the notation $\mathrm{Tr}_{\hat{Q}^2}$ is explained in Example B.1.33.

Proposition 1.3.29 *Let $\psi_1 = \mathrm{Tr}_{\hat{Q}^2}$. Then, ψ_1 is an n.s.f. weight on $B(H)^+$ such that*

$$\psi_1\big((\iota \otimes \omega)(W(x \otimes 1)W^*)\big) = \omega(1)\psi(x)$$

for all $x \in \mathfrak{M}^+_{\psi_1}$ and $\omega \in B(H)^+_*$.

The above Proposition 1.3.29 thus plays a crucial role in the proofs of the results below. Thanks to the existence of the weight ψ_1, we can apply typical quantum group techniques in the framework of modular multiplicative unitaries.

For this reason, we can now prove some important properties. Although they are standard in the quantum group theory, it seems that they never appeared before in the setting of multiplicative unitaries. So, it is worthwhile to mention them. The proofs we give are based on the proofs of the counterparts in quantum group theory. Of course, it is necessary to make some adaptations.

We proceed with the Lemma 1.3.30 below.

Lemma 1.3.30 *Let $x \in B(H)$. If $W(x \otimes 1)W^* = 1 \otimes x$, then $x \in \mathbb{C}1$.*

Proof. Take $x \in B(H)$. We consider the normal *-homomorphisms

a) $\Phi : B(H) \to B(H \otimes H) : x \mapsto W(x \otimes 1)W^*$,

b) $\tau_t : B(H) \to B(H) : x \mapsto Q^{-2it}xQ^{2it}$ when $t \in \mathbb{R}$,

c) $\hat{\tau}_t : B(H) \to B(H) : x \mapsto \hat{Q}^{-2it}x\hat{Q}^{2it}$ when $t \in \mathbb{R}$.

It is clear that (τ_t) and $(\hat{\tau}_t)$ are strongly continuous one parameter groups of *-automorphisms on $B(H)$. The used notations do not cause any confusion.

Now, suppose that $\Phi(x) = 1 \otimes x$. For all $n \in \mathbb{N}$, we define elements $x_n, y_n \in B(H)$ by the formulas

$$x_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 s^2) \hat{\tau}_{-s}(x)\, ds$$

and

$$y_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 s^2) \tau_{-s}(x)\, ds.$$

The integrals are taken in the strong topology.

From the theory of analytic continuations, we then get that the elements x_n are analytic with respect to $\hat{\tau}$; see Lemma B.2.27.

By Definition 1.3.7, we know that W and $\hat{Q} \otimes Q$ strongly commute. This clearly implies that

$$\Phi \hat{\tau}_t(y) = (\hat{\tau}_t \mathbin{\bar{\otimes}} \tau_t)\Phi(y)$$

for all $y \in B(H)$ and $t \in \mathbb{R}$.

Hence, for all $n \in \mathbb{N}$, we have that

$$\Phi(x_n) = 1 \otimes y_n. \tag{1.5}$$

We consider the n.s.f. weight ψ_1 on $B(H)^+$ as defined in Proposition 1.3.29.

Let σ be the modular automorphism group of ψ_1. From Lemma B.1.34, it ensues that we have $\sigma_t(y) = \hat{\tau}_{-t}(y)$ for all $y \in B(H)$ and $t \in \mathbb{R}$.

Now, take $d \in \mathfrak{M}_{\psi_1}^+$ such that $\psi_1(d) = 1$.

Take $n \in \mathbb{N}$. We know that $x_n \in D(\hat{\tau}_{-\frac{i}{2}}) = D(\sigma_{\frac{i}{2}})$. The Tomita-Takesaki theory in Theorem B.1.31 then gives that $dx_n \in \mathfrak{M}_{\psi_1}$.

Let $\omega \in B(H)_*$. We get from Equation (1.5) that

$$(\iota \mathbin{\bar{\otimes}} \omega)\Phi(dx_n) = (\iota \mathbin{\bar{\otimes}} y_n\omega)\Phi(d).$$

Because $dx_n \in \mathfrak{M}_{\psi_1}$, the right invariance property of the weight ψ_1 gives that

$$(\iota \mathbin{\bar{\otimes}} \omega)\Phi(dx_n) \in \mathfrak{M}_{\psi_1} \qquad \text{and} \qquad (\iota \mathbin{\bar{\otimes}} y_n\omega)\Phi(d) \in \mathfrak{M}_{\psi_1}$$

and

$$\begin{aligned}\psi_1(dx_n)\omega(1) &= \psi_1\big((\iota \mathbin{\bar{\otimes}} \omega)\Phi(dx_n)\big) \\ &= \psi_1\big((\iota \mathbin{\bar{\otimes}} y_n\omega)\Phi(d)\big) = \psi_1(d)(y_n\omega)(1) = \omega(y_n).\end{aligned}$$

Notice that we here thus rely on Proposition 1.3.29.

The above calculation shows that $y_n = \psi_1(dx_n)1$ for every $n \in \mathbb{N}$.

It is standard to check that $y_n \to x$ in the strong topology. This implies that $\psi_1(dx_n)$ is a Cauchy sequence and thus $\psi_1(dx_n) \to \lambda$ for a number $\lambda \in \mathbb{C}$.

We can conclude that $x = \lambda 1$. ∎

We rephrase Lemma 1.3.30 in terms of the comultiplication.

Proposition 1.3.31 *Let $x \in M$. Assume that $\Phi(x) = x \otimes 1$ or $\Phi(x) = 1 \otimes x$. Then, we have that $x \in \mathbb{C}1$.*

Proof. The second case is an immediate consequence of Lemma 1.3.30. Using the commutation relation $\Phi R = \dot{\sigma}(R \otimes R)\Phi$, the first case can easily be deduced from the second one. ∎

The next Corollary 1.3.32 is to be expected.

Corollary 1.3.32 *We have that $M \cap \hat{M}' = \mathbb{C}1$ and $\hat{M} \cap M' = \mathbb{C}1$.*

Proof. By duality, it is sufficient to prove one result.

Suppose that $x \in M \cap \hat{M}'$. We know from Remark 1.3.14 that $W \in \hat{M} \bar{\otimes} M$. From the formula $\Phi(x) = W(x \otimes 1)W^*$, we thus get that $\Phi(x) = x \otimes 1$.

Proposition 1.3.31 now implies that $x \in \mathbb{C}1$. ∎

From Corollary 1.3.32, we can derive the following result.

Corollary 1.3.33 *The linear spaces $M'\hat{M}$ and $\hat{M}'M$ are dense in $B(H)$ with respect to the σ-strong* topology.*

Proof. By duality, it is again sufficient to prove one result. We use $\mathfrak{B}$ to denote the σ-strong* closure of $M'\hat{M}$.

From Remark 1.3.14, we know that $W \in \hat{M} \bar{\otimes} M$. We hence have

$$W^*(M' \bar{\otimes} 1)W \subseteq (1 \bar{\otimes} M')'. \tag{1.6}$$

Further, we get from Remark 1.3.14 that $W(M \bar{\otimes} 1)W^* \subseteq M \bar{\otimes} M$. It is then not difficult to deduce that

$$W^*(M' \bar{\otimes} 1)W \subseteq (M \bar{\otimes} 1)'. \tag{1.7}$$

Equations (1.6) and (1.7) together imply that

$$W^*(M' \bar{\otimes} 1)W \subseteq (M \bar{\otimes} M')' = M' \bar{\otimes} M. \tag{1.8}$$

The equality $(M \bar{\otimes} M')' = M' \bar{\otimes} M$ follows from the commutation theorem for tensor products of von Neumann algebras; see e.g. [119, Theorem IV.5.9].

Because W is a unitary, we get from Equation (1.8) that

$$W^*(M' \bar{\otimes} 1) \subseteq (M' \bar{\otimes} M)W^*. \tag{1.9}$$

From Theorem 1.3.9, we get that $\hat{M} = [\,(\iota \otimes \omega)W \mid \omega \in B(H)_*]''$. If we combine this result with Equation (1.9), we find that

$$\hat{M}M' \subseteq \mathfrak{B}.$$

This implies that $\mathfrak{B}$ is a von Neumann algebra.

It is easy to check that $\mathfrak{B}' = M \cap \hat{M}'$. Corollary 1.3.32 then gives that $\mathfrak{B}' = \mathbb{C}1$. We conclude that $\mathfrak{B} = B(H)$. ∎

The next Lemma 1.3.34 is used in the proof of Proposition 1.3.36 below.

Lemma 1.3.34 *We have that*

$$\operatorname{span}\big\{(x \otimes 1)W(1 \otimes y) \mid x, y \in B(H)\big\}$$

is dense in $B(H \otimes H)$ with respect to the σ-strong* topology.

Proof. Let $\xi, \eta \in H$. Since W is a multiplicative unitary, we get that

$$1 \otimes (\iota \otimes \omega_{\xi,\eta})(W) = (\iota \otimes \iota \otimes \omega_{\xi,\eta})(W_{13}^* W_{12}^* W_{23} W_{12}). \tag{1.10}$$

Now, let $\{\,\xi_i \mid i \in I\}$ be an orthonormal basis for H. We get from Lemma B.4.2 that we have

$$\begin{aligned}(\iota \otimes \iota \otimes \omega_{\xi,\eta})&(W_{13}^* W_{12}^* W_{23} W_{12}) \\ &= \sum_{i \in I} (\iota \otimes \iota \otimes \omega_{\eta,\xi_i})(W_{12}W_{13})^* (\iota \otimes \iota \otimes \omega_{\xi,\xi_i})(W_{23}W_{12})\end{aligned} \tag{1.11}$$

with convergence in the σ-strong* topology.

From Equations (1.10) and (1.11), we then find that

$$1 \otimes (\iota \otimes \omega_{\xi,\eta})(W) = \sum_{i \in I} (\iota \otimes \iota \otimes \omega_{\xi_i,\eta})(W_{13}^*) W^* (\iota \otimes \iota \otimes \omega_{\xi,\xi_i})(W_{23}) W \tag{1.12}$$

with convergence in the σ-strong* topology.

For any subset $X \subseteq B(H \otimes H)$, we use the notation $[X]^{\sigma\text{-strong}^*}$ to denote the closure of $\operatorname{span} X$ in the σ-strong* topology.

Equation (1.12) implies that

$$1 \mathbin{\bar{\otimes}} \hat{M} \subseteq [\,(p^* \otimes 1)W^*(1 \otimes q)W \mid p, q \in \hat{M}\,]^{\sigma\text{-strong}^*}. \tag{1.13}$$

From Remark 1.3.14, we know that $W \in \hat{M} \mathbin{\bar{\otimes}} M$. It is then not difficult to deduce from Equation (1.13) that we have

$$\big(B(H) \mathbin{\bar{\otimes}} M'\big)\big(1 \mathbin{\bar{\otimes}} \hat{M}\big) \subseteq [\,(a \otimes 1)W^*(1 \otimes b)W \mid a, b \in B(H)\,]^{\sigma\text{-strong}^*}. \tag{1.14}$$

If we combine Equation (1.14) with Corollary 1.3.33, we can derive that

$$B(H \otimes H) = [\,(a \otimes 1)W^*(1 \otimes b)W \mid a, b \in B(H)\,]^{\sigma\text{-strong}^*}.$$

We can then conclude that

$$B(H \otimes H) = [\,(a \otimes 1)W^*(1 \otimes b) \mid a, b \in B(H)\,]^{\sigma\text{-strong}^*}. \tag{1.15}$$

It is possible to extend R and $\hat{R}$ to involutive, normal *-anti-isomorphisms of $B(H)$. This follows from the results in [162, 114]. For the remainder of the proof, we use R and $\hat{R}$ to also denote these extensions.

We know from Proposition 1.3.25 that $(\hat{R} \otimes R)(W) = W$. If we apply $\hat{R} \bar{\otimes} R$ to Equation (1.15), we thus get

$$B(H \otimes H) = [\,(1 \otimes y)W^*(x \otimes 1) \mid x, y \in B(H)\,]^{\sigma\text{-strong}^*}.$$

This ends the proof of the lemma. ∎

We mention the following consequence of Lemma 1.3.34. It is clearly related to the regularity properties of W.

Corollary 1.3.35 *We have that*

$$\mathcal{C}(W) = [\,(\omega \otimes \iota)(\Sigma W) \mid \omega \in B(H)_*]$$

is dense in $B(H)$ with respect to the σ-strong* topology.

Proof. This can be deduced from Lemma 1.3.34 by using the same techniques as in the proof of [4, Proposition 3.2]. ∎

The next Proposition 1.3.36 is quite standard in the setting of regular multiplicative unitaries [4]. It is called (left and right) *reducibility* for W.

We remark that the reducibility property below is related to the faithfulness of the Haar weights on the bi-C*-algebra (A, Φ); see [39].

Proposition 1.3.36 *Let $\omega \in B(H)_*^+$ and $x \in M^+$. If $(\omega \otimes \iota)\Phi(x) = 0$ or $(\iota \otimes \omega)\Phi(x) = 0$, then we have either $x = 0$ or $\omega = 0$.*

Proof. By using the commutation relation $\Phi R = \dot{\sigma}(R \otimes R)\Phi$, it is not difficult to see that it is sufficient to prove one of the two considered cases.

Let $\omega \in B(H)_*^+$ and $x \in M^+$. Assume that $(\iota \otimes \omega)\Phi(x) = 0$. Because any $\rho \in B(H)_*^+$ is the (infinite) sum of positive vector functionals, we can assume that there exists a vector $\xi \in H$ such that $\omega = \omega_\xi$.

Take $y \in M$ such that $x = y^* y$. Let $P \in B(H)$ be the projection on $\mathbb{C}\xi$.

The assumptions of the proposition yield that we have

$$(\iota \otimes \omega_\xi)\big(W(y^* \otimes 1)(y \otimes 1)W^*\big) = 0.$$

This is equivalent to saying that

$$(y \otimes 1)W^*(1 \otimes P) = 0. \tag{1.16}$$

Let $a, b \in B(H)$. From Equation (1.16), it immediately follows that

$$(y \otimes 1)\big((1 \otimes a)W^*(b \otimes 1)\big)(1 \otimes P) = 0. \tag{1.17}$$

If we combine Equation (1.17) with Lemma 1.3.34, we get

$$(y \otimes 1)c(1 \otimes P) = 0.$$

for all $c \in B(H \otimes H)$. This implies that either $y = 0$ or $P = 0$. ■

The Corollary 1.3.38 is a C*-algebra version of Proposition 1.3.36. We make use of the next Lemma 1.3.37 in order to give a short proof.

We recall that the functionals $\omega\rho$ are defined in Notation 1.1.3.

Lemma 1.3.37 *Let $\omega \in A^*$ and $\rho \in B(H)_*$. There exist normal functionals $\theta_1, \theta_2 \in B(H)_*$ such that $\theta_1|_A = \rho\omega$ and $\theta_2|_A = \omega\rho$.*

In the case that $\omega \in A^*_+$ and $\rho \in B(H)^+_*$, we can take $\theta_1, \theta_2 \in B(H)^+_*$.

Proof. Using the commutation relation $\Phi R = \dot{\sigma}(R \otimes R)\Phi$, it is again sufficient to prove one of the two considered cases.

Take GNS-constructions (K_1, π_1, ξ_1) and (K_2, π_2, ξ_2) for ω, respectively ρ.

We know from Theorem 1.3.9 that $W \in M(B_0(H) \otimes A)$. Hence, we can define a unitary $U \in B(K_2) \otimes B(K_1)$ by the formula $U = (\pi_2 \otimes \pi_1)(W)$.

Now, we can define a normal functional $\theta_1 \in B(H)_*$ by setting

$$\theta_1(x) = (\omega_{\xi_2} \otimes \omega_{\xi_1})\big(U(\pi_2(x) \otimes 1)U^*\big).$$

It is not difficult to check that $\theta_1|_A = \rho\omega$. ■

Corollary 1.3.38 *Let $\omega \in A^*_+$ and $x \in A^+$. Assume that $(\omega \otimes \iota)\Phi(x) = 0$ or $(\iota \otimes \omega)\Phi(x) = 0$. Then we either have $x = 0$ or $\omega = 0$.*

Proof. We only consider the first result. It is clear that the second one can be proved in a completely similar way.

Assume that $(\omega \otimes \iota)\Phi(x) = 0$. Take $\rho \in B(H)^+_*$ such that $\rho(1) = 1$. We have that

$$(\omega\rho \otimes \iota)\Phi(x) = (\rho \otimes \iota)\Phi\big((\omega \otimes \iota)\Phi(x)\big) = 0.$$

Lemma 1.3.37 gives that there exists a $\theta \in B(H)_*^+$ such that $\theta|_A = \omega\rho$.

We clearly have that $(\theta \otimes \iota)\Phi(x) = 0$. Hence, Proposition 1.3.36 implies that either $x = 0$ or $\theta = 0$. If $\theta = 0$, then also $\omega = 0$. ∎

The Corollary 1.3.38 has a direct consequence in the compact case.

Corollary 1.3.39 *If A has a unit, then (A, Φ) is a compact quantum group.*

Proof. Assume that A is unital. From Theorem 1.3.9, we then get that (A, Φ) is a compact quantum group in the sense of Woronowicz. Using Remark 1.1.39 and Corollary 1.3.38, we find that the Haar state is faithful. ∎

We will need the Lemma 1.3.40 below.

Lemma 1.3.40 *Let $P \in M$ be a projection such that one of the four following conditions hold:*

1. $\Phi(P) \leq P \otimes 1$,
2. $\Phi(P) \leq 1 \otimes P$,
3. $\Phi(P) \geq P \otimes 1$,
4. $\Phi(P) \geq 1 \otimes P$.

Then, we have $P = 0$ or $P = 1$.

Proof. It clearly is enough to prove the third case. Using the commutation relation $\Phi R = \dot{\sigma}(R \otimes R)\Phi$, it is not difficult to deduce the fourth case. The two other cases then follow by replacing P with $1 - P$.

Take a projection $P \in M$ and assume that $W(P \otimes 1)W^* \geq P \otimes 1$. We then have that $(P \otimes 1)W(P \otimes 1)W^* = P \otimes 1$. This yields that

$$(P \otimes 1)W(P \otimes 1) = (P \otimes 1)W. \tag{1.18}$$

Take $\xi \in D(Q)$ and $\eta \in D(Q^{-1})$. Applying $\iota \mathbin{\bar{\otimes}} \omega_{Q\xi, Q^{-1}\eta}$ to the above equality, we get that

$$P(\iota \mathbin{\bar{\otimes}} \omega_{Q\xi, Q^{-1}\eta})(W)P = P(\iota \mathbin{\bar{\otimes}} \omega_{Q\xi, Q^{-1}\eta})(W). \tag{1.19}$$

It is possible to extend R and $\hat{R}$ to involutive, normal *-anti-isomorphisms of $B(H)$. This follows from the results in [162, 114]. For the remainder of the proof, we use R and $\hat{R}$ to also denote these extensions.

Applying $\hat{R}$ to the above Equation (1.19), we get

$$\hat{R}(P)\hat{R}\big((\iota \mathbin{\bar{\otimes}} \omega_{Q\xi, Q^{-1}\eta})(W)\big)\hat{R}(P) = \hat{R}\big((\iota \mathbin{\bar{\otimes}} \omega_{Q\xi, Q^{-1}\eta})(W)\big)\hat{R}(P). \tag{1.20}$$

From the dual version of Proposition 1.3.23, we know that

$$\hat{R}\big((\iota \mathbin{\bar{\otimes}} \omega_{Q\xi, Q^{-1}\eta})(W)\big) = (\iota \mathbin{\bar{\otimes}} \omega_{\xi, \eta})W^*. \tag{1.21}$$

Equations (1.20) and (1.21) together yield

$$\hat{R}(P)(\iota \mathbin{\bar{\otimes}} \omega_{\xi,\eta})(W^*)\hat{R}(P) = (\iota \mathbin{\bar{\otimes}} \omega_{\xi,\eta})(W^*)\hat{R}(P). \tag{1.22}$$

Therefore, we can conclude that

$$(\hat{R}(P) \otimes 1)W^*(\hat{R}(P) \otimes 1) = W^*(\hat{R}(P) \otimes 1).$$

We now apply $\hat{R} \mathbin{\bar{\otimes}} R$ to Equation 1.22. Remembering from Proposition 1.3.25 that $(\hat{R} \mathbin{\bar{\otimes}} R)(W) = W$, we then directly find that

$$(P \otimes 1)W^*(P \otimes 1) = (P \otimes 1)W^*.$$

By taking the adjoint, we get that

$$(P \otimes 1)W(P \otimes 1) = W(P \otimes 1). \tag{1.23}$$

If we combine Equations (1.18) and (1.23), we see that $W(P \otimes 1) = (P \otimes 1)W$ and thus

$$\Phi(P) = P \otimes 1.$$

From Proposition 1.3.31, we now get that $P \in \mathbb{C}1$. This implies that we either have $P = 0$ or $P = 1$. ■

We below prove important corollaries of the above Lemma 1.3.40. These results handle about normal weights that are left or right invariant.

The properties are first formulated in the von Neumann algebra framework. Not so surprisingly, this gives some more elegant results than the C*-algebra case. Further below, we also discuss the C*-algebraic counterparts.

We state the Definition 1.3.41 of left and right invariance for weights on M^+. This is of course the von Neumann algebraic version of Definition 1.1.4.

Definition 1.3.41 *Let φ be a weight on M^+.*

- *The weight φ is called* left invariant *if $\varphi\big((\omega \otimes \iota)\Phi(a)\big) = \omega(1)\varphi(x)$ for all $x \in \mathfrak{M}_\varphi^+$ and $\omega \in M_*^+$. It is called* strongly *left invariant if this equality can be extended to hold for all $x \in M^+$ and $\omega \in M_*^+$;*
- *The weight φ is called* right invariant *if $\varphi\big((\iota \otimes \omega)\Phi(x)\big) = \omega(1)\varphi(x)$ for all $x \in \mathfrak{M}_\varphi^+$ and $\omega \in M_*^+$. It is called* strongly *right invariant if this equality can be extended to hold for all $x \in M^+$ and $\omega \in M_*^+$.*

The unitary antipode R interchanges left and right invariance.

Proposition 1.3.42 *Let η be a weight on M^+. If η is left invariant, then ηR is right invariant. If η is right invariant, then ηR is left invariant.*

We have the same properties in the case of strong invariance.

Proof. Assume for instance that η is left invariant. From Theorem 1.3.9, we know that we have $\Phi R = \dot{\sigma}(R \otimes R)\Phi$.

Take $x \in \mathfrak{M}^+_{\eta R}$. We then have

$$\begin{aligned}\eta R\big((\iota \otimes \omega)\Phi(x)\big) &= \eta R\big((\omega \otimes \iota)(R \otimes R)\big(\Phi R(x)\big)\big) \\ &= \eta\big((\omega R \otimes \iota)(\Phi R(x))\big) = \omega R(1)\eta(R(x)) = \omega(1)\eta R(x).\end{aligned}$$

This gives that ηR is right invariant.

All the other results can be proved in a completely similar way. ∎

The next Proposition 1.3.43 states the important property that non-zero, normal left invariant weights on (M, Φ) are automatically faithful.

Proposition 1.3.43 *Let η be a non-zero, normal weight on M^+. Suppose that we have for all $x \in M^+$ with $\eta(x) = 0$ and $\omega \in M^+_*$ that*

$$\eta\big((\omega \,\bar{\otimes}\, \iota)\Phi(x)\big) = \omega(1)\eta(x)$$

Then, we have that η is faithful.

Proof. Put $N = \{x \in M \mid \eta(x^*x) = 0\}$. Then N is a σ-weakly closed left ideal in M. So, there exists a projection $P \in M$ such that $N = MP$.

We thus have $P \in N$. The left invariance property of η gives that

$$\eta\Big(\big((\omega \,\bar{\otimes}\, \iota)\Phi(P)\big)^*\big((\omega \,\bar{\otimes}\, \iota)\Phi(P)\big)\Big) \leq \|\omega\|\eta\big((\omega \,\bar{\otimes}\, \iota)\Phi(P^*P)\big) = \|\omega\|^2\eta(P^*P) = 0$$

for every $\omega \in M^+_*$. The used inequality infers from Lemma B.4.1.

The property $(\omega \,\bar{\otimes}\, \iota)\Phi(P) \in N$ directly follows from the above calculation.

For all $\omega \in M_*$, we thus have $(\omega \,\bar{\otimes}\, \iota)\Phi(P)P = (\omega \,\bar{\otimes}\, \iota)\Phi(P)$.

It then ensues that

$$\Phi(P)(1 \otimes P) = \Phi(P).$$

From Lemma 1.3.40, we now get that either $P = 0$ or $P = 1$. ∎

It is clear that Proposition 1.3.42 can be applied to directly deduce the right invariant version of the above Proposition 1.3.43.

Proposition 1.3.44 *Let η be a non-zero, normal weight on M^+. Suppose that we have for all $x \in M^+$ with $\eta(x) = 0$ and $\omega \in M^+_*$ that*

$$\eta\big((\iota \,\bar{\otimes}\, \omega)\Phi(x)\big) = \omega(1)\eta(x)$$

Then, we have that η is faithful.

We now prove Proposition 1.3.45. This property is quite similar to Proposition 1.3.43, but is about left invariant weights and semi-finiteness.

Proposition 1.3.45 *Let η be a normal weight on M^+ that is left invariant. Then, either η is semi-finite or $\mathfrak{M}_\eta^+ = \{0\}$.*

Proof. Let N be the σ-weak closure of $\mathfrak{N}_\eta$. Then N is a σ-weakly closed left ideal in M. So, there exists a projection $P \in M$ such that $N = MP$.

We thus have $P \in N$. Let $(x_i)_{i\in I}$ be a net in $\mathfrak{N}_\eta$ such that $x_i \to P$ in the σ-weak topology. The left invariance property of η gives that

$$\eta\Big(\big((\omega \mathbin{\bar{\otimes}} \iota)\Phi(x_i)\big)^*\big(\omega \mathbin{\bar{\otimes}} \iota)\Phi(x_i)\big)\Big) \leq \|\omega\|\eta\big((\omega \mathbin{\bar{\otimes}} \iota)\Phi(x_i^* x_i)\big) = \|\omega\|^2\eta(x_i^* x_i) < +\infty$$

for all $i \in I$ and $\omega \in M_*^+$. Here, we again use Lemma B.4.1.

The above calculation implies that $(\omega \mathbin{\bar{\otimes}} \iota)\Phi(x_i) \in N$ for all $i \in I$. This result yields that $(\omega \mathbin{\bar{\otimes}} \iota)\Phi(x_i)P = (\omega \mathbin{\bar{\otimes}} \iota)\Phi(x_i)$ for all $i \in I$.

For all $\omega \in M_*$, we thus have $(\omega \mathbin{\bar{\otimes}} \iota)\Phi(P)P = (\omega \mathbin{\bar{\otimes}} \iota)\Phi(P)$.

It then ensues that

$$\Phi(P)(1 \otimes P) = \Phi(P).$$

From Lemma 1.3.40, we now get that either $P = 0$ or $P = 1$. ∎

We again also state the right invariant version.

Proposition 1.3.46 *Let η be a normal weight on M^+ that is right invariant. Then, either η is semi-finite or $\mathfrak{M}_\eta^+ = \{0\}$.*

The next Definition 1.3.47 is introduced for notational convenience.

Definition 1.3.47 *We define a faithful, normal weight ψ_∞ on M^+ by*

$$\psi_\infty(x) = \begin{cases} 0 & \text{if } x = 0, \\ +\infty & \text{if } x \neq 0. \end{cases}$$

The earlier Proposition 1.3.36 trivially implies the next property.

Proposition 1.3.48 *The weight ψ_∞ is strongly left and strongly right invariant.*

The next Proposition 1.3.49 classifies the normal weights which are left or right invariant. It is a direct consequence of Propositions 1.3.43 to 1.3.46.

It turns out that invariant weights are either very well behaved (i.e. n.s.f. weights making (M, Φ) into a quantum group) or completely trivial.

Proposition 1.3.49 *Let η be a normal weight on M^+. Assume that η is left or right invariant. Then, we are in one of the following cases:*

1. *We have that $\eta = 0$,*

2. *We have that $\eta = \psi_\infty$,*
3. *We have that η is an n.s.f. weight.*

The above Proposition 1.3.49 infers the next result.

Corollary 1.3.50 *Let η be a normal weight on M^+. Further, assume that η is left or right invariant. Then, we have that η is an n.s.f. weight if and only if there exists an element $x \in M^+$ such that $\eta(x) = 1$.*

The Proposition 1.3.49 can also be used to prove that every normal, left invariant weight is automatically strongly left invariant.

Proposition 1.3.51 *Let η be a normal weight on M^+ that is left invariant. Then η is strongly left invariant.*

Proof. Let η be a normal, left invariant weight on M. According to Proposition 1.3.49, we have to consider three cases.

In the case that $\eta = 0$, the strong invariance is trivial. If $\eta = \psi_\infty$, then we get from Proposition 1.3.48 that η is strongly invariant.

Assume that η is an n.s.f. weight. It then ensues from Theorem 1.3.9 that (M, Φ) is a von Neumann algebraic quantum group with left Haar weight η.

We can hence use the von Neumann algebraic version of Theorem 1.1.11 to prove that the n.s.f. weight η satisfies the strong invariance property. ∎

The right invariant version of course also holds.

Proposition 1.3.52 *Let η be a normal weight on M^+ that is right invariant. Then η is strongly right invariant.*

We then finally mention a uniqueness property. The Proposition 1.3.54 below generalizes in a (very) small matter the corresponding property in the quantum group case. It is only mentioned for the sake of completeness.

The Proposition 1.3.54 almost directly follows from Proposition 1.3.49 together with Theorem 1.1.10. Further, it is clear that the corresponding property for right invariant weights holds in a completely similar form.

Notation 1.3.53 *Let $\lambda = +\infty$. For every weight η on M^+, we set $\lambda\eta = \psi_\infty$.*

Proposition 1.3.54 *Let η_1 and η_2 be two non-zero, normal weights on M^+. Suppose that η_1 and η_2 are both left invariant.*

Then, there exists a $\lambda \in]0, +\infty]$ such that $\eta_1 = \lambda\eta_2$.

We now also prove a result for the C*-algebra case. It is clear that Proposition 1.3.55 below is the C*-algebraic counterpart of Proposition 1.3.43. We do not know if there holds a C*-algebra version of Proposition 1.3.45.

The Proposition 1.3.55 quite easily ensues from Proposition 1.3.38. The relation between these two propositions is meticulously explored in [39].

Proposition 1.3.55 *Let η be a non-zero, lower semi-continuous weight on A^+. Suppose that we have for all $x \in A^+$ with $\eta(x) = 0$ and $\omega \in A^*_+$ that*

$$\eta\big((\omega \otimes \iota)\Phi(x)\big) = \omega(1)\eta(x)$$

Then, we have that η is faithful.

Proof. Take $a \in A^+$ with $\eta(a) = 0$. It is clear that $a \in \mathfrak{M}^+_\eta$. For all $\rho \in A^*_+$, we then have

$$\eta\big((\rho \otimes \iota)\Phi(a)\big) = \rho(1)\eta(a) = 0.$$

Using Notation B.1.10, we then see that $\omega\big((\rho \otimes \iota)\Phi(a)\big) = 0$ for all $\omega \in \mathcal{F}_\eta$ and $\rho \in A^*_+$. This implies that

$$(\iota \otimes \omega)\Phi(a) = 0 \tag{1.24}$$

for all $\omega \in \mathcal{F}_\eta$.

Suppose that $a \neq 0$. From Equation (1.24) and Proposition 1.3.38, we then get that $\mathcal{F}_\eta = \{0\}$. Proposition B.1.12 now yields that $\eta = 0$. ∎

We also state the right invariant version.

Proposition 1.3.56 *Let η be a non-zero, lower semi-continuous weight on A^+. Suppose that we have for all $x \in A^+$ with $\eta(x) = 0$ and $\omega \in A^*_+$ that*

$$\eta\big((\iota \otimes \omega)\Phi(x)\big) = \omega(1)\eta(x)$$

Then, we have that η is faithful.

Strongly modular multiplicative unitaries

We end the Section 1.3 by giving a few more results regarding the manageability theory. We introduce in Definition 1.3.57 below a notion of *strong modularity* for multiplicative unitaries. This new condition is slightly stronger than the 'usual' modularity. It has the advantage that we automatically get a nice implementation of the unitary antipode R; see Proposition 1.3.70.

We think that this new condition defines a quite natural category wherein some formulas are more refined than in the modular case.

For strongly modular multiplicative unitaries, it is possible to simplify the known proofs of several basic results (cf. Remark 1.3.69). It is however not the case that the strong modularity leads up to more powerful properties.

The strong modularity is thus not a real renewal. It only revamps the modularity in a small degree. Even so, the strong modularity has some plus-point over modularity. The Proposition 1.3.70 displays a tangible formula for the unitary antipodes R and $\hat{R}$. This can be useful in concrete examples.

The Definition 1.3.57 below of course resembles Definition 1.3.7.

Definition 1.3.57 *Let $W \in B(H \otimes H)$ be a multiplicative unitary. We then say that W is* strongly modular *if there exist two strictly positive operators Q and $\hat{Q}$ on H and two anti-unitary operators I and $\hat{I}$ on H such that*

1. *We have the two following relations:*
   (a) $W(\hat{Q} \otimes Q)W^* = \hat{Q} \otimes Q$,
   (b) $(\hat{I} \otimes I)W^*(\hat{I}^* \otimes I^*) = W$.
2. *For all $x, z \in H$ and $u \in D(Q)$, $y \in D(Q^{-1})$, we have*
$$\big\langle W(z \otimes y), x \otimes u \big\rangle = \big\langle W^*(z \otimes IQu), x \otimes IQ^{-1}y \big\rangle.$$

The next Proposition 1.3.58 is used a few times below.

Proposition 1.3.58 *Let $(W, Q, \hat{Q}, I, \hat{I})$ be a quintuple satisfying the conditions in Definition 1.3.57. These conditions then also hold for $(W, Q, \hat{Q}, I^*, \hat{I}^*)$.*

Proof. Consider a quintuple $(W, Q, \hat{Q}, I, \hat{I})$ which satisfies the conditions in Definition 1.3.57. Because I and $\hat{I}$ are anti-unitaries, it is clear that
$$(\hat{I}^* \otimes I^*)W^*(\hat{I} \otimes I) = W.$$
Let $r \in D(\hat{Q})$, $v \in D(\hat{Q}^{-1})$ and $w \in D(Q)$, $s \in D(Q^{-1})$. Then, we have
$$\begin{aligned}\langle W(r \otimes s), v \otimes w \rangle &= \langle W(\hat{Q}r \otimes Qs), \hat{Q}^{-1}v \otimes Q^{-1}w \rangle \\ &= \langle W^*(\hat{Q}r \otimes Iw), \hat{Q}^{-1}v \otimes Is \rangle.\end{aligned}$$
We hence get that
$$\langle W(\hat{Q}^{-1}v \otimes Is), \hat{Q}r \otimes Iw \rangle = \langle W^*(v \otimes w), r \otimes s \rangle. \tag{1.25}$$
It is clear that this Equation (1.25) can easily be extended to hold in the more general case that $s, w \in H$.

Take $x \in D(\hat{Q})$, $z \in D(\hat{Q}^{-1})$ and $u \in D(Q)$, $y \in D(Q^{-1})$. Equation (1.25) gives that we have
$$\begin{aligned}\langle W(z \otimes y), x \otimes u \rangle &= \langle W(\hat{Q}^{-1}z \otimes II^*Q^{-1}y), \hat{Q}x \otimes II^*Qu \rangle \\ &= \langle W^*(z \otimes I^*Qu), x \otimes I^*Q^{-1}y \rangle\end{aligned}$$
By continuity, this equality can be extended to elements $x, z \in H$. ∎

The next Proposition 1.3.59 justifies the used terminology. The Condition (1b) in Definition 1.3.57 is not needed to have this property. We do not know if there exist strongly modular multiplicative unitaries that are not modular.

The connection between the notions of modularity and strong modularity is further investigated in Proposition 1.3.73 below.

Proposition 1.3.59 *Let $W \in B(H \otimes H)$ be a multiplicative unitary operator. If W is strongly modular, then it is also modular.*

Proof. Let $\overline{H}$ be the conjugate Hilbert space of H and let L be the canonical anti-unitary operator $L : H \to \overline{H} : \xi \mapsto \overline{\xi}$. Then, define a unitary operator $\breve{W}$ by the formula

$$\breve{W} = (L \otimes I)W(L^* \otimes I^*) = (L\hat{I}^* \otimes 1)W^*(\hat{I}L^* \otimes 1). \tag{1.26}$$

Take $x, z \in H$ and $u \in D(Q)$, $y \in D(Q^{-1})$. Proposition 1.3.58 implies that

$$\begin{aligned}\langle W(z \otimes y), x \otimes u\rangle &= \langle W^*(z \otimes I^*Qu), x \otimes I^*Q^{-1}y\rangle \\ &= \langle z \otimes I^*Qu, (L^* \otimes I^*)\breve{W}(L \otimes I)(x \otimes I^*Q^{-1}y)\rangle \\ &= \langle \breve{W}(\overline{x} \otimes Q^{-1}y), \overline{z} \otimes Qu\rangle.\end{aligned}$$

The quadruple $(W, \breve{W}, Q, \hat{Q})$ thus satisfies the conditions in Definition 1.3.7. ■

We have the following duality property.

Proposition 1.3.60 *Let W be a multiplicative unitary. In the case that W is strongly modular, also $\Sigma W^* \Sigma$ is strongly modular.*

The two operators Q and $\hat{Q}$ appearing in Definition 1.3.57 then exchange their positions. Similarly, the two operators I and $\hat{I}$ switch places.

Proof. We denote $\hat{W} = \Sigma W^* \Sigma$. We know from Proposition 1.3.2 that $\hat{W}$ is a multiplicative unitary. It is easy to check that

$$\hat{W}^*(Q \otimes \hat{Q})\hat{W} = Q \otimes \hat{Q} \qquad \text{and} \qquad (I^* \otimes \hat{I}^*)\hat{W}^*(I \otimes \hat{I}) = \hat{W}.$$

Take $x \in D(Q)$, $z \in D(Q^{-1})$ and $u \in D(\hat{Q})$, $y \in D(\hat{Q})$. We have that

$$\begin{aligned}\langle \hat{W}(z \otimes y), x \otimes u\rangle &= \langle \hat{Q}^{-1}y \otimes Q^{-1}z, W(\hat{Q}u \otimes Qx)\rangle \\ &= \langle \hat{Q}^{-1}y \otimes I^*x, W^*(\hat{Q}u \otimes I^*z)\rangle \\ &= \langle (\hat{I}^* \otimes I^*)W^*(\hat{I} \otimes I)(\hat{Q}^{-1}y \otimes I^*x), \hat{Q}u \otimes I^*z\rangle \\ &= \langle \hat{I}\hat{Q}u \otimes z, W^*(\hat{I}\hat{Q}^{-1}y \otimes x)\rangle \\ &= \langle \hat{W}^*(z \otimes \hat{I}\hat{Q}u), x \otimes \hat{I}\hat{Q}^{-1}y\rangle.\end{aligned}$$

The second equality follows from Proposition 1.3.58.

It is now easy to deduce the case $z, u \in H$ by using a continuity argument. ■

We observe that (the proof of) Proposition 1.3.59 shows that the Notation 1.3.61 below does not contradict Notation 1.3.15 in any respect.

The Notation 1.3.61 is introduced to avoid long formulations.

Notation 1.3.61 *Consider the modular multiplicative unitary W that we have fixed in the earlier Notation 1.3.15.*

For the rest of the section, we assume that W is strongly modular.

We keep applying the Notation 1.3.15. Further, we also fix the two anti-unitary operators I and $\hat{I}$ related to W in the sense of Definition 1.3.57.

The next Lemma 1.3.62 holds because I and $\hat{I}$ are anti-unitary.

Lemma 1.3.62 *We have that*

$$(\hat{I} \otimes I)W^*(\hat{I}^* \otimes I^*) = W, \qquad (\hat{I}^* \otimes I^*)W^*(\hat{I} \otimes I) = W,$$
$$(\hat{I} \otimes I)W(\hat{I}^* \otimes I^*) = W^*, \qquad (\hat{I}^* \otimes I^*)W(\hat{I} \otimes I) = W^*.$$

The following five lemmas can be useful in calculations. They are all related to the Condition (2) in Definition 1.3.57; see Remark 1.3.68.

We first give a short proof of Lemmas 1.3.63 and 1.3.64.

Lemma 1.3.63 *For all $x \in D(\hat{Q}^{-1})$, $z \in D(\hat{Q})$ and $y, u \in H$, we have*

$$\big\langle W(z \otimes y), x \otimes u \big\rangle = \big\langle W^*(\hat{Q}z \otimes Iu), \hat{Q}^{-1}x \otimes Iy \big\rangle.$$

Proof. Take $x \in D(\hat{Q}^{-1})$, $z \in D(\hat{Q})$ and $y \in D(Q)$, $u \in D(Q^{-1})$. It is direct to check that

$$\begin{aligned}\langle W(z \otimes y), x \otimes u \rangle &= \langle W(\hat{Q}z \otimes Qy), \hat{Q}^{-1}x \otimes Q^{-1}u \rangle \\ &= \langle W(\hat{Q}z \otimes Iu), \hat{Q}^{-1}x \otimes Iy \rangle.\end{aligned}$$

By a continuity argument, the proof is finished. ∎

Lemma 1.3.64 *For all $x, z \in H$ and $u \in I^*D(Q^{-1})$, $y \in I^*D(Q)$, we have*

$$\big\langle W(z \otimes y), x \otimes u \big\rangle = \big\langle W^*(z \otimes Q^{-1}Iu), x \otimes QIy \big\rangle.$$

Proof. Take $x, z \in H$ and $u \in I^*D(Q^{-1})$, $y \in I^*D(Q)$. We have that

$$\begin{aligned}\langle W(z \otimes y), x \otimes u \rangle &= \langle W(\hat{I}^*z \otimes I^*y), \hat{I}^*x \otimes I^*u \rangle \\ &= \langle W^*(x \otimes Q^{-1}I^*y), z \otimes QI^*u \rangle.\end{aligned}$$

The last equality follows from Lemma 1.3.63 applied to the dual multiplicative unitary $\Sigma W^* \Sigma$ (cf. Lemma 1.3.66 below). ∎

From Proposition 1.3.60, it follows that the three lemmas below are simple consequences of Definition 1.3.57 and Lemmas 1.3.63 and 1.3.64.

Lemma 1.3.65 *For all $x \in D(\hat{Q}^{-1})$, $z \in D(\hat{Q})$ and $y, u \in H$ we have*

$$\big\langle W(z \otimes y), x \otimes u \big\rangle = \big\langle W^*(\hat{I}\hat{Q}^{-1}x \otimes y), \hat{I}\hat{Q}z \otimes u \big\rangle.$$

Lemma 1.3.66 *For all $x, z \in H$ and $y \in D(Q^{-1})$, $u \in D(Q)$, we have*

$$\big\langle W(z \otimes y), x \otimes u \big\rangle = \big\langle W^*(\hat{I}x \otimes Q^{-1}y), \hat{I}z \otimes Qu \big\rangle.$$

Lemma 1.3.67 *For all $x \in \hat{I}^* D(\hat{Q}^{-1})$, $z \in \hat{I}^* D(\hat{Q})$ and $y, u \in H$, we have*

$$\big\langle W(z \otimes y), x \otimes u \big\rangle = \big\langle W^*(\hat{Q}\hat{I}z \otimes u), \hat{Q}^{-1}\hat{I}x \otimes y \big\rangle.$$

The next Remark 1.3.68 alludes to the above lemmas.

Remark 1.3.68 *The Condition* (2) *in Definition 1.3.57 can be replaced by either one of the properties in Lemmas 1.3.63 to 1.3.67.*

This of course only holds under the assumption of Conditions (1a) *and* (1b)*.*

In the case of Lemma 1.3.63, the equivalence with Condition (2) *is implicitly proved in the last part of the proof of Proposition 1.3.73.*

Maybe, this is a good place to include the next Remark 1.3.69. The carrying out of this remark can be done by using Propositions 1.3.21 and 1.3.70.

The Remark 1.3.69 has of course has no great value.

Remark 1.3.69 *In the case of strongly modular multiplicative unitaries, several of the calculations in [162] can be simplified in a great extent. For instance, the main formula*

$$S\big((\omega \otimes \iota)W\big) = (\omega \otimes \iota)W^*$$

is now an almost direct consequence of the easy Lemma 1.3.63.

The next Proposition 1.3.70 shows the advantage of strongly modular multiplicative unitaries over modular ones.

We obtain the property that the unitary antipode R in the strong modular case is implemented by the anti-unitary operator I.

Proposition 1.3.70 *We have that*

$$R(x) = Ix^*I^* = I^*x^*I \qquad \textit{and} \qquad \hat{R}(y) = \hat{I}y^*\hat{I}^* = \hat{I}^*y^*\hat{I}$$

for all $x \in M$ and $y \in \hat{M}$.

Proof. We prove the first formula. The second one can then be deduced by using the duality property in Proposition 1.3.60.

Let $\omega \in B(H)_*$. Proposition 1.3.59 gives that W is modular. Hence, we get from Proposition 1.3.19 that

$$R\big((\omega \otimes \iota)W\big) = (\omega^\top \otimes \iota)\breve{W}^*. \tag{1.27}$$

In this formula, $\breve{W}$ is the unitary defined by Equation (1.26).

Define $\omega_{\hat{R}} \in B(H)_*$ by $\omega_{\hat{R}}(x) = \omega(\hat{I}^* x^* \hat{I})$.

We have that

$$I\big((\omega \otimes \iota)W\big)^* I^* = (\omega_{\hat{R}} \otimes \iota)\big((\hat{I} \otimes I)W^*(\hat{I}^* \otimes I^*)\big) = (\omega_{\hat{R}} \otimes \iota)W. \tag{1.28}$$

Further, we have

$$(\omega^\top \otimes \iota)\breve{W}^* = (\omega^\top \otimes \iota)\big((L\hat{I}^* \otimes 1)W(\hat{I}L^* \otimes 1)\big) = (\omega_{\hat{R}} \otimes \iota)W. \tag{1.29}$$

The three Equations (1.27), (1.28) and (1.29) together imply that we have for all $\omega \in B(H)_*$ that

$$R\big((\omega \otimes \iota)W\big) = I\big((\omega \otimes \iota)W\big)^* I^*. \tag{1.30}$$

We know from Remark 1.3.14 that R is a normal *-anti-automorphism on M. The Equation (1.30) thus infers that we have for all $x \in M$ that

$$R(x) = I x^* I^*.$$

Because $R^2 = \iota$, we also get that $R(x) = I^* x^* I$ for all $x \in M$. ∎

The following generalization of Proposition 1.3.58 is a direct consequence.

Corollary 1.3.71 *Let $(W, Q, \hat{Q}, I, \hat{I})$ be a quintuple satisfying all the conditions in Definition 1.3.57. These conditions then also hold for the three quintuples $(W, Q, \hat{Q}, I^*, \hat{I}^*)$, $(W, Q, \hat{Q}, I, \hat{I}^*)$ and $(W, Q, \hat{Q}, I^*, \hat{I})$.*

Proof. This result follows from a combination of Propositions 1.3.58 and 1.3.70 together with Propositions 1.3.59 and 1.3.25. ∎

We can also deduce the following variant of Lemma 1.3.62.

Lemma 1.3.72 *We have that*

$$(\hat{I} \otimes I^*)W^*(\hat{I}^* \otimes I) = W, \qquad (\hat{I}^* \otimes I)W^*(\hat{I} \otimes I^*) = W,$$
$$(\hat{I} \otimes I^*)W(\hat{I}^* \otimes I) = W^*, \qquad (\hat{I}^* \otimes I)W(\hat{I} \otimes I^*) = W^*.$$

It turns out that the implementation of R is precisely the property that makes the 'strong' in strong modularity. We can prove Proposition 1.3.73.

We remark that, for once, we make the exception not to follow Notation 1.3.61.

Proposition 1.3.73 *Let W be a modular multiplicative unitary. Working with the notations from Notation 1.3.15, we have that W is strongly modular if and only if there exist anti-unitary operators I and $\hat{I}$ such that*

$$R(x) = IxI^* \qquad \text{and} \qquad \hat{R}(y) = \hat{I}y\hat{I}^* \tag{1.31}$$

for all $x \in M$ and $y \in \hat{M}$.

Proof. If W is strongly modular, it follows from Proposition 1.3.59 that W is modular. From Proposition 1.3.70, we get the implementation of R and $\hat{R}$.

Now, assume that W is modular and that R and $\hat{R}$ are implemented as in the Equation (1.31). It directly follows from Definition 1.3.7 that

$$W(\hat{Q} \otimes Q)W^* = \hat{Q} \otimes Q.$$

Proposition 1.3.25 implies that we have

$$(\hat{I} \otimes I)W^*(\hat{I}^* \otimes I^*) = W.$$

Take $r \in D(\hat{Q}^{-1})$, $v \in D(\hat{Q})$ and $s \in D(Q^{-1})$, $w \in D(Q)$. Then, we have

$$\langle W(v \otimes s), r \otimes w\rangle = \langle(\omega_{v,r} \otimes \iota)Ws, w\rangle.$$

If we use Proposition 1.3.23, we get that

$$\begin{aligned}\langle W(v \otimes s), r \otimes w\rangle &= \langle I^*(\omega_{\hat{Q}^{-1}r,\hat{Q}v} \otimes \iota)WIs, w\rangle \\ &= \langle W^*(\hat{Q}v \otimes Iw), \hat{Q}^{-1}r \otimes Is\rangle.\end{aligned}$$

Hence, for $x \in D(\hat{Q})$, $z \in D(\hat{Q}^{-1})$ and $u \in D(Q)$, $y \in D(Q^{-1})$, we have that

$$\begin{aligned}\langle W(z \otimes y), x \otimes u\rangle &= \langle W(\hat{Q}^{-1}z \otimes Q^{-1}y), \hat{Q}x \otimes Qu\rangle \\ &= \langle W^*(z \otimes IQu), x \otimes IQ^{-1}y\rangle.\end{aligned}$$

By continuity, we can extend this result to $x, z \in H$. ∎

The implementation of the unitary antipode can be used to prove some more formulas. We mention two simple properties.

Proposition 1.3.74 *Let $\xi, \eta \in H$. We have that*

$$R\big((\omega_{\xi,\eta} \otimes \iota)W\big) = (\omega_{\hat{I}\eta,\hat{I}\xi} \otimes \iota)W = (\omega_{\hat{I}^*\eta,\hat{I}^*\xi} \otimes \iota)W.$$

Proof. It is clear that $\omega_{\hat{I}\eta,\hat{I}\xi}(x) = \omega_{\hat{I}^*\eta,\hat{I}^*\xi}(x) = \omega_{\xi,\eta}(\hat{R}(x))$ for all $x \in B(H)$. Hence, an application of Proposition 1.3.26 ends the proof. ∎

Lemma 1.3.75 *Let $\xi \in D(\hat{Q}^{-1})$ and $\eta \in D(\hat{Q})$. Then, we have*

$$(\omega_{\xi,\eta} \otimes \iota)W^* = (\omega_{\hat{I}\hat{Q}\eta, \hat{I}\hat{Q}^{-1}\xi} \otimes \iota)W.$$

Proof. Take $\xi \in D(\hat{Q}^{-1})$ and $\eta \in D(\hat{Q})$. We have that

$$\begin{aligned}(\omega_{\xi,\eta} \otimes \iota)W^* &= S\big((\omega_{\xi,\eta} \otimes \iota)W\big) \\ &= R\big((\omega_{\hat{Q}^{-1}\xi, \hat{Q}\eta} \otimes \iota)W\big) = (\omega_{\hat{I}\hat{Q}\eta, \hat{I}\hat{Q}^{-1}\xi} \otimes \iota)W\end{aligned}$$

In the above calculation, we successively use Theorem 1.3.9, Proposition 1.3.74 and Proposition 1.3.21. ∎

The next Proposition 1.3.76 is about the implementation of the comultiplication. It can easily be deduced from Definition 1.3.57.

Proposition 1.3.76 *Define $V \in M(A \otimes B_0(H))$ by $V = (I \otimes I)\Sigma W^* \Sigma (I^* \otimes I^*)$. Then, V is a multiplicative unitary. For all $x \in A$, we have that*

$$\Phi(x) = W(x \otimes 1)W^* = V^*(1 \otimes x)V.$$

We end this Section 1.3 by a remark about the strong modularity of the regular representation of a locally compact quantum group.

The properties in Remark 1.3.77 follow from the results in [66].

Remark 1.3.77 *Let W be the (left or right) regular representation of a locally compact quantum group (A, Φ). Then W is strongly modular.*

Moreover, we can take $Q = \hat{Q}$ and also $I^* = I$ and $\hat{I}^* = \hat{I}$.

Further, we have the commutation relations $IN^tI = N^{-t}$ and $\hat{I}N^t\hat{I} = N^{-t}$ for all $t \in \mathbb{R}$. We have $I\hat{I} = \nu^{\frac{i}{4}}\hat{I}I$ where ν is the scaling constant of (A, Φ).

1.4 A formula for the Haar weights

From the results in Section 1.3, we see that the theory of modular multiplicative unitaries is not too far away from the theory of locally compact quantum groups. The only gulf between these two branches of quantum group theory sits in the existence of Haar weights. From the principal Theorem 1.3.9, we know that a bi-C*-algebra (A, Φ) associated to a modular multiplicative unitary W is only a Haar weight away from being a locally compact quantum group.

But still, the Haar weights actually are the most indispensable tool in the Kustermans-Vaes theory. The possible lack of invariant weights is thus quite a serious problem. In what follows, we create a guideline of how to seek for Haar weights within the framework of modular multiplicative unitaries.

As far as we know, there has not been much research in this direction. There probably only is the recent paper [166] of Woronowicz. However, the results in this thesis are significantly stronger than the ones formulated in [166].

The main result in this Section 1.4 is Theorem 1.4.30. Starting from a modular multiplicative unitary W, it describes a family of faithful, normal weights ψ_q that are strongly right invariant on the associated bi-C*-algebra (A, Φ).

In this way, we then obtain a technique that is useful to find a formula for the (right) Haar weight in specific examples of quantum groups. As it is already said above, we will apply this technique in Chapter 2 to construct the Haar weights on the quantum $E(2)$ group and its dual; see Section 2.7.

The results in this Section 1.4 show that the framework of modular multiplicative unitaries is well-suited to study right invariance of weights on a bi-C*-algebra. We prove that a modular multiplicative unitary satisfying some extra conditions gives rise to a locally compact quantum group. In particular, the found results are helpful to apply the construction scheme stated on page 38.

In the proofs, we are inspired by the technique used by A. Van Daele in [148] to construct the Haar weights on the quantum $ax+b$-group and the quantum $az+b$-group. The general idea underlying this technique is presented in [134].

Although it is not the most general setting (cf. Remark 1.4.33), we prefer to work in the framework of modular multiplicative unitaries. This seems to be a natural choice. At the end of the section, we treat more general cases.

The next Notation 1.4.1 recalls the conventions about the notations.

Notation 1.4.1 *For the remainder of the Section 1.4, we again use the modular multiplicative unitary W that is fixed in Notation 1.3.15.*

We also again make use of the standard notations for the related objects.

We also fix an operator $\hat{N}$ which implements the scaling group $\hat{\tau}$. It is explained in Remark 1.4.3 that such an operator $\hat{N}$ always exists.

Notation 1.4.2 *Let $\hat{N}$ be a strictly positive operator on H such that*

$$\hat{\tau}(y) = \hat{N}^{-it} y \hat{N}^{it}$$

for all $y \in \hat{M}$.

The next Remark 1.4.3 is thus important to observe.

Remark 1.4.3 *The existence of an operator $\hat{N}$ satisfying the assumptions in the Notation 1.4.2 is guaranteed by Proposition 1.3.19.*

This Proposition 1.3.19 infers that the choice $\hat{N} = \hat{Q}^2$ is always correct.

The next Definition 1.4.4 defines an n.s.f. weight ψ_1 that is of crucial importance. This weight ψ_1 has a leading role in this Section 1.4.

We remark that the notation $\mathrm{Tr}_{\hat{N}}$ is explained in Example B.1.33.

Definition 1.4.4 *We define the* n.s.f. weight ψ_1 *on* $B(H)^+$ *by* $\psi_1 = \mathrm{Tr}_{\hat{N}}$.

We use $(H_{\psi_1}, \pi_{\psi_1}, \Lambda_{\psi_1})$ *to denote the* GNS-construction *of* ψ_1.

Furthermore, we denote by ∇_{ψ_1} *the* modular operator *and by* J_{ψ_1} *the* modular conjugation *of the n.s.f. weight* ψ_1.

We also introduce a family of deformations ψ_q of the weight ψ_1. Further in the section, we prove that (in the case $q \neq 0$) all the weights ψ_q are faithful, normal weights that are strongly right invariant; see Theorem 1.4.18.

Definition 1.4.5 *For all* $q \in \hat{M}'$, *we define a normal weight* ψ_q *on* $B(H)^+$ *by setting that*

$$\psi_q(x) = \psi_1(q^* x q) = \mathrm{Tr}_{\hat{N}}(q^* x q)$$

The Lemma 1.4.6 below contains a remark about the use of the notation ψ_q. It is important not to forget this convention.

Lemma 1.4.6 *Let* $q \in \hat{M}'$. *The restriction of* ψ_q *to* M^+ *is a normal weight. The restriction of* ψ_q *to* A^+ *is lower semi-continuous.*

We use ψ_q *to also denote these restrictions.*

The next Remark 1.4.7 discusses the goals of the Section 1.4.

Remark 1.4.7 *The main purpose of this Section 1.4 is to seek for a right Haar weight* ψ *on the bi-*C^**-algebra* (A, Φ). *In the case that this search is successful, we get that* (A, Φ) *is a locally compact quantum group.*

It is not known if it is always possible to construct Haar weights. The existence of Haar weights is only guaranteed in some special cases. We know e.g. from the Corollary 1.3.39 that a Haar state always exists if A *is unital.*

Nevertheless, the weights ψ_q *defined in Definition 1.4.5 are good candidates. This follows from the results below. We will prove that* ψ_q *is a Haar weight under the condition that there exists an element* $x \in A^+$ *such that* $\psi_q(x) = 1$.

We first point our attention to the original weight ψ_1. The Lemma 1.4.8 below ensues from the theory of n.s.f. weights; see Lemma B.1.34.

Lemma 1.4.8 *The modular automorphism group* σ *of the n.s.f. weight* ψ_1 *is given by*

$$\sigma_t : B(H) \to B(H) : x \mapsto \hat{N}^{it} x \hat{N}^{-it}.$$

The next Definition 1.4.9 introduces one more notation.

Definition 1.4.9 *For every element $x \in D(\sigma_{\frac{i}{2}})$, we define a bounded operator $\pi'_{\psi_1}(x) \in B(H_{\psi_1})$ by the formula*

$$\pi'_{\psi_1}(x) = J_{\psi_1} \pi_{\psi_1}\big(\sigma_{\frac{i}{2}}(x)\big)^* J_{\psi_1}.$$

The property in Proposition 1.4.10 below explicates the relation of the above Definition 1.4.9 with the *-representation π_{ψ_1}. We mention that it is a standard result from the Tomita-Takesaki theory; see Theorem B.1.31.

The Proposition 1.4.10 also gives a motivation for the used notation.

Proposition 1.4.10 *Let $x \in D(\sigma_{\frac{i}{2}})$. For all $y \in \mathfrak{N}_{\psi_1}$, we have that $yx \in \mathfrak{N}_{\psi_1}$ and*

$$\pi'_{\psi_1}(x)\Lambda_{\psi_1}(y) = \Lambda_{\psi_1}(yx).$$

We now prove the Lemmas 1.4.11 and 1.4.12. These two lemmas are crucial steps towards the invariance of the weights ψ_q. In fact, they are the key results in our invariance proof. This becomes clear in the proof of Proposition 1.4.13.

First, we look at the close relation between the modular automorphism group σ and the scaling group $\hat{\tau}$ of the dual bi-C*-algebra $(\hat{A}, \hat{\Phi})$.

Although the Lemma 1.4.11 is almost trivial to prove, it is really this property that eventually gives the (right) invariance of the weights ψ_q.

Lemma 1.4.11 *Let $t \in \mathbb{R}$. For all $x \in \hat{M}$, we have $\sigma_t(x) = \hat{\tau}_{-t}(x)$.*

Proof. This follows from Notation 1.4.2 and Lemma 1.4.8. ∎

The Lemma 1.4.12 gives a precise meaning to the map $\pi'_{\psi_1} \circ \hat{S}^{-1}$.

Lemma 1.4.12 *Consider the injective, normal and unital *-homomorphism γ defined by*

$$\gamma : \hat{M} \to B(H_{\psi_1}) : x \mapsto \gamma(x) = J_{\psi_1} \pi_{\psi_1}\big(\hat{R}(x^*)\big) J_{\psi_1}.$$

For all $x \in D(\hat{S}^{-1})$, we have that

$$\gamma(x) = \pi'_{\psi_1}\big(\hat{S}^{-1}(x)\big).$$

Proof. Take $x \in D(\hat{S}^{-1})$. From Lemma 1.4.11, it follows that $\hat{S}^{-1}(x) \in D(\sigma_{\frac{i}{2}})$. The operator $\pi'_{\psi_1}(\hat{S}^{-1}(x))$ is thus well-defined; see Definition 1.4.9.

Using Lemma 1.4.11, we see that

$$\sigma_{\frac{i}{2}}(\hat{S}^{-1}(x)) = \sigma_{\frac{i}{2}}\big(\hat{\tau}_{\frac{i}{2}} \hat{R}(x)\big) = \sigma_{\frac{i}{2}}\big(\sigma_{-\frac{i}{2}}(\hat{R}(x))\big) = \hat{R}(x).$$

We hence have

$$\pi'_{\psi_1}\big(\hat{S}^{-1}(x)\big) = J_{\psi_1}\pi_{\psi_1}\big(\sigma_{\frac{i}{2}}(\hat{S}^{-1}(x))\big)^* J_{\psi_1} = J_{\psi_1}\pi_{\psi_1}\big(\hat{R}(x)\big)^* J_{\psi_1} = \gamma(x).$$

This ends the proof of the lemma. ∎

The next Proposition 1.4.13 states a basic form of the right invariance of the weights ψ_q. It is formulated for the weights ψ_q considered on $B(H)^+$. This gives a property that is stronger than just right invariance.

We observe that we used the result below in its full generality in our calculations in Section 1.3 (at least the special case $q = 1$; see Proposition 1.3.29).

Proposition 1.4.13 *Let $q \in \hat{M}'$. For all $x \in \mathfrak{M}^+_{\psi_q}$ and $\omega \in B(H)^+_*$, we have that*

$$\psi_q\big((\iota \otimes \omega)(W(x \otimes 1)W^*)\big) = \omega(1)\psi_q(x).$$

Proof. Take $x \in \mathfrak{M}^+_{\psi_q}$ and $\omega \in B(H)^+_*$. We denote $\Phi(x) = W(x \otimes 1)W^*$.

We consider a GNS-construction (K, π, ξ) for ω. Then $\pi : B(H) \to B(K)$ is a normal and unital *-homomorphism. We use the notation $W_\pi = (\iota \mathbin{\bar{\otimes}} \pi)W$.

From Remark 1.3.14, we get $W \in \hat{M} \mathbin{\bar{\otimes}} M$. Thus, W_π is a unitary in $\hat{M} \mathbin{\bar{\otimes}} B(K)$.

Take an orthonormal basis $(\xi_i)_{i \in I}$ in the Hilbert space K. It then infers from Lemma B.4.2 that we have

$$\begin{aligned}
(\iota \mathbin{\bar{\otimes}} \omega)\Phi(x) &= (\iota \mathbin{\bar{\otimes}} \omega_{\xi,\xi})(W_\pi(x \otimes 1)W_\pi^*) \\
&= \sum_{i \in I} (\iota \mathbin{\bar{\otimes}} \omega_{\xi_i,\xi})(W_\pi)x(\iota \mathbin{\bar{\otimes}} \omega_{\xi,\xi_i})(W_\pi^*) \\
&= \sum_{i \in I} (\iota \mathbin{\bar{\otimes}} \omega_{\xi_i,\xi}\pi)(W)x(\iota \mathbin{\bar{\otimes}} \omega_{\xi,\xi_i}\pi)(W^*)
\end{aligned}$$

with convergence in the σ-strong* topology on $B(H)$.

Using the above result, we see that we have

$$\begin{aligned}
\psi_q\big((\iota \mathbin{\bar{\otimes}} \omega)\Phi(x)\big) &= \psi_q\Big(\sum_{i \in I} (\iota \mathbin{\bar{\otimes}} \omega_{\xi_i,\xi}\pi)(W)x(\iota \mathbin{\bar{\otimes}} \omega_{\xi,\xi_i}\pi)(W^*)\Big) \\
&= \sum_{i \in I} \psi_1\Big(q^*(\iota \mathbin{\bar{\otimes}} \omega_{\xi_i,\xi}\pi)(W)x(\iota \mathbin{\bar{\otimes}} \omega_{\xi,\xi_i}\pi)(W^*)q\Big) \\
&= \sum_{i \in I} \psi_1\Big(\Big(x^{\frac{1}{2}}q(\iota \mathbin{\bar{\otimes}} \omega_{\xi,\xi_i}\pi)W^*\Big)^*\Big(x^{\frac{1}{2}}q(\iota \mathbin{\bar{\otimes}} \omega_{\xi,\xi_i}\pi)W^*\Big)\Big).
\end{aligned}$$

In the last step of this calculation, we use that $q \in \hat{M}'$.

We will prove below that $x^{\frac{1}{2}}q(\iota \,\bar{\otimes}\, \omega_{\xi,\xi_i}\pi)W^* \in \mathfrak{N}_{\psi_1}$. If we already use this result here, we get from the above calculation that

$$\psi_q\big((\iota \,\bar{\otimes}\, \omega)\Phi(x)\big) = \sum_{i\in I} \big\|\Lambda_{\psi_1}\big(x^{\frac{1}{2}}q(\iota \,\bar{\otimes}\, \omega_{\xi,\xi_i}\pi)W^*\big)\big\|^2. \tag{1.32}$$

This ends the first part of the proof.

From Theorem 1.3.9 (applied to $\Sigma W^*\Sigma$), we get that $(\iota \,\bar{\otimes}\, \omega_{\xi,\xi_i}\pi)W \in D(\hat{S}^{-1})$ and

$$\hat{S}^{-1}\big((\iota \,\bar{\otimes}\, \omega_{\xi,\xi_i}\pi)W\big) = (\iota \,\bar{\otimes}\, \omega_{\xi,\xi_i}\pi)W^*. \tag{1.33}$$

Using Lemma 1.4.11, we see that this implies that $(\iota \,\bar{\otimes}\, \omega_{\xi,\xi_i}\pi)W^* \in D(\sigma_{\frac{i}{2}})$.

It is clear that $x^{\frac{1}{2}}q \in \mathfrak{N}_{\psi_1}$. Hence, we find $x^{\frac{1}{2}}q(\iota \,\bar{\otimes}\, \omega_{\xi,\xi_i}\pi)W^* \in \mathfrak{N}_{\psi_1}$ and

$$\Lambda_{\psi_1}\big(x^{\frac{1}{2}}q(\iota \,\bar{\otimes}\, \omega_{\xi,\xi_i}\pi)W^*\big) = \pi'_{\psi_1}\big((\iota \,\bar{\otimes}\, \omega_{\xi,\xi_i}\pi)W^*\big)\Lambda_{\psi_1}(x^{\frac{1}{2}}q). \tag{1.34}$$

This follows from Proposition 1.4.10.

Equations (1.33) and (1.34) together with Lemma 1.4.12 yield that

$$\Lambda_{\psi_1}\big(x^{\frac{1}{2}}q(\iota \,\bar{\otimes}\, \omega_{\xi,\xi_i}\pi)W^*\big) = \gamma\big((\iota \,\bar{\otimes}\, \omega_{\xi,\xi_i}\pi)W\big)\Lambda_{\psi_1}(x^{\frac{1}{2}}q). \tag{1.35}$$

Now, we define a unitary $W_2 \in B(H_{\psi_1} \otimes K)$ by $W_2 = (\gamma \,\bar{\otimes}\, \iota)W_\pi$. With this notation, we can rewrite Equation (1.35) as

$$\Lambda_{\psi_1}\big(x^{\frac{1}{2}}q(\iota \,\bar{\otimes}\, \omega_{\xi,\xi_i}\pi)W^*\big) = \big((\iota \,\bar{\otimes}\, \omega_{\xi,\xi_i})W_2\big)\Lambda_{\psi_1}(x^{\frac{1}{2}}q). \tag{1.36}$$

The two Equations (1.32) and (1.36) together give the invariance of ψ_q. By using these results, we can make the computation

$$\begin{aligned}\psi_q\big((\iota \,\bar{\otimes}\, \omega)\Phi(x)\big) &= \sum_{i\in I} \big\|\Lambda_{\psi_1}\big(x^{\frac{1}{2}}q(\iota \,\bar{\otimes}\, \omega_{\xi,\xi_i}\pi)W^*\big)\big\|^2 \\ &= \sum_{i\in I} \big\|\big((\iota \,\bar{\otimes}\, \omega_{\xi,\xi_i})W_2\big)\Lambda_{\psi_1}(x^{\frac{1}{2}}q)\big\|^2 = \big\|W_2\big(\Lambda_{\psi_1}(x^{\frac{1}{2}}q) \otimes \xi\big)\big\|^2 \\ &= \langle \xi, \xi\rangle\, \psi_1(q^*xq) = \omega(1)\psi_q(x).\end{aligned}$$

The proof of the proposition is thus completed. ∎

Now that we have proved the above Proposition 1.4.13, we can make use of the results in Section 1.3. This leads up to interesting new properties.

The results below are formulated for the weights ψ_q considered either on M^+ or A^+. The context makes it clear which case we are working in.

The next Theorem 1.4.14 states that the weights ψ_q are strongly right invariant. This property is a direct consequence of Proposition 1.3.51.

Theorem 1.4.14 *Let $q \in \hat{M}'$. The weight ψ_q is strongly right invariant, i.e., for all $x \in M^+$ and $\omega \in M_*^+$, we have that*

$$\psi_q\big((\iota \otimes \omega)\Phi(x)\big) = \omega(1)\psi_q(x).$$

From Proposition 1.3.49, we get the following classification result.

Proposition 1.4.15 *Let $q \in \hat{M}'$. Only the three possibilities below occur:*

1. *We have that $\psi_q = 0$,*
2. *We have that $\psi_q = \psi_\infty$,*
3. *We have that ψ_q is an n.s.f. weight.*

The first possibility in Proposition 1.4.15 can be excluded in the case $q \neq 0$. It only requires a short proof to obtain this property.

Proposition 1.4.16 *Take $q \in \hat{M}'$. We have that $\psi_q = 0$ if and only if $q = 0$. In the case $q \neq 0$, we have that ψ_q is a faithful, normal weight.*

Proof. Assume that $\psi_q = 0$. Then, we have that $\psi_1(q^* x^* x q) = 0$ for all $x \in M$. From Definition 1.4.4, we know that ψ_1 is an n.s.f. weight on $B(H)^+$. We thus get that $xq = 0$ for all $x \in M$. This implies that $q = 0$.

It ensues from Proposition 1.4.15 that any $\psi_q \neq 0$ is faithful and normal. ■

The Proposition 1.4.15 also provides us with a necessary and sufficient condition for ψ_q to be an n.s.f. weight (cf. Corollary 1.3.50).

Proposition 1.4.17 *Let $q \in \hat{M}'$. Then, ψ_q is an n.s.f. weight if and only if there exists an element $x \in M^+$ such that $\psi_q(x) = 1$.*

For $q \neq 0$, we can formulate a strong version of Theorem 1.4.14. It follows from a combination of Theorem 1.4.14 and Propositions 1.4.15 and 1.4.16.

The next Theorem 1.4.18 is the main result of Section 1.4.

Theorem 1.4.18 *Take $q \in \hat{M}'$ with $q \neq 0$. Then ψ_q is a faithful, normal weight that is strongly right invariant, i.e., for all $x \in M^+$ and $\omega \in M_*^+$, we have*

$$\psi_q\big((\iota \mathbin{\bar{\otimes}} \omega)\Phi(x)\big) = \omega(1)\psi_q(x).$$

The Definition 1.4.19 below defines for every weight ψ_q a close related weight φ_q. From the Proposition 1.3.42, we get that every 'right' property about ψ_q has a left invariant version concerning φ_q.

The weights φ_q and ψ_q of course have similar standard properties.

Definition 1.4.19 *For all $q \in \hat{M}'$, we define a normal weight φ_q on M^+ by setting that*

$$\varphi_q(x) = \psi_q(R(x)).$$

Further, we use φ_q to also denote its restriction to A^+.

We then get the very nice Corollary 1.4.20 below.

Corollary 1.4.20 *Assume that there exist elements $q \in \hat{M}'$ and $x \in M^+$ such that $\psi_q(x) = 1$. Then, (M, Φ) is a von Neumann algebraic quantum group.*

Moreover, ψ_q is the right Haar weight and φ_q is the left Haar weight.

Proof. The Remark 1.3.14 gives that Φ is a normal and unital *-homomorphism on the von Neumann algebra M such that Φ is coassociative, i.e., we have

$$(\Phi \,\bar{\otimes}\, \iota)\Phi = (\iota \,\bar{\otimes}\, \Phi)\Phi.$$

From the assumptions and Proposition 1.4.17, we immediately get that ψ_q is an n.s.f. weight on M^+. From Theorem 1.4.18, it follows that ψ_q is strongly right invariant in the sense that

$$\psi_q\big((\omega \,\bar{\otimes}\, \iota)\Phi(x)\big) = \omega(1)\psi_q(x)$$

for all $x \in M^+$ and $\omega \in M_*^+$.

Again from Remark 1.3.14, we know that R is an injective, normal and unital *-automorphism on M. This implies that $\varphi_q = \psi_q R$ is an n.s.f. weight on M^+. We get from Proposition 1.3.42 that φ_q is strongly left invariant. ■

Using Corollary 1.4.20, we can examine what are the chances for (M, Φ) to be a von Neumann algebraic quantum group.

Remark 1.4.21 *We consider the set $\mathfrak{J} = \{\, xq \mid x \in M, q \in \hat{M}' \}$.*

From Corollary 1.4.20, we see that it is sufficient to prove that $\mathfrak{J} \cap \mathfrak{N}_{\psi_1} \neq \{0\}$ in order to conclude that (M, Φ) is a von Neumann algebraic quantum group.

The Corollary 1.3.32 implies that the *-algebra generated by $\mathfrak{J}$ is strongly dense in $B(H)$. Definition 1.4.4 gives that also $\mathfrak{N}_{\psi_1}$ is strongly dense in $B(H)$.

We now also prove the C*-algebraic counterparts of all the above properties.

From Lemma 1.4.6 and Proposition 1.4.16, we get the following result.

Proposition 1.4.22 *Take $q \in \hat{M}'$. If $q \neq 0$, we have that ψ_q is a faithful, lower semi-continuous weight.*

Also in the C*-algebra case, we have a classification result. It can be deduced from Proposition 1.4.15 by copying the proof of [68, Proposition 1.6].

Theorem 1.4.23 *Let $q \in \hat{M}'$. Only the three possibilities below occur:*

1. *We have that $\psi_q = 0$,*
2. *We have that $\psi_q = \psi_\infty$,*
3. *We have that ψ_q is a faithful KMS-weight.*

The next Proposition 1.4.24 is similar to Proposition 1.4.17. It is evidently a direct consequence of Proposition 1.4.23.

Proposition 1.4.24 *Let $q \in \hat{M}'$. Then, ψ_q is a faithful KMS-weight if and only if there exists an element $x \in A^+$ such that $\psi_q(x) = 1$.*

It is not difficult to translate the main Theorems 1.4.14 and 1.4.18 into the C*-algebra setting. We get completely similar results.

First, we prove the strong right invariance of the weights ψ_q.

Theorem 1.4.25 *Let $q \in \hat{M}'$. The weight ψ_q is strongly right invariant, i.e., for all $x \in A^+$ and $\omega \in A^*_+$, we have*

$$\psi_q\big((\iota \otimes \omega)\Phi(x)\big) = \omega(1)\psi_q(x).$$

Proof. According to Proposition 1.4.15, we have to consider three cases. In the case $\psi_q = 0$, the result is of course trivial. If $\psi_q = \psi_\infty$, the strong invariance follows directly from Corollary 1.3.38.

In the case that ψ_q is an n.s.f. weight on M^+, we can use Lemma 1.3.37 to deduce from Theorem 1.4.14 that ψ_q is strongly right invariant. ∎

The stronger (and more interesting) case below follows from Theorem 1.4.25 combined with Proposition 1.4.22.

Theorem 1.4.26 *Take $q \in \hat{M}'$ with $q \neq 0$. Then, we have that ψ_q is a faithful, lower semi-continuous weight that is strongly right invariant, i.e., for all $x \in A^+$ and $\omega \in A^*_+$, we have*

$$\psi_q\big((\iota \otimes \omega)\Phi(x)\big) = \omega(1)\psi_q(x).$$

We compare the above Theorem 1.4.26 with the results of Woronowicz.

Remark 1.4.27 *S.L. Woronowicz proves in [166, Theorem 1.1] a weakened form of Theorem 1.4.25 in the special case $\hat{N} = \hat{Q}^2$ and $q = 1$. We have thus fashioned this property into Theorem 1.4.26. Together with Proposition 1.4.24, this Theorem 1.4.26 constitutes a combination which we genuinely contemplate as a substantial improvement to Woronowicz' results in [166].*

We will demonstrate in Remark 2.7.20 that the formula of Woronowicz does not always give a Haar weight. It is possible that $\mathfrak{M}^+_{\mathrm{Tr}_{\hat{Q}^2}} \cap M^+ = \{0\}$.

For $q \in \hat{M}'$, it is not difficult to generalize the methods of Woronowicz to prove the right invariance of the weight ψ_q informally defined by $\psi_q(x) = \mathrm{Tr}_{\hat{Q}^2}(q^* x q)$. The computations in [166] can be adapted in a straightforward way in order to *hold in the more general case. Hence, using the methods of Woronowicz, we can prove a version of Theorem 1.4.25 in the special case* $\hat{N} = \hat{Q}^2$.

We know that Woronowicz, independently, also found this latter generalization of [166, Theorem 1.1]. He mentioned this in a private communication we had at a CIRM conference in Marseille in June 2004.

We also formulate the C*-algebra version of Corollary 1.4.20.

Proposition 1.4.28 *Assume that there exist elements* $q \in \hat{A}'$ *and* $x \in A^+$ *such that* $\psi_q(x) = 1$. *Then,* (A, Φ) *is a locally compact quantum group.*

Moreover, ψ_q *is the right Haar weight and* φ_q *is the left Haar weight.*

Proof. We can repeat the proof of [68, Proposition 1.6] to deduce from Corollary 1.4.20 that (A, Φ) is a locally compact quantum group.

By proceeding in this way, we automatically get that ψ_q and φ_q are respectively the right and the left Haar weight on (A, Φ). ∎

The next Remark 1.4.29 is about the conditions in Proposition 1.4.28.

Remark 1.4.29 *Notice that it is not a stronger assumption to require that there exists an element* $x \in A^+$ *such that* $\psi_q(x) = 1$.

It is equivalent either to allow elements $x \in A^+$ *or elements* $x \in M^+$.

Suppose that $y \in M^+$ *and* $\psi_q(y) = 1$. *We denote* $z = R(y^{\frac{1}{2}})$. *It is clear that we can take* $\omega \in B(H)_*^+$ *such that* $\omega(1) = 1$ *and* $\omega(z) \neq 0$.

We define an element $y_\omega \in A^+$ *by the formula*

$$y_\omega = R\Big((\omega z \otimes \iota)(W^*)^*(\omega z \otimes \iota)(W^*)\Big).$$

From Proposition 1.3.28, it follows that $y_\omega \neq 0$.

Further, we have

$$\begin{aligned}
\psi_q(y_\omega) &= \psi_q R\Big((\omega z \otimes \iota)(W^*)^*(\omega z \otimes \iota)(W^*)\Big) \\
&= \psi_q R\Big((\omega \otimes \iota)((z \otimes 1)W^*)^*(\omega \otimes \iota)((z \otimes 1)W^*)\Big) \\
&\leq \psi_q R\Big(\|\omega\|(\omega \otimes \iota)\big(W(z^*z \otimes 1)W^*\big)\Big) \\
&= \psi_q R\big((\omega \otimes \iota)\Phi(z^*z)\big) = \psi_q(y) = 1.
\end{aligned}$$

So, we find $\psi_q(y_\omega) < +\infty$.

Because ψ_q *is faithful, we get that* $0 < \psi_q(y_\omega) < +\infty$.

The next Theorem 1.4.30 summarizes the above results. It shows that if we have a modular multiplicative unitary at hand, we are probably (more than) halfway in the construction of a locally compact quantum group.

Theorem 1.4.30 *Fix a modular multiplicative unitary $W \in B(H \otimes H)$ together with all its related objects. Take a strictly positive operator $\hat{N}$ on H such that we have $\hat{\tau}_t = \hat{N}^{-it} x \hat{N}^{it}$ for all $x \in \hat{A}$.*

For every $q \in \hat{M}'$ with $q \neq 0$, we can consider a faithful, normal weight ψ_q on M^+ informally defined by $\psi(x) = \mathrm{Tr}_{\hat{N}}(q^* x q)$ when $x \in M^+$.

Assume that there exist elements $q \in \hat{M}'$ and $x \in M^+$ such that $\psi_q(x) = 1$. Then, the following properties hold:

1. *We have that (A, Φ) is a locally compact quantum group with right Haar weight ψ_q and left Haar weight φ_q,*
2. *We have that (M, Φ) is a von Neumann algebraic quantum group with right Haar weight ψ_q and left Haar weight φ_q.*

The following two results can be useful for compact quantum groups.

Lemma 1.4.31 *Let $q \in \hat{M}' \cap \mathfrak{N}_{\psi_1}$. Then ψ_q is a normal linear functional.*

Proof. Take $x \in B(H)^+$. We have that

$$\psi_q(x) = \psi_1(q^* x q) \leq \|x\| \psi_1(q^* q) < +\infty$$

Hence ψ_q is everywhere defined. ∎

Corollary 1.4.32 *Suppose that A is unital and that $q \in \hat{M}' \cap \mathfrak{N}_{\psi_1}$ with $q \neq 0$. If we denote $\lambda = \frac{1}{\|\psi_q\|}$, we have that $\lambda \psi_q$ is the Haar state on (A, Φ).*

Proof. This follows from Lemma 1.4.31 and Theorem 1.4.18. ∎

The Remarks 1.4.33 and 11.4.34 are about a generalization of our technique.

Remark 1.4.33 *The result of Proposition 1.4.13 above can be proved to hold in a more general setting. This generalized property is however much less natural than the results concerning modular multiplicative unitaries.*

Let K be a Hilbert space. Fix a unitary $U \in B(H \otimes K)$.

Let ψ_1 be an n.s.f. weight on $B(H)^+$ and denote its modular automorphism group with σ. Assume that $(\iota \otimes \omega)U^* \in D(\sigma_{\frac{i}{2}})$ for all $\omega \in B(H)_*$.

Further, let $(H_{\psi_1}, \pi_{\psi_1}, \Lambda_{\psi_1})$ be the GNS-construction of ψ_1 and assume that there exists a unitary $\tilde{U} \in B(H_{\psi_1} \otimes K)$ such that

$$\Lambda_{\psi_1}\big(y(\iota \otimes \omega)U^*\big) = \big((\iota \otimes \omega)\tilde{U}\big)\Lambda_{\psi_1}(y)$$

for all $y \in \mathfrak{N}_{\psi_1}$ and $\omega \in B(K)_*$.

Let $q \in B(H)$ be a bounded operator that belongs to $\{(\iota \otimes \omega)U \mid \omega \in B(K)_*\}'$. Then define a normal weight ψ_q on $B(H)^+$ by $\psi_q(x) = \psi_1(q^* x q)$. It is possible to prove that

$$\psi_q\big((\iota \,\bar{\otimes}\, \omega)(U(x \otimes 1)U^*)\big) = \omega(1)\psi_q(x)$$

for all $x \in \mathfrak{M}^+_{\psi_q}$ and $\omega \in B(K)_*$.

Remark 1.4.34 *The condition in the above Remark 1.4.33 is quite unnatural. It is satisfied for all unitaries $U \in B(H \otimes K)$ such that, for all $\omega \in B(K)_*$, we have that $(\iota \otimes \omega)U \in D(\hat{S}^{-1})$ and*

$$\hat{S}^{-1}\big((\iota \otimes \omega)U\big) = (\iota \otimes \omega)U^*. \tag{1.37}$$

For instance, if $U \in B(H \otimes K)$ is a unitary satisfying $U_{12}W_{23} = W_{23}U_{13}U_{12}$, then the Condition (1.37) *is a consequence of [162, Theorem 1.6].*

1.5 The affiliation relation

In this Section 1.5, we overview some basic results concerning the notion of elements *affiliated* to a C*-algebra A. We include this survey to keep the text self-contained. The concept of an affiliated element plays a very important role in our study of the quantum $E(2)$ group in Chapter 2.

The affiliation relation was introduced by Baaj in [3]. It is however Woronowicz who elaborated a full study of affiliated elements [157, 161]. We mainly follow his approach in the treatment below. We do not include any proof.

We further mention that the affiliation relation also provides us with a very efficient approach to the *functional calculus* of normal operators. We refer to the complete Section B.3 for more explication on this matter.

It is most convenient to work with concrete C*-algebras. We therefore introduce the following Notation 1.5.1.

Notation 1.5.1 *We fix a non-degenerate C*-algebra $A \subseteq B(H)$. Further, we fix a non-degenerate C*-algebra $B \subseteq B(K)$.*

The multiplier algebra

We first give a short study of the *multiplier algebra* $M(A)$. We look at the *strict topology* on $M(A)$ and give its basic properties.

The undermentioned results about $M(A)$ are quite standard and can e.g. be found in [72, 152, 161, 60, 81, 15, 122, 104].

The most easy way to introduce $M(A)$ is as follows.

Definition 1.5.2 *The unital C^*-algebra $M(A)$ is defined by*

$$M(A) = \{x \in B(H) \mid xA \subseteq A \text{ and } Ax \subseteq A\}.$$

Further, $M(A)$ is called the multiplier algebra *of A.*

The next three Lemmas 1.5.3, 1.5.4 and 1.5.5 are quite trivial.

Lemma 1.5.3 *We have $A \subseteq M(A)$.*

Lemma 1.5.4 *We have $M(A) = A$ if and only if A is unital.*

Lemma 1.5.5 *We have that $M(A) \otimes M(B) \subseteq M(A \otimes B)$.*

The Definition 1.5.6 below equips $M(A)$ with the *strict topology.*

Definition 1.5.6 *The* strict topology *on $M(A)$ is the weakest topology such that, for all $a \in A$, we have that the mappings*

$$M(A) \to A : x \mapsto xa \quad \text{and} \quad M(A) \to A : x \mapsto ax$$

are continuous.

Hence, a net $(x_i)_{i \in I}$ in $M(A)$ strictly converges to $x \in M(A)$ if and only if

$$\|x_i a - xa\| \to 0 \quad \text{and} \quad \|ax_i - ax\| \to 0$$

for all $a \in A$.

The next Lemma 1.5.7 is again immediately clear.

Lemma 1.5.7 *The strict topology on $M(A)$ is weaker than the norm topology on $M(A)$. These two topologies coincide if and only if A is unital.*

It takes more effort to prove the following property.

Lemma 1.5.8 *The multiplier algebra $M(A)$ is the completion of A with respect to the strict topology. This implies that A is strictly dense in $M(A)$.*

We mention the two basic Examples 1.5.9 and 1.5.10.

Example 1.5.9 *Let X be a locally compact Hausdorff space. We consider the commutative C^*-algebra* $\mathrm{C}_0(X)$*.*

We then have that $M(\mathrm{C}_0(X)) = \mathrm{C}_b(X)$.

The strict topology on $M(\mathrm{C}_0(X))$ lies in between the norm topology and the topology of uniform convergence on compact subsets.

If X is non-compact, then the three mentioned topologies differ.

Example 1.5.10 *Let H be a Hilbert space. We have $M(B_0(H)) = B(H)$.*

The strict topology on $M(B_0(H))$ is stronger than the σ-strong* topology. It is strictly stronger if H is infinite dimensional.

The two mentioned topologies always coincide on bounded subsets.

The following Definition 1.5.11 is very important.

Definition 1.5.11 *Let $\rho : A \to M(B)$ be a linear map. We call ρ* strict *if it is both norm bounded and strictly continuous on bounded subsets of A.*

The next Proposition 1.5.12 is proved in Section 7 of [60].

Proposition 1.5.12 *Let $\rho : A \to M(B)$ be a strict linear map. There exists a unique linear mapping $\bar{\rho} : M(A) \to M(B)$ that extends ρ.*

We have that $\bar{\rho}$ is norm bounded and $\|\bar{\rho}\| = \|\rho\|$. Further, we have that $\bar{\rho}$ is strictly continuous on bounded subsets of $M(A)$.

It is easy to check the following result.

Lemma 1.5.13 *Let $C \subseteq B(L)$ be a non-degenerate C^*-algebra. Assume that $\rho : A \to M(B)$ and $\theta : B \to M(C)$ are two strict linear maps.*

Then, we also have that $\bar{\theta} \circ \rho : A \to M(C)$ is a strict linear map.

The Definitions 1.5.14 and 1.5.14 define two categories of linear maps. These are basic examples of strict linear maps; see Proposition 1.5.18.

Definition 1.5.14 *Let $\pi : A \to M(B)$ be a *-homomorphism. Then, π is called* non-degenerate *if we have that $\pi(A)B$ is dense in B.*

Definition 1.5.15 *Let $R : A \to M(B)$ be a *-anti-homomorphism. Then, R is called* non-degenerate *if we have that $R(A)B$ is dense in B.*

The Lemma 1.5.16 below can be useful. It is of course no surprise that we have a completely similar result for *-anti-homomorphisms.

We remark that any approximate unit (e_α) in A strictly converges to 1.

Lemma 1.5.16 *Let $\theta : A \to M(B)$ be a *-homomorphism. The four conditions below are equivalent:*

1. *θ is non-degenerate,*
2. *For any approximate unit (e_α), we have that $(\theta(e_\alpha))$ strictly converges to 1,*
3. *There is a bounded net (e_α) in A such that $(\theta(e_\alpha))$ strictly converges to 1,*
4. *The set $\{\, \theta(a)b \mid a \in A, b \in B\}$ is dense in B.*

In the remainder of this Section 1.5, we give a lot of useful information about non-degenerate *-homomorphisms.

First, we give a remark about *-anti-homomorphisms.

Remark 1.5.17 *For non-degenerate *-anti-homomorphisms, we have counterparts of all the properties stated below. These results are used in the thesis, but are not explicitly stated in the treatment below.*

The next Proposition 1.5.18 is a standard property.

Proposition 1.5.18 *Let $\pi : A \to M(B)$ be a non-degenerate *-homomorphism. We then have that π is a strict linear map.*

Moreover, the extension $\bar{\pi} : M(A) \to M(B)$ is a unital *-homomorphism.

The Lemma 1.5.19 below gathers some useful information.

Lemma 1.5.19 *Let $\pi : A \to M(B)$ be a non-degenerate *-homomorphism and let $\bar{\pi} : M(A) \to M(B)$ be its unique extension.*

We have the following properties:

- *If π is injective on A, then $\bar{\pi}$ is injective on $M(A)$,*
- *If $\pi(A) \supseteq B$, then $\bar{\pi}$ is strictly continuous on $M(A)$,*
- *If π is a *-isomorphism from A to B, then $\bar{\pi}$ is a strictly continuous *-isomorphism from $M(A)$ to $M(B)$.*

Also the following Remark 1.5.20 is certainly interesting.

Remark 1.5.20 *Consider a *-representation $\pi : A \to B(H)$. Then, we have that π is a non-degenerate *-representation if and only if $\pi : A \to M(B_0(H))$ is a non-degenerate *-homomorphism.*

Let $C \subseteq B(L_1)$ and $D \subseteq B(L_2)$ be two non-degenerate C^*-algebras.

If $\pi_1 : A \to M(C)$ and $\pi_2 : B \to M(D)$ are non-degenerate *-homomorphisms, then also $\pi_1 \otimes \pi_2 : A \otimes B \to M(C \otimes D)$ is non-degenerate.

For continuous linear functionals, we have the very strong result of Proposition 1.5.22. It is a consequence of the Proposition 1.5.21 below.

Proposition 1.5.21 *We have*

$$\begin{aligned} A^* &= \{a\omega b \mid \omega \in A^*, a, b \in A\}, \\ A^*_+ &= \{a\omega a^* \mid \omega \in A^*_+, a \in A\}, \end{aligned}$$

The next Proposition 1.5.22 was first proved in [122].

Proposition 1.5.22 *Let $\omega : A \to \mathbb{C}$ be a continuous linear functional. We have that ω is a strict linear map.*

Moreover, the extension $\bar{\omega} : M(A) \to \mathbb{C}$ is strictly continuous.

Further, also the slice maps are strict linear maps.

Lemma 1.5.23 *Let $\omega : A \to \mathbb{C}$ be a continuous linear functional. Then, the slice maps $\omega \otimes \iota : A \otimes B \to B$ and $\iota \otimes \omega : A \otimes B \to A$ are strict.*

Finally, we introduce the Notation 1.5.24 to simplify notations. The uniqueness part of Proposition 1.5.12 makes that there is not caused any confusion.

The Notation 1.5.24 is applied throughout the whole thesis.

Notation 1.5.24 *Let $\rho : A \to M(B)$ be a strict linear map. We then use ρ to also denote the extension $\bar{\rho} : M(A) \to M(B)$.*

The affiliation relation

We now go further and define the set of elements *affiliated* to A. We mention that we keep using the Notation 1.5.1. This means that we always work with two concrete C*-algebras $A \subseteq B(H)$ and $B \subseteq B(K)$.

The results below are proved e.g. in [157, 161, 59, 72, 167].

First, we introduce the notion of the *z-transform* of a closed operator.

Definition 1.5.25 *Let T be a closed operator on H. We define $z_T \in B(H)$ by setting*

$$z_T = T(1 + T^*T)^{-\frac{1}{2}}.$$

We call z_T the z-transform *of T.*

The next Lemma 1.5.26 gathers some basic (but non-trivial) properties. It shows that the z-transform z_T contains full information about T.

Lemma 1.5.26 *Let S, T be two closed operators on H. Then, the following properties hold:*

- *We have $\|z_T\| \leq 1$,*
- *We have that $T \in B(H)$ if and only if $\|z_T\| < 1$,*
- *We have $T = z_T(1 - z_T^* z_T)^{-\frac{1}{2}}$,*
- *We have $S = T$ if and only if $z_S = z_T$.*

The next Definition 1.5.27 is the central issue in this Section 1.5. It makes use of the z-transform to introduce the *affiliation relation*.

We refer to [157, 161] for a full elaboration of this subject.

Definition 1.5.27 *Let T be a closed operator on H. We say that T is* affiliated *to A if and only if*

1. *$z_T \in M(A)$,*
2. *$(1 + T^*T)^{-\frac{1}{2}} A$ is dense in A.*

We then write $T \,\eta\, A$.

The next Notation 1.5.28 is standard.

Notation 1.5.28 *We use A^η to denote the set of elements affiliated to A.*

It is not difficult to prove the three lemmas below.

Lemma 1.5.29 *We have $M(A) = \{T \,\eta\, A \mid T \in B(H)\}$.*

Lemma 1.5.30 *We have $A \subseteq M(A) \subseteq A^\eta$.*

Lemma 1.5.31 *We have $M(A) = A^\eta$ if and only if A is unital.*

The Lemmas 1.5.32 and 1.5.33 are less trivial than they look.

Lemma 1.5.32 *Let $T \,\eta\, A$ and $\gamma \in \mathbb{C}_0$. Then, we have $\gamma T \,\eta\, A$.*

Lemma 1.5.33 *Let $T \,\eta\, A$ and $x \in M(A)$. Then, we have $(T + x) \,\eta\, A$.*
If x is invertible in $M(A)$, then we also have $Tx, xT \,\eta\, A$.

The next Lemma 1.5.34 is a basic result. It shows that the affiliation relation well-behaves with respect to the adjoint operation.

Lemma 1.5.34 *Let T be a closed operator on H. If $T\,\eta\,A$, then we have that $T^*\,\eta\,A$ and $z_{T^*} = z_T^*$.*
Further, we have $T^*T\,\eta\,A$ and $TT^*\,\eta\,A$.

We then have the Lemma 1.5.35 below.

Lemma 1.5.35 *Let $T\,\eta\,A$. The properties below hold:*

- *We have that T is normal if and only if z_T is normal,*
- *We have that T is self-adjoint if and only if z_T is self-adjoint,*
- *We have that T is positive if and only if z_T is positive,*

There are also some useful results concerning the tensor product of two affiliated elements; see [167, Theorem 6.1] and [169, Proposition A.1].

Proposition 1.5.36 *Assume that $S\,\eta\,A$ and $T\,\eta\,B$. Then, we have*

$$S \otimes T \;\eta\; A \otimes B.$$

Proposition 1.5.37 *Let S, T be two non-zero, normal operators on H and K. If $S \otimes T\,\eta\,A \otimes B$, then we have $S\,\eta\,A$ and $T\,\eta\,B$.*

For normal operators T, we can also use the functional calculus to find a simple characterization of the affiliation relation.

Proposition 1.5.38 *Let T be a normal operator on H. Then, the following conditions are equivalent:*

1. *We have $T\,\eta\,A$,*
2. *We have that $\big\{f(T) \mid f \in \mathrm{C}_0(\sigma(T))\big\}$ is a subset of $M(A)$ containing an approximate unit for A.*

Below, we will endow A^η with a natural topology. We first look at the behavior of the z-transform with respect to the strict topology on $M(A)$.

Proposition 1.5.39 *Let (x_n) be a sequence in $M(A)$ and $x \in A$. The following two conditions are equivalent:*

1. *We have that $x_n \to x$ in the strict topology,*
2. *The sequence (x_n) is bounded and we have $z_{x_n} \to z_x$ in the strict topology.*

Proposition 1.5.40 *Let (x_i) be a net in $M(A)$ and $x \in A$. If $x_i \to x$ in the strict topology, then we also have $z_{x_i} \to z_x$ in the strict topology.*

If the net (x_i) is bounded, these two convergence conditions are equivalent.

We now define the *η-topology.* The Definition 1.5.41 is taken from [161], where this topology is called the *topology of almost uniform convergence*.

The Definition 1.5.41 is based on the Propositions 1.5.39 and 1.5.40.

Definition 1.5.41 *The η-*topology *on A^η is the weakest topology which makes the z-transform*

$$A^\eta \to M(A) : T \mapsto z_T$$

strictly continuous.

Hence, a net $(T_i)_{i\in I}$ in A^η converges to $T \,\eta\, A$ if and only if

$$z_{T_i} \to z_T$$

in the strict topology.

The next Lemma 1.5.42 is a consequence of Proposition 1.5.40.

Lemma 1.5.42 *The η-topology restricted to $M(A)$ is weaker than the strict topology. These two topologies coincide on bounded subsets of $M(A)$.*

It takes some effort to prove the following property.

Lemma 1.5.43 *Let $T \,\eta\, A$. Then, there exists a sequence (T_n) in $M(A)$ such that $T_n \to T$ in the η-topology.*

This result implies that A is dense in A^η with respect to the η-topology.

We also mention the next result.

Lemma 1.5.44 *Let (T_i) be a net in A^η and $T \,\eta\, A$. Assume that $T_i \to T$ in the η-topology. Then, we have that $T_i^* \to T^*$ in the η-topology.*

We now further explore the Examples 1.5.9 and 1.5.10. These two main cases are used a few times in Chapter 2.

It is again easy to handle the commutative case.

Example 1.5.45 *Let X be a locally compact Hausdorff space. We consider the commutative C*-algebra $\mathrm{C}_0(X)$.*

We then have that $\mathrm{C}_0(X)^\eta = \mathrm{C}(X)$. The η-topology on $\mathrm{C}_0(X)^\eta$ coincides with the topology of uniform convergence on compact subsets.

The next Example 1.5.46 shows that that A^η can behave quite badly with respect to the sum and the product of two affiliated elements.

Example 1.5.46 *Let H be a Hilbert space. We then have that $B_0(H)^\eta$ is equal to the set of closed operators on H.*

We now again point our attention to non-degenerate *-homomorphisms. It turns out that such a non-degenerate *-homomorphism can be extended to the set of affiliated elements. This gives us some very useful properties.

In the next Definition 1.5.47, we make use of Proposition 1.5.18.

Definition 1.5.47 *Let $\pi : A \to M(B)$ be a non-degenerate *-homomorphism. For every $S\,\eta\,A$, there exists a unique element $T\,\eta\,B$ such that $\pi(z_S) = z_T$.*

We then write $\pi(S) = T$.

We have to be quite careful when we use the extension of π to A^η. Luckily, the following results make it somewhat easier to work with π.

The Propositions 1.5.48 and 1.5.49 collect some basic properties.

Proposition 1.5.48 *Let $\pi : A \to M(B)$ be a non-degenerate *-homomorphism. For every $T\,\eta\,A$, we have*

$$\pi(T^*) = \pi(T)^*, \qquad \pi(T^*T) = \pi(T)^*\pi(T), \qquad \pi(TT^*) = \pi(T)\pi(T)^*.$$

Proposition 1.5.49 *Let $\pi : A \to M(B)$ be a non-degenerate *-homomorphism. Let $T\,\eta\,A$. The properties below hold:*

1. *If T is normal, then $\pi(T)$ is normal,*
2. *If T is self-adjoint, then $\pi(T)$ is self-adjoint,*
3. *If T is positive, then $\pi(T)$ is positive,*
4. *If T is strictly positive, then $\pi(T)$ is strictly positive.*

Also the Remarks 1.5.50 and 1.5.51 contain some useful properties.

Remark 1.5.50 *Let $C \subseteq B(L_1)$ and $D \subseteq B(L_2)$ be non-degenerate C^*-algebras.*

We consider non-degenerate *-homomorphisms $\pi : A \to M(B)$, $\theta : B \to M(C)$ and $\pi_1 : A \to M(C)$, $\pi_2 : B \to M(D)$.

Then, also $\theta\pi : A \to M(C)$ is non-degenerate. It is easy to check that we have $(\theta\pi)(T) = \theta(\pi(T))$ for all $T\,\eta\,A$.

Further, we have that $\pi_1 \otimes \pi_2 : A \otimes B \to M(C \otimes D)$ is non-degenerate and that $(\pi_1 \otimes \pi_2)(S \otimes T) = \pi_1(S) \otimes \pi_2(T)$ for all $S\,\eta\,A$ and $T\,\eta\,B$.

Remark 1.5.51 *Let $\pi : A \to M(B)$ be a non-degenerate *-homomorphism that is injective. Lemma 1.5.19 gives that π is also injective on $M(A)$.*

It is then easy to deduce that π is also injective on A^η.

The next Proposition 1.5.52 states that non-degenerate *-homomorphisms are continuous on the set of affiliated elements.

Proposition 1.5.52 *Let $\pi : A \to M(B)$ be a non-degenerate *-homomorphism. Then, we have that the map*

$$\pi : A^\eta \to B^\eta : T \mapsto \pi(T)$$

is continuous in the η-topology.

For every net (T_i) in A^η and $T \,\eta\, A$ such that $T_i \to T$ in the η-topology, we thus have that

$$\pi(T_i) \to \pi(T)$$

in the η-topology.

C*-algebras generated by affiliated elements

The last topic we focus on are the C*-algebras which are generated by affiliated elements. All results below are taken from [161].

If we have a finite set of bounded operators on a Hilbert space, we can consider the smallest C*-algebra which contains all these operators. We then say that this C*-algebra is *generated* by the given set of bounded operators.

S.L. Woronowicz gave in [161] a definition for the C*-algebra A to be generated by a finite number of (possibly unbounded) affiliated elements.

The definition of Woronowicz is stated in Definition 1.5.53 below. It starts with the C*-algebra A and affiliated elements $T_1, T_2, \ldots, T_n \,\eta\, A$ and then gives a condition under which we say that A is generated by $T_1, T_2, \ldots, T_n \,\eta\, A$.

In the bounded case, Definition 1.5.53 coincides with the usual notion as it is mentioned above. The approach is however quite different.

For the interested reader, we mention that it is also possible to define when the C*-algebra A is generated by a *quantum family of generators.* We do not go further into this matter and refer to [161] for a detailed treatment.

We now state the main Definition 1.5.53.

Definition 1.5.53 *Let $T_1, T_2, \ldots, T_n \,\eta\, A$. We then say that A is generated by $T_1, T_2, \ldots, T_n \,\eta\, A$ if for any non-degenerate C*-algebra $C \subseteq B(L)$ and any non-degenerate *-representation $\pi : A \to B(L)$, we have*

$$\begin{pmatrix} \pi(T_i) \,\eta\, C \text{ for any} \\ i = 1, 2, \ldots, n \end{pmatrix} \Longrightarrow \big(\pi : A \to M(C) \text{ is non-degenerate}\big)\,.$$

At first sight, it looks like the Definition 1.5.53 is given in a laborious manner. The method of working is quite opposite to the intuition.

It is however not possible to work the other way round. There is no effective procedure to construct a C*-algebra out of a given set of affiliated elements.

In each separate case, we thus have to invent an appropriate C*-algebra and prove a posteriori that it is indeed the C*-algebra we seek for. Even worse is the fact that the existence of such a C*-algebra is not guaranteed.

The Proposition 1.5.54 is an excellent expedient when one wants to prove that the C*-algebra A is generated by affiliated elements $T_1, T_2, \ldots, T_n \,\eta\, A$.

Proposition 1.5.54 *Let $T_1, T_2, \ldots, T_n \,\eta\, A$. We define a set $\Gamma \subseteq M(A)$ by*

$$\Gamma = \big\{(1 + T_i^* T_i)^{-1} \mid i = 1, 2, \ldots, n\big\} \cup \big\{(1 + T_i T_i^*)^{-1} \mid i = 1, 2, \ldots, n\big\}.$$

Assume that the following two conditions are satisfied:

1. *We have that $T_1, T_2, \ldots, T_n \,\eta\, A$ separate non-degenerate *-representations of A, i.e., if we consider two such maps $\pi_1, \pi_2 : A \to B(L)$ which are different, then $\pi_1(T_i) \neq \pi_2(T_i)$ for some $i = 1, 2, \ldots, n$,*
2. *There exist elements $r_1, r_2, \ldots, r_k \in \Gamma$ such that $r_1 r_2 \cdots r_k \in A$.*

Then A is generated by $T_1, T_2, \ldots, T_n \,\eta\, A$.

We explicate the meaning of Definition 1.5.53 in the commutative case.

Example 1.5.55 *Let X be a locally compact Hausdorff space. We consider the commutative C*-algebra $\mathrm{C}_0(X)$.*

From Example 1.5.45, we know that $\mathrm{C}_0(X)^\eta = \mathrm{C}(X)$.

Let $f_1, f_2, \ldots, f_n \,\eta\, \mathrm{C}_0(X)$. Then, $\mathrm{C}_0(X)$ is generated by $f_1, f_2, \ldots, f_n$ if and only if they separate points of X and

$$\lim_{\lambda \to \infty} \sum_{i=1}^{N} |f_i(\lambda)|^2 = +\infty.$$

This latter condition is superfluous (and meaningless) if X is compact.

Chapter 2

The quantum $E(2)$ group and its dual

This Chapter 2 constitutes the heart of the thesis. It encompasses all details regarding the construction of the quantum $E(2)$ group. In particular, we give a proof of the fact that this example fits in with the framework of Kustermans and Vaes. We display a very accurate and detailed treatment.

We construct the quantum $E(2)$ group (A, Φ) following the 'atomic' construction procedure as explained in the Method 1 of Section 1.2. More precisely, we apply the scheme on page 38. We start with a multiplicative unitary W and then define (A, Φ) as the associated bi-C*-algebra. From the results in Section 1.4, it will ensue quite easily that (A, Φ) is a locally compact quantum group.

It should be observed that this method of working is one of the main differences of our approach in comparison with the work of Woronowicz. He introduces the bi-C*-algebra (A, Φ) in [159] without making use of W.

The central property of the Chapter 2 is thus that (A, Φ) satisfies the conditions in Definition 1.1.6. We state this main result in Theorem 2.4.3. It is the starting point of a profound study of all the aspects of the quantum $E(2)$ group.

In Sections 2.5 to 2.8, we investigate the structure of (A, Φ) as a locally compact quantum group. The information in these sections provides us with a solid base for further research on the typical features of this example. This becomes clear already in Chapter 3 where we study its amenability properties.

The representation theory of (A, Φ) is unravelled in Chapter 4. The Chapter 5 is used to mention a few open problems as an incentive for future research.

We also treat the quantum $\hat{E}(2)$ group $(\hat{A}, \hat{\Phi})$. For the construction of this example, we operate completely similar to the case of the quantum $E(2)$ group. In Section 2.8, we show that $(\hat{A}, \hat{\Phi})$ is the opposite dual of (A, Φ).

In the Introduction and in Chapter 1, we already focussed on the history of the quantum $E(2)$ group. We do not repeat these things so that we can put all our attention to the mathematics. We only recall that the quantum $E(2)$ group was first introduced and studied by S.L. Woronowicz [157, 158, 159, 149] and that the Haar weights on this example were first obtained by S. Baaj [2, 1].

Throughout the Chapter 2, we mainly concentrate on the quantum $E(2)$ group as a C*-algebraic quantum group. We are very accurate in all of the calculations. The von Neumann algebra case is considered separately in Section 2.9.

We recall (cf. the Introduction) that we have fixed a number μ with $0 < \mu < 1$. This μ is used as the *deformation parameter* in our construction.

Because this Chapter 2 is quite elaborated, we first overview its sections.

Section 2.1 - The Hopf *-algebra level. In this Section 2.1, we present a dual pair of Hopf *-algebras. They describe the quantum groups $E(2)$ and $\hat{E}(2)$ in an algebraic way. Although this part is not really necessary for the rest of the thesis, it is rather easy and very instructive. It will help to understand the formulas in the further Sections 2.2 to 2.8. As such, this level is mainly used as an algebraic intuition for later results. It will not be used to give proofs on the analytic level. The polar decomposition of the antipode is one of the results that can already be obtained in this purely algebraic context.

Section 2.2 - Operator equalities and special functions. This is the start of the Hilbert space level. We here deal with some technical details. Most of these results concern the function F_μ introduced in Definition 2.2.1. We focus on the relation of F_μ with certain pairs of operators and we give crucial properties of its Fourier coefficients. The mentioned results are taken from [158, 168, 1]. They are of the utmost importance for later calculations.

Section 2.3 - The multiplicative unitary. The results in this section are crucial. We describe the main tool in our construction. We will construct a *multiplicative unitary* W and prove that it is manageable. The important unitary operator W will appear many times throughout this thesis. It will be a main actor both in the construction procedure and in the study of the amenability properties of the quantum $E(2)$ group (cf. Chapter 3).

Section 2.4 - The definition of the quantum $E(2)$ group. This short Section 2.4 is central in Chapter 2. It is used to fix the data of the operator algebra level. We give a (possible) definition for the quantum $E(2)$ group (A, Φ). Further, we state that (A, Φ) is a locally compact quantum group. In a complete similar way, also the quantum $\hat{E}(2)$ group $(\hat{A}, \hat{\Phi})$ is introduced.

Section 2.5 - The C*-algebras and the comultiplications. We here have a closer look at the *C*-algebras* and the *comultiplications*. It is shown that the C*-algebras A and $\hat{A}$ can be obtained by taking suitable functions of the generators. They are also described as crossed products. Further, we characterize the comultiplications Φ and $\hat{\Phi}$ in terms of the generators.

Section 2.6 - The antipodes. In the Section 2.6, we study the *antipodes* and their properties. We describe the action of S and $\hat{S}$ on the generators. These results are then used to prove that both S and $\hat{S}$ are unbounded.

Section 2.7 - The Haar weights. We here construct the *Haar weights* and describe their main properties. By applying the technique introduced in the Section 1.4, we are able to prove the invariance of the Haar weights by just a very short calculation. It turns out that the quantum $E(2)$ group is unimodular while the quantum $\hat{E}(2)$ group is non-unimodular.

Section 2.8 - Features of the quantum $E(2)$ group. This Section 2.8 compiles several short studies. We treat in detail all the features of the quantum $E(2)$ group known from the general theory. We describe the *modular elements* and the *regular representations*. Also, we look at *regularity properties* and we study some *duality results*. We describe closed quantum subgroups and construct an ergodic (left) action γ of the quantum $E(2)$ group. At the end of the section, we state Theorem 2.8.54 which summarizes the main results.

Section 2.9 - The von Neumann algebra case. Finally, we also look somewhat closer at the von Neumann algebra level. We give a short study of the quantum $E(2)$ group as a von Neumann algebraic quantum group.

2.1 The Hopf *-algebra level

This Section 2.1 is the start of our detailed built-up of the quantum $E(2)$ group. As it is said above, we apply the Method 1 in Section 1.2 in order to construct the quantum $E(2)$ group in a transparent way.

We thus build up the quantum $E(2)$ group as an 'atomic' example. It is explained in Section 1.2 that this involves that our treatment is split up into three levels. First, there is the *Hopf *-algebra level* which is the content of this Section 2.1. The *Hilbert space level* and the *C*-algebra level* (and *von Neumann algebra level*) are treated in the subsequent Sections 2.2 to 2.9.

The main issue in this Ph.D.-thesis lies in the study of the quantum $E(2)$ group as a C*-algebraic quantum group. The Hopf *-algebra level we only give as an intuitive motivation for later results. Nonetheless, we believe that it is certainly instructive first to consider the easy algebraic counterpart of the much more difficult results on the operator algebra level.

In this Section 2.1, we begin our study of the quantum $E(2)$ group by defining two Hopf *-algebras (A, Δ) and $(B, \hat{\Delta})$. We give their main properties and, in particular, we show that they form a dual pair of Hopf *-algebras.

Since the results in this Section 2.1 are not needed to treat the C*-algebra case, we will not linger over details. However, we do treat all the results with a reasonable care. We will not omit any precision.

For the theory of dual pairs of Hopf *-algebras, we refer to [143]. Notice that we use Δ to denote the comultiplication on the Hopf *-algebra level, whereas in the C*-algebra framework we use Φ for the comultiplication.

The dual pair of Hopf *-algebras that is behind the example of the quantum $E(2)$ group is the well-known pair in the Proposition 2.1.1 below.

The proof of Proposition 2.1.1 is (almost) trivial; its content certainly is not. For the duality between (A, Δ) and $(B, \hat{\Delta})$, we refer to Proposition 2.1.4.

We again recall that $0 < \mu < 1$.

Proposition 2.1.1 *Consider the unital *-algebra A over $\mathbb{C}$ that is generated by a unitary element c and a normal element d satisfying $cd = \mu dc$.*

The *-algebra A can be made into a Hopf *-algebra when a comultiplication Δ is defined by

$$\Delta(c) = c \otimes c \qquad \textit{and} \qquad \Delta(d) = c \otimes d + d \otimes c^*.$$

The counit ε is given by $\varepsilon(c) = 1$ and $\varepsilon(d) = 0$. The antipode S is given by $S(c) = c^*$, $S(d) = -\mu^{-1}d$ and $S(d^*) = -\mu d^*$.

Further, consider the unital *-algebra B over $\mathbb{C}$ that is generated by a self-adjoint invertible element a and a normal element b satisfying $ab = \mu ba$.

The *-algebra B can be made into a Hopf *-algebra when a comultiplication $\hat{\Delta}$ is defined through the formulas

$$\hat{\Delta}(a) = a \otimes a \qquad \textit{and} \qquad \hat{\Delta}(b) = a \otimes b + b \otimes a^{-1}.$$

The counit $\hat{\varepsilon}$ is given by $\hat{\varepsilon}(a) = 1$ and $\hat{\varepsilon}(b) = 0$, while the antipode $\hat{S}$ is determined by $\hat{S}(a) = a^{-1}$, $\hat{S}(b) = -\mu^{-1}b$ and $\hat{S}(b^*) = -\mu b^*$.

The next Remark 2.1.2 explains some choices.

Remark 2.1.2 *As it is common usage in the theory of dual pairs of Hopf *-algebras, we have defined the comultiplication $\hat{\Delta}$ dual to Δ.*

For transparency, we will make this choice also in Section 2.4 when we define the quantum $E(2)$ group in the C*-algebra framework.

This method of working is different from the normal approach in the theory of locally compact quantum groups, where we usually flip the comultiplication on the dual quantum group (cf. Proposition 1.1.28).

There are two motives for making this choice. First, as we have mentioned above, it enhances the connection between the Hopf *-algebra level and the C*-algebra level. We moreover get the advantage that our formulas then match with the pristine formulas that Woronowicz displayed in the articles [159, 149].

The algebra A in Proposition 2.1.1 can be considered – on the Hopf *-algebra level – as the quantum analogue of the group $E(2)$ of motions on the plane. This fact is explained in the Remark 2.1.3 below. Further, the algebra B is the Hopf *-algebra underlying the quantum $\hat{E}(2)$-group.

The idea behind the construction of the quantum $E(2)$ group is similar to the case of the quantum $SU(2)$ group (cf. Example 1.2.10). It will however turn out to be much more difficult to perform the lifting to the C*-level.

The Remark 2.1.3 is taken from [157, Section 3].

Remark 2.1.3 *S.L. Woronowicz found its inspiration for the definition of the Hopf *-algebra (A, Δ) in a deformation of the $E(2)$ group.*

First, consider the matrix group $E(2)$ that is defined as follows:

$$E(2) = \left\{ \begin{pmatrix} v & n \\ 0 & \bar{v} \end{pmatrix} \in M_2(\mathbb{C}) \;\middle|\; v, n \in \mathbb{C} \text{ such that } |v| = 1 \right\}.$$

Then consider the functions $c, d \in \mathrm{C}(E(2))$ defined by

$$c \begin{pmatrix} v & n \\ 0 & \bar{v} \end{pmatrix} = v \qquad \text{and} \qquad d \begin{pmatrix} v & n \\ 0 & \bar{v} \end{pmatrix} = n$$

for all $v, n \in \mathbb{C}$ such that $|v| = 1$.

Now, let A_0 be the unital *-subalgebra of $\mathrm{C}(E(2))$ generated by c and d. We can define a *-homomorphism $\Delta : A_0 \to A_0 \otimes A_0$ by the formula $\Delta(f)(s,t) = f(st)$ when $f \in A_0$ and $s, t \in E(2)$. This *-homomorphism Δ is coassociative.

It is easy to check that (A_0, Δ) is a Hopf *-algebra. We have that

$$\Delta(c) = c \otimes c \qquad \text{and} \qquad \Delta(d) = c \otimes d + d \otimes c^*.$$

This indicates where the idea for the Hopf *-algebra (A, Δ) in Proposition 2.1.1 comes from. It is however a non-trivial property that it is possible to define a deformation such as it is done in the Proposition 2.1.1.

The relation with the group of motions of the plane is explained as follows. Let $g \in E(2)$ be the (unique) element with $c(g) = v$ and $d(g) = n$. Then, we associate with g the transformation $\tilde{g}$ of $\mathbb{C}$ defined by the formula

$$\tilde{g} : \mathbb{C} \to \mathbb{C} : \xi \mapsto v^2 \xi + vn. \tag{2.1}$$

The map $g \mapsto \tilde{g}$ defines a *-homomorphism from $E(2)$ into the group of all transformations of $\mathbb{C}$ which preserve orientation and Euclidian distance.

We have that the kernel of this *-homomorphism is a normal subgroup of $E(2)$ that is isomorphic to $\mathbb{Z}_2$. Thus, we get that $E(2)$ is in fact the two-fold covering of the group of motions on the two-dimensional Euclidian plane.

The pairing between the two Hopf *-algebras A and B is described in the next Proposition 2.1.4. It is a standard task to prove this result (cf. [143]). Only the non-degeneracy of the pairing is not immediately clear. This property will follow once we have calculated the full pairing; see below.

Proposition 2.1.4 *The pair (A, B) becomes a dual pair of Hopf *-algebras when a pairing is defined by*

$$\begin{array}{ccc} \langle c, a\rangle = \mu^{\frac{1}{2}}, & \langle d, a\rangle = 0, & \langle d^*, a\rangle = 0, \\ \langle c, b\rangle = 0, & \langle d, b\rangle = z, & \langle d^*, b\rangle = 0, \\ \langle c, b^*\rangle = 0, & \langle d, b^*\rangle = 0, & \langle d^*, b^*\rangle = -\mu\bar{z} \end{array}$$

where $z \in \mathbb{C}$ is arbitrary. This pairing is non-degenerate if $z \neq 0$.

Although the Hopf *-algebras A and B are certainly not difficult, it is by no means trivial to realize them on the operator algebra level. This will become clear in the following sections where we describe in a precise manner the construction of both of the quantum groups $E(2)$ and $\hat{E}(2)$ on the C*-algebra level.

One of the main tools in the construction of the quantum $E(2)$ group in the C*-algebra framework is the *multiplicative unitary.* We introduce this unitary operator W in Definition 2.3.9. The formula that we use in this definition is due to Woronowicz; see [159, Theorem 2.1]. In what follows, we give an algebraic motivation to the formula for W in the crucial Definition 2.3.9.

We introduce a completion $B \bar{\otimes} A$ of $B \otimes A$. For a detailed treatment of how to work correctly in this *-algebra, we refer to [143].

Notation 2.1.5 *We use the notation $B \bar{\otimes} A$ to denote the completion of $B \otimes A$ with respect to the weak topology induced by $A \otimes B$.*

The next Definition 2.1.6 is quite crucial.

Definition 2.1.6 *We denote by W the element in $B \bar{\otimes} A$ that 'carries' the duality between the two Hopf *-algebras A and B. This means that*

$$\langle W, x \otimes y\rangle = \langle x, y\rangle$$

for all $x \in A$ and $y \in B$.

The element W corresponds to the *multiplicative unitary* on the C*-algebra level. Hence, we want W to satisfy the pentagonal equation

$$W_{12}W_{13}W_{23} = W_{23}W_{12}$$

where we used the leg-numbering notation.

Applying the slice maps $\langle x, \cdot\,\rangle \otimes \iota \otimes \langle\,\cdot\,, y\rangle$ on the above pentagonal equation, we see that it is equivalent with the commutation rules

$$yx = x_{(1)}\langle x_{(2)}, y_{(1)}\rangle y_{(2)} \tag{2.2}$$

when $x \in A$ and $y \in B$. In this formula, we use the Sweedler notation.

Notice that, for $x \in A$, we mean by $\langle x, \cdot\,\rangle$ the functional $B \to \mathbb{C} : t \mapsto \langle x, t\rangle$. Similarly, we define linear maps $\langle\,\cdot\,, y\rangle : A \to \mathbb{C} : s \mapsto \langle s, y\rangle$ when $y \in B$.

We can easily calculate the commutation rules (2.2) between the generators. The most important ones are collected in the Proposition 2.1.7 below.

Proposition 2.1.7 *We have that*

$$ac = \mu^{\frac{1}{2}} ca, \qquad ad = \mu^{-\frac{1}{2}} da,$$
$$bc = \mu^{\frac{1}{2}} cb, \qquad bd = \mu^{-\frac{1}{2}} db + zca^{-1}, \qquad bd^* = \mu^{\frac{1}{2}} d^* b.$$

One should compare this result with Proposition 2.3.6. There, we will consider the above commutation relations on the Hilbert space level.

We now want to obtain a formula for W in terms of the generators. For this purpose, we calculate the *full pairing*.

For the rest of the Section 2.1, we make use of the Notation 2.1.8.

Notation 2.1.8 *We denote* $\mathbf{x} = cd$ *and* $\mathbf{y} = ab$.

The next Proposition 2.1.9 displays a detailed description of the full pairing.

Proposition 2.1.9 *We have that*

$$\Delta(\mathbf{x}) = c^2 \otimes cd + cd \otimes 1 = c^2 \otimes \mathbf{x} + \mathbf{x} \otimes 1,$$
$$\Delta(\mathbf{y}) = a^2 \otimes ab + ab \otimes 1 = a^2 \otimes \mathbf{y} + \mathbf{y} \otimes 1.$$

It is immediate to check that $\{\mathbf{x}^p(\mathbf{x}^*)^q c^r \mid p, q \in \mathbb{N}, r \in \mathbb{Z}\}$ *is a basis for* A *and that* $\{\mathbf{y}^{p'}(\mathbf{y}^*)^{q'} a^{r'} \mid p', q' \in \mathbb{N}, r' \in \mathbb{Z}\}$ *is a basis for* B.

The full pairing on these basis elements is given by

$$\big\langle \mathbf{x}^p(\mathbf{x}^*)^q c^r, \mathbf{y}^{p'}(\mathbf{y}^*)^{q'} a^{r'}\big\rangle = \langle \mathbf{x}^p, \mathbf{y}^{p'}\rangle \langle (\mathbf{x}^*)^q, (\mathbf{y}^*)^{q'}\rangle \langle c^r, a^{r'}\rangle \tag{2.3}$$

where

$$\langle \mathbf{x}^p, \mathbf{y}^{p'}\rangle = \delta_{p,p'} [p, \mu^2]!\, \langle \mathbf{x}, \mathbf{y}\rangle^p = \delta_{p,p'} [p, \mu^2]!\, (\mu^{\frac{3}{2}} z)^p,$$
$$\langle (\mathbf{x}^*)^q, (\mathbf{y}^*)^{q'}\rangle = \delta_{q,q'} [q, \mu^{-2}]!\, \langle \mathbf{x}^*, \mathbf{y}^*\rangle^q = \delta_{q,q'} [q, \mu^{-2}]!\, (-\mu^{\frac{3}{2}} \overline{z})^q,$$
$$\langle c^r, a^{r'}\rangle = \mu^{\frac{1}{2} r r'}.$$

Hereby, we use the following notations. We define $[0, w]! = 1$ *and*

$$[p, w] = \frac{1 - w^p}{1 - w}, \qquad [p, w]! = [1, w][2, w] \cdots [p, w]$$

when $p \in \mathbb{N}_0$ *and* $w \in \mathbb{C}$ *with* $w \neq 1$.

The calculation of the full pairing is rather long, but standard. We remark that this pairing is calculated before; see e.g. [53, Proposition 3.2].

At first sight, it might seem more natural to search for formulas in terms of the numbers $\langle c^p d^q (d^*)^r, a^{p'} b^{q'} (b^*)^{r'} \rangle$ (as is done in [53]). This is of course equivalent to the full pairing above. The reason of the change of generators will become clear later, when we derive a formula for the multiplicative unitary W.

By using the results in Proposition 2.1.9, one can apply standard techniques to prove the non-degeneracy of the considered pairing. See [143] for details.

Further, the above description of the full pairing leads us to the formula for the element W that is displayed in the Lemma 2.1.10 below.

The result in Lemma 2.1.10 is mainly due to the Equation (2.3).

Lemma 2.1.10 *We have that*

$$W = G_{\mu^2}(\mathbf{y} \otimes \mathbf{x}) G_{\mu^{-2}}(\mathbf{y}^* \otimes \mathbf{x}^*) \chi(a \otimes 1, 1 \otimes c) \tag{2.4}$$

where χ is the bicharacter on $\mathbb{R}(\mu^{\frac{1}{2}}) \times S^1$ defined by

$$\chi(\mu^{\frac{1}{2}n}, t) = t^n$$

and where G_{μ^2} and $G_{\mu^{-2}}$ are the functions on $\mathbb{C}$ defined by

$$G_{\mu^2}(z) = \sum_{q=0}^{\infty} \frac{1}{[q, \mu^2]!} \Big(\frac{z}{\langle x, y \rangle} \Big)^q, \qquad G_{\mu^{-2}}(z) = \sum_{q=0}^{\infty} \frac{1}{[q, \mu^{-2}]!} \Big(\frac{z}{\langle x^*, y^* \rangle} \Big)^q.$$

Proof. As said above, we consider $B \bar{\otimes} A$ equipped with the weak topology induced by $A \otimes B$. This means that a net $(x_i) \in B \bar{\otimes} A$ converges to a limit $x \in B \bar{\otimes} A$ if and only if

$$\langle x_i, y \rangle \to \langle x, y \rangle \tag{2.5}$$

for all $y \in A \otimes B$.

From this angle, we see that Equation (2.5) can be applied in order to study the convergence of series of elements in $B \otimes A$.

Let $p, q, p', q' \in \mathbb{N}$ and $r, r' \in \mathbb{Z}$. We then have

$$\begin{aligned}
\langle W, \mathbf{x}^p (\mathbf{x}^*)^q c^r \otimes \mathbf{y}^{p'} (\mathbf{y}^*)^{q'} a^{r'} \rangle &= \langle \mathbf{x}^p (\mathbf{x}^*)^q c^r, \mathbf{y}^{p'} (\mathbf{y}^*)^{q'} a^{r'} \rangle \\
&= \langle \mathbf{x}^p, \mathbf{y}^{p'} \rangle \langle (\mathbf{x}^*)^q, (\mathbf{y}^*)^{q'} \rangle \langle c^r, a^{r'} \rangle \\
&= \langle W, \mathbf{x}^p \otimes \mathbf{y}^{p'} \rangle \langle W, (\mathbf{x}^*)^q \otimes (\mathbf{y}^*)^{q'} \rangle \langle W, c^r \otimes a^{r'} \rangle.
\end{aligned}$$

The formulas in Proposition 2.1.9 imply that

$$\begin{aligned}
\langle (\mathbf{y} \otimes \mathbf{x})^q, \mathbf{x}^p \otimes \mathbf{y}^{p'} \rangle &= \langle \mathbf{x}^p, \mathbf{y}^q \rangle \langle \mathbf{x}^q, \mathbf{y}^{p'} \rangle \\
&= \delta_{p,q} \delta_{p',q} [q, \mu^2]! \, \langle \mathbf{x}, \mathbf{y} \rangle^q [p', \mu^2]! \langle \mathbf{x}, \mathbf{y} \rangle^{p'} \delta_{p,q} \delta_{p,p'} ([p, \mu^2]!)^2 \langle \mathbf{x}, \mathbf{y} \rangle^{2p}.
\end{aligned}$$

Hence, we get

$$\Big\langle \frac{1}{[q,\mu^2]!}\Big(\frac{\mathbf{y}\otimes\mathbf{x}}{\langle \mathbf{x},\mathbf{y}\rangle}\Big)^q, \mathbf{x}^p\otimes\mathbf{y}^{p'}\Big\rangle = \delta_{p,q}\delta_{p,p'}[p,\mu^2]!\,\langle \mathbf{x},\mathbf{y}\rangle^p = \delta_{p,q}\langle \mathbf{x}^p,\mathbf{y}^{p'}\rangle.$$

This formula together with Equation (2.3) gives that $G_{\mu^2}(\mathbf{y}\otimes\mathbf{x})$ is well-defined in $B\,\bar{\otimes}\,A$ and that

$$\big\langle G_{\mu^2}(\mathbf{y}\otimes\mathbf{x}), \mathbf{x}^p\otimes\mathbf{y}^{p'}\big\rangle = \langle \mathbf{x}^p,\mathbf{y}^{p'}\rangle = \langle W, \mathbf{x}^p\otimes\mathbf{y}^{p'}\rangle.$$

Completely similar, we find that $G_{\mu^{-2}}(\mathbf{y}^*\otimes\mathbf{x}^*)$ is well-defined in $B\,\bar{\otimes}\,A$ and

$$\big\langle G_{\mu^{-2}}(\mathbf{y}^*\otimes\mathbf{x}^*), (\mathbf{x}^*)^q\otimes(\mathbf{y}^*)^{q'}\big\rangle = \langle (\mathbf{x}^*)^q, (\mathbf{y}^*)^{q'}\rangle = \langle W, (\mathbf{x}^*)^q\otimes(\mathbf{y}^*)^{q'}\rangle.$$

Further, we have that

$$\langle a^q\otimes c^{q'}, c^r\otimes a^{r'}\rangle = \mu^{\frac{1}{2}rq}\mu^{\frac{1}{2}r'q'}$$

and hence

$$\big\langle (a\otimes 1)^q(1\otimes c)^{q'}, c^r\otimes a^{r'}\big\rangle = (\mu^{\frac{1}{2}r})^q(\mu^{\frac{1}{2}r'})^{q'}.$$

This implies that also $\chi(a\otimes 1, 1\otimes c)$ is well-defined in $B\,\bar{\otimes}\,A$ and

$$\big\langle \chi(a\otimes 1,1\otimes c), c^r\otimes a^{r'}\big\rangle = \chi(\mu^{\frac{1}{2}r},\mu^{\frac{1}{2}r'}) = \mu^{\frac{1}{2}rr'} = \langle c^r, a^{r'}\rangle = \langle W, c^r\otimes a^{r'}\rangle.$$

Using Equation (2.3) and all the results above, we find

$$\begin{aligned}
&\big\langle G_{\mu^2}(\mathbf{y}\otimes\mathbf{x})G_{\mu^{-2}}(\mathbf{y}^*\otimes\mathbf{x}^*)\chi(a\otimes 1,1\otimes c), \mathbf{x}^p(\mathbf{x}^*)^q c^r\otimes \mathbf{y}^{p'}(\mathbf{y}^*)^{q'}a^{r'}\big\rangle\\
&\quad = \langle G_{\mu^2}(\mathbf{y}\otimes\mathbf{x}), \mathbf{x}^p\otimes\mathbf{y}^{p'}\rangle\langle G_{\mu^{-2}}(\mathbf{y}^*\otimes\mathbf{x}^*), (\mathbf{x}^*)^q\otimes(\mathbf{y}^*)^{q'}\rangle\langle \chi(a\otimes 1,1\otimes c), c^r\otimes a^{r'}\rangle\\
&\quad = \langle W, \mathbf{x}^p\otimes\mathbf{y}^{p'}\rangle\langle W, (\mathbf{x}^*)^q\otimes(\mathbf{y}^*)^{q'}\rangle\langle W, c^r\otimes a^{r'}\rangle\\
&\quad = \big\langle W, \mathbf{x}^p(\mathbf{x}^*)^q c^r\otimes\mathbf{y}^{p'}(\mathbf{y}^*)^{q'}a^{r'}\big\rangle.
\end{aligned}$$

We thus obtain that

$$W = G_{\mu^2}(\mathbf{y}\otimes\mathbf{x})G_{\mu^{-2}}(\mathbf{y}^*\otimes\mathbf{x}^*)\chi(a\otimes 1,1\otimes c).$$

This ends the proof of the lemma. ∎

We will below rewrite Equation (2.4) in a form that is more allied to the results on the operator algebra level appearing later in this Chapter 2. For this, we need some formulas concerning the functions G_{μ^2} and $G_{\mu^{-2}}$.

Throughout the thesis, we make extensively use of the function F_μ as defined below. It is a very powerful tool in our calculations. We refer to Section 2.2 and Appendix A for more information about this important function.

Definition 2.1.11 ([158]) *We consider the function F_μ on $\mathbb{C}$ that is defined in the following way. For $z\in\mathbb{C}\setminus\{-\mu^{-2k}\mid k\in\mathbb{N}\}$, we set*

$$F_\mu(z) = \prod_{k=0}^{\infty}\frac{1+\mu^{2k}\overline{z}}{1+\mu^{2k}z}. \tag{2.6}$$

We further set $F_\mu(z) = -1$ when $z\in\{-\mu^{-2k}\mid k\in\mathbb{N}\}$.

The next Lemma 2.1.12 is quite obvious.

Lemma 2.1.12 *For almost all $z \in \mathbb{C}$, we have*

$$F_\mu\Big(\frac{\mu^2-1}{\langle x,y\rangle}\,z\Big) = G_{\mu^2}(z)G_{\mu^{-2}}(\overline{z}).$$

Proof. Let $z \in \mathbb{C} \setminus \big\{\frac{\langle x,y\rangle}{1-\mu^2}\,\mu^{-2k} \mid k \in \mathbb{N}\big\}$. Some easy computations yield

$$G_{\mu^2}(z) = \prod_{k=0}^{\infty} \frac{1}{1+\mu^{2k}t} \qquad \text{with} \;\; t = (\mu^2-1)\frac{z}{\langle x,y\rangle},$$
$$G_{\mu^{-2}}(z) = \prod_{k=0}^{\infty} (1+\mu^{2k}s) \qquad \text{with} \;\; s = (1-\mu^2)\frac{z}{\langle x^*,y^*\rangle}.$$

From Equation (2.6), it then follows that

$$F_\mu\Big(\frac{\mu^2-1}{\langle \mathbf{x},\mathbf{y}\rangle}\,z\Big) = G_{\mu^2}(z)G_{\mu^{-2}}(\overline{z}).$$

This ends the proof of the lemma. ∎

It is clear that Notation 2.1.13 below is founded upon Lemma 2.1.12. We use it to skirt a direct definition of elements like $F_\mu(y \otimes x)$.

Notation 2.1.13 *We denote*

$$F_\mu\Big(\frac{\mu^2-1}{\langle \mathbf{x},\mathbf{y}\rangle}\,(\mathbf{y}\otimes\mathbf{x})\Big) = G_{\mu^2}(\mathbf{y}\otimes\mathbf{x})G_{\mu^{-2}}(\mathbf{y}^*\otimes\mathbf{x}^*).$$

The Proposition 2.1.14 below gives a complete description of W in terms of the generators a,b,c,d. By this, the study of the element W is concluded.

It is precisely the elegant formula in Equation (2.7) that we will use to define the multiplicative unitary W on the C*-algebra level; see Definition 2.3.9.

The Lemma 2.1.10 is the immediate cause for the following result.

Proposition 2.1.14 *We have that*

$$W = F_\mu\Big(\frac{\mu^2-1}{\langle \mathbf{x},\mathbf{y}\rangle}(\mathbf{y}\otimes\mathbf{x})\Big)\chi(a\otimes 1, 1\otimes c) = F_\mu\Big(\frac{\mu^{-\frac{3}{2}}(\mu^2-1)}{z}(ab\otimes cd)\Big)\chi(a\otimes 1, 1\otimes c).$$

If we use the natural choice $z = \mu^{-\frac{3}{2}}(\mu^2-1)$, we get

$$W = F_\mu(ab\otimes cd)\chi(a\otimes 1, 1\otimes c). \tag{2.7}$$

In the remainder of this Section 2.1, we look at a few properties of the Hopf *-algebras A and B. They are included as an algebraic motivation for later results. All the formulas below will be recovered on the operator algebra level.

The Proposition 2.1.15 collects some results about the antipode S. In particular, it gives that we have a *polar decomposition* $S = R\tau_{-\frac{i}{2}}$.

The fact that there already exists a polar decomposition on the Hopf *-algebra level is not so uncommon. For instance, this is always the case if we are working within the category of algebraic quantum groups; see [62].

Proposition 2.1.15 *There is an involutive *-anti-homomorphism R on A and an analytic one-parameter group $\{\tau_t \mid t \in \mathbb{R}\}$ of *-automorphisms on A such that $S = R\tau_{-\frac{i}{2}} = \tau_{-\frac{i}{2}}R$. For all $z \in \mathbb{C}$, we have that R and τ_z commute.*

For all $x \in A$ and $t \in \mathbb{R}$, we have

$$\Delta(\tau_t(x)) = (\tau_t \otimes \tau_t)\Delta(x).$$

Further, we have for all $x \in A$ that

$$\Delta(R(x)) = \dot{\sigma}(R \otimes R)\Delta(x)$$

where $\dot{\sigma}$ denotes the flip *from $A \otimes A$ to itself given by $\dot{\sigma}(x \otimes y) = y \otimes x$.*

For all $t \in \mathbb{R}$, we have

$$\tau_t(c) = c \qquad \textit{and} \qquad \tau_t(d) = \mu^{-2it}d.$$

We have that

$$R(c) = c^* \qquad \textit{and} \qquad R(d) = -d.$$

Proof. First, we consider the one-parameter group τ. For $t \in \mathbb{R}$, we define $\tau_t(c) = c$ and $\tau_t(d) = \mu^{-2it}d$. It is easy to verify that these formulas yield a one-parameter group of *-automorphisms on the Hopf *-algebra A.

This one-parameter group τ is analytic in the sense that, for all $x \in A$ and all linear functionals f on A, the map $t \mapsto f(\tau_t(x))$ is analytic. We hereby mean that the map $t \mapsto f(\tau_t(x))$ is the restriction to $\mathbb{R}$ of an entire map.

We can then extend τ to $\mathbb{C}$ by using analytic continuation. It is clear that, for all $z \in \mathbb{C}$, we have

$$\tau_z(c) = c \qquad \text{and} \qquad \tau_z(d) = \mu^{-2iz}d.$$

These formulas imply that $\Delta(\tau_z(x)) = (\tau_z \otimes \tau_z)\Delta(x)$ for all $x \in A$ and $z \in \mathbb{C}$. Further, it also follows that τ_z commutes with S for every $z \in \mathbb{C}$.

Define $R = S\tau_{\frac{i}{2}}$. Since S commutes with $\tau_{\frac{i}{2}}$, we also have $R = \tau_{\frac{i}{2}}S$. Hence, we get that $S = R\tau_{-\frac{i}{2}} = \tau_{-\frac{i}{2}}R$.

Because S is an anti-homomorphism and $\tau_{\frac{i}{2}}$ is a homomorphism, it is clear that R is an anti-homomorphism. As S flips the comultiplication and τ leaves it invariant, also R will flip the comultiplication.

Now, we look at the behavior of R with respect to the involution. We have the general property

$$R(x^*) = S(\tau_{\frac{i}{2}}(x^*)) = S(\tau_{-\frac{i}{2}}(x)^*) = S^{-1}(\tau_{-\frac{i}{2}}(x))^* = R^{-1}(x)^*$$

so that $R(x^*) = R^{-1}(x)^*$ for all $x \in A$.

We also have that

$$\begin{aligned} R^2(c) &= S^2\tau_i(c) = S^2(c) = c, \\ R^2(d) &= S^2\tau_i(d) = \mu^2 S^2(d) = \mu^2 S(-\mu^{-1}d) = -\mu S(d) = -\mu(\mu^{-1}d) = d, \\ R^2(d^*) &= S^2\tau_i(d^*) = S^2(\tau_{-i}(d)^*) = \mu^{-2}S^2(d^*) = d^*. \end{aligned}$$

We get that $R^2 = \iota$ and R is thus an anti-automorphism. Together with the property $R(x^*) = R^{-1}(x)^*$, we see that R is also a *-map. ∎

The next Proposition 2.1.16 is about the antipode $\hat{S}$.

Proposition 2.1.16 *There is an involutive *-anti-homomorphism $\hat{R}$ on B and an analytic one-parameter group $\{\hat{\tau}_t \mid t \in \mathbb{R}\}$ of *-automorphisms on B such that $\hat{S} = \hat{R}\hat{\tau}_{-\frac{i}{2}} = \hat{\tau}_{-\frac{i}{2}}\hat{R}$. For all $z \in \mathbb{C}$, we have that $\hat{R}$ and $\hat{\tau}_z$ commute.*

For all $y \in B$ and $t \in \mathbb{R}$, we have

$$\hat{\Delta}(\hat{\tau}_t(y)) = (\hat{\tau}_t \otimes \hat{\tau}_t)\hat{\Delta}(y).$$

Further, we have for all $y \in B$ that

$$\hat{\Delta}(\hat{R}(y)) = \dot{\sigma}(\hat{R} \otimes \hat{R})\hat{\Delta}(y)$$

where $\dot{\sigma}$ denotes the flip from $B \otimes B$ to itself.

For all $t \in \mathbb{R}$, we have

$$\hat{\tau}_t(a) = a \qquad \text{and} \qquad \hat{\tau}_t(b) = \mu^{-2it}b$$

We have that

$$\hat{R}(a) = a^{-1} \qquad \text{and} \qquad \hat{R}(b) = -b.$$

The theory of dual pairs of Hopf *-algebras shows that Proposition 2.1.17 follows directly from the pairing; see e.g. [143]. The corresponding formulas on the operator algebra level can be found in Theorems 2.6.11 and 2.6.39.

We note that in Equation (2.8) we make use of the unique extensions of $\iota \otimes S$ and $\hat{S} \otimes \iota$ to closed linear maps on the *-algebra $B \bar{\otimes} A$.

Proposition 2.1.17 *We have that*

$$(\iota \otimes S)W = (\hat{S} \otimes \iota)W = W^*. \tag{2.8}$$

From Remark 2.1.3, we know that the $E(2)$ group has a natural action on $\mathbb{C}$. This leads up to the construction of a *coaction* γ of (A, Δ).

The next Proposition 2.1.18 is taken from [157].

Proposition 2.1.18 *Let C be the unital *-algebra over $\mathbb{C}$ that is generated by a single element ξ satisfying $\xi\xi^* = \mu^2\xi^*\xi$.*

We consider the *-homomorphism $\boldsymbol{\gamma} : C \to A \otimes C$ determined by the formula

$$\boldsymbol{\gamma}(\xi) = c^2 \otimes \xi + cd \otimes 1.$$

This definition is of course inspired by the Equation (2.1)*.*

It is not difficult to check that

$$(\Delta \otimes \iota)\boldsymbol{\gamma} = (\iota \otimes \boldsymbol{\gamma})\boldsymbol{\gamma}.$$

This means that $\boldsymbol{\gamma}$ is a left coaction *of (A, Δ) on C.*

By identifying ξ with cd, we can consider C as a *-subalgebra of A. We then have that the action $\boldsymbol{\gamma}$ coincides with $\Delta|_C$.

To end the Section 2.1, we mention the Proposition 2.1.19. The non-existence of Haar functionals on the Hopf *-algebra level is not unexpected.

Proposition 2.1.19 *There does not exist a non-zero positive integral on A which is either left or right invariant.*

We have a completely similar result for the Hopf *-algebra B.

Proof. Let φ be a non-zero positive left invariant integral on A.

From the general theory (see e.g. [147]), we know that the integral φ is faithful in the sense that $\varphi(a^*a) = 0$ implies $a = 0$.

We have that

$$\varphi(cd)1 = (\iota \otimes \varphi)\Delta(cd) = (\iota \otimes \varphi)(c^2 \otimes cd + cd \otimes 1) = \varphi(cd)c^2 + \varphi(1)cd.$$

This yields

$$\varphi(cd)(1 - c^2) = \varphi(1)cd. \tag{2.9}$$

Multiplying on the left by c^* and on the right by c gives that

$$\varphi(cd)(1 - c^2) = \varphi(1)dc. \tag{2.10}$$

If we now combine the Equations (2.9) and (2.10), we get

$$\varphi(1)cd = \varphi(1)dc.$$

Since $cd = \mu dc$ and $0 < \mu < 1$, we find that $\varphi(1) = 0$. This clearly contradicts the faithfulness property of φ.

We conclude that there exists no non-zero positive left invariant integral on A. The other results can be proven by using the exactly same technique. ∎

2.2 Operator equalities and special functions

This Section 2.2 deals with a few technical results. Two special functions F_μ and χ are introduced and investigated. Both these functions play a very crucial role in the theory of the quantum $E(2)$ group.

In the next Section 2.3, we use the functions F_μ and χ to define the multiplicative unitary W; see Definition 2.3.9. Further, the properties of F_μ are used in the Theorem 2.3.30 to prove the manageability of W.

The Section 2.2 encompasses several important results that are relentlessly used in later calculations. It is mainly included to keep the text self-contained and therefore does not contain many proofs.

We observe that most of the mentioned properties are proved in Appendix A where we give an accurate study of the function F_μ and its Fourier coefficients. However, we also include some results from [158, 168] without proof. It is always designated in a clear way where one can find full details.

The next Definition 2.2.1 first appeared in [158]. Notice that the product in Equation (2.11) converges because we have $0 < \mu < 1$.

Definition 2.2.1 ([158]) *We consider the function F_μ on $\mathbb{C}$ that is defined in the following way. For all $z \in \mathbb{C} \setminus \{-\mu^{-2k} \mid k \in \mathbb{N}\}$, we set*

$$F_\mu(z) = \prod_{k=0}^{\infty} \frac{1 + \mu^{2k}\overline{z}}{1 + \mu^{2k} z} \tag{2.11}$$

Further, we set $F_\mu(z) = -1$ when $z \in \{-\mu^{-2k} \mid k \in \mathbb{N}\}$.

We define a bicharacter $\chi \in \mathrm{C}_b(\mathbb{R}(\mu^{\frac{1}{2}}) \times S^1)$ by setting that

$$\chi(\mu^{\frac{1}{2}k}, z) = z^k$$

when $k \in \mathbb{Z}$ and $z \in S^1$.

Especially the function F_μ plays a vital role in this Ph.D.-thesis. This function is investigated thoroughly by Woronowicz in [158, 168] and by Baaj in [1].

As it is said above, we use this technical Section 2.2 to elucidate (some of) the known results about F_μ and we refer to Appendix A for proofs.

First, we have the next Lemma 2.2.2 containing some basic properties.

Lemma 2.2.2 ([158]) *The function F_μ is not continuous on $\mathbb{C}$. But we do have that the restriction of F_μ to $\bar{\mathbb{C}}^\mu$ is continuous.*

We also use the notation F_μ to denote this restriction.

For all $z \in \mathbb{C}$, we have $F_\mu(\overline{z}) = \overline{F_\mu(z)}$ and $|F_\mu(z)| = 1$. Hence, $F_\mu \in \mathrm{C}_b(\bar{\mathbb{C}}^\mu)$.

The function F_μ has the following asymptotical behavior.

Lemma 2.2.3 ([158]) *For $z \in \bar{\mathbb{C}}^\mu$, we have $F_\mu(\mu^2 z)F_\mu(z^{-1}) = \left(\frac{z}{|z|}\right)^{\log_\mu(|z|)+1}$. Thus, for $z \in \bar{\mathbb{C}}^\mu$ large enough, we get that*

$$F_\mu(z) \approx \left(\frac{z}{|z|}\right)^{\log_\mu(|z|)-1}.$$

We make use of the Fourier transform of F_μ with respect to the angle variable. The Fourier coefficients of F_μ and their properties are studied very intensively in Appendix A. For now, we only mention the most interesting results.

We introduce a notation for the Fourier coefficients.[1] The uniform convergence of the Fourier series in Definition 2.2.4 below is proved in Lemma A.8.

Definition 2.2.4 ([1, 2]) *Let $k, n \in \mathbb{Z}$. Then, we define a number $B(k,n)$ by*

$$B(k,n) = \frac{1}{2\pi}\int_0^{2\pi} F_\mu(\mu^n e^{it})e^{-ikt}\,dt.$$

We thus have by definition that $B(k,n)$ is the k^{th} Fourier coefficient of the continuous function $S^1 \to \mathbb{C} : z \mapsto F_\mu(\mu^n z)$.

For all $z \in S^1$, we have that

$$F_\mu(\mu^n z) = \sum_{k\in\mathbb{Z}} B(k,n)z^k. \tag{2.12}$$

Further, the series in Equation (2.12) has uniform convergence on S^1.

S. Baaj discovered in [1] some properties about the numbers $B(k,n)$ which turn out to be very useful in computations. We collect the most important ones in Propositions 2.2.5 and 2.2.6. Proofs can be found in Appendix A.

The Proposition 2.2.5 below is quite basic. There are however needed some very ingenious tricks in order to prove Proposition 2.2.6.

Proposition 2.2.5 ([1, 2]) *We have the following properties:*

1. *$B(k,n) \in \mathbb{R}$ and $|B(k,n)| \leq 1$ for all $k,n \in \mathbb{Z}$,*
2. *$\lim_{n\to+\infty} B(k,n) = \delta_{k,0}$ uniformly in k.*

Proposition 2.2.6 ([1, 2]) *Let $k, n \in \mathbb{Z}$. We have that*

$$\begin{aligned} B(k,n) &= (-\mu)^k B(-k, n-k) \\ &= B(k-n+1, -n+2) = (-\mu)^{k-n+1}B(n-1, k+1). \end{aligned}$$

[1]The B comes from Baaj who first studied these numbers.

The next Remark 2.2.7 is about where the above properties are used.

Remark 2.2.7 *The set of equalities in Proposition 2.2.6 is the most important property of the numbers $B(k,n)$ (at least for our purposes). For instance, it is used in the proof of the manageability of the multiplicative unitary W.*

The formula $\lim_{n\to+\infty} B(k,n) = \delta_{k,0}$ is also important. It is used in Section 3.1 to prove (strong) amenability of the quantum $E(2)$ group and its dual.

Further, also the regularity properties of W are studied by means of this property.

We now collect some results of Woronowicz involving the function F_μ and certain pairs of normal operators. These results are a main tool when studying the properties of the multiplicative unitary W. We do not give any proof.

First, we define a set of pairs (R, S) of normal operators satisfying in a strong sense the relations $RS = \mu^{-2}SR$ and $R^*S = SR^*$. We often deal with such pairs of operators when studying the quantum $E(2)$ group.

The next Definition 2.2.8 is explored in the papers [158, 168].

Definition 2.2.8 ([158]) *We denote by D_μ the set of all pairs (R, S) of normal operators on a Hilbert space K such that R, S and their polar decompositions $R = u_R|R|$ and $S = u_S|S|$ satisfy the following five conditions:*

1. $\ker R = \ker S = \{0\}$,
2. $u_R u_S = u_S u_R$,
3. $u_R^*|S|u_R = \mu|S|$,
4. $u_S|R|u_S^* = \mu|R|$,
5. $|R|$ *and* $|S|$ *strongly commute.*

Except for its last result, it is a simple task to prove the Lemma 2.2.9 below. The properties in Lemma 2.2.9 are useful basic results.

More information about $R \dotplus S$ can be found in Theorem 2.2.11.

Lemma 2.2.9 *Let $(R, S) \in D_\mu$. The operators RS, $R^{-1}S$, RS^{-1} and $R \dotplus S$ are well-defined. Moreover, $R^{-1}S$ and RS^{-1} are normal and RS is not normal.*

The operator $R \dotplus S$ is normal if and only if $\sigma(R^{-1}S) \subseteq \bar{\mathbb{C}}^\mu$.

It is easy to check the Lemma 2.2.10. From Theorem 2.4 in [168], it follows that this property more or less characterizes the pairs of operators in D_μ.

Lemma 2.2.10 *Let $(R, S) \in D_\mu$. Then, we have*

$$RS = \mu^{-2}SR \qquad \text{and} \qquad R^*S = SR^*.$$

The Theorems 2.2.11 and 2.2.12 show the great importance of the function F_μ. These deep results will play a major role in later calculations.

We first state the Theorem 2.2.11 about pairs (R, S) belonging to D_μ.

Theorem 2.2.11 ([158],[168]) *Let $(R, S) \in D_\mu$. We have that $(S^*, R^*) \in D_\mu$. Further, the following conditions are equivalent:*

1. *$R + S$ admits a normal extension,*
2. *$R \dotplus S$ is normal,*
3. *$\sigma(R^{-1}S) \subseteq \bar{\mathbb{C}}^\mu$,*
4. *$\sigma(RS^{-1}) \subseteq \bar{\mathbb{C}}^\mu$.*

If one of the above conditions is satisfied, then we have

1. *$R \dotplus S = F_\mu(R^{-1}S) R F_\mu(R^{-1}S)^*$,*
2. *$R \dotplus S = F_\mu(RS^{-1})^* S F_\mu(RS^{-1})$,*
3. *$(R \dotplus S)^* = R^* \dotplus S^*$.*

Hence, in this case, $R \dotplus S$ is unitarily equivalent to both R and S.

Finally, if $A \subseteq B(H)$ is a non-degenerate C^*-algebra and $R, S \,\eta\, A$, then we also have that

$$R \dotplus S \,\eta\, A.$$

The remarkable Theorem 2.2.12 is the most crucial feature of the function F_μ. It gives that F_μ is some kind of *quantum exponential function.*

Theorem 2.2.12 ([158]) *Let $(R, S) \in D_\mu$ and $\sigma(R), \sigma(S) \subseteq \bar{\mathbb{C}}^\mu$ (in this case the condition $\sigma(R^{-1}S) \subseteq \bar{\mathbb{C}}^\mu$ is fulfilled automatically). Then, we have*

$$F_\mu(R \dotplus S) = F_\mu(R) F_\mu(S). \tag{2.13}$$

The Proposition 2.2.13 states that F_μ is essentially the only bounded function satisfying the character property in Equation (2.13). We mention this uniqueness result for completeness. We will make no use of it in this thesis.

Proposition 2.2.13 ([158]) *Let $(R, S) \in D_\mu$. Assume that f is a measurable function on $\bar{\mathbb{C}}^\mu$ such that both f and $\frac{1}{f}$ are bounded.*

We then have that the two following conditions are equivalent:

1. *$f(R \dotplus S) = f(R) f(S)$,*
2. *There is a number $t \in \bar{\mathbb{C}}^\mu$ such that $f(z) = F_\mu(tz)$ for almost all $z \in \bar{\mathbb{C}}^\mu$.*

In the two Propositions 2.2.14 and 2.2.15, we investigate the relation between the special functions F_μ and χ and the affiliation relation.

We use the techniques from [165, Section 5] to give a proof of these properties.

Proposition 2.2.14 ([158]) *Let $A \subseteq B(K)$ be a non-degenerate C^*-algebra. Let Q be a normal operator on K such that $\sigma(Q) \subseteq \bar{\mathbb{C}}^\mu$.*

The two statements below are equivalent:

1. *For any $z \in \bar{\mathbb{C}}^\mu$ we have $F_\mu(zQ) \in M(A)$. Moreover, the mapping*
$$\bar{\mathbb{C}}^\mu \to M(A) : z \mapsto F_\mu(zQ)$$
is strictly continuous,
2. *We have that $Q \,\eta\, A$.*

Proof. We consider the commutative C*-algebra $B = C_0(\bar{\mathbb{C}}^\mu)$. It follows from Examples 1.5.9 and 1.5.45 that $M(B) = C_b(\bar{\mathbb{C}}^\mu)$ and $B^\eta = C(\bar{\mathbb{C}}^\mu)$.

We define a function $g \,\eta\, B$ by
$$g : \bar{\mathbb{C}}^\mu \to \mathbb{C} : x \mapsto x.$$
Using Example 1.5.55, we see that B is generated by $g \,\eta\, B$.

For every $\gamma \in \bar{\mathbb{C}}^\mu$, we define a function $F^\gamma \in M(B)$ by
$$F^\gamma : \bar{\mathbb{C}}^\mu \to S^1 : x \mapsto F_\mu(\gamma x).$$
Then F^γ is a unitary element in $M(B)$.

It is easy to check that $F^\gamma \to F^{\gamma_0}$ uniformly on compact subsets when $\gamma \to \gamma_0$. For bounded subsets of $M(B)$, the topology of uniform convergence on compact subsets coincides with the strict topology. Therefore, the mapping
$$\bar{\mathbb{C}}^\mu \to M(B) : \gamma \mapsto F^\gamma \tag{2.14}$$
is strictly continuous.

For all $k, n \in \mathbb{Z}$, we define a function $f_{k,n} \in M(B)$ as follows. For every $z \in \bar{\mathbb{C}}^\mu$, we set
$$f_{k,n}(z) = \frac{1}{2\pi} \int_0^{2\pi} F_\mu(\mu^n z e^{i\theta}) e^{-ik\theta}\, d\theta. \tag{2.15}$$
Thus, we have by definition that $f_{k,n}(z)$ is the k^{th} Fourier coefficient of the function $S^1 \to \mathbb{C} : t \mapsto F_\mu(\mu^n z t)$. This e.g. means that $f_{k,n}(1) = B(k,n)$.

Using the asymptotical behavior of the function F_μ (see Lemma 2.2.3), it is not so difficult to prove that $f_{k,n} \in B$ for all $k, n \in \mathbb{Z}$.

Further, we can use the asymptotical behavior of F_μ to show (with some effort) that the set $\{f_{k,n} \mid k, n \in \mathbb{Z}\}$ separates points of $\bar{\mathbb{C}}^\mu$.

We hence get that $\{f_{k,n} \mid k, n \in \mathbb{Z}\}$ is a subset of B that separates points of $\bar{\mathbb{C}}^\mu$. The Stone-Weierstrass theorem implies that the *-algebra generated by $\{f_{k,n} \mid k, n \in \mathbb{Z}\}$ is dense in B. We use B_0 to denote this *-algebra.

Now, let Q be an operator satisfying the assumptions of the theorem. Then, the functional calculus

$$\pi : B \to B(H) : f \mapsto f(Q)$$

is a non-degenerate *-homomorphism. We have that $\pi(g) = Q$.

First, assume that $Q \,\eta\, A$. Because $g \,\eta\, B$ generates the C*-algebra B, we can then consider π as a non-degenerate *-homomorphism from B into $M(A)$. This gives that $F_\mu(\gamma Q) = F^\gamma(Q)$ belongs to $M(A)$ for all $\gamma \in \bar{\mathbb{C}}^\mu$.

The continuity in (2.14) gives that the mapping

$$\bar{\mathbb{C}}^\mu \to M(A) : \gamma \mapsto F_\mu(\gamma Q)$$

is strictly continuous.

Conversely, assume that $F_\mu(\gamma Q) \in M(A)$ for every $\gamma \in \bar{\mathbb{C}}^\mu$ and that the mapping

$$\bar{\mathbb{C}}^\mu \to M(A) : \gamma \mapsto F_\mu(\gamma Q) \tag{2.16}$$

is strictly continuous.

In this case, we have for all $k, n \in \mathbb{Z}$ that the mapping

$$[0, 2\pi] \to M(A) : \theta \mapsto F_\mu(\mu^n e^{i\theta} Q) e^{-ik\theta}$$

is strictly continuous.

Integrating over θ, we get that $f_{k,n}(Q) \in M(A)$ for all $k, n \in \mathbb{Z}$. Because B_0 is dense in B, we obtain that $f(Q) \in M(A)$ for every $f \in B$.

Using the continuity (2.16), it is not difficult to show that $f_{0,n}(Q) \to 1$ in the strict topology on $M(A)$ if $n \to +\infty$.

We can thus conclude that $\{\, f(Q) \mid f \in B\}$ is a subset of $M(A)$ containing an approximate unit for A. This property implies that we can consider π as a non-degenerate *-homomorphism from B to $M(A)$.

Applying π to $g \,\eta\, B$, we obtain that $Q \,\eta\, A$. ∎

Proposition 2.2.15 *Let $A \subseteq B(K)$ be a non-degenerate C*-algebra. Let Q be a strictly positive operator on K such that $\sigma(Q) \subseteq \bar{\mathbb{R}}(\mu^{\frac{1}{2}})$.*

The two statements below are equivalent:

1. *For any $z \in S^1$ we have $\chi(Q, z) \in M(A)$. Moreover, the mapping*

$$S^1 \to M(A) : z \mapsto \chi(Q, z)$$

is strictly continuous,

2. *We have that $Q \,\eta\, A$ and $Q^{-1} \,\eta\, A$.*

Proof. We consider the commutative C*-algebra $D = \mathrm{C}_0(\mathbb{R}(\mu^{\frac{1}{2}}))$. It follows from Examples 1.5.9 and 1.5.45 that $M(D) = \mathrm{C}_b(\mathbb{R}(\mu^{\frac{1}{2}}))$ and $D^\eta = \mathrm{C}(\mathbb{R}(\mu^{\frac{1}{2}}))$.

We define two functions $h_1, h_2 \,\eta\, D$ by

$$h_1 : \mathbb{R}(\mu^{\frac{1}{2}}) \to \mathbb{C} : x \mapsto x \qquad \text{and} \qquad h_2 : \mathbb{R}(\mu^{\frac{1}{2}}) \to \mathbb{C} : x \mapsto x^{-1}.$$

Using Example 1.5.55, we see that D is generated by $h_1, h_2 \,\eta\, D$.

For every $z \in S^1$, we define a function $G^z \in M(D)$ by

$$G^z : \mathbb{R}(\mu^{\frac{1}{2}}) \to S^1 : x \mapsto \chi(x, z).$$

Then G^z is a unitary element in $M(D)$.

It is easy to check that $G^z \to G^{z_0}$ uniformly on compact subsets when $z \to z_0$. For bounded subsets of $M(D)$, the topology of uniform convergence on compact subsets coincides with the strict topology. Therefore, the mapping

$$S^1 \to M(D) : z \mapsto G^z \tag{2.17}$$

is strictly continuous.

By definition, we have for $z \in S^1$ that $\chi(Q, z) = G^z(Q)$.

For all $k, n \in \mathbb{Z}$ with $n \neq 0$, we define a function $g_{k,n} \in M(D)$ as follows. For every $x \in \mathbb{R}(\mu^{\frac{1}{2}})$, we set

$$g_{k,n}(x) = \frac{1}{2\pi} \int_0^{2\pi} \chi(x, e^{\frac{i\theta}{n}}) e^{-ik\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} G^{e^{\frac{i\theta}{n}}}(x) e^{-ik\theta}\, d\theta. \tag{2.18}$$

Thus, $g_{k,n}(x)$ is the k^{th} Fourier coefficient of the function $S^1 \to \mathbb{C} : z \mapsto \chi(x, z^{\frac{1}{n}})$.

Let $k, n \in \mathbb{Z}$ with $n \neq 0$. It is then easy to prove that $g_{k,n} \in D$. Further, it is direct to check that $g_{k,1}$ is the characteristic function of the set $\{\mu^{\frac{1}{2}k}\}$.

We hence find that $\{g_{k,n} \mid k, n \in \mathbb{Z}, n \neq 0\}$ is a subset of D separating points of $\mathbb{R}(\mu^{\frac{1}{2}})$. The Stone-Weierstrass theorem gives that the *-algebra generated by $\{g_{k,n} \mid k, n \in \mathbb{Z}, n \neq 0\}$ is dense in D. We use D_0 to denote this *-algebra.

Now, let Q be an operator satisfying the assumptions of the theorem. Then, the functional calculus

$$\pi : D \to B(H) : f \mapsto f(Q)$$

is a non-degenerate *-homomorphism. We have $\pi(h_1) = Q$ and $\pi(h_2) = Q^{-1}$.

First, assume that $Q \,\eta\, A$ and $Q^{-1} \,\eta\, A$. Then, because $h_1, h_2 \,\eta\, D$ generate the C*-algebra D, we can consider π as a non-degenerate *-homomorphism from D into $M(A)$. This gives that $\chi(Q, z) = \pi(G^z)$ belongs to $M(A)$ for every $z \in S^1$.

The continuity in (2.17) gives that the mapping

$$S^1 \to M(A) : z \mapsto \chi(Q, z)$$

is strictly continuous.

Conversely, assume that $\chi(Q,z) \in M(A)$ for every $z \in S^1$ and that the mapping

$$S^1 \to M(A) : z \mapsto \chi(Q,z) \tag{2.19}$$

is strictly continuous.

Then, we have that $\pi(G^{e^{i\theta}}) \in M(A)$ for every $\theta \in [0,2\pi]$. For all $k,n \in \mathbb{Z}$ with $n \neq 0$, the mapping

$$[0,2\pi] \to M(A) : \theta \mapsto G^{e^{\frac{i\theta}{n}}}(Q)e^{-ik\theta}$$

is strictly continuous.

Integrating over θ, we obtain that $g_{k,n}(Q) \in M(A)$ for all $k,n \in \mathbb{Z}$ with $n \neq 0$. Because D_0 is dense in D, we find that $g(Q) \in M(A)$ for every $g \in D$.

Using the continuity (2.19), it is not difficult to show that $g_{0,n}(Q) \to 1$ in the strict topology on $M(A)$ if $n \to +\infty$.

We can thus conclude that $\{\, g(Q) \mid g \in D\}$ is a subset of $M(A)$ containing an approximate unit for A. This property implies that we can consider π as a non-degenerate *-homomorphism from D to $M(A)$.

Applying π to $h_1, h_2 \,\eta\, D$, we obtain that $Q \,\eta\, A$ and $Q^{-1} \,\eta\, A$. ∎

2.3 The multiplicative unitary

In Section 2.1, we have given a complete description of the quantum $E(2)$ group on the Hopf *-algebra level. Our goal is to lift these results to the operator algebra level. This will be done in the subsequent Sections 2.4 to 2.9. First, we introduce the material needed to perform the lifting procedure.

Thus, in this Section 2.3, we are working on the *Hilbert space level* which is the intermediate step between the easy Hopf *-algebra level and the much more difficult C*-algebra level. This C*-algebra level starts in the Section 2.4.

The emphasis of the Section 2.3 lies on the construction of the *multiplicative unitary* W. We describe this indispensable tool and its most important features. In particular, we prove that W is strongly modular.

This Section 2.3 is crucial for later reasoning. It is precisely the manageability property of W that allows us to apply the technique in Section 1.4. The results in the Sections 1.3 and 1.4 then make it possible to bring in the manageability theory in the construction of the quantum $E(2)$ group.

As a first step, we represent the generators of the Hopf *-algebras A and B as operators on a Hilbert space. We use these operators in Definition 2.3.9 to define the multiplicative unitary W. Further, they are used in the Section 2.5 to describe the C*-algebras of the quantum $E(2)$ group and its dual.

We begin by defining the Hilbert space H where the generating operators will be working on. This is done in the Notation 2.3.1 below. The subspace H_0 is important as a core for unbounded operators.

The Notation 2.3.1 is kept fixed for the rest of the thesis.

Notation 2.3.1 *We consider the Hilbert space $\ell^2(\mathbb{Z})$ and the tensor product Hilbert space $H = \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$.*

Further, we fix an orthonormal basis $\{e_k \mid k \in \mathbb{Z}\}$ of the Hilbert space $\ell^2(\mathbb{Z})$. Thus $\{e_k \otimes e_l \mid k, l \in \mathbb{Z}\}$ is an orthonormal basis for H.

We use the notation

$$H_0 = \Big\{ \sum_{k,l} a_{k,l} e_k \otimes e_l \mid a_{k,l} \in \mathbb{C},\ a_{k,l} \neq 0 \textit{ for finitely many } k, l \Big\}.$$

Then H_0 is a dense subspace of H.

The next Notation 2.3.2 is introduced to shorten notations.

Notation 2.3.2 *Let $k, l, m, n \in \mathbb{Z}$. We then use $\omega_{k,l,m,n}$ to denote the vector functional $\omega_{e_k \otimes e_l, e_m \otimes e_n}$. Further, we abbreviate $\omega_{k,l,k,l}$ to $\omega_{k,l}$.*

The Definition 2.3.3 below describes the operators involved. Notice that we here use the same notations for the generators as on the algebraic level. From now on, we only work in the operator algebra framework. There can hence be no confusion with aforementioned notations.

The operators a, b, c, d introduced in the next Definition 2.3.3 will thus appear as the generators of the quantum $E(2)$ group and its dual.

It is explained in the next Section 2.5 how c, d 'generate' the quantum $E(2)$ group and how a, b 'generate' the quantum $\hat{E}(2)$ group.

The notations in Definition 2.3.3 remain fixed for the rest of the thesis.

Definition 2.3.3 *We define a bounded operator s and a closed operator m on $\ell^2(\mathbb{Z})$ as follows. For $k \in \mathbb{Z}$, we set*

$$s e_k = e_{k+1} \qquad \textit{and} \qquad m e_k = \mu^k e_k.$$

By definition, we take $\operatorname{span}\{e_k \mid k \in \mathbb{Z}\}$ *as a core for m.*

We have that s is a unitary operator and that m is a strictly positive operator. It is easy to check that

$$ms = \mu s m.$$

Further, we define operators a, b, c, d on H by

$$a = m^{-\frac{1}{2}} \otimes m, \qquad b = m^{\frac{1}{2}} \otimes s,$$
$$c = s \otimes s, \qquad d = s \otimes m^{-1}.$$

The next Lemma 2.3.4 rallies some basic properties.

Lemma 2.3.4 *The operator c is unitary and the operator a is strictly positive. The operators b and d both are non-singular and normal.*

Further, the adjoints of a, b, c, d are given by

$$a^* = m^{-\frac{1}{2}} \otimes m, \qquad b^* = m^{\frac{1}{2}} \otimes s^*,$$
$$c^* = s^* \otimes s^*, \qquad d^* = s^* \otimes m^{-1}.$$

The subspace H_0 is a core for the generators a, b, c, d and their adjoints.

For all $z \in \mathbb{C}$, we have

$$a^z = m^{-\frac{1}{2}z} \otimes m^z.$$

Moreover, we have that H_0 is a core for a^z.

The polar decompositions of a, c and their adjoints are trivial. Furthermore, we have the polar decompositions

$$b = (1 \otimes s)(m^{\frac{1}{2}} \otimes 1) \qquad \text{and} \qquad d = (s \otimes 1)(1 \otimes m^{-1}).$$

The polar decompositions of b^*, d^* can be deduced directly from these formulas.

It is easy to compute that $\sigma(s) = S^1$ and $\sigma(m) = \bar{\mathbb{R}}^\mu$. The results in [37] can then be used to deduce the Lemma 2.3.5 below.

The Lemma 2.3.5 is important when applying the functional calculus.

Lemma 2.3.5 *The spectra of the operators a, b, c, d are given by*

$$\sigma(a) = \bar{\mathbb{R}}(\mu^{\frac{1}{2}}), \qquad \sigma(b) = \bar{\mathbb{C}}(\mu^{\frac{1}{2}}), \qquad \sigma(c) = S^1, \qquad \sigma(d) = \bar{\mathbb{C}}^\mu.$$

From Definition 2.3.3, we know that $ms = \mu sm$. Using this basic result, we can find several commutation rules between the generating operators. We list the principal ones in Proposition 2.3.6 below (cf. [158, Section 5]).

From Definition 2.3.3, we know that $ms = \mu sm$. Using this basic result, we can find several commutation rules between the generating operators. We list the principal ones in the Proposition 2.3.6 (cf. [158, Section 5]).

We recall that all products of operators (e.g. like $a^z b a^{-z}$) are considered as closed operators. This convention is explained in the Introduction.

The resemblance between the Propositions 2.3.6 and 2.1.7 is clear.

Proposition 2.3.6 *We have that*

$$cd = \mu dc \qquad \text{and} \qquad ab = \mu ba.$$

It hereby should be observed that the last commutation rule actually is a shortened notation meaning that we have $a^{it} b a^{-it} = \mu^{it} b$ for all $t \in \mathbb{R}$.

Further, we remark that we even have $a^z b a^{-z} = \mu^z b$ for all $z \in \mathbb{C}$.

If we use $b = u|b|$ to denote the polar decomposition of b, we have that $au = \mu ua$ while the operators a and $|b|$ strongly commute.

We also have that

1. $ac = \mu^{\frac{1}{2}} ca$ *and* $ad = \mu^{-\frac{1}{2}} da$,
2. $bc = \mu^{\frac{1}{2}} cb$, $bd = \mu^{\frac{3}{2}} db$ *and* $bd^* = \mu^{\frac{1}{2}} d^* b$,
3. $bd - \mu^{-\frac{1}{2}} db = \mu^{-\frac{3}{2}} (\mu^2 - 1) ca^{-1}$.

By the commutation rule $ad = \mu^{-\frac{1}{2}} da$, we again mean that $a^{it} da^{-it} = \mu^{-\frac{1}{2}it} d$ for all $t \in \mathbb{R}$. Also here, we actually have $a^z da^{-z} = \mu^{-\frac{1}{2}z} d$ for all $z \in \mathbb{C}$.

Finally, if we use $d = v|d|$ to denote the polar decomposition of d, we have that $av = \mu^{-\frac{1}{2}} va$ while the operators a and $|d|$ strongly commute.

We make a small Remark 2.3.7 about the above commutation results.

Remark 2.3.7 *Surely the main outcome of Proposition 2.3.6 is that we have the commutation relations $cd = \mu dc$ and $ab = \mu ba$. They will give that the pair (c, d) generates the quantum $E(2)$ group while the pair (a, b) (more or less) generates the quantum $\hat{E}(2)$ group. See Theorems 2.5.3 and 2.5.24.*

The fact that (three of) the generating operators are unbounded is close related to the property that the quantum $E(2)$ group and its dual are non-compact. This non-compactness is proved in the Theorems 2.5.2 and 2.5.21.

Using the two main commutation rules $cd = \mu dc$ and $ab = \mu ba$ together with some spectral conditions, one can also prove that the pairs (c, d) and (a, b) are (in some sense) universal. This universality is explored in Section 4.2.

It is easy to check that the operators cd and ab are not normal. We however do have the normality result in Lemma 2.3.8 below.

This Lemma 2.3.8 is needed to make correct the Definition 2.3.9 below.

Lemma 2.3.8 *We have that $ab \otimes cd$ is a non-singular and normal operator. Its spectrum $\sigma(ab \otimes cd)$ is contained in $\bar{\mathbb{C}}^\mu$.*

Proof. We have that

$$ab \otimes cd = 1 \otimes ms \otimes s^2 \otimes sm^{-1}.$$

The polar decomposition of $ab \otimes cd$ is given by

$$ab \otimes cd = (1 \otimes s \otimes s^2 \otimes s)|ab \otimes cd| = (1 \otimes s \otimes s^2 \otimes s)\big(\mu(1 \otimes m \otimes 1 \otimes m^{-1})\big).$$

It is immediate to check that $s \otimes s$ commutes with $m \otimes m^{-1}$. This implies that also $(1 \otimes s \otimes s^2 \otimes s)$ and $|ab \otimes cd|$ commute.

Using the above polar decomposition, it is direct to prove that the operator $ab \otimes cd$ is non-singular and normal.

The formula $|ab \otimes cd| = \mu(1 \otimes m \otimes 1 \otimes m^{-1})$ gives that $\sigma(|ab \otimes cd|) = \bar{\mathbb{R}}^{\mu}$. ∎

Now, we are ready to define the unitary operator W. The Definition 2.3.9 below is clearly inspired by the Proposition 2.1.14. This definition of W is due to S.L. Woronowicz and was first introduced and studied in [159].

Definition 2.3.9 ([159]) *We define a unitary operator $W \in B(H \otimes H)$ by*

$$W = F_\mu(ab \otimes cd)\chi(a \otimes 1, 1 \otimes c).$$

We then immediately state the main Theorem 2.3.10. This result is nothing more than Proposition 2.3.21 below in the special case $\gamma = 1$. We postpone the proof until we formulate the more general result of Proposition 2.3.21.

The next Theorem 2.3.10 is the central property of this Section 2.3.

Theorem 2.3.10 *The unitary W is multiplicative, i.e., it satisfies the* pentagonal equation

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

The importance of the unitary operator W can hardly be overestimated. It will be the main actor in our construction of the quantum $E(2)$ group.

The multiplicativity of W will thus follow from a more general result. In Proposition 2.3.21 below, we define a family of unitary operators (including W) that are *adapted* to W. This general result is needed later in the thesis.

The next Terminology 2.3.11 is justified by Theorem 2.3.10 and by the properties of W that we prove below. Further, also the results in Section 2.1 show that it is appropriate to use this terminology.

This Terminology 2.3.11 is of course not so important.

Terminology 2.3.11 *For the rest of the thesis, the unitary operator W is called* the *multiplicative unitary of the quantum $E(2)$ group.*

From Lemma 2.2.2, we get that $F_\mu(\overline{z}) = \overline{F_\mu(z)}$ for all $z \in \bar{\mathbb{C}}^{\mu}$. We further know that χ is a bicharacter. We can then deduce the Lemma 2.3.12 below.

Lemma 2.3.12 *We have that*

1. $F_\mu(ab \otimes cd)^* = F_\mu\big((ab \otimes cd)^*\big)$,
2. $\chi(a \otimes 1, 1 \otimes c) = \chi(a^{-1} \otimes 1, 1 \otimes c^*)$,
3. $\chi(a \otimes 1, 1 \otimes c)^* = \chi(a \otimes 1, 1 \otimes c^*) = \chi(a^{-1} \otimes 1, 1 \otimes c)$.

We now compute the action of both W and W^* on basis vectors. The next Proposition 2.3.13 displays all the related results.

The formulas below turn out to be very powerful in later calculations.

Proposition 2.3.13 *Let $k, l, m, n \in \mathbb{Z}$. Then, the following equalities hold:*

1. $\chi(a \otimes 1, 1 \otimes c)(e_k \otimes e_l \otimes e_m \otimes e_n) = e_k \otimes e_l \otimes e_{m-k+2l} \otimes e_{n-k+2l}$,
2. $\chi(a \otimes 1, 1 \otimes c)^*(e_k \otimes e_l \otimes e_m \otimes e_n) = e_k \otimes e_l \otimes e_{m+k-2l} \otimes e_{n+k-2l}$,
3. $F_\mu(ab \otimes cd)(e_k \otimes e_l \otimes e_m \otimes e_n) = \sum_{t \in \mathbb{Z}} B(t, l-n+1)(e_k \otimes e_{l+t} \otimes e_{m+2t} \otimes e_{n+t})$,
4. $F_\mu(ab \otimes cd)^*(e_k \otimes e_l \otimes e_m \otimes e_n) = \sum_{t \in \mathbb{Z}} B(t, l-n+1)(e_k \otimes e_{l-t} \otimes e_{m-2t} \otimes e_{n-t})$.

Hence, we have that

5. $W(e_k \otimes e_l \otimes e_m \otimes e_n)$
$$= \sum_{t \in \mathbb{Z}} B(t, k-l-n+1)(e_k \otimes e_{l+t} \otimes e_{m-k+2l+2t} \otimes e_{n-k+2l+t}),$$
6. $W^*(e_k \otimes e_l \otimes e_m \otimes e_n) = \sum_{t \in \mathbb{Z}} B(t, l-n+1)(e_k \otimes e_{l-t} \otimes e_{m+k-2l} \otimes e_{n+k-2l+t})$.

Proof. Let $k, l, m, n \in \mathbb{Z}$. The Lemma 2.3.4 gives that a is strictly positive and that c is unitary. Clearly, $e_k \otimes e_l \otimes e_m \otimes e_n$ is an eigenvector of $a \otimes 1$ with eigenvalue $\mu^{-\frac{1}{2}k+l}$. We hence have

$$\begin{aligned} \chi(a \otimes 1, 1 \otimes c)(e_k \otimes e_l \otimes e_m \otimes e_n) &= \chi\big(\mu^{-\frac{1}{2}k+l}(1 \otimes 1), 1 \otimes c\big)(e_k \otimes e_l \otimes e_m \otimes e_n) \\ &= (1 \otimes c)^{-k+2l}(e_k \otimes e_l \otimes e_m \otimes e_n) = e_k \otimes e_l \otimes e_{m-k+2l} \otimes e_{n-k+2l}. \end{aligned}$$

The polar decomposition of the normal operator $ab \otimes cd$ is given by

$$ab \otimes cd = (1 \otimes s \otimes s^2 \otimes s)|ab \otimes cd|$$

where $|ab \otimes cd| = \mu(1 \otimes m \otimes 1 \otimes m^{-1})$.

We see that $e_k \otimes e_l \otimes e_m \otimes e_n$ is an eigenvector of $|ab \otimes cd|$ with eigenvalue μ^{l-n+1}.

Thus, using the Fourier transform of F_μ (see Definition 2.2.4), we get

$$\begin{aligned} F_\mu(ab \otimes cd)(e_k \otimes e_l \otimes e_m \otimes e_n) &= F_\mu\big(\mu^{l-n+1}(1 \otimes s \otimes s^2 \otimes s)\big)(e_k \otimes e_l \otimes e_m \otimes e_n) \\ &= \sum_{t \in \mathbb{Z}} B(t, l-n+1)(1 \otimes s \otimes s^2 \otimes s)^t(e_k \otimes e_l \otimes e_m \otimes e_n) \\ &= \sum_{t \in \mathbb{Z}} B(t, l-n+1)(e_k \otimes e_{l+t} \otimes e_{m+2t} \otimes e_{n+t}). \end{aligned}$$

Now, a trivial calculation yields the given formula for $W(e_k \otimes e_l \otimes e_m \otimes e_n)$.

Remembering the Lemma 2.3.12, we can compute the three remaining formulas in a completely similar way. ∎

The next Corollary 2.3.14 is a direct consequence of Proposition 2.3.13. It is mainly used in calculations in the Sections 2.7 and 3.1.

Corollary 2.3.14 *Let $k, l, m, n, p, q \in \mathbb{Z}$. We have*

1. $\big((\omega_{k,l,m,n} \otimes \iota)W\big)(e_p \otimes e_q) = B(n-l, k-l-q+1)\delta_{k,m}(e_{p-k+2n} \otimes e_{q-k+l+n})$,
2. $\big((\omega_{k,l,m,n} \otimes \iota)W^*\big)(e_p \otimes e_q) = B(n-l, n-q+1)\delta_{k,m}(e_{p+k-2n} \otimes e_{q+k-l-n})$,
3. $\big((\iota \otimes \omega_{k,l,m,n})W\big)(e_p \otimes e_q)$
$= B(p-2q-l+n, p-q-l+1)\delta_{-k+2l+m-2n,p-2q}(e_p \otimes e_{p-q-l+n})$,
4. $\big((\iota \otimes \omega_{k,l,m,n})W^*\big)(e_p \otimes e_q)$
$= B(-p+2q+l-n, q-n+1)\delta_{k-m,p-2q}(e_p \otimes e_{p-q-l+n})$.

The next two Lemmas 2.3.15 and 2.3.16 describe some basic relations between the multiplicative unitary W and the generating operators a, b, c, d.

We rephrase these results in Section 2.5 in order to obtain a formula for the action of the comultiplications on the quantum $E(2)$ group and its dual.

We make use of Theorem 2.2.11 to give correct proofs.

Lemma 2.3.15 *We have that*

$$W(c \otimes 1)W^* = c \otimes c, \qquad W(c^* \otimes 1)W^* = c^* \otimes c^*,$$
$$W(d \otimes 1)W^* = c \otimes d \dotplus d \otimes c^*, \qquad W(d^* \otimes 1)W^* = c^* \otimes d^* \dotplus d^* \otimes c.$$

Proof. A straightforward calculation (using Proposition 2.3.13) yields

$$\chi(a \otimes 1, 1 \otimes c)(c \otimes 1)\chi(a \otimes 1, 1 \otimes c)^* = c \otimes c. \tag{2.20}$$

Definition 2.3.3 gives that $ms = \mu sm$. From this, it follows that

$$(c \otimes c)(ab \otimes cd) = (ab \otimes cd)(c \otimes c).$$

The Fuglede-Putnam theorem implies that we also have

$$(c \otimes c)F_\mu(ab \otimes cd) = F_\mu(ab \otimes cd)(c \otimes c). \tag{2.21}$$

Equations (2.20) and (2.21) together give that

$$W(c \otimes 1)W^* = c \otimes c.$$

We find the formula $W(c^* \otimes 1)W^* = c^* \otimes c^*$ by taking the adjoint.

Recalling that H_0 is a core for the normal operator d (cf. Lemma 2.3.4), we can use Proposition 2.3.13 to prove that

$$\chi(a \otimes 1, 1 \otimes c)(d \otimes 1)\chi(a \otimes 1, 1 \otimes c)^* = d \otimes c^*. \tag{2.22}$$

Now, we define normal operators R and S by

$$R = d \otimes c^* \qquad \text{and} \qquad S = c \otimes d.$$

It is direct to check that $(R,S) \in D_\mu$ and $R^{-1}S = ab \otimes cd$.

Theorem 2.2.11 (together with Lemma 2.3.8) yields

$$F_\mu(R^{-1}S)RF_\mu(R^{-1}S)^* = R \dotplus S$$

and hence

$$F_\mu(ab \otimes cd)(d \otimes c^*)F_\mu(ab \otimes cd)^* = c \otimes d \dotplus d \otimes c^*. \tag{2.23}$$

From the Equations (2.22) and (2.23) together, we get that

$$W(d \otimes 1)W^* = c \otimes d \dotplus d \otimes c^*.$$

Theorem 2.2.11 gives that $(R \dotplus S)^* = R^* \dotplus S^*$. Hence, taking the adjoint of the above formula, we find that

$$W(d^* \otimes 1)W^* = c^* \otimes d^* \dotplus d^* \otimes c$$

This ends the proof of the lemma. ■

Lemma 2.3.16 *We have that*

$$W^*(1 \otimes a)W = a \otimes a, \qquad W^*(1 \otimes a^{-1})W = a^{-1} \otimes a^{-1},$$
$$W^*(1 \otimes b)W = a \otimes b \dotplus b \otimes a^{-1}, \qquad W^*(1 \otimes b^*)W = a \otimes b^* \dotplus b^* \otimes a^{-1}.$$

Proof. It is easy to check that $F_\mu(ab \otimes cd)(H_0 \odot H_0)$ is a subset of $D(1 \otimes a)$. Further, we know from Lemma 2.3.4 that H_0 is a core for the strictly positive operator a. We can hence use Proposition 2.3.13 to prove that

$$F_\mu(ab \otimes cd)^*(1 \otimes a)F_\mu(ab \otimes cd) = 1 \otimes a. \tag{2.24}$$

Similarly, we find the commutation rule

$$\chi(a \otimes 1, 1 \otimes c)^*(1 \otimes a)\chi(a \otimes 1, 1 \otimes c) = a \otimes a. \tag{2.25}$$

Equations (2.24) and (2.25) together give that

$$W^*(1 \otimes a)W = a \otimes a.$$

The formula $W^*(1 \otimes a^{-1})W = a^{-1} \otimes a^{-1}$ follows immediately from this result.

Now, we define normal operators R and S by

$$R = ab \otimes a^{-1}c^2 \qquad \text{and} \qquad S = 1 \otimes b.$$

Some easy computations yield that $(R,S) \in D_\mu$ and $RS^{-1} = ab \otimes cd$.

Theorem 2.2.11 (together with Lemma 2.3.8) gives that

$$F_\mu(RS^{-1})^* S F_\mu(RS^{-1}) = R \dot{+} S$$

and hence

$$F_\mu(ab \otimes cd)^*(1 \otimes b) F_\mu(ab \otimes cd) = ab \otimes a^{-1}c^2 \dot{+} 1 \otimes b. \tag{2.26}$$

With similar techniques as above, we can use Proposition 2.3.13 to prove that

- $\chi(a \otimes 1, 1 \otimes c)^*(ab \otimes a^{-1}c^2)\chi(a \otimes 1, 1 \otimes c) = b \otimes a^{-1}$,
- $\chi(a \otimes 1, 1 \otimes c)^*(1 \otimes b)\chi(a \otimes 1, 1 \otimes c) = a \otimes b$.

Now, Equation (2.26) together with these commutation relations yield

$$W^*(1 \otimes b)W = a \otimes b \dot{+} b \otimes a^{-1}. \tag{2.27}$$

We define normal operators P, Q by $P = a \otimes b$ and $Q = b \otimes a^{-1}$. Then, we have that $(Q, P) \in D_\mu$. Using Equation (2.27), we see that $P \dot{+} Q$ is normal. It now follows from Theorem 2.2.11 that $(P \dot{+} Q)^* = P^* \dot{+} Q^*$.

Hence, taking the adjoint of the formula in Equation (2.27), we find

$$W^*(1 \otimes b^*)W = a \otimes b^* \dot{+} b^* \otimes a^{-1}$$

This ends the proof of the lemma. ■

The next Remark 2.3.17 is worthwhile to mention.

Remark 2.3.17 *If we combine the results in (the proofs of) the above lemmas with Lemma 3.2 in [158], we find that the subspace $H_0 \odot H_0$ is a core for the normal operators*

$$c \otimes d \dot{+} d \otimes c^* \qquad \text{and} \qquad a \otimes b \dot{+} b \otimes a^{-1}$$

and their adjoints. Hence, we have that $H_0 \odot H_0$ is a core for all the operators appearing in the Lemmas 2.3.15 and 2.3.16.

We will however not need this result in any calculation.

The techniques used in the proofs above can also be applied to prove the general pentagonal equation. This will be done in Proposition 2.3.21.

The Lemma 2.3.18 is already a first step towards the Proposition 2.3.21.

Lemma 2.3.18 *We have that*

$$(1 \otimes W)(ab \otimes cd \otimes 1)(1 \otimes W^*) = ab \otimes cd \otimes 1 \dot{+} ab \otimes c^2 \otimes cd.$$

Proof. From Equations (2.20) and (2.22), we find

$$\chi(a \otimes 1, 1 \otimes c)(cd \otimes 1)\chi(a \otimes 1, 1 \otimes c)^* = cd \otimes 1.$$

This implies that

$$(1 \otimes W)(ab \otimes cd \otimes 1)(1 \otimes W^*) = F_\mu(1 \otimes ab \otimes cd)(ab \otimes cd \otimes 1)F_\mu(1 \otimes ab \otimes cd)^*.$$

Define normal operators R and S by

$$R = ab \otimes cd \otimes 1 \qquad \text{and} \qquad S = ab \otimes c^2 \otimes cd.$$

It is not difficult to check that $(R, S) \in D_\mu$ and $R^{-1}S = 1 \otimes ab \otimes cd$. By using Lemma 2.3.8, we see that we have $\sigma(R^{-1}S) \subseteq \bar{\mathbb{C}}^\mu$.

An application of Theorem 2.2.11 gives

$$F_\mu(R^{-1}S)RF_\mu(R^{-1}S)^* = R \dotplus S.$$

This means precisely that

$$F_\mu(1 \otimes ab \otimes cd)(ab \otimes cd \otimes 1)F_\mu(1 \otimes ab \otimes cd)^* = ab \otimes cd \otimes 1 \dotplus ab \otimes c^2 \otimes cd.$$

Now, the lemma follows from the above results. ■

The Corollary 2.3.19 then gives a second intermediate step.

Corollary 2.3.19 *For all $\gamma \in \bar{\mathbb{C}}^\mu$, we have*

$$F_\mu\big((1 \otimes W)(\gamma ab \otimes cd \otimes 1)(1 \otimes W^*)\big) = F_\mu(\gamma ab \otimes cd \otimes 1)F_\mu(\gamma ab \otimes c^2 \otimes cd).$$

Proof. The result is trivial if $\gamma = 0$. Let $\gamma \in \mathbb{C}^\mu$. Then, we define normal operators R and S by

$$R = \gamma ab \otimes cd \otimes 1 \qquad \text{and} \qquad S = \gamma ab \otimes c^2 \otimes cd.$$

From the proof of Lemma 2.3.18, we get that $(R, S) \in D_\mu$. From Lemma 2.3.8, it follows that we have $\sigma(R), \sigma(S) \subseteq \bar{\mathbb{C}}^\mu$.

The Proposition 2.2.12 then gives that

$$F_\mu(R \dotplus S) = F_\mu(R)F_\mu(S)$$

which means precisely that

$$F_\mu(\gamma ab \otimes cd \otimes 1 \dotplus \gamma ab \otimes c^2 \otimes cd) = F_\mu(\gamma ab \otimes cd \otimes 1)F_\mu(\gamma ab \otimes c^2 \otimes cd).$$

If we combine this result with Lemma 2.3.18, we get

$$F_\mu\big((1 \otimes W)(\gamma ab \otimes cd \otimes 1)(1 \otimes W^*)\big) = F_\mu(\gamma ab \otimes cd \otimes 1)F_\mu(\gamma ab \otimes c^2 \otimes cd).$$

This gives the corollary. ■

We are now ready to prove Proposition 2.3.21 below. This crucial result describes a whole family of unitary operators $V(\gamma)$ that are *adapted* to W.

The Proposition 2.3.21 is thus a (small) generalization of Theorem 2.3.10.

Definition 2.3.20 *For every $\gamma \in \bar{\mathbb{C}}^\mu$, we define a unitary $V(\gamma) \in B(H \otimes H)$ by the formula*

$$V(\gamma) = F_\mu(\gamma ab \otimes cd)\chi(a \otimes 1, 1 \otimes c).$$

Proposition 2.3.21 *Let $\gamma \in \bar{\mathbb{C}}^\mu$. We have that $V(\gamma)$ is* adapted *to W in the sense that it satisfies the pentagonal equation*

$$W_{23}V_{12}(\gamma) = V_{12}(\gamma)V_{13}(\gamma)W_{23}.$$

Proof. Take $\gamma \in \bar{\mathbb{C}}^\mu$. For notational convenience, we denote shortly $\boldsymbol{\chi}$ for $\chi(a \otimes 1, 1 \otimes c)$ and $\boldsymbol{F}(\gamma)$ for $F_\mu(\gamma ab \otimes cd)$. We hence have that

$$\begin{aligned} V_{12}(\gamma) &= \boldsymbol{F}_{12}(\gamma)\boldsymbol{\chi}_{12} = F_\mu(\gamma ab \otimes cd \otimes 1)\chi(a \otimes 1 \otimes 1, 1 \otimes c \otimes 1), \\ V_{13}(\gamma) &= \boldsymbol{F}_{13}(\gamma)\boldsymbol{\chi}_{13} = F_\mu(\gamma ab \otimes 1 \otimes cd)\chi(a \otimes 1 \otimes 1, 1 \otimes 1 \otimes c), \\ V_{23}(\gamma) &= \boldsymbol{F}_{23}(\gamma)\boldsymbol{\chi}_{23} = F_\mu(\gamma 1 \otimes ab \otimes cd)\chi(1 \otimes a \otimes 1, 1 \otimes 1 \otimes c). \end{aligned}$$

Notice that $V(1) = W$.

First, we split up the pentagonal equation

$$W_{23}V_{12}(\gamma) = V_{12}(\gamma)V_{13}(\gamma)W_{23}. \tag{2.28}$$

into two easier pentagonal equations, namely Equations (2.29) and (2.30).

To obtain this, we rewrite Equation (2.28) as

$$\boldsymbol{F}_{23}(1)\boldsymbol{\chi}_{23}\boldsymbol{F}_{12}(\gamma)\boldsymbol{\chi}_{12} = \boldsymbol{F}_{12}(\gamma)\boldsymbol{\chi}_{12}\boldsymbol{F}_{13}(\gamma)\boldsymbol{\chi}_{13}\boldsymbol{F}_{23}(1)\boldsymbol{\chi}_{23}$$

and move the $\boldsymbol{\chi}$'s to the right.

With the techniques used to prove Lemmas 2.3.15 and 2.3.16, one can also find the following commutation relations:

$$\begin{aligned} &\boldsymbol{\chi}(cd \otimes 1)\boldsymbol{\chi}^* = cd \otimes 1, \qquad &&\boldsymbol{\chi}(ab \otimes 1)\boldsymbol{\chi}^* = ab \otimes c^2, \\ &\boldsymbol{\chi}(1 \otimes cd)\boldsymbol{\chi}^* = a^2 \otimes cd, \qquad &&\boldsymbol{\chi}(a^2 \otimes ab)\boldsymbol{\chi}^* = 1 \otimes ab. \end{aligned}$$

By applying the above equalities and the Fuglede-Putnam theorem, we get that we have for every $\gamma \in \bar{\mathbb{C}}^\mu$ that

$$\begin{aligned} &\boldsymbol{\chi}_{23}\boldsymbol{F}_{12}(\gamma) = \boldsymbol{F}_{12}(\gamma)\boldsymbol{\chi}_{23}, \qquad &&\boldsymbol{\chi}_{12}\boldsymbol{F}_{13}(\gamma) = F_\mu(\gamma ab \otimes c^2 \otimes cd)\boldsymbol{\chi}_{12}, \\ &\boldsymbol{\chi}_{13}\boldsymbol{F}_{23}(\gamma) = F_\mu(\gamma a^2 \otimes ab \otimes cd)\boldsymbol{\chi}_{13}, \qquad &&\boldsymbol{\chi}_{12}F_\mu(\gamma a^2 \otimes ab \otimes cd) = \boldsymbol{F}_{23}(\gamma)\boldsymbol{\chi}_{12}. \end{aligned}$$

These commutation relations yield

$$W_{23}V_{12}(\gamma) = \boldsymbol{F}_{23}(1)\boldsymbol{\chi}_{23}\boldsymbol{F}_{12}(\gamma)\boldsymbol{\chi}_{12} = \boldsymbol{F}_{23}(1)\boldsymbol{F}_{12}(\gamma)\boldsymbol{\chi}_{23}\boldsymbol{\chi}_{12}.$$

Further, they also give that

$$\begin{aligned}V_{12}(\gamma)V_{13}(\gamma)W_{23} &= \boldsymbol{F}_{12}(\gamma)\boldsymbol{\chi}_{12}\boldsymbol{F}_{13}(\gamma)\boldsymbol{\chi}_{13}\boldsymbol{F}_{23}(1)\boldsymbol{\chi}_{23}\\ &= \boldsymbol{F}_{12}(\gamma)F_\mu(\gamma ab\otimes c^2\otimes cd)\boldsymbol{\chi}_{12}F_\mu(a^2\otimes ab\otimes cd)\boldsymbol{\chi}_{13}\boldsymbol{\chi}_{23}\\ &= \boldsymbol{F}_{12}(\gamma)F_\mu(\gamma ab\otimes c^2\otimes cd)\boldsymbol{F}_{23}(1)\boldsymbol{\chi}_{12}\boldsymbol{\chi}_{13}\boldsymbol{\chi}_{23}.\end{aligned}$$

Using Proposition 2.3.13, it is easy to check that $\boldsymbol{\chi}$ satisfies the pentagonal equation

$$\boldsymbol{\chi}_{12}\boldsymbol{\chi}_{13}\boldsymbol{\chi}_{23} = \boldsymbol{\chi}_{23}\boldsymbol{\chi}_{12}. \tag{2.29}$$

Hence, to complete the proof, we have to prove that $\boldsymbol{F}$ satisfies the following pentagonal equation:

$$\boldsymbol{F}_{12}(\gamma)F_\mu(\gamma ab\otimes c^2\otimes cd)\boldsymbol{F}_{23}(1) = \boldsymbol{F}_{23}(1)\boldsymbol{F}_{12}(\gamma). \tag{2.30}$$

We have that

$$\begin{aligned}\boldsymbol{F}_{23}(1)\boldsymbol{F}_{12}(\gamma)\boldsymbol{F}_{23}(1)^* &= F_\mu\big(\boldsymbol{F}_{23}(1)(\gamma ab\otimes cd\otimes 1)\boldsymbol{F}_{23}(1)^*\big)\\ &= F_\mu\big(\boldsymbol{F}_{23}(1)\boldsymbol{\chi}_{23}(\gamma ab\otimes cd\otimes 1)\boldsymbol{\chi}_{23}^*\boldsymbol{F}_{23}(1)^*\big)\\ &= F_\mu\big((1\otimes W)(\gamma ab\otimes cd\otimes 1)(1\otimes W^*)\big).\end{aligned}$$

Applying Corollary 2.3.19, we see that this calculation implies that

$$\boldsymbol{F}_{23}(1)\boldsymbol{F}_{12}(\gamma)\boldsymbol{F}_{23}(1)^* = \boldsymbol{F}_{12}(\gamma)F_\mu(\gamma ab\otimes c^2\otimes cd).$$

From this, Equation (2.30) follows immediately. ∎

The next Remark 2.3.22 is about the proof of Theorem 2.3.10.

Remark 2.3.22 *With $\gamma = 1$, we have by definition that $V(\gamma) = W$. In this case, the Proposition 2.3.21 reads as*

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

This precisely is the pentagonal equation in Theorem 2.3.10.

The Proposition 2.3.23 is also about the operators $V(\gamma)$. This result is more or less dual to Proposition 2.3.21. It is not used in later calculations.

Proposition 2.3.23 *Let $\gamma \in \bar{\mathbb{C}}^\mu$. We have that $V(\gamma)$ satisfies the extended pentagonal equation*

$$V_{23}(\gamma)W_{12} = W_{12}V_{13}(\gamma)V_{23}(\gamma).$$

Proof. Take $\gamma \in \bar{\mathbb{C}}^\mu$. Similar to the proof of Proposition 2.3.21, we denote shortly $\boldsymbol{\chi}$ for $\chi(a\otimes 1, 1\otimes c)$ and $\boldsymbol{F}(\gamma)$ for $F_\mu(\gamma ab\otimes cd)$.

From the commutation relations formulated in the proof of Proposition 2.3.21, we immediately find that

$$V_{23}(\gamma)W_{12} = \boldsymbol{F}_{12}(\gamma)\boldsymbol{\chi}_{23}\boldsymbol{F}_{12}(1)\boldsymbol{\chi}_{12} = \boldsymbol{F}_{23}(\gamma)\boldsymbol{F}_{12}(1)\boldsymbol{\chi}_{23}\boldsymbol{\chi}_{12}$$

and also that

$$\begin{aligned} W_{12}V_{13}(\gamma)V_{23}(\gamma) &= \boldsymbol{F}_{12}(1)\boldsymbol{\chi}_{12}\boldsymbol{F}_{13}(\gamma)\boldsymbol{\chi}_{13}\boldsymbol{F}_{23}(\gamma)\boldsymbol{\chi}_{23} \\ &= \boldsymbol{F}_{12}(1)F_\mu(\gamma ab \otimes c^2 \otimes cd)\boldsymbol{\chi}_{12}F_\mu(\gamma a^2 \otimes ab \otimes cd)\boldsymbol{\chi}_{13}\boldsymbol{\chi}_{23} \\ &= \boldsymbol{F}_{12}(1)F_\mu(\gamma ab \otimes c^2 \otimes cd)\boldsymbol{F}_{23}(\gamma)\boldsymbol{\chi}_{12}\boldsymbol{\chi}_{13}\boldsymbol{\chi}_{23}. \end{aligned}$$

The Equation (2.29) gives that $\boldsymbol{\chi}$ satisfies the pentagonal equation

$$\boldsymbol{\chi}_{12}\boldsymbol{\chi}_{13}\boldsymbol{\chi}_{23} = \boldsymbol{\chi}_{23}\boldsymbol{\chi}_{12}.$$

Thus, to complete the proof, it suffices to prove that $\boldsymbol{F}$ satisfies the following pentagonal equation:

$$\boldsymbol{F}_{12}(1)F_\mu(\gamma ab \otimes c^2 \otimes cd)\boldsymbol{F}_{23}(\gamma) = \boldsymbol{F}_{23}(\gamma)\boldsymbol{F}_{12}(1). \tag{2.31}$$

We have that

$$\boldsymbol{F}_{12}(1)^*\boldsymbol{F}_{23}(\gamma)\boldsymbol{F}_{12}(1) = F_\mu\big(\boldsymbol{F}_{12}(1)^*(\gamma 1 \otimes ab \otimes cd)\boldsymbol{F}_{12}(1)\big). \tag{2.32}$$

Now, we define normal operators R and S by

$$R = \gamma ab \otimes c^2 \otimes cd \qquad \text{and} \qquad S = \gamma 1 \otimes ab \otimes cd.$$

It is not difficult to check that $(R, S) \in D_\mu$ and $RS^{-1} = ab \otimes cd \otimes 1$. By using Lemma 2.3.8, we see that we have $\sigma(RS^{-1}) \subseteq \bar{\mathbb{C}}^\mu$.

An application of Theorem 2.2.11 gives

$$F_\mu(RS^{-1})^* S F_\mu(RS^{-1}) = R \dotplus S.$$

This means precisely that

$$F_\mu(ab \otimes cd \otimes 1)^*(\gamma 1 \otimes ab \otimes cd)F_\mu(ab \otimes cd \otimes 1) = \gamma ab \otimes c^2 \otimes cd \dotplus \gamma 1 \otimes ab \otimes cd.$$

Together with Equation (2.32), this result yields

$$\boldsymbol{F}_{12}(1)^*\boldsymbol{F}_{23}(\gamma)\boldsymbol{F}_{12}(1) = F_\mu(\gamma ab \otimes c^2 \otimes cd \dotplus 1 \otimes \gamma ab \otimes cd). \tag{2.33}$$

Using Lemma 2.3.8, we see that $\sigma(R), \sigma(S) \subseteq \bar{\mathbb{C}}^\mu$. So, Proposition 2.2.12 gives that we have

$$F_\mu(R \dotplus S) = F_\mu(R)F_\mu(S)$$

which means that

$$F_\mu(\gamma ab \otimes c^2 \otimes cd \dotplus 1 \otimes \gamma ab \otimes cd) = F_\mu(\gamma ab \otimes c^2 \otimes cd)F_\mu(1 \otimes \gamma ab \otimes cd). \tag{2.34}$$

If we combine the Equations (2.33) and (2.34), we find

$$\boldsymbol{F}_{12}(1)^*\boldsymbol{F}_{23}(\gamma)\boldsymbol{F}_{12}(1) = F_\mu(\gamma ab \otimes c^2 \otimes cd)\boldsymbol{F}_{23}(\gamma).$$

From this, Equation (2.31) follows immediately. ∎

The following Remark 2.3.24 is a simple comment.

Remark 2.3.24 *The results in Propositions 2.3.21 and 2.3.23 are not genuinely stronger than the pentagonal equation.*

They can be directly deduced from the pentagonal equation by multiplying on the left and on the right with appropriate operators. For this, we can e.g. make use of the *-automorphisms

$$\rho_{k,t} : B(H) \to B(H) : x \mapsto (s^{-2k} \otimes s^k m^{it})x(s^{2k} \otimes m^{-it} s^{-k});$$
$$\rho'_{k,t} : B(H) \to B(H) : x \mapsto (m^{it} \otimes s^k m^{-it})x(m^{-it} \otimes m^{it} s^{-k})$$

where $k \in \mathbb{Z}$ and $t \in \mathbb{R}$. They should be applied to the first, respectively third, leg of the pentagonal equation in order to find the extended equations.

We now look at the manageability properties of W. These play a crucial role in the rest of the Chapter 2. The Theorem 2.3.30 is the central result.

The Definition 2.3.25 defines three important operators. These operators are the ones that make W strongly modular.

It is explained below how N and $I, \hat{I}$ are related to W.

Definition 2.3.25 *We define $N = m^2 \otimes m^{-2}$. Thus, N is a strictly positive operator on H. Further, we define two anti-unitary operators I and $\hat{I}$ on H by requiring that we have*

- $I(e_k \otimes e_l) = (-1)^{k+l}(e_{-k+2l} \otimes e_l)$,
- $\hat{I}(e_k \otimes e_l) = (-1)^l(e_k \otimes e_{k-l})$ *and* $\hat{I}^*(e_k \otimes e_l) = (-1)^{k+l}(e_k \otimes e_{k-l})$.

when $k, l \in \mathbb{Z}$.

We split up the manageability proof of W into three parts. Each of the following Propositions 2.3.26, 2.3.27 and 2.3.28 has individual significance.

The Proposition 2.3.26 says that W strongly commutes with $N \otimes N$.

Proposition 2.3.26 *We have that*

$$W(N \otimes N) = (N \otimes N)W.$$

Proof. Let $t \in \mathbb{R}$. It is easy to check that

$$(N \otimes N)^{it}(ab \otimes cd) = (ab \otimes cd)(N \otimes N)^{it}.$$

This gives that $N \otimes N$ and $ab \otimes cd$ strongly commute.

By using the properties of the functional calculus, we then find

$$(N \otimes N)F_\mu(ab \otimes cd) = F_\mu(ab \otimes cd)(N \otimes N). \tag{2.35}$$

Further, we have that N commutes with both a and c. Hence, we get that $N \otimes N$ strongly commutes with $\chi(a \otimes 1, 1 \otimes c)$. This implies that

$$(N \otimes N)\chi(a \otimes 1, 1 \otimes c) = \chi(a \otimes 1, 1 \otimes c)(N \otimes N). \tag{2.36}$$

Equations (2.35) and (2.36) together yield that W and $N \otimes N$ commute. ∎

The next Proposition 2.3.27 gives another commutation property.

Proposition 2.3.27 *We have that*

$$(\hat{I} \otimes I)W^*(\hat{I}^* \otimes I) = (\hat{I}^* \otimes I)W^*(\hat{I} \otimes I) = W.$$

Proof. Let $k, l, m, n \in \mathbb{Z}$. By using the formulas in Proposition 2.3.13, it follows from a straightforward calculation that we have

$$\begin{aligned}
&(\hat{I} \otimes I)W^*(\hat{I}^* \otimes I)(e_k \otimes e_l \otimes e_m \otimes e_n) \\
&\qquad = \sum_{t \in \mathbb{Z}} B(t, k-l-n+1)(e_k \otimes e_{l+t} \otimes e_{m-k+2l+2t} \otimes e_{n-k+2l+t}) \\
&\qquad = W(e_k \otimes e_l \otimes e_m \otimes e_n).
\end{aligned}$$

This proves that

$$(\hat{I} \otimes I)W^*(\hat{I}^* \otimes I) = W.$$

The other formula can be proved in a completely similar way. ∎

We then also have the Proposition 2.3.28 below.

Proposition 2.3.28 *Let $x, z \in H$ and $u \in D(N^{\frac{1}{2}})$, $y \in D(N^{-\frac{1}{2}})$. Then, we have that*

$$\langle W(z \otimes y), x \otimes u \rangle = \langle W^*(z \otimes IN^{\frac{1}{2}}u), x \otimes IN^{-\frac{1}{2}}y \rangle.$$

Proof. We have that H_0 is a core for both $N^{\frac{1}{2}}$ and $N^{-\frac{1}{2}}$. Hence, it is sufficient to check the above formula for basis vectors.

Take $k, l, m, n, r, s, x, y \in \mathbb{Z}$. Applying the Proposition 2.3.13, one can find by a straightforward calculation that

$$\begin{aligned}
&\langle W(e_r \otimes e_s \otimes e_m \otimes e_n), e_k \otimes e_l \otimes e_x \otimes e_y \rangle \\
&\qquad = B(l-s, k-n-s+1)\delta_{k,r}\delta_{k-2l-m,x}\delta_{k-l-n,s-y}
\end{aligned} \tag{2.37}$$

and also that

$$\begin{aligned}
&\langle W^*\big(e_r \otimes e_s \otimes IN^{\frac{1}{2}}(e_x \otimes e_y)\big), e_k \otimes e_l \otimes IN^{-\frac{1}{2}}(e_m \otimes e_n) \rangle \\
&\qquad = (-\mu)^{l-s} B(s-l, k-l-n+1)\delta_{k,r}\delta_{k-2l-m,x}\delta_{k-l-n,s-y}.
\end{aligned} \tag{2.38}$$

Proposition 2.2.6 gives that

$$B(l-s, k-n-s+1) = (-\mu)^{l-s} B(s-l, k-l-n+1).$$

The Expressions (2.37) and (2.38) are thus equal. ■

We then arrive at the Proposition 2.3.29 below.

Proposition 2.3.29 *The quintuple $(W, N^{\frac{1}{2}}, N^{\frac{1}{2}}, I, \hat{I})$ satisfies Definition 1.3.57.*

Proof. This follows from the three Propositions 2.3.26, 2.3.27 and 2.3.28. ■

The next Theorem 2.3.30 rounds out the manageability study of W. It more or less sums up the main results in this Section 2.3.

It is clear that the Theorem 2.3.30 just picks out the most relevant information of Proposition 2.3.29. So, it does not really impart a new result.

This Theorem 2.3.30 is a fundamental in the Chapter 2.

Theorem 2.3.30 *The multiplicative unitary W is strongly modular.*

We recall that the fact that W is strongly modular makes it possible to apply the modularity theory as expounded in the Sections 1.3 and 1.4. This of course expedites the search for more properties of W and its related objects.

By combining the Proposition 1.3.59 and the (proof of) Theorem 2.3.30, we get that the Corollary 2.3.31 below holds.

Corollary 2.3.31 *We have that W is manageable.*

By using the Corollary 1.3.35, it is relatively easy to prove that the multiplicative unitary W is semi-regular, but not regular.

The Proposition 2.3.33 collects the regularity properties of W.

Definition 2.3.32 *We define an algebra $\mathcal{C}(W)$ by*

$$\mathcal{C}(W) = [(\iota \otimes \omega)(\Sigma W) \mid \omega \in B(H)_*].$$

Proposition 2.3.33 *We have that $\mathcal{C}(W)$ is a C^*-algebra. Moreover, we have the (strict) inclusions $B_0(H) \subsetneq \mathcal{C}(W) \subsetneq B(H)$.*

This means that W is semi-regular, but not regular.

Proof. Take $k, l, m, n \in \mathbb{Z}$. Using Proposition 2.3.13, we can easily calculate the action of the operator $(\iota \otimes \omega_{k,l,m,n})(\Sigma W)$. For all $p, q \in \mathbb{Z}$, we have

$$(\iota \otimes \omega_{k,l,m,n})(\Sigma W)(e_p \otimes e_q) = B(l-m+n, l-m+q+1)\delta_{p,m}(e_{k-m+2n} \otimes e_{l-m+n+q}).$$

By applying Proposition A.9, it is straightforward to deduce from this formula that $(\iota \otimes \omega_{k,l,m,n})(\Sigma W) \in B_0(H)$ if and only if $m \neq l+n$.

We define an operator $T_{k,l,m,n}$ by

$$T_{k,l,m,n} = (\iota \otimes \omega_{k-2l+m+2n-2,m+n-1,m,l-n+1})(\Sigma W).$$

It is easy to see that

$$\{T_{k,l,m,n} \mid k,l,m,n \in \mathbb{Z}\} = \{(\iota \otimes \omega_{k,l,m,n})(\Sigma W) \mid k,l,m,n \in \mathbb{Z}\}.$$

The action of the operator $T_{k,l,m,n}$ is given by

$$T_{k,l,m,n}(e_p \otimes e_q) = B(l, n+q)\delta_{p,m}(e_k \otimes e_{l+q})$$

when $p, q \in \mathbb{Z}$.

It is easy to prove that $T_{k,l,m,n} \in B_0(\ell^2(\mathbb{Z})) \otimes B(\ell^2(\mathbb{Z}))$ for all $k,l,m,n \in \mathbb{Z}$. We can then deduce that we have $\mathcal{C}(W) \subsetneq B(H)$.

The formulas in Proposition 2.2.6 imply that $T^*_{k,l,m,n} = (-\mu)^l T_{m,-l,k,n-l}$. This gives that $\mathcal{C}(W)$ is a C*-algebra.

From Corollary 1.3.35, it ensues that $\mathcal{C}(W)$ acts irreducibly on H. Further, the above results imply that $\mathcal{C}(W) \cap B_0(H) \neq \emptyset$.

The property $B_0(H) \subseteq \mathcal{C}(W)$ now follows from standard operator theory. ∎

2.4 The definition of the quantum $E(2)$ group

This central Section 2.4 starts up the operator algebra level. Its main purpose is to fix the data of the quantum $E(2)$ group (A, Φ) and its dual.

From Theorem 2.3.30, we know that the multiplicative unitary W is (strongly) modular. As it is explained in Section 1.3, this result provides us with a quick way of defining two bi-C*-algebras via W; see Theorem 1.3.9.

We apply this method of working to define the quantum $E(2)$ group (A, Φ) on the C*-algebra level. This is done in the Definition 2.4.1 below.

It is clear that Theorems 2.4.3 and 2.4.4 form the core of Chapter 2. We repeat that they can also be obtained from the results of S. Baaj in [1].

In the Sections 2.5 to 2.8, we give a detailed study of all the aspects of the quantum $E(2)$ group. We show in Section 2.5 that Definition 2.4.1 below defines the same bi-C*-algebra (A, Φ) as the one defined by Woronowicz in [159].

Thanks to Theorem 1.3.9, we can give the next Definition 2.4.1.

This Definition 2.4.1 defines the *quantum* $E(2)$ *group.*

Definition 2.4.1 *We define a bi-C*-algebra* (A, Φ) *by setting*

- $A = [\,(\omega \otimes \iota)W \mid \omega \in B(H)_*]$,
- $\Phi : A \to M(A \otimes A) : x \mapsto W(x \otimes 1)W^*$.

We call (A, Φ) *the* quantum $E(2)$ group.

We also define a second bi-C*-algebra $(\hat{A}, \hat{\Phi})$. Again, the Definition 2.4.2 is justified by Theorem 1.3.9 (applied to $\Sigma W^*\Sigma$ instead of W).

Notice that we define $\hat{\Phi}$ in a slightly different manner. In contrast with Theorem 1.3.9 (and the general theory), we here take $\hat{\Phi}$ not flipped.

The Definition 2.4.2 is dual to Definition 2.4.1.

Definition 2.4.2 *We define a bi-C*-algebra* $(\hat{A}, \hat{\Phi})$ *by setting*

- $\hat{A} = [\,(\iota \otimes \omega)W \mid \omega \in B(H)_*]$,
- $\hat{\Phi} : \hat{A} \to M(\hat{A} \otimes \hat{A}) : y \mapsto W^*(1 \otimes y)W$.

We call $(\hat{A}, \hat{\Phi})$ *the* quantum $\hat{E}(2)$ group.

It is now a quite simple task to prove that (A, Φ) and $(\hat{A}, \hat{\Phi})$ are indeed locally compact quantum groups in the sense of Kustermans and Vaes. The technique expounded in Section 1.4 does all the hard work.

In the proofs of Theorems 2.4.3 and 2.4.4 below, we invoke on properties proved later in Section 2.7. For this reason, we can be very concise.

The Theorem 2.4.3 is about the quantum $E(2)$ group.

Theorem 2.4.3 *The pair* (A, Φ) *is a locally compact quantum group.*

Proof. By Definition 2.4.1, we have that (A, Φ) is a bi-C*-algebra. It follows from Theorem 1.3.9 that

$$A \otimes A = [\Phi(A)(A \otimes 1)] = [\Phi(A)(1 \otimes A)].$$

Theorem 2.7.9 gives that there exists a faithful KMS-weight ψ on (A, Φ) that is strongly left and strongly right invariant. ∎

The Theorem 2.4.4 states the dual result.

Theorem 2.4.4 *The pair* $(\hat{A}, \hat{\Phi})$ *is a locally compact quantum group.*

Proof. By Definition 2.4.2, we have that $(\hat{A}, \hat{\Phi})$ is a bi-C*-algebra. It follows from Theorem 1.3.9 that

$$\hat{A} \otimes \hat{A} = [\hat{\Phi}(\hat{A})(\hat{A} \otimes 1)] = [\hat{\Phi}(\hat{A})(1 \otimes \hat{A})].$$

From Theorems 2.7.22 and 2.7.23, we find that $(\hat{A}, \hat{\Phi})$ possesses a left Haar weight $\hat{\varphi}$ and a right Haar weight $\hat{\psi}$. ∎

2.5 The C*-algebras and the comultiplications

In this Section 2.5, we investigate the bi-C*-algebras (A, Φ) and $(\hat{A}, \hat{\Phi})$. We give different characterizations of the C*-algebras A and $\hat{A}$. Further, we prove several formulas that describe how Φ and $\hat{\Phi}$ act on basic elements.

The techniques in this Section 2.5 are inspired by the methods that Woronowicz uses in his recent studies of quantum group examples [165, 169]. It need not be said that Woronowicz' older results about the quantum $E(2)$ group [159] serve even more as an important source of inspiration.

The Section 2.5 is divided into two parts. First, we consider the quantum $E(2)$ group. Secondly, the quantum $\hat{E}(2)$ group is considered.

We remark that also the Sections 2.6 and 2.7 are built up in this way. This is done in order not to skip from one subject to another.

The quantum $E(2)$ group

We first consider the C*-algebra A. We provide a description of A in terms of the generators c, d and we prove that A is generated by $c, d \,\eta\, A$. Further, we also portray how to write A as a crossed product.

The next Proposition 2.5.1 is a consequence of Theorem 1.3.9.

Proposition 2.5.1 *The two properties below hold:*

1. *We have that A is a (separable) non-degenerate C*-algebra in $B(H)$,*
2. *We have $W \in M(B_0(H) \otimes A)$.*

As a first task, we prove that we can find the C*-algebra A by taking appropriate functions of the generating operators c and d. We write everything in a form that is suitable for further reasoning and calculations.

The following Theorem 2.5.2 is important. It says that we actually study the same C*-algebra as Woronowicz does so in [159].

We have to be quite diligent in order to give a rigorous proof.

Theorem 2.5.2 *We have that*

$$A = \Big[\sum_{k \in \mathbb{Z}} c^k f_k(d) \;\Big|\; f_k \in \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \text{ for every } k \text{ and } f_k \neq 0 \text{ for finitely many } k \Big].$$

Thus, we get that A is a non-unital C-algebra.*

Proof. We define B as

$$B = \Big[\sum_{k \in \mathbb{Z}} c^k f_k(d) \;\Big|\; f_k \in \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \text{ for every } k \text{ and } f_k \neq 0 \text{ for finitely many } k \Big].$$

Hence, we have to proof that $A = B$.

Because of the commutation relations in Proposition 2.3.6, we get that B is a (separable) C^*-algebra. It is clear that B acts non-degenerately on H.

The definition of B infers that $c \in M(B)$ and $d \,\eta\, B$. Further, c and d separate non-degenerate *-representations of B. It is easy to check that $(1 + d^*d)^{-1} \in A$. Now, the Proposition 1.5.54 gives that B is generated by $c, d \,\eta\, B$.

Since c is a unitary, we get $c, cd \,\eta\, B$. We know that a and ab are closed operators acting on H. Thus $a, ab \,\eta\, B_0(H)$ (cf. Example 1.5.46). Using Proposition 1.5.36, we then get that the normal operators $a \otimes 1, 1 \otimes c$ and $ab \otimes cd$ are affiliated to $B_0(H) \otimes B$. From this, we find that $W \in M(B_0(H) \otimes B)$.

Thus, we get that $(\omega \otimes \iota)W \in M(B)$ for all $\omega \in B(H)_*$. This then gives that A is a C^*-subalgebra of $M(B)$. From this, we deduce that $AB \subseteq B$.

The above results give that W is a unitary in $M(B_0(H) \otimes B)$. This implies that we have $W(B_0(H) \otimes A) = B_0(H) \otimes A$.

We get that the set

$$\{W(m \otimes y) \mid m \in B_0(H), y \in B\} \tag{2.39}$$

is linearly dense in $B_0(H) \otimes B$.

Now, let $\omega \in B(H)_*$, $m \in B_0(H)$ and $y \in B$. We have $m\omega \in B(H)_*$. It is clear that

$$(\omega \otimes \iota)\big(W(m \otimes y)\big) = \big((m\omega \otimes \iota)W\big)y.$$

We hence get that $(\omega \otimes \iota)\big(W(m \otimes y)\big)$ belongs to AB.

Take $\omega \in B(H)_*$ with $\omega \neq 0$. Applying the slice map $\omega \otimes \iota$ to the elements in Equation (2.39), we see that

$$\big\{(\omega \otimes \iota)\big(W(m \otimes y)\big) \mid m \in B_0(H), y \in B\big\}$$

is linearly dense in $(\omega \otimes \iota)\big(B_0(H) \otimes B\big) = B$.

In particular, we can conclude that AB is dense in B.

We proceed by proving that $c \in M(A)$ and $d \,\eta\, A$.

Proposition 2.5.1 implies that $W \in M(B_0(H) \otimes A)$. From Lemma 2.3.15, we know that

$$1 \otimes c = (c^* \otimes 1)W(c \otimes 1)W^*.$$

Thus, we see that $1 \otimes c \in M(B_0(H) \otimes A)$ and hence $c \in M(A)$.

Lemma 2.3.8 gives that $ab \otimes cd$ is a non-singular, normal operator. We now use Proposition 2.2.14 to prove that $(ab \otimes cd)\,\eta\, B_0(H) \otimes A$.

For this, we anew consider the operators $V(\gamma)$ as introduced in Definition 2.3.20. It is clear that $V(\gamma) \in B(H \otimes H) = M(B_0(H) \otimes B_0(H))$ for every $\gamma \in \bar{\mathbb{C}}^\mu$.

The Proposition 2.2.14 gives that the mapping

$$\bar{\mathbb{C}}^\mu \to M(B_0(H) \otimes B_0(H)) : \gamma \mapsto V(\gamma)$$

is strictly continuous.

Hence, also the mapping

$$\bar{\mathbb{C}}^\mu \to M(B_0(H) \otimes B_0(H) \otimes A) : \gamma \mapsto V_{12}(\gamma) \tag{2.40}$$

is strictly continuous.

From Proposition 2.3.21, we know that the unitary operators $V(\gamma)$ are adapted to W. This implies that we have

$$V_{13}(\gamma) = V_{12}(\gamma)^* W_{23} V_{12}(\gamma) W_{23}^*$$

for every $\gamma \in \bar{\mathbb{C}}^\mu$. Using this formula and remembering that $W \in M(B_0(H) \otimes A)$, we find from Equation (2.40) that the mapping

$$\bar{\mathbb{C}}^\mu \to M(B_0(H) \otimes B_0(H) \otimes A) : \gamma \mapsto V_{13}(\gamma)$$

is strictly continuous.

This implies that also the mapping

$$\bar{\mathbb{C}}^\mu \to M(B_0(H) \otimes A) : \gamma \mapsto V(\gamma)V(0)^* \tag{2.41}$$

is strictly continuous.

For every $\gamma \in \bar{\mathbb{C}}^\mu$, we have

$$V(\gamma)V(0)^* = F_\mu(\gamma ab \otimes cd)\chi(a \otimes 1, 1 \otimes c)\chi(a \otimes 1, 1 \otimes c)^* = F_\mu(\gamma ab \otimes cd).$$

Thus, Equation (2.41) precisely means that the mapping

$$\bar{\mathbb{C}}^\mu \to M(B_0(H) \otimes A) : \gamma \mapsto F_\mu(\gamma ab \otimes cd)$$

is strictly continuous.

An application of Proposition 2.2.14 gives that

$$(ab \otimes cd)\,\eta\, B_0(H) \otimes A. \tag{2.42}$$

We define a unitary operator $r \in M(B_0(H))$ by $r = 1 \otimes s^*$. Also, we define a strictly positive operator n on H by $n = 1 \otimes \mu m$. We have $n\,\eta\, B_0(H)$.

From Equation (2.42), we find that $(r \otimes c^*)(ab \otimes cd)\,\eta\, B_0(H) \otimes A$. This exactly means that

$$(n \otimes d)\,\eta\, B_0(H) \otimes A.$$

Proposition 1.5.37 now implies that $d\,\eta\, A$.

We then look at the embedding

$$j : B \to B(H).$$

The above results give that

$$j(c) \in M(A) \qquad \text{and} \qquad j(d)\,\eta\,A.$$

We know from the beginning of the proof that the C*-algebra B is generated by $c, d\,\eta\,B$. Thus, we have that j is a non-degenerate *-homomorphism from B to $M(A)$. We find that BA is dense in A.

If we combine the arguments above, we get that AB is dense in B and that BA is dense in A. These results together yield that $A = B$. ∎

The Theorem 2.5.3 states the property that A is generated by the pair (c, d). One can infer this result from the proof of Theorem 2.5.2.

This Theorem 2.5.3 is very useful, especially to prove uniqueness results.

Theorem 2.5.3 *We have $c \in M(A)$ and $d\,\eta\,A$. The operators $c, d\,\eta\,A$ separate non-degenerate *-representations of A.*

We have that the C-algebra A is generated by $c, d\,\eta\,A$.*

The next Lemma 2.5.4 follows directly from Theorem 2.5.2.

Lemma 2.5.4 *We have that*

$$M(A) \supseteq \Big[\sum_{k\in\mathbb{Z}} c^k f_k(d) \,\big|\, f_k \in \mathrm{C}_b(\bar{\mathbb{C}}^\mu) \textit{ for every } k \textit{ and } f_k \neq 0 \textit{ for finitely many } k\Big],$$
$$A^\eta \supseteq \big\{c^k f(d) \,|\, k \in \mathbb{Z}, f \in \mathrm{C}(\bar{\mathbb{C}}^\mu)\big\}.$$

The Proposition 2.5.7 then displays how A can be written as a crossed product. We refer to the Section 4.1 for proofs of the results below.

Definition 2.5.5 *We define an action $\alpha : \mathbb{Z} \to \mathrm{Aut}(\mathrm{C}_0(\bar{\mathbb{C}}^\mu))$ by*

$$\alpha_n : \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \to \mathrm{C}_0(\bar{\mathbb{C}}^\mu) : f \mapsto f(\mu^n \cdot).$$

Lemma 2.5.6 *We have that $(\mathrm{C}_0(\bar{\mathbb{C}}^\mu), \mathbb{Z}, \alpha)$ is a C*-dynamical system.*

Proposition 2.5.7 *We have $A \cong \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \ltimes_\alpha \mathbb{Z}$.*

This Proposition 2.5.7 has the next useful consequence.

Corollary 2.5.8 *We have that A is a type I C*-algebra and is thus nuclear.*

We recall the standard properties of the comultiplication Φ on A. They follow from the general manageability theory; see Section 1.3.

The three Propositions 2.5.9, 2.5.10 and 2.5.11 follow from a direct application of Theorem 1.3.9 and Proposition 1.3.18.

Proposition 2.5.9 *We can consider $\Phi : A \to M(A \otimes A)$ as a non-degenerate *-homomorphism. Moreover, the results below hold:*

1. *We have that Φ is* coassociative, *i.e., $(\Phi \otimes \iota)\Phi = (\iota \otimes \Phi)\Phi$,*
2. *The linear spaces $\Phi(A)(1 \otimes A)$ and $\Phi(A)(A \otimes 1)$ are dense in $A \otimes A$.*

Proposition 2.5.10 *We have that Φ is characterized as the only non-degenerate *-homomorphism from A to $M(A \otimes A)$ such that*

$$(\iota \otimes \Phi)W = W_{12}W_{13}.$$

Proposition 2.5.11 *Let $\omega \in B(H)_*$. We then have*

$$\Phi\big((\omega \otimes \iota)W\big) = (\omega \otimes \iota \otimes \iota)(W_{12}W_{13}).$$

We calculate the action of Φ on the generating elements. The results found in the Proposition 2.5.12 below completely fit in with Proposition 2.1.1.

It is important to remark that the formulas in the Equation (2.43) are used by S.L. Woronowicz in [159] as the defining relations for the comultiplication Φ on A. We refer to Remark 4.2.5 for more explanation about this matter.

The formulas in Proposition 2.5.12 ensue from Lemma 2.3.15. The uniqueness part is a direct consequence of Theorem 2.5.3.

Proposition 2.5.12 *We have that Φ is characterized as the only non-degenerate *-homomorphism from A to $M(A \otimes A)$ such that*

$$\Phi(c) = c \otimes c \qquad \textit{and} \qquad \Phi(d) = c \otimes d \dotplus d \otimes c^*. \tag{2.43}$$

Furthermore, we have

$$\Phi(c^*) = c^* \otimes c^*, \qquad \Phi(d^*) = c^* \otimes d^* \dotplus d^* \otimes c,$$
$$\Phi(cd) = c^2 \otimes cd \dotplus cd \otimes 1.$$

The next Remark 2.5.13 is just a detail.

Remark 2.5.13 *The existence of a comultiplication Φ satisfying the formulas in Equation (2.43) is mainly due to the spectral condition $\sigma(d) \subseteq \bar{\mathbb{C}}^\mu$.*

The role of this condition is explained in more detail in Remark 4.2.6.

From Proposition 2.5.12, we find how Φ acts on basic elements.

Proposition 2.5.14 *Let* $\sum c^k f_k(d) \in A$. *Then, we have*

$$\Phi\big(\sum c^k f_k(d)\big) = \sum (c^k \otimes c^k) f_k(c \otimes d \dotplus d \otimes c^*).$$

We now study the *semigroup behavior* of the quantum $E(2)$ group. This quite uncanny property was discovered by Woronowicz in [160].

The result in Proposition 2.5.15 below is both intriguing and whimsical.

Proposition 2.5.15 ([160]) *We have that*

$$\Phi(A) \subseteq A \otimes A.$$

Proof. We again consider the functions $f_{k,n} \in \mathrm{C}_0(\bar{\mathbb{C}}^\mu)$ as defined in the proof of Proposition 2.2.14. Recall that, for $z \in \bar{\mathbb{C}}^\mu$, we have

$$f_{k,n}(z) = \frac{1}{2\pi} \int_0^{2\pi} F_\mu(\mu^n e^{it} z) e^{-ikt}\, dt. \tag{2.44}$$

Let $k, l \in \mathbb{Z}$. It is easy to check that

$$f_{k,n}(e^{i\theta} x) = e^{ik\theta} f_{k,n}(x) \tag{2.45}$$

for all $\theta \in [0, 2\pi]$ and $x \in \bar{\mathbb{C}}^\mu$.

Let $k, l, m, n \in \mathbb{Z}$. We have that $f_{0,0}(d)(e_k \otimes e_l) = B(0, -l)(e_k \otimes e_l)$. From the formulas in Proposition 2.3.13, it then follows that

$$\begin{aligned}
\Phi\big(f_{0,0}(d)\big)(e_k \otimes e_l \otimes e_m \otimes e_n) &= W(f_{0,0}(d) \otimes 1)W^*(e_k \otimes e_l \otimes e_m \otimes e_n) \\
&= \sum_{t\in\mathbb{Z}} \sum_{s\in\mathbb{Z}} B(t, l-n+1) B(0, -l+t) B(s+t, l-n+1)(e_k \otimes e_{l+s} \otimes e_{m+2s} \otimes e_{n+s}) \\
&= \sum_{s\in\mathbb{Z}} \Big(\sum_{t\in\mathbb{Z}} B(t, l-n+1) B(0, -l+t) B(s+t, l-n+1)\Big)(e_k \otimes e_{l+s} \otimes e_{m+2s} \otimes e_{n+s}).
\end{aligned}$$

Using Lemma A.14, we see that the above calculation implies

$$\begin{aligned}
&\Phi\big(f_{0,0}(d)\big)(e_k \otimes e_l \otimes e_m \otimes e_n) \\
&\qquad = \sum_{s\in\mathbb{Z}} B(-s, -l-s) B(s, -n)(e_k \otimes e_{l+s} \otimes e_{m+2s} \otimes e_{n+s}).
\end{aligned} \tag{2.46}$$

Now, take $x \in \mathbb{C}^\mu$ and write $x = \mu^p e^{i\varphi}$ with $p \in \mathbb{Z}$ and $\varphi \in [0, 2\pi[$. Let $\hat{F}_p$ as introduced in Definition A.5. We consider the analytic function

$$h_x : \{z \in \mathbb{C} \mid \operatorname{Im} z > \log \mu\} \to \mathbb{C} : \theta \mapsto \hat{F}_p(e^{i(\theta+\varphi)})$$

Also, we define $h_0 = 1$.

Let $x \in \bar{\mathbb{C}}^\mu$. Using the properties of $\hat{F}_p$ (see Lemma A.6), we get that h_x is an analytic extension of the function $\mathbb{R} \to \mathbb{C} : \theta \mapsto F_\mu(e^{i\theta}x)$.

Take $a \in [1, \mu^{-1}[$. Using the Definition A.5 of $\hat{F}_p$, one can prove (with quite some effort) that there exists a constant $M_a \geq 0$ such that, for all $x \in \bar{\mathbb{C}}^\mu$, we have $|h_x(z)| \leq M_a$ for every z on the line $L_a := \{\theta \in \mathbb{C} \mid \operatorname{Im}\theta = -\log a\}$.

Let $q \in \mathbb{Z}$ and $z \in \bar{\mathbb{C}}^\mu$. It is clear that $|f_{q,0}(z)| \leq 1$. Shifting the integration contour in (2.44) down in the complex plane, we get the inequality

$$|f_{q,0}(z)| \leq M_a a^{-q}.$$

Hence, for every $a \in [1, \mu^{-1}[$ and $q \in \mathbb{Z}$, we have

$$\left\| f_{-q,0}(d)c^q \otimes c^q f_{q,0}(d) \right\| \leq M_a a^{-|q|}.$$

This clearly implies that the series

$$\sum_{q \in \mathbb{Z}} f_{-q,0}(d)c^q \otimes c^q f_{q,0}(d)$$

is norm-converging.

Using the Equality (2.45), it is easy to calculate that

$$\begin{aligned}\Big(\sum_{q\in\mathbb{Z}} f_{-q,0}(d)c^q \otimes c^q f_{q,0}(d)\Big)&(e_k \otimes e_l \otimes e_m \otimes e_n) \\ &= \sum_{q\in\mathbb{Z}} B(-q,-l-q)B(q,-n)(e_k \otimes e_{l+q} \otimes e_{m+2q} \otimes e_{n+q}).\end{aligned} \tag{2.47}$$

Equations (2.46) and (2.47) together give that

$$\Phi\big(f_{0,0}(d)\big) = \sum_{q\in\mathbb{Z}} f_{-q,0}(d)c^q \otimes c^q f_{q,0}(d). \tag{2.48}$$

For all $q, r \in \mathbb{Z}$, we have $f_{q,r}(d) = f_{q,0}(\mu^r d) = c^{-r} f_{q,0}(d) c^r$. Thus, we can deduce from Equation (2.48) that

$$\Phi\big(f_{0,r}(d)\big) = \sum_{q\in\mathbb{Z}} f_{-q,r}(d)c^q \otimes c^q f_{q,r}(d) \tag{2.49}$$

where the sum on the right hand side is norm-converging.

From Equation (2.49), it follows that $\Phi\big(f_{0,r}(d)\big) \in A \otimes A$ for all $r \in \mathbb{Z}$.

Further, we have that $\big(f_{0,r}(d)\big)$ is an approximate unit in A. This property is shown in the proof of Proposition 2.2.14.

Let $a \in A$. We have $af_{0,r}(d) \to a$ and hence $\Phi(af_{0,r}(d)) \to \Phi(a)$ if $r \to +\infty$. The above results imply that $\Phi(af_{0,r}(d)) \in A \otimes A$ for every $r \in \mathbb{Z}$.

We conclude that $\Phi(a) \in A \otimes A$. ■

It is thought-out to use the term 'semigroup behavior'.

Remark 2.5.16 *The reason to call the property in Proposition 2.5.15* semigroup behavior *is to be found in Remark 1.2.3.*

Let G be a locally compact group that is non-compact. We consider the quantum group $(\mathrm{C}_0(G), \Phi)$ defined in Example 1.2.1. Then, we have

$$\Phi\big(\mathrm{C}_0(G)\big) \cup \big(\mathrm{C}_0(G) \otimes \mathrm{C}_0(G)\big) = \{0\}.$$

This property also holds for other classes of non-compact quantum groups.

On the other hand, for a lot of non-compact semigroups H (that are not groups) we have $\Phi\big(\mathrm{C}_0(H)\big) \subseteq \mathrm{C}_0(H) \otimes \mathrm{C}_0(H)$. This is e.g. the case if $H = \mathbb{N}$.

The quantum $\hat{E}(2)$ group

Now, we come to the quantum $\hat{E}(2)$ group. We deduce similar properties as for the quantum $E(2)$ group. However, it has to be said that the structure in this dual case is more difficult. We hence have to make some extra effort.

We first look at the C*-algebra $\hat{A}$.

The two Propositions 2.5.17 and 2.5.18 ensue from Theorem 1.3.9.

Proposition 2.5.17 *The two properties below hold:*

1. *We have that $\hat{A}$ is a (separable) non-degenerate C*-algebra in $B(H)$,*
2. *We have $W \in M(\hat{A} \otimes B_0(H))$.*

Proposition 2.5.18 *We have $W \in M(\hat{A} \otimes A)$.*

We prove in the Theorem 2.5.21 below that we can find the C*-algebra $\hat{A}$ by taking appropriate functions of the generating operators a and b. First, we discuss the joint functional calculus of the pair $(a, |b|)$.

We define a subset $E \subseteq \mathbb{R}(\mu^{\frac{1}{2}}) \times \bar{\mathbb{R}}(\mu^{\frac{1}{2}})$.

Notation 2.5.19 *We denote $E = \big\{(p, q) \in \mathbb{R}(\mu^{\frac{1}{2}}) \times \bar{\mathbb{R}}(\mu^{\frac{1}{2}}) \mid pq \in \bar{\mathbb{R}}^{\mu}\big\}$.*

The set E is introduced for technical reasons.

Remark 2.5.20 *The Proposition 2.3.6 gives that a and $|b|$ strongly commute. We can thus consider the joint functional calculus of the pair $(a, |b|)$.*

For $g \in \mathrm{C}\big(\mathbb{R}(\mu^{\frac{1}{2}}) \times \bar{\mathbb{R}}(\mu^{\frac{1}{2}})\big)$, it is shown in [59] how to define $g(a, |b|)$.

By some easy calculations, it can be shown that the element $g(a, |b|)$ is completely determined by the restriction of g to E.

For the rest of the thesis, we then always consider elements $g(a,|b|)$ *for* $g \in \mathrm{C}(E)$. *The joint functional calculus of* $(a,|b|)$ *is injective on* $\mathrm{C}(E)$.

It is sometimes more convenient to fall back on the joint functional calculus of $(a,|b|)$ *on* $C\big(\mathbb{R}(\mu^{\frac{1}{2}}) \times \bar{\mathbb{R}}(\mu^{\frac{1}{2}})\big)$. *In particular, we advise the reader to use this latter one when checking computations. This probably is more handy.*

The next Theorem 2.5.21 provides a possible description of $\hat{A}$ in terms of the operators a and b. Its proof is based on Propositions 2.2.14 and 2.2.15. We recall that $b = u|b|$ is the polar decomposition of the normal operator b.

We notice that Woronowicz in [159] describes $\hat{A}$ in a (slightly) different way. The two approaches are of course equivalent. Also, see Remark 2.5.25.

We refer to Remark 2.5.22 for more information about the restriction on the elements $g_k(a,|b|)$ that appears in the description of $\hat{A}$ given below.

Theorem 2.5.21 *We have that*

$$\hat{A} = \Big[\sum_{k\in\mathbb{Z}} u^k g_k(a,|b|) \;\Big|\; g_k \in \mathrm{C}_0(E) \textit{ for every } k,\ g_k \neq 0 \textit{ for finitely many } k \\ \textit{and } g_k(s,0) = 0 \textit{ for all } s \in \mathbb{R}(\mu^{\frac{1}{2}}) \textit{ if } k \neq 0\Big].$$

Thus, we get that $\hat{A}$ *is a non-unital C*-algebra.*

Proof. We define $\hat{B}$ as

$$\hat{B} = \Big[\sum_{k\in\mathbb{Z}} u^k g_k(a,|b|) \;\Big|\; g_k \in \mathrm{C}_0(E) \text{ for every } k,\ g_k \neq 0 \text{ for finitely many } k \\ \text{and } g_k(s,0) = 0 \text{ for all } s \in \mathbb{R}(\mu^{\frac{1}{2}}) \text{ if } k \neq 0\Big].$$

Hence, we have to proof that $\hat{A} = \hat{B}$.

Thanks to the commutation relations in Proposition 2.3.6, we have that $\hat{B}$ is a (separable) C*-algebra. It is clear that $\hat{B}$ acts non-degenerately on H.

The definition of $\hat{B}$ implies that $a, a^{-1}, b\,\eta\,\hat{B}$. Further, the operators a and b separate non-degenerate *-representations of $\hat{B}$. It is easy to check that we have $(1+a^2)^{-1}(1+a^{-2})^{-1}(1+b^*b)^{-1} \in \hat{B}$. Now, Proposition 1.5.54 gives that $\hat{B}$ is generated by the affiliated elements $a, a^{-1}, b\,\eta\,\hat{B}$.

Because $a, |b|\,\eta\,\hat{B}$ strongly commute, it follows from Lemma B.3.19 that we have $a|b|\,\eta\,\hat{B}$. Using this property, one can easily prove that $ab\,\eta\,\hat{B}$.

We thus have $a, ab\,\eta\,\hat{A}$. We know that c and cd are closed operators acting on H. Thus $c, cd\,\eta\,B_0(H)$ (cf. Example 1.5.46). Using Proposition 1.5.36, we get that the normal operators $a \otimes 1, 1 \otimes c$ and $ab \otimes cd$ are affiliated to $\hat{B} \otimes B_0(H)$. From this, we find that $W \in M(\hat{B} \otimes B_0(H))$.

Thus, we get that $(\iota \otimes \omega)W \in M(\hat{B})$ for all $\omega \in B(H)_*$. This then gives that $\hat{A}$ is a C*-subalgebra of $M(\hat{B})$. From this, we deduce that $\hat{A}\hat{B} \subseteq \hat{B}$.

The above results give that W is a unitary in $M(\hat{B} \otimes B_0(H))$. This implies that we have $W(\hat{B} \otimes B_0(H)) = \hat{B} \otimes B_0(H)$. We get that the set

$$\{W(y \otimes m) \mid m \in B_0(H), y \in \hat{B}\} \tag{2.50}$$

is linearly dense in $\hat{B} \otimes B_0(H)$.

Now, let $\omega \in B(H)_*$, $m \in B_0(H)$ and $y \in \hat{B}$. We have $m\omega \in B(H)_*$. It is clear that

$$(\iota \otimes \omega)\big(W(y \otimes m)\big) = \big((\iota \otimes m\omega)W\big)y.$$

We hence get that $(\iota \otimes \omega)\big(W(y \otimes m)\big)$ belongs to $\hat{A}\hat{B}$.

Take $\omega \in B(H)_*$ with $\omega \neq 0$. Applying the slice map $\iota \otimes \omega$ to the elements in Equation (2.50), we see that

$$\big\{(\iota \otimes \omega)\big(W(y \otimes m)\big) \mid m \in B_0(H), y \in \hat{B}\big\}$$

is linearly dense in $(\iota \otimes \omega)\big(\hat{B} \otimes B_0(H)\big) = \hat{B}$.

In particular, we can conclude that $\hat{A}\hat{B}$ is dense in $\hat{B}$.

We proceed by proving that $a, a^{-1}, b \,\eta\, M(\hat{A})$.

Lemma 2.3.8 gives that $ab \otimes cd$ is a non-singular, normal operator. We now use Proposition 2.2.14 to prove that $(ab \otimes cd) \,\eta\, \hat{A} \otimes B_0(H)$.

For this, we anew consider the operators $V(\gamma)$ as introduced in Definition 2.3.20. It is clear that $V(\gamma) \in B(H \otimes H) = M(B_0(H) \otimes B_0(H))$ for every $\gamma \in \bar{\mathbb{C}}^\mu$.

Proposition 2.2.14 gives that the mapping

$$\bar{\mathbb{C}}^\mu \to M(B_0(H) \otimes B_0(H)) : \gamma \mapsto V(\gamma)$$

is strictly continuous.

Hence, also the mapping

$$\bar{\mathbb{C}}^\mu \to M(\hat{A} \otimes B_0(H) \otimes B_0(H)) : \gamma \mapsto V_{23}(\gamma) \tag{2.51}$$

is strictly continuous.

From Proposition 2.3.21, we know that the unitary operators $V(\gamma)$ are adapted to W. Let $\gamma \in \bar{\mathbb{C}}^\mu$. If we write $\hat{W} = \Sigma W^* \Sigma$ and $\hat{V}(\gamma) = \Sigma V(\gamma) \Sigma$, we can then infer that we have

$$\hat{V}_{23}(\gamma)\hat{W}_{12} = \hat{W}_{12}\hat{V}_{13}(\gamma)\hat{V}_{23}(\gamma). \tag{2.52}$$

This is a standard result; see [4].

From the Equation (2.52), we get that

$$V_{13}(\gamma) = W_{12}V_{23}(\gamma)V_{23}(\gamma)^* W_{12}^*$$

for every $\gamma \in \bar{\mathbb{C}}^\mu$. Using this formula and remembering that $W \in M(\hat{A} \otimes B_0(H))$, we find from Equation (2.51) that the mapping

$$\bar{\mathbb{C}}^\mu \to M(\hat{A} \otimes B_0(H) \otimes B_0(H)) : \gamma \mapsto V_{13}(\gamma) \tag{2.53}$$

is strictly continuous.

This implies that also the mapping

$$\bar{\mathbb{C}}^\mu \to M(\hat{A} \otimes B_0(H)) : \gamma \mapsto V(\gamma)V(0)^* \tag{2.54}$$

is strictly continuous.

For every $\gamma \in \bar{\mathbb{C}}^\mu$, we have

$$V(\gamma)V(0)^* = F_\mu(\gamma ab \otimes cd)\chi(a \otimes 1, 1 \otimes c)\chi(a \otimes 1, 1 \otimes c)^* = F_\mu(\gamma ab \otimes cd).$$

Thus, Equation (2.54) precisely means that the mapping

$$\bar{\mathbb{C}}^\mu \to M(\hat{A} \otimes B_0(H)) : \gamma \mapsto F_\mu(\gamma ab \otimes cd)$$

is strictly continuous.

An application of Proposition 2.2.14 gives that

$$(ab \otimes cd)\, \eta\, \hat{A} \otimes B_0(H). \tag{2.55}$$

Now, we use Proposition 2.2.15 to prove that $a, a^{-1}\, \eta\, \hat{A}$.

The Lemma 2.3.4 gives that a and a^{-1} are strictly positive operators. It is clear that $\chi(a, z) \in B(H) = M(B_0(H))$ for every $z \in S^1$.

Proposition 2.2.15 gives that the mapping

$$S^1 \to M(B_0(H)) : z \mapsto \chi(a, z)$$

is strictly continuous.

Hence, also the mapping

$$S^1 \to M(\hat{A} \otimes B_0(H)) : z \mapsto 1 \otimes \chi(a, z) \tag{2.56}$$

is strictly continuous.

We again shortly denote $\boldsymbol{\chi}$ for $\chi(a \otimes 1, 1 \otimes c)$. Equation (2.53) (with $\gamma = 0$) gives that $\boldsymbol{\chi} \in M(\hat{A} \otimes B_0(H))$. Together with the continuity in Equation (2.56), this then implies that the mapping

$$S^1 \to M(\hat{A} \otimes B_0(H)) : z \mapsto \boldsymbol{\chi}^*\big(1 \otimes \chi(a, z)\big)\boldsymbol{\chi}\big(1 \otimes \chi(a, z)^*\big) \tag{2.57}$$

is strictly continuous.

Let $z \in S^1$. An easy calculation (using Proposition 2.3.13) yields that

$$\boldsymbol{\chi}^*\big(1 \otimes \chi(a, z)\big)\boldsymbol{\chi}\big(1 \otimes \chi(a, z)^*\big) = \chi(a, z) \otimes 1. \tag{2.58}$$

From Equations (2.57) and (2.58) together, it follows that the mapping

$$S^1 \to M(\hat{A} \otimes B_0(H)) : z \mapsto \chi(a, z) \otimes 1$$

is strictly continuous.

We thus have that the mapping

$$S^1 \to M(\hat{A}) : z \mapsto \chi(a, z)$$

is strictly continuous.

An application of Proposition 2.2.15 now yields $a, a^{-1} \,\eta\, \hat{A}$.

Next, we combine several of the above results to prove that $b \,\eta\, \hat{A}$.

We define a unitary operator $\hat{r} \in M(B_0(H))$ by $\hat{r} = s^2 \otimes s$. Also, we define a strictly positive operator $\hat{n}$ on H by $\hat{n} = 1 \otimes m$. We have $\hat{n} \,\eta\, B_0(H)$.

The above results imply that $a^{-1} \,\eta\, \hat{A}$. We thus have $(a^{-1} \otimes \hat{n}) \,\eta\, \hat{A} \otimes B_0(H)$; see Proposition 1.5.36. Equation (2.55) gives that $(ab \otimes cd) \,\eta\, \hat{A} \otimes B_0(H)$.

It is not difficult to check that $a^{-1} \otimes \hat{n}$ and $ab \otimes cd$ strongly commute. We hence find $(a^{-1} \otimes \hat{n})(ab \otimes cd) \,\eta\, \hat{A} \otimes B_0(H)$. This precisely means that

$$(b \otimes \mu\hat{r}) \,\eta\, \hat{A} \otimes B_0(H).$$

Proposition 1.5.37 now implies that $b \,\eta\, \hat{B}$.

We then look at the embedding

$$j : \hat{B} \to B(H).$$

The above results give that

$$j(a), j(a^{-1}), j(b) \,\eta\, \hat{A}.$$

We know from the beginning of the proof that the C*-algebra $\hat{B}$ is generated by $a, a^{-1}, b \,\eta\, \hat{B}$. Thus, we have that j is a non-degenerate *-homomorphism from $\hat{B}$ to $M(\hat{A})$. We find that $\hat{B}\hat{A}$ is dense in $\hat{A}$.

If we combine the arguments above, we get that $\hat{A}\hat{B}$ is dense in $\hat{B}$ and that $\hat{B}\hat{A}$ is dense in $\hat{A}$. These results together give that $\hat{A} = \hat{B}$. ∎

The next Remark 2.5.22 alludes to Theorem 2.5.21.

Remark 2.5.22 *The extra condition that $g_k(s, 0) = 0$ for all $s \in \mathbb{R}(\mu^{\frac{1}{2}})$ if $k \neq 0$ is necessary because we want elements in $\hat{A}$ to be* continuous *functions of a and b. In this way, we have that Theorem 2.5.24 below holds. Without this extra condition, we would have that $u \in M(\hat{A})$.*

We try to give some more explanation about this feature. It is clear that we have to concentrate on elements of the form $u^k f(|b|)$ where $f \in \mathrm{C}_0(\mathbb{R}(\mu^{\frac{1}{2}}))$.

Consider the function h defined by

$$h : \mathbb{C} \to \mathbb{C} : x \mapsto \begin{cases} \frac{z}{|z|} & \textit{if } z \neq 0; \\ 0 & \textit{if } z = 0. \end{cases}$$

By using the Borel calculus, we get that $h(b) = u$.

For all $f \in \mathrm{C}_0(\bar{\mathbb{C}}(\mu^{\frac{1}{2}}))$ and $k \in \mathbb{Z}$, we then have $u^k f(|b|) = h^k f(b)$. It is obvious that we have $h^k f \in \mathrm{C}_0(\bar{\mathbb{C}}(\mu^{\frac{1}{2}}))$ if and only if $f(0) = 0$.

In order to have the property that the C-algebra $\hat{A}$ is generated by the affiliated elements $a, a^{-1}, b \,\eta\, \hat{A}$, it is certainly desirable that the elements $a, b \,\eta\, \hat{A}$ separate non-degenerate *-representations of $\hat{A}$; see Proposition 1.5.54.*

Let $\pi : A \to B(K)$ be a non-degenerate *-representation. By working as above, we see that the value of $\pi(b)$ determines the value of all the elements $\pi(u^k f(|b|))$ where $f \in \mathrm{C}_0(\bar{\mathbb{R}}(\mu^{\frac{1}{2}}))$ and $f(0) = 0$. We however do not get information about $\pi(u^k f(|b|))$ if the condition $f(0) = 0$ is not satisfied.

The next Lemma 2.5.23 follows directly from Theorem 2.5.21.

Lemma 2.5.23 *We have that*

$$M(\hat{A}) \supseteq \Big[\sum_{k \in \mathbb{Z}} u^k g_k(a, |b|) \,\Big|\, g_k \in \mathrm{C}_b(E) \text{ for every } k, g_k \neq 0 \text{ for finitely many } k$$
$$\text{and } g_k(s, 0) = 0 \text{ for all } s \in \mathbb{R}(\mu^{\frac{1}{2}}) \text{ if } k \neq 0 \Big],$$

$$\hat{A}^\eta \supseteq \big\{ u^k g(a, |b|) \mid g \in \mathrm{C}\big(E\big) \text{ and } g(s, 0) = 0 \text{ for every } s \in \mathbb{R}(\mu^{\frac{1}{2}}) \text{ if } k \neq 0 \big\}.$$

We now state the important property that $\hat{A}$ is generated by the triple (a, a^{-1}, b). One can infer this result from the proof of Theorem 2.5.21.

The Theorem 2.5.24 is very useful, especially to prove uniqueness results.

Theorem 2.5.24 *We have that $a, a^{-1}, b \,\eta\, \hat{A}$. The two operators $a, b \,\eta\, \hat{A}$ separate non-degenerate *-representations of $\hat{A}$.*

We have that the C-algebra $\hat{A}$ is generated by $a, a^{-1}, b \,\eta\, \hat{A}$.*

The next Remark 2.5.25 accompanies Theorem 2.5.24.

Remark 2.5.25 *One can easily check that $\log a, b \,\eta\, \hat{A}$ and that the C*-algebra $\hat{A}$ is generated by the affiliated elements $\log a, b \,\eta\, \hat{A}$.*

It is thus possible to generate $\hat{A}$ by two affiliated elements. This is (almost) the choice made by S.L. Woronowicz in the original article [159].

From the proof of Proposition 2.5.21, we know that $a|b|, ab \,\eta\, \hat{A}$. Further, it is not difficult to check that we have $u \notin M(\hat{A})$.

It is possible to describe $\hat{A}$ as a C*-subalgebra of a crossed product. The results below are proved in Section 4.1.

We first have some definitions and basic results.

Definition 2.5.26 *We define an action $\beta : \mathbb{Z} \to \mathrm{Aut}(\mathrm{C}_0(E))$ by*

$$\beta_n : \mathrm{C}_0(E) \to \mathrm{C}_0(E) : g \mapsto g(\mu^{-n} \cdot, \cdot).$$

Lemma 2.5.27 *We have that $(\mathrm{C}_0(E), \mathbb{Z}, \beta)$ is a C*-dynamical system.*

Definition 2.5.28 *Let $\hat{A}_{ex} \supsetneq \hat{A}$ be the C*-algebra defined by*

$$\hat{A}_{ex} = \Big[\sum_{k \in \mathbb{Z}} u^k g_k(a, |b|) \;\Big|\; g_k \in \mathrm{C}_0(E) \text{ for every } k \text{ and } g_k \neq 0 \text{ for finitely many } k \Big].$$

We denote by $\hat{A}_{cp}$ the C-subalgebra of $\mathrm{C}_0(E) \ltimes_\beta \mathbb{Z}$ generated by the elements $g \in \mathrm{K}(\mathbb{Z}, \mathrm{C}_0(E))$ that satisfy $g(n)(s, 0) = 0$ for every $s \in \mathbb{R}(\mu^{\frac{1}{2}})$ if $n \neq 0$.*

Next, we state the following properties.

Proposition 2.5.29 *We have $\hat{A} \cong \hat{A}_{cp}$ and $\hat{A}_{ex} \cong \mathrm{C}_0(E) \ltimes_\beta \mathbb{Z}$.*

Corollary 2.5.30 *We have that $\hat{A}$ is a type I C*-algebra and is thus nuclear.*

We recall the standard properties of the comultiplication $\hat{\Phi}$ on $\hat{A}$. They follow from the general manageability theory; see Section 1.3.

Since we have flipped the comultiplication, we have to be somewhat careful to adapt the results from Theorem 1.3.9 in order to get correct formulas.

We have the following three Propositions 2.5.31, 2.5.32 and 2.5.33.

Proposition 2.5.31 *We can consider $\hat{\Phi} : \hat{A} \to M(\hat{A} \otimes \hat{A})$ as a non-degenerate *-homomorphism. Moreover, the results below hold:*

1. *We have that $\hat{\Phi}$ is* coassociative, *i.e., $(\hat{\Phi} \otimes \iota)\hat{\Phi} = (\iota \otimes \hat{\Phi})\hat{\Phi}$,*
2. *The linear spaces $\hat{\Phi}(\hat{A})(1 \otimes \hat{A})$ and $\Phi(\hat{A})(\hat{A} \otimes 1)$ are dense in $\hat{A} \otimes \hat{A}$.*

Proposition 2.5.32 *We have that $\hat{\Phi}$ is characterized as the only non-degenerate *-homomorphism from $\hat{A}$ to $M(\hat{A} \otimes \hat{A})$ such that*

$$(\hat{\Phi} \otimes \iota)W = W_{13}W_{23}.$$

Proposition 2.5.33 *Let $\omega \in B(H)_*$. We then have*

$$\hat{\Phi}\big((\iota \otimes \omega)W\big) = (\iota \otimes \iota \otimes \omega)(W_{13}W_{23}).$$

We calculate the action of $\hat{\Phi}$ on the generating elements. The results found in the Proposition 2.5.34 below completely fit in with Proposition 2.1.1.

The formulas in Proposition 2.5.34 ensue from Lemma 2.3.16. The uniqueness part is a direct consequence of Theorem 2.5.24.

Proposition 2.5.34 *We have that $\hat{\Phi}$ is characterized as the only non-degenerate *-homomorphism from $\hat{A}$ to $M(\hat{A} \otimes \hat{A})$ such that*

$$\hat{\Phi}(a) = a \otimes a \qquad \text{and} \qquad \hat{\Phi}(b) = a \otimes b \dotplus b \otimes a^{-1}.$$

Furthermore, we have

$$\hat{\Phi}(a^{-1}) = a^{-1} \otimes a^{-1}, \qquad \hat{\Phi}(b^*) = a \otimes b^* \dotplus b^* \otimes a^{-1},$$
$$\hat{\Phi}(ab) = a^2 \otimes ab \dotplus ab \otimes 1.$$

Proof. Only the formula for $\hat{\Phi}(ab)$ needs some extra explanation. We use the notation $K_0 = \text{span}\{e_k \otimes e_l \otimes e_m \mid k, l, m \in \mathbb{Z}\}$.

It is clear that we have

$$\hat{\Phi}(ab) = W^*(1 \otimes ab)W.$$

This formula can be rewritten as

$$\hat{\Phi}(ab) = W^*(1 \otimes a)WW^*(1 \otimes b)W.$$

We then get from Lemma 2.3.16 that

$$\hat{\Phi}(ab) = (a \otimes a)(a \otimes b \dotplus b \otimes a^{-1}). \tag{2.59}$$

From this, we see that $H_0 \odot H_0 \subseteq D\big(\hat{\Phi}(ab)\big)$.

It is not difficult to check that $a^2 \otimes ab \dotplus ab \otimes 1$ is well-defined. Further, it is then clear that we have $H_0 \odot H_0 \subseteq D(a^2 \otimes ab \dotplus ab \otimes 1)$.

We again use the short notations $\boldsymbol{\chi} = \chi(a \otimes 1, 1 \otimes c)$ and $\boldsymbol{F}(\gamma) = F_\mu(\gamma ab \otimes cd)$. Then, we have that

$$\begin{aligned}
\hat{\Phi}(ab) \otimes cd &= (W^* \otimes 1)(1 \otimes ab \otimes cd)(W \otimes 1) \\
&= \boldsymbol{\chi}_{12}^* \boldsymbol{F}_{12}(1)^*(1 \otimes ab \otimes cd)\boldsymbol{F}_{12}(1)\boldsymbol{\chi}_{12} \\
&= \boldsymbol{\chi}_{12}^*(ab \otimes c^2 \otimes cd \dotplus 1 \otimes ab \otimes cd)\boldsymbol{\chi}_{12} \\
&= \boldsymbol{\chi}_{12}^*(ab \otimes c^2 \otimes cd)\boldsymbol{\chi}_{12} \dotplus \boldsymbol{\chi}_{12}^*(1 \otimes ab \otimes cd)\boldsymbol{\chi}_{12} \\
&= ab \otimes 1 \otimes cd \dotplus a^2 \otimes ab \otimes cd.
\end{aligned}$$

The third equality follows from Equation (2.33). The last equality can easily be calculated by applying Proposition 2.3.13.

Some basic calculations give that $K_0 \subseteq D\big((a^2 \otimes ab \dotplus ab \otimes 1) \otimes cd\big)$ and that

$$(a^2 \otimes ab \dotplus ab \otimes 1) \otimes cd \subseteq ab \otimes 1 \otimes cd \dotplus a^2 \otimes ab \otimes cd. \tag{2.60}$$

We define two normal operators R and S by

$$R = ab \otimes 1 \otimes cd \qquad \text{and} \qquad S = a^2 \otimes ab \otimes cd.$$

It is easy to check that $(R, S) \in D_\mu$. We have that K_0 is a core for R and that K_0 is invariant under spectral projections of $|S|$. The Lemma 3.2 in [158] now implies that K_0 is a core for the normal operator $R \dotplus S$.

By combining this result with Equation (2.60), we get

$$(a^2 \otimes ab \dotplus ab \otimes 1) \otimes cd = ab \otimes 1 \otimes cd \dotplus a^2 \otimes ab \otimes cd.$$

We thus have

$$\hat{\Phi}(ab) \otimes cd = (a^2 \otimes ab \dotplus ab \otimes 1) \otimes cd.$$

It is now direct to deduce that

$$\hat{\Phi}(ab) = a^2 \otimes ab \dotplus ab \otimes 1.$$

This ends the proof. ∎

From Proposition 2.5.34, we find how $\hat{\Phi}$ acts on basic elements.

Notation 2.5.35 *We set $w = W^*(1 \otimes u)W$. Then, $w \in B(H \otimes H)$ is a unitary and we have that*

$$a \otimes b \dotplus b \otimes a^{-1} = w|a \otimes b \dotplus b \otimes a^{-1}|$$

is the polar decomposition of the normal operator $a \otimes b \dotplus b \otimes a^{-1}$.

Proposition 2.5.36 *Let $\sum u^k g_k(a, |b|) \in \hat{A}$. Then, we have*

$$\Phi\big(\sum u^k g_k(a, |b|)\big) = \sum w^k g_k(a \otimes a, |a \otimes b \dotplus b \otimes a^{-1}|).$$

2.6 The antipodes

In this Section 2.6, we take a next step in our description of the quantum group structure of the quantum $E(2)$ group and its dual. We give a detailed study of the *antipode* on both of the quantum groups $E(2)$ and $\hat{E}(2)$.

We make again use of the results in the Section 1.3. The basic properties of the antipodes S and $\hat{S}$ directly ensue from the manageability theory.

Further, we calculate the action of the antipodes on the generating elements. We find an intuition in the formulas of Proposition 2.1.15 where we have studied the antipodes and their polar decompositions on the Hopf *-algebra level.

We recall Definition 2.3.25 and introduce two more operators.

Definition 2.6.1 *We define $N = m^2 \otimes m^{-2}$. Thus, N is a strictly positive operator on H. Further, we define two anti-unitary operators I and $\hat{I}$ on H by requiring that we have*

- *$I(e_k \otimes e_l) = (-1)^{k+l}(e_{-k+2l} \otimes e_l)$,*
- *$\hat{I}(e_k \otimes e_l) = (-1)^l(e_k \otimes e_{k-l})$ and $\hat{I}^*(e_k \otimes e_l) = (-1)^{k+l}(e_k \otimes e_{k-l})$.*

when $k, l \in \mathbb{Z}$.

Definition 2.6.2 *We define two strictly positive operators $\hat{N}$ and $\hat{N}_0$ on H by the formulas $\hat{N} = m^{-2} \otimes m^2$ and $\hat{N}_0 = 1 \otimes m^2$.*

It is direct to check the results in the Lemma 2.6.3 below.

Lemma 2.6.3 *The next two properties hold:*

- *We have $I^* = I$ and $I^2 = 1$.*
- *Let $t \in \mathbb{R}$. Then, we have $IN^{it}I = N^{it}$ and $IN^tI = N^{-t}$.*

The formulas in Lemma 2.6.4 are typical for N and I. In a manner, the Definition 2.6.1 is chosen in such a way that Lemma 2.6.4 holds.

This Lemma 2.6.4 can be proved by some very simple calculations.

Lemma 2.6.4 *We have that*

$$Ic^*I = c^* \quad \text{and} \quad Id^*I = -d.$$

Further, we have

$$N^{-it}cN^{it} = c \quad \text{and} \quad N^{-it}dN^{it} = \mu^{-2it}d.$$

The Lemmas 2.6.5 and 2.6.6 are then about the operators $\hat{N}$, $\hat{N}_0$ and $\hat{I}$.

Lemma 2.6.5 *The next two properties hold:*

- *We have $\hat{I}^2 = 1$.*
- *Let $t \in \mathbb{R}$. Then, we have $\hat{I}\hat{N}^{it}\hat{I}^* = \hat{N}_0^{it}$ and $\hat{I}\hat{N}^t\hat{I}^* = \hat{N}_0^{-t}$.*

Lemma 2.6.6 *We have that*

$$\hat{I}a\hat{I}^* = \hat{I}^*a\hat{I} = a^{-1} \quad \text{and} \quad \hat{I}b^*\hat{I}^* = \hat{I}^*b^*\hat{I} = -b.$$

Further, we have

$$\hat{N}^{-it}a\hat{N}^{it} = \hat{N}_0^{-it}a\hat{N}_0^{it} = a \quad \text{and} \quad \hat{N}^{-it}b\hat{N}^{it} = \hat{N}_0^{-it}b\hat{N}_0^{it} = \mu^{-2it}b.$$

We also restate the Proposition 2.3.29 about the strong modularity property of the multiplicative unitary W.

The Proposition 2.6.7 plays a crucial role in this Section 2.6.

Proposition 2.6.7 *The quintuple $(W, N^{\frac{1}{2}}, N^{\frac{1}{2}}, I, \hat{I})$ satisfies Definition 1.3.57.*

The quantum $E(2)$ group

First, we consider the *antipode* S of the quantum $E(2)$ group (A, Φ). As usual, we define S through its polar decomposition. It turns out that S is an unbounded linear map. We prove this property in Proposition 2.6.32.

From Proposition 2.6.7, we know that W is strongly modular. It makes that we can fully rely on the modularity theory as elaborated in Section 1.3.

We start below with the definitions. Because W is *strongly* modular, we dispose of implementations both of the unitary antipode R and the scaling group (τ_t). This makes it possible to give explicit formulas for R and (τ_t).

Since we make use of the results in Section 1.3 very often, we do not provide an explicit reference for each separate property. It is easy to find out the exact places where the corresponding general properties are stated.

We first define the *unitary antipode* R.

Definition 2.6.8 *We consider the* **-anti-automorphism* R *on* A *defined by*

$$R : A \to A : x \mapsto I x^* I.$$

We have that $R^2 = \iota$.

Next, we come to the *scaling group* (τ_t).

Definition 2.6.9 *Let* $t \in \mathbb{R}$. *Then, we consider the* **-automorphism* τ_t *on* A *defined by*

$$\tau_t : A \to A : x \mapsto N^{-it} x N^{it}.$$

We have that (τ_t) *is a norm continuous one-parameter group on* A.

The *antipode* S is then defined as follows.

Definition 2.6.10 *We define*

$$S = R\tau_{-\frac{i}{2}} = \tau_{-\frac{i}{2}} R.$$

The next Theorem 2.6.11 encompasses the basic properties of S.

Theorem 2.6.11 *We have that* $S : D(S) \subseteq A \to A$ *is a closed linear operator such that* $\{(\omega \otimes \iota)W \mid \omega \in B(H)_*\}$ *is a core for* S *and*

$$S\big((\omega \otimes \iota)W\big) = (\omega \otimes \iota)W^*$$

for every $\omega \in B(H)_*$.
Further, the properties below hold:

- S *is densely defined and has dense range,*
- S *is injective and* $S^{-1} = R\tau_{\frac{i}{2}} = \tau_{\frac{i}{2}} R$,
- S *is anti-multiplicative: for* $x, y \in D(S)$, *we have that* $xy \in D(S)$ *and* $S(xy) = S(y)S(x)$,
- *For all* $x \in D(S)$, *we have* $S(x)^* \in D(S)$ *and* $S(S(x)^*)^* = x$,
- $S^2 = \tau_{-i}$.

We have the following commutation relations:

- $R\tau_z = \tau_z R$ *for all* $z \in \mathbb{C}$,
- $\Phi\tau_t = (\tau_t \otimes \tau_t)\Phi$ *for all* $t \in \mathbb{R}$,
- $\Phi R = \dot{\sigma}(R \otimes R)\Phi$,
- $RS = SR$,
- $\tau_t S = S\tau_t$ *for all* $t \in \mathbb{R}$.

In the next Lemma 2.6.12, the strict closure of S is introduced.

Lemma 2.6.12 *The antipode S is strictly closed on A. We can extend S to a strictly closed linear map $\bar{S}$ on $M(A)$. Then, $\bar{S}$ is the strict closure of S.*

We know from Remark 1.1.20 that we can extend τ to a strictly continuous one-parameter group $\bar{\tau}$ on $M(A)$. By definition, we have

$$\bar{S} = R\bar{\tau}_{-\frac{i}{2}} = \bar{\tau}_{-\frac{i}{2}}R.$$

The extension $\bar{S}$ satisfies the strict version of Theorem 2.6.11.

We now describe the action of R, τ and S. In the first place, we look at the action on elements $(\omega \otimes \iota)W$. We can then apply the results from Section 1.3.

We first state the next Proposition 2.6.13 about the action of S.

Proposition 2.6.13 *Let $\omega \in A^*$. We then have $(\omega \otimes \iota)W \in D(\bar{S})$ and*

$$\bar{S}\big((\omega \otimes \iota)W\big) = (\omega \otimes \iota)W^*. \tag{2.61}$$

Proof. We get from Proposition 3.2.8 that (A, Φ) is a universal quantum group. From Remark 9.1 in [61], it then ensues that Equation (2.61) holds. ∎

The case of the unitary antipode R is considered as second one.

Lemma 2.6.14 *Let $\xi \in D(N^{\frac{1}{2}})$ and $\eta \in D(N^{-\frac{1}{2}})$. Then, we have*

$$R\big((\omega_{\xi,\eta} \otimes \iota)W\big) = (\omega_{N^{\frac{1}{2}}\xi, N^{-\frac{1}{2}}\eta} \otimes \iota)W^*.$$

Lemma 2.6.15 *Let $\omega \in B(H)_*$. Define $\omega_{\hat{R}} \in B(H)_*$ by $\omega_{\hat{R}}(x) = \omega(\hat{I}x^*\hat{I}^*)$ for $x \in B(H)$. We have that*

$$R\big((\omega \otimes \iota)W\big) = (\omega_{\hat{R}} \otimes \iota)W.$$

Lemma 2.6.16 *Let $\xi, \eta \in H$. Then, we have*

$$R\big((\omega_{\xi,\eta} \otimes \iota)W\big) = (\omega_{\hat{I}\eta,\hat{I}\xi} \otimes \iota)W = (\omega_{\hat{I}^*\eta,\hat{I}^*\xi} \otimes \iota)W.$$

The action of the scaling group (τ_t) is described in the next two lemmas.

Lemma 2.6.17 *Let $t \in \mathbb{R}$. For all $\omega \in B(H)_*$, we have that*

$$\tau_t\big((\omega \otimes \iota)W\big) = (\omega_t \otimes \iota)W$$

where $\omega_t \in B(H)_*$ is defined by $\omega_t(x) = \omega(N^{it}xN^{-it})$.

Lemma 2.6.18 *Let $z \in \mathbb{C}$. Take $\xi \in D(N^{-\mathrm{Im} z})$ and $\eta \in D(N^{\mathrm{Im} z})$. Then, we have that $(\omega_{\xi,\eta} \otimes \iota)W \in D(\tau_z)$. Further, we have*

$$\tau_z\big((\omega_{\xi,\eta} \otimes \iota)W\big) = (\omega_{N^{-iz}\xi, N^{-i\bar{z}}\eta} \otimes \iota)W.$$

By using Proposition B.2.10, we also get the next Lemma 2.6.19.

Lemma 2.6.19 *If $\xi, \eta \in H_0$, then $(\omega_{\xi,\eta} \otimes \iota)W$ is analytic with respect to τ. For every $z \in \mathbb{C}$, the set* $\operatorname{span}\{(\omega_{\xi,\eta} \otimes \iota)W \mid \xi, \eta \in H_0\}$ *is a core for τ_z.*

We further focus on the action on the basic elements $\sum c^k f_k(d)$. Since this case is not standard, it is more interesting. In Proposition 2.6.32, we apply the found results to prove important properties of the antipode S.

One should notice that all formulas below are in complete correspondence with their algebraic counterparts (cf. Proposition 2.1.15).

The next Proposition 2.6.20 describes the action of R. It follows almost directly from a combination of Lemma 2.6.4 and Theorem 2.5.3.

Proposition 2.6.20 *We have that R is characterized as the only non-degenerate *-anti-homomorphism from A to $M(A)$ such that*

$$R(c) = c^* \qquad \text{and} \qquad R(d) = -d.$$

Let $\sum c^k f_k(d) \in A$. Then, we have

$$R\big(\sum c^k f_k(d)\big) = \sum c^{-k} f_k(-\mu^k d).$$

The Proposition 2.6.21 below also follows from Lemma 2.6.4 and Theorem 2.5.3. It describes the action of the scaling group (τ_t).

Proposition 2.6.21 *Let $t \in \mathbb{R}$. We have that τ_t is characterized as the only non-degenerate *-homomorphism from A to $M(A)$ such that*

$$\tau_t(c) = c \qquad \text{and} \qquad \tau_t(d) = \mu^{-2it} d.$$

Let $\sum c^k f_k(d) \in A$. Then, we have

$$\tau_t\big(\sum c^k f_k(d)\big) = \sum c^k f_k(\mu^{-2it} d).$$

In what follows, we construct basic elements that are analytic for the scaling group τ. These results are used to calculate the action of S. We make use of the standard properties from the theory of analytic continuations.

The Section B.3 contains a short overview of the theory of analytic continuations together with an explanation of the regarding notations and terminology.

We first consider the elements $f(c)$. It is direct to check the Lemma 2.6.22 below. We recall that $\bar{\tau}$ is the strictly continuous extension of τ to $M(A)$.

Lemma 2.6.22 *Let $f \in \mathrm{C}(S^1)$. Then, we have that $f(c)$ is strictly analytic with respect to τ. For all $z \in \mathbb{C}$, we have $\bar{\tau}_z(f(c)) = f(c)$.*

We also look at elements of the form $f(d)$. Here, the situation is (a lot) more complicated. We can only prove some partial results.

First, we introduce some notations.

Notation 2.6.23 *Let $k \in \mathbb{Z}$. We use the notation $S_k = \{z \in \bar{\mathbb{C}}^\mu \mid |z| = \mu^k\}$. For every $n \in \mathbb{Z}$, we define a function $h_{k,n} : S_k \to \mathbb{C}$ by $h_{k,n}(z) = z^n$.*

For all $n \in \mathbb{Z}$, we introduce the following sets:

$$D_{n,1} = \big\{ f \in \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \mid f \textit{ is constant on every } S_k \big\},$$
$$D_{n,2} = \big\{ f \in \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \mid \textit{For every } k, \textit{ we have } f|_{S_k} = h_{k,n} \textit{ or } f|_{S_k} = 0 \big\},$$
$$D_{n,3} = \big\{ f \in \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \mid \textit{For every } k, \textit{ we have } f|_{S_k} = h_{k,-n} \textit{ or } f|_{S_k} = 0 \big\}.$$

The *-algebra generated by the functions in $D_{n,1} \cup D_{n,2} \cup D_{n,3}$ is denoted by D_n.

The next Lemma 2.6.24 can be proved by using standard techniques.

Lemma 2.6.24 *Let $n \in \mathbb{N}$. If $f \in D_n$, then $f(d)$ and $f(d)^*$ are analytic with respect to τ. For every $z \in \mathbb{C}$, we have the following properties:*

1. *If $f \in D_{n,1}$, we have $\tau_z(f(d)) = f(d)$,*
2. *If $f \in D_{n,2}$, we have $\tau_z(f(d)) = \mu^{-2inz} f(d)$ and $\tau_z(f(d)^*) = \mu^{2inz} f(d)^*$,*
3. *If $f \in D_{n,3}$, we have $\tau_z(f(d)) = \mu^{2inz} f(d)$ and $\tau_z(f(d)^*) = \mu^{-2inz} f(d)^*$.*

We make the following Remark 2.6.25 only once. A similar remark can also be made to Lemma 2.6.27 and Propositions 2.6.29 and 2.6.30

Remark 2.6.25 *If we would replace in Notation 2.6.23 the C*-algebra $\mathrm{C}_0(\bar{\mathbb{C}}^\mu)$ by its multiplier algebra $\mathrm{C}_b(\bar{\mathbb{C}}^\mu)$, then the conclusions of Lemma 2.6.24 remain true under the condition that we replace τ by its strict closure $\bar{\tau}$.*

We know that every map τ_z is linear and multiplicative. Hence, we can combine all the above results to construct more elements that are analytic for τ.

Again, we first introduce some notations.

Notation 2.6.26 *Let D be the *-algebra generated by $\bigcup_{n \in \mathbb{N}} D_n$. Then, we have that D is a dense subset of $\mathrm{C}_0(\bar{\mathbb{C}}^\mu)$. We define a *-algebra $\mathbf{D}$ by*

$$\mathbf{D} = \big\{ \sum c^k f_k(d) \mid f_k \in D \textit{ for every } k \textit{ and } f_k \neq 0 \textit{ for finitely many } k \big\}.$$

It is clear that $\mathbf{D}$ is dense in A.

We can now use Proposition B.2.10 to deduce the next result.

Lemma 2.6.27 *We have that $\mathbf{D}$ is a dense *-subalgebra of A. Every element in $\mathbf{D}$ is analytic with respect to τ.*

For every $z \in \mathbb{C}$, we have that $\mathbf{D}$ is a core for τ_z.

We can use Proposition 2.6.20 to translate Lemmas 2.6.22, 2.6.24 and 2.6.27 into properties of the antipode S and its strict closure $\bar{S}$.

It is immediate to deduce the three following propositions.

Proposition 2.6.28 *Let $f \in \mathrm{C}(S^1)$. Then, $f(c) \in D(\bar{S})$ and $\bar{S}(f(c)) = f(c^*)$. In particular, we have that $1, c, c^* \in D(\bar{S})$ with*

$$\bar{S}(1) = 1, \qquad \bar{S}(c) = c^* \qquad \text{and} \qquad \bar{S}(c^*) = c.$$

Proposition 2.6.29 *Let $n \in \mathbb{N}$. If $f \in D_n$, then $f(d), f(d)^* \in D(S)$. We have the following properties:*

1. *If $f \in D_{n,1}$, we have $S(f(d)) = f(d)$,*
2. *If $f \in D_{n,2}$, we have $S(f(d)) = (-\mu)^{-n} f(d)$ and $S(f(d)^*) = (-\mu)^n f(d)^*$,*
3. *If $f \in D_{n,3}$, we have $S(f(d)) = (-\mu)^n f(d)$ and $S(f(d)^*) = (-\mu)^{-n} f(d)^*$.*

Proposition 2.6.30 *We have that $\mathbf{D}$ is a core for the antipode S.*

We then calculate the action of S on the generators. Loosely speaking, we find that we have $S(c) = c^*$, $S(d) = -\mu^{-1} d$ and $S(d^*) = -\mu d^*$.

The Proposition 2.6.31 improves the intuition about the antipode S.

Proposition 2.6.31 *We have that $c, c^* \in D(\bar{S})$ with*

$$\bar{S}(c) = c^* \qquad \text{and} \qquad \bar{S}(c^*) = c.$$

Further, there exists a sequence (d_k) in $D(S)$ such that

$$d_k \to d \qquad \text{and} \qquad S(d_k) \to -\mu^{-1} d$$

and also

$$d_k^* \to d^* \qquad \text{and} \qquad S(d_k^*) \to -\mu d^*$$

with convergence in the η-topology.

Proof. The results about c and c^* are already stated in Proposition 2.6.28.

Let $k \in \mathbb{N}$. Then, we consider the function $f_k \in \mathrm{C}_0(\bar{\mathbb{C}}^\mu)$ defined by

$$f_k : \bar{\mathbb{C}}^\mu \to \mathbb{C} : z \mapsto \begin{cases} z & \text{if } |z| \leq k; \\ 0 & \text{if } |z| > k. \end{cases}$$

We define an element $d_k \in A$ by $d_k = f_k(d)$.

Proposition 2.6.29 gives that $d_k \in D(S)$ and $S(d_k) = -\mu^{-1} d_k$ for every k.

It is not difficult to prove that $d_k \to d$ and $S(d_k) \to -\mu^{-1} d$ in the η-topology.

The results about d^* are completely similar. ∎

From the above results, we can derive some interesting properties. We state the most important features of the antipode S in Proposition 2.6.32 below.

The behavior of S implies that the quantum $E(2)$ group is not a Kac algebra.

The Proposition 2.6.32 concludes our study of the antipode S.

Proposition 2.6.32 *We have the following properties:*

1. *The antipode S is an unbounded linear map,*
2. *We have that $D(S) \neq A$,*
3. *The antipode S is not *-preserving,*
4. *We have that $S^2 \neq \iota$.*

Proof. Let $n \in \mathbb{N}$. Then, we consider the function $h_n \in \mathrm{C}_0(\bar{\mathbb{C}}^\mu)$ defined by

$$h_n : \bar{\mathbb{C}}^\mu \to \mathbb{C} : z \mapsto \begin{cases} z^n & \text{if } z \in S^1; \\ 0 & \text{if } z \notin S^1. \end{cases}$$

We define an element $y_n \in A$ by $y_n = h_n(d)$.

Proposition 2.6.29 gives that $y_n \in D(S)$ and $S(y_n) = (-\mu)^{-n} y_n$ for every n.

For every $n \in \mathbb{N}$, we have that $\|y_n\| = 1$ while $\|S(y_n)\| = \mu^{-n}$. We hence find that S is an unbounded linear map.

The closed graph theorem now implies that $D(S) \neq A$.

Using Proposition 2.6.29, we find that $y_1, y_1^* \in D(S)$ with $S(y_1) = -\mu^{-1} y_1$ and $S(y_1^*) = -\mu y_1^*$. From this, we then get that S is not *-preserving.

Also, we see that $y_1 \in D(S^2)$ and $S^2(y_1) = \mu^{-2} y_1$. This yields $S^2 \neq \iota$. ∎

The quantum $\hat{E}(2)$ group

We now construct the antipode $\hat{S}$ of the quantum $\hat{E}(2)$ group $(\hat{A}, \hat{\Phi})$. Similar to the case of the quantum $E(2)$ group, we define $\hat{S}$ through its polar decomposition $\hat{S} = \hat{R}\hat{\tau}_{-\frac{i}{2}}$. Every property of S has a dual counterpart related to $\hat{S}$.

The Proposition 2.6.7 again makes it possible to apply the results in Section 1.3. From the modularity theory, we directly get a series of standard formulas.

We first define the *unitary antipode* $\hat{R}$.

Definition 2.6.33 *We consider the *-anti-automorphism $\hat{R}$ on $\hat{A}$ defined by*

$$\hat{R} : \hat{A} \to \hat{A} : x \mapsto \hat{I} x^* \hat{I}^*.$$

We have that $\hat{R}^2 = \iota$.

Because $\hat{R}^2 = \iota$, the next Lemma 2.6.34 holds.

Lemma 2.6.34 *We have $\hat{R}(x) = \hat{I}^* x^* \hat{I}$ for all $x \in \hat{A}$.*

Next, we come to the *scaling group* $(\hat{\tau}_t)$.

Definition 2.6.35 *Let $t \in \mathbb{R}$. Then, we consider the *-automorphism $\hat{\tau}_t$ on $\hat{A}$ defined by*

$$\hat{\tau}_t : \hat{A} \to \hat{A} : x \mapsto \hat{N}^{-it} x \hat{N}^{it}.$$

We have that $(\hat{\tau}_t)$ is a norm continuous one-parameter group on $\hat{A}$.

It is important to keep in mind the next Remark 2.6.36.

Remark 2.6.36 *The scaling group $\hat{\tau}$ as introduced in Definition 2.6.35 is a modification of the scaling group $\hat{\tau}$ which is studied in Section 1.3.*

Since we have defined $\hat{\Phi}$ as the opposite comultiplication, it is necessary that also the scaling group $\hat{\tau}$ undergoes a change. With respect to the results in Section 1.3, we have to replace $\hat{\tau}_t$ by $\hat{\tau}_{-t}$. This coincides with a change from N to $\hat{N}$.

These changes also have their effect on the formulas describing the action of $\hat{S}$. The results below are of course adapted to the chosen setting.

From Lemma 2.6.6 and Theorem 2.5.24, we find that also $\hat{N}_0$ can be used to implement the scaling group $\hat{\tau}$. This result is stated in Lemma 2.6.37.

We remark that the operator $\hat{N}_0$ is used in Definition 2.7.15 to implement the modular automorphism group σ of the Haar weight ψ on (A, Φ).

Lemma 2.6.37 *Let $t \in \mathbb{R}$. For all $x \in \hat{A}$, we have that*

$$\hat{\tau}_t(x) = \hat{N}_0^{-it} x \hat{N}_0^{it}.$$

The *antipode* $\hat{S}$ is then defined as follows.

Definition 2.6.38 *We define*

$$\hat{S} = \hat{R}\hat{\tau}_{-\frac{i}{2}} = \hat{\tau}_{-\frac{i}{2}}\hat{R}.$$

The next Theorem 2.6.39 encompasses the basic properties of S.

Theorem 2.6.39 *We have that $\hat{S} : D(\hat{S}) \subseteq \hat{A} \to \hat{A}$ is a closed linear operator such that $\{(\iota \otimes \omega)W \mid \omega \in B(H)_*\}$ is a core for $\hat{S}$ and*

$$\hat{S}\big((\iota \otimes \omega)W\big) = (\iota \otimes \omega)W^*$$

for every $\omega \in B(H)_*$.

Further, the properties below hold:

- *$\hat{S}$ is densely defined and has dense range,*
- *$\hat{S}$ is injective and $\hat{S}^{-1} = \hat{R}\hat{\tau}_{\frac{i}{2}} = \hat{\tau}_{\frac{i}{2}}\hat{R}$,*

- *$\hat{S}$ is anti-multiplicative: for $x, y \in D(\hat{S})$, we have that $xy \in D(\hat{S})$ and $\hat{S}(xy) = \hat{S}(y)\hat{S}(x)$,*
- *For all $x \in D(\hat{S})$, we have $\hat{S}(x)^* \in D(\hat{S})$ and $\hat{S}(\hat{S}(x)^*)^* = x$,*
- *$\hat{S}^2 = \hat{\tau}_{-i}$.*

We have the following commutation relations:

- *$\hat{R}\hat{\tau}_z = \hat{\tau}_z\hat{R}$ for all $z \in \mathbb{C}$,*
- *$\hat{\Phi}\hat{\tau}_t = (\hat{\tau}_t \otimes \hat{\tau}_t)\hat{\Phi}$ for all $t \in \mathbb{R}$,*
- *$\hat{\Phi}\hat{R} = \dot{\sigma}(\hat{R} \otimes \hat{R})\hat{\Phi}$,*
- *$\hat{R}\hat{S} = \hat{S}\hat{R}$,*
- *$\hat{\tau}_t\hat{S} = \hat{S}\hat{\tau}_t$ for all $t \in \mathbb{R}$.*

The next Remark 2.6.40 is about the strict closure of $\hat{S}$.

Remark 2.6.40 *The antipode $\hat{S}$ is strictly closed on $\hat{A}$. Similar to the results in the Lemma 2.6.12, we can extend $\hat{S}$ to a strictly closed linear map on $M(A)$. We will use $\hat{S}$ to also denote this extension.*

For $\hat{S}$ considered on $M(\hat{A})$, the strict version of Proposition 2.6.39 holds.

In the results below, it takes little effort to see where we work with $\hat{S}$ on $\hat{A}$ and where with its strict closure $\hat{S}$ on $M(\hat{A})$.

Most of the time, it does not really matter as almost all properties can be phrased both for the elements in $\hat{A}$ and for those in $M(\hat{A})$.

We now describe the action of $\hat{R}$, $\hat{\tau}$ and $\hat{S}$. In the first place, we look at the action on elements $(\iota \otimes \omega)W$. We can then apply the results from Section 1.3.

We first state the next Proposition 2.6.41 about the action of $\hat{S}$.

Proposition 2.6.41 *Let $\omega \in \hat{A}^*$. We then have $(\iota \otimes \omega)W \in D(\hat{S})$ and*

$$\hat{S}\big((\iota \otimes \omega)W\big) = (\iota \otimes \omega)W^*. \tag{2.62}$$

Proof. We get from Proposition 3.2.30 that $(\hat{A}, \hat{\Phi})$ is a universal quantum group. From Remark 9.1 in [61], it then ensues that Equation (2.62) holds. ∎

The case of the unitary antipode $\hat{R}$ is considered as second one.

Lemma 2.6.42 *Let $\xi \in D(\hat{N}^{\frac{1}{2}})$ and $\eta \in D(\hat{N}^{-\frac{1}{2}})$. Then, we have*

$$R\big((\iota \otimes \omega_{\xi,\eta})W\big) = (\iota \otimes \omega_{\hat{N}^{\frac{1}{2}}\xi,\hat{N}^{-\frac{1}{2}}\eta})W^*.$$

Lemma 2.6.43 *Let $\omega \in B(H)_*$. Define $\omega_R \in B(H)_*$ by $\omega_R(x) = \omega(Ix^*I)$ for $x \in B(H)$. We have that*

$$\hat{R}\big((\iota \otimes \omega)W\big) = (\iota \otimes \omega_R)W.$$

Lemma 2.6.44 *Let $\xi, \eta \in H$. Then, we have*

$$\hat{R}\big((\iota \otimes \omega_{\xi,\eta})W\big) = (\iota \otimes \omega_{I\eta,I\xi})W.$$

The action of the scaling group $(\hat{\tau}_t)$ is described in the next two lemmas.

Lemma 2.6.45 *Let $t \in \mathbb{R}$. For all $\omega \in B(H)_*$, we have that*

$$\hat{\tau}_t\big((\iota \otimes \omega)W\big) = (\iota \otimes \omega_t)W$$

where $\omega_t \in B(H)_*$ is defined by $\omega_t(x) = \omega(\hat{N}^{it} x \hat{N}^{-it})$.

Lemma 2.6.46 *Let $z \in \mathbb{C}$. Take $\xi \in D(\hat{N}^{-\operatorname{Im} z})$ and $\eta \in D(\hat{N}^{\operatorname{Im} z})$. Then, we have that $(\iota \otimes \omega_{\xi,\eta})W \in D(\hat{\tau}_z)$. Further, we have*

$$\hat{\tau}_z\big((\iota \otimes \omega_{\xi,\eta})W\big) = (\iota \otimes \omega_{\hat{N}^{-iz}\xi,\hat{N}^{-i\bar{z}}\eta})W.$$

By using Proposition B.2.10, we also get the next Lemma 2.6.47.

Lemma 2.6.47 *If $\xi, \eta \in H_0$, then $(\iota \otimes \omega_{\xi,\eta})W$ is analytic with respect to $\hat{\tau}$. For every $z \in \mathbb{C}$, the set* $\operatorname{span}\{(\iota \otimes \omega_{\xi,\eta})W \mid \xi, \eta \in H_0\}$ *is a core for $\hat{\tau}_z$.*

We also mention the Proposition 2.6.48 below.

Proposition 2.6.48 *Let $t \in \mathbb{R}$. We have that*

1. $(\hat{R} \otimes R)(W) = W$,
2. $(\hat{\tau}_{-t} \otimes \tau_t)(W) = W$.

We further focus on the action on the basic elements $\sum u^k g_k(a, |b|)$. The formulas below completely correspond with the results in Proposition 2.1.15.

The next Proposition 2.6.49 describes the action of $\hat{R}$. It follows almost directly from a combination of Lemma 2.6.6 and Theorem 2.5.24.

Proposition 2.6.49 *We have that $\hat{R}$ is characterized as the only non-degenerate *-anti-homomorphism from $\hat{A}$ to $M(\hat{A})$ such that*

$$\hat{R}(a) = a^{-1} \qquad \textit{and} \qquad \hat{R}(b) = -b.$$

Let $\sum u^k g_k(a, |b|) \in \hat{A}$. Then, we have

$$\hat{R}\big(\sum u^k g_k(a, |b|)\big) = \sum (-u)^k g_k(\mu^{-k} a^{-1}, |b|).$$

The Proposition 2.6.50 below also follows from Lemma 2.6.6 and Theorem 2.5.24. It describes the action of the scaling group $(\hat{\tau}_t)$.

Proposition 2.6.50 *Let $t \in \mathbb{R}$. We have that $\hat{\tau}_t$ is characterized as the only non-degenerate *-homomorphism from $\hat{A}$ to $M(\hat{A})$ such that*

$$\hat{\tau}_t(a) = a \qquad \text{and} \qquad \hat{\tau}_t(b) = \mu^{-2it} b.$$

Let $\sum u^k g_k(a, |b|) \in \hat{A}$. Then, we have

$$\hat{\tau}_t\big(\sum u^k g_k(a, |b|)\big) = \sum \mu^{-2itk} u^k g_k(a, |b|).$$

Similar to the case of the quantum $E(2)$ group, we construct basic elements in $D(\hat{S})$ and calculate the action of $\hat{S}$ on these elements. And again, we will use these results later on to study the behavior of the antipode $\hat{S}$.

First, we prove some results regarding the scaling group $\hat{\tau}$. The Lemma 2.6.51 is a quite easy consequence of the Propositions 2.6.50 and B.2.10.

The strict version of Lemma 2.6.51 also holds.

Lemma 2.6.51 *Let $\sum u^k g_k(a, |b|) \in \hat{A}$. We have that $\sum u^k g_k(a, |b|)$ is analytic with respect to $\hat{\tau}$. For $z \in \mathbb{C}$, we have*

$$\hat{\tau}_z\big(\sum u^k g_k(a, |b|)\big) = \sum \mu^{-2izk} u^k g_k(a, |b|).$$

Moreover, the *-algebra

$$\Big\{ \sum_{k \in \mathbb{Z}} u^k g_k(a, |b|) \;\Big|\; g_k \in \mathrm{C}_0(E) \text{ for every } k,\ g_k \neq 0 \text{ for finitely many } k \text{ and } g_k(s, 0) = 0 \text{ for every } s \in \mathbb{R}(\mu^{\frac{1}{2}}) \text{ if } k \neq 0 \Big\}$$

is a core for $\hat{\tau}_z$.

We also look at elements of the form $f(b)$. Here, the situation is (a lot) more complicated. We can only prove some partial results.

First, we introduce some notations.

Notation 2.6.52 *Let $k \in \mathbb{Z}$. Then, we define $\hat{S}_k = \{z \in \bar{\mathbb{C}}(\mu^{\frac{1}{2}}) \mid |z| = \mu^{\frac{1}{2}k}\}$. For every $n \in \mathbb{Z}$, we define a function $\hat{h}_{k,n} : \hat{S}_k \to \mathbb{C}$ by $\hat{h}_{k,n}(z) = z^n$.*

For all $n \in \mathbb{Z}$, we introduce the following sets:

$$\hat{D}_{n,1} = \big\{ f \in \mathrm{C}_b\big(\bar{\mathbb{C}}(\mu^{\frac{1}{2}})\big) \mid f \text{ is constant on every } \hat{S}_k \big\},$$
$$\hat{D}_{n,2} = \big\{ f \in \mathrm{C}_b\big(\bar{\mathbb{C}}(\mu^{\frac{1}{2}})\big) \mid \text{For every } k, \text{ we have } f|_{\hat{S}_k} = \hat{h}_{k,n} \text{ or } f|_{\hat{S}_k} = 0 \big\},$$
$$\hat{D}_{n,3} = \big\{ f \in \mathrm{C}_b\big(\bar{\mathbb{C}}(\mu^{\frac{1}{2}})\big) \mid \text{For every } k, \text{ we have } f|_{\hat{S}_k} = \hat{h}_{k,-n} \text{ or } f|_{\hat{S}_k} = 0 \big\},$$

The *-algebra generated by the functions in $\hat{D}_{n,1} \cup \hat{D}_{n,2} \cup \hat{D}_{n,3}$ is denoted by $\hat{D}_n$.

The next Lemma 2.6.53 can be proved by using standard techniques.

Lemma 2.6.53 *Let $n \in \mathbb{N}$. If $f \in \hat{D}_n$, then $f(b)$ and $f(b)^*$ are strictly analytic with respect to $\hat{\tau}$. For every $z \in \mathbb{C}$, we have the following properties:*

1. *If $f \in \hat{D}_{n,1}$, we have $\hat{\tau}_z(f(b)) = f(b)$,*
2. *If $f \in \hat{D}_{n,2}$, we have $\hat{\tau}_z(f(b)) = \mu^{-2inz} f(b)$ and $\hat{\tau}_z(f(b)^*) = \mu^{2inz} f(b)^*$,*
3. *If $f \in \hat{D}_{n,3}$, we have $\hat{\tau}_z(f(b)) = \mu^{2inz} f(b)$ and $\hat{\tau}_z(f(b)^*) = \mu^{-2inz} f(b)^*$.*

The next Remark 2.6.54 is just a detail.

Remark 2.6.54 *For $f \in \mathrm{C}_b(\bar{\mathbb{C}}(\mu^{\frac{1}{2}}))$, we have that $f(b) \in \hat{A}$ if and only if $f = 0$. There is needed only a small calculation to prove this result.*

This property shows that in the Lemma 2.6.53, we are obliged to work with the scaling group $\hat{\tau}$ considered on $M(\hat{A})$.

This remark also applies to Proposition 2.6.56 and 2.6.57.

We can then use Proposition 2.6.49 to translate Lemmas 2.6.51 and 2.6.53 into properties of the antipode $\hat{S}$ and its strict closure.

It is immediate to deduce the two following propositions.

Proposition 2.6.55 *Let $\sum u^k g_k(a, |b|) \in \hat{A}$. We have $\sum u^k g_k(a, |b|) \in D(\hat{S})$ and*

$$\hat{S}\big(\sum u^k g_k(a, |b|)\big) = \sum (-\mu)^{-k} u^k g_k(\mu^{-k} a^{-1}, |b|).$$

Moreover, the *-algebra

$$\Big\{ \sum_{k \in \mathbb{Z}} u^k g_k(a, |b|) \;\Big|\; g_k \in \mathrm{C}_0(E) \text{ for every } k,\ g_k \neq 0 \text{ for finitely many } k$$
$$\text{and } g_k(s, 0) = 0 \text{ for every } s \in \mathbb{R}(\mu^{\frac{1}{2}}) \text{ if } k \neq 0 \Big\}$$

is a core for $\hat{S}$.

Proposition 2.6.56 *Let $n \in \mathbb{N}$. If $f \in \hat{D}_n$, then $f(b), f(b)^* \in D(\hat{S})$. We have the following properties:*

1. *If $f \in \hat{D}_{n,1}$, we have $\hat{S}(f(b)) = f(b)$,*
2. *If $f \in \hat{D}_{n,2}$, we have $\hat{S}(f(b)) = (-\mu)^{-n} f(b)$ and $\hat{S}(f(b)^*) = (-\mu)^{n} f(b)^*$,*
3. *If $f \in \hat{D}_{n,3}$, we have $\hat{S}(f(b)) = (-\mu)^{n} f(b)$ and $\hat{S}(f(b)^*) = (-\mu)^{-n} f(b)^*$.*

We now calculate the action of $\hat{S}$ on the generators. Loosely speaking, we find that we have $\hat{S}(a) = a^{-1}$, $\hat{S}(b) = -\mu^{-1}b$ and $\hat{S}(b^*) = -\mu b^*$.

The Proposition 2.6.57 improves the intuition about the antipode $\hat{S}$.

Proposition 2.6.57 *There exists a sequence (a_k) in $D(\hat{S})$ with every a_k invertible and such that*

$$a_k \to a \qquad \text{and} \qquad \hat{S}(a_k) \to a^{-1}$$

and also

$$a_k^{-1} \to a^{-1} \qquad \text{and} \qquad \hat{S}(a_k^{-1}) \to a$$

with convergence in the η-topology.

Further, there exists a sequence (b_k) in $D(\hat{S})$ such that

$$b_k \to b \qquad \text{and} \qquad \hat{S}(b_k) \to -\mu^{-1}b$$

and also

$$b_k^* \to b^* \qquad \text{and} \qquad \hat{S}(b_k^*) \to -\mu b^*$$

with convergence in the η-topology.

Proof. Let $k \in \mathbb{N}$. Then, we consider the function $f_k \in \mathrm{C}_b(\mathbb{R}(\mu^{\frac{1}{2}}))$ defined by

$$f_k : \mathbb{R}(\mu^{\frac{1}{2}}) \to \mathbb{C} : z \mapsto \begin{cases} z & \text{if } \frac{1}{k} \leq |z| \leq k; \\ 1 & \text{if } |z| < \frac{1}{k} \text{ or } |z| > k. \end{cases}$$

We define an element $a_k \in M(\hat{A})$ by $a_k = f_k(a)$.

Now, we can use (the strict version of) Proposition 2.6.55 to get the formulated convergence results about the sequence (a_k).

For $k \in \mathbb{N}$, we also consider the function $h_k \in \mathrm{C}_0\big(\bar{\mathbb{C}}(\mu^{\frac{1}{2}})\big)$ defined by

$$h_k : \bar{\mathbb{C}}(\mu^{\frac{1}{2}}) \to \mathbb{C} : z \mapsto \begin{cases} z & \text{if } |z| \leq k; \\ 0 & \text{if } |z| > k. \end{cases}$$

We define an element $b_k \in M(\hat{A})$ by $b_k = h_k(b)$.

Proposition 2.6.56 gives that $b_k \in D(S)$ and $\hat{S}(b_k) = -\mu^{-1}b_k$ for every k.

It is not difficult to prove that $b_k \to b$ and $\hat{S}(b_k) \to -\mu^{-1}b$ in the η-topology.

The results about b^* are completely similar. ■

From the above results, we can derive some interesting properties. We state the most important features of the antipode $\hat{S}$ in Proposition 2.6.58 below.

The behavior of $\hat{S}$ implies that the quantum $\hat{E}(2)$ group is not a Kac algebra.

The Proposition 2.6.32 concludes our study of the antipode S.

Proposition 2.6.58 *We have the following properties:*

1. *The antipode $\hat{S}$ is an unbounded linear map,*
2. *We have that $D(\hat{S}) \neq \hat{A}$,*

3. *The antipode $\hat{S}$ is not *-preserving,*
4. *We have that $\hat{S}^2 \neq \iota$.*

Proof. Let $n \in \mathbb{N}$. Then, we consider the function $h_n \in \mathrm{C}_0\big(\bar{\mathbb{C}}(\mu^{\frac{1}{2}})\big)$ defined by

$$h_n : \bar{\mathbb{C}}(\mu^{\frac{1}{2}}) \to \mathbb{C} : z \mapsto \begin{cases} z^n & \text{if } z \in S^1; \\ 0 & \text{if } z \notin S^1. \end{cases}$$

We define an element $y_n \in M(\hat{A})$ by $y_n = h_n(b)$.

Proposition 2.6.56 gives that $y_n \in D(\hat{S})$ and $\hat{S}(y_n) = (-\mu)^{-n} y_n$ for every n.

For every $n \in \mathbb{N}$, we have that $\|y_n\| = 1$ while $\|\hat{S}(y_n)\| = \mu^{-n}$. We hence find that $\hat{S}$ is an unbounded linear map.

The closed graph theorem now implies that $D(\hat{S}) \neq A$.

Using Proposition 2.6.56, we find that $y_1, y_1^* \in D(\hat{S})$ with $\hat{S}(y_1) = -\mu^{-1} y_1$ and $\hat{S}(y_1^*) = -\mu y_1^*$. From this, we then get that $\hat{S}$ is not *-preserving.

Also, we see that $y_1 \in D(\hat{S}^2)$ and $\hat{S}^2(y_1) = \mu^{-2} y_1$. This yields $\hat{S}^2 \neq \iota$. ∎

2.7 The Haar weights

In this Section 2.7, we give an intensive study of the Haar weights on the quantum $E(2)$ group and its dual. The results from Section 1.4 play a crucial role in our calculations. In this way, we give an illustration of their usefulness.

The Haar weights are indispensable objects in the study of a quantum group example. From the general theory (cf. Section 1.1), we know that it is difficult to underestimate their importance. It is thus necessary to pay enough attention to the related part of the quantum group structure of (A, Φ) and $(\hat{A}, \hat{\Phi})$.

As it is said above, we apply the technique from Section 1.4 in our construction of the Haar weights. For this reason, there is only required a very short calculation to prove their invariance. Besides the invariance proof, we also provide a lot of further information about the Haar weights.

The Haar weight ψ on (A, Φ) is both left and right invariant. This precisely means that the quantum $E(2)$ group is unimodular. The quantum $\hat{E}(2)$ group is non-unimodular. In this case, we thus have that the left Haar weight $\hat{\varphi}$ and the right Haar weight $\hat{\psi} = \hat{\varphi}\hat{R}$ are different from each other.

We again mention that we are not the first ones to construct the Haar weights on the quantum $E(2)$ group and its dual. This credit should go to S. Baaj who already found a formula for the Haar weights in 1992; see [1, 2].

We asses our technique to prove the invariance of the Haar weights as more direct and easier than the method applied by Baaj. From the results in Section 1.4, we get a clear view of what is going on. The invariance proof given in [1] heavily depends on the typical structure of (A, Φ). For instance, it makes use of extended properties of the numbers $B(k, n)$ studied in Appendix A.

At a few points in this Section 2.7, we make use of von Neumann algebra theory. We therefore introduce the Notation 2.7.1 below.

Notation 2.7.1 *We denote $M = A''$ and $\hat{M} = \hat{A}''$.*

Also the next Notation 2.7.2 is important.

Notation 2.7.2 *We use* Tr *to denote the canonical trace on $B(H)^+$.*

The next Lemma 2.7.3 is crucial in our calculations.

Lemma 2.7.3 *Let $x \in M$ and $p, q \in \mathbb{Z}$. Then, we have*

$$\langle x(e_p \otimes e_q), e_p \otimes e_q \rangle = \langle x(e_0 \otimes e_q), e_0 \otimes e_q \rangle.$$

Proof. Let $k, l, m, n \in \mathbb{Z}$. From Corollary 2.3.14, it follows that

$$\big\langle (\omega_{k,l,m,n} \otimes \iota)W(e_p \otimes e_q), e_p \otimes e_q \big\rangle = B(0, l - q + 1)\delta_{k,m}\delta_{l,n}\delta_{k,2l}.$$

By Definition 2.4.1, $\big\{(\omega_{k,l,m,n} \otimes \iota)W \mid k, l, m, n \in \mathbb{Z}\big\}$ is weakly dense in M. ■

We also need the dual version of this result.

Lemma 2.7.4 *Let $x \in \hat{M}$ and $p, q \in \mathbb{Z}$. There exists a unique vector $\eta \in H$ such that*

$$x(e_p \otimes e_q) = e_p \otimes \eta.$$

Proof. Let $k, l, m, n \in \mathbb{Z}$. From Corollary 2.3.14, it follows that

$$\begin{aligned}(\iota \otimes \omega_{k,l,m,n})&W(e_p \otimes e_q) \\ &= e_p \otimes B(p - 2q - l + n, p - q - l + 1)\delta_{-k+2l+m-2n,p-2q}\, e_{p-q-l+n}.\end{aligned}$$

By Definition 2.4.2, $\big\{(\iota \otimes \omega_{k,l,m,n})W \mid k, l, m, n \in \mathbb{Z}\big\}$ is weakly dense in $\hat{M}$. ■

The quantum $E(2)$ group

Similar as in Sections 2.5 and 2.6, we first look at the case of the quantum $E(2)$ group. The Theorem 1.4.26 infers that there exists a family of weights ψ_q on (A, Φ) that are strongly right invariant. Without being industrious, we can find a $q \in \hat{M}'$ such that ψ_q is a right Haar weight. We then denote $\psi = \psi_q$. As said above, it turns out that ψ is both left and right invariant.

The results below complete the proof of Theorem 2.4.3. They also show that the method of Woronowicz [166] does not give the Haar weight (cf. Remark 2.7.20). This then indicates the value of the technique in Section 1.4.

In this Section 2.7, we make use of several notations as presented in Section B.1. We e.g. recall that the notation Tr_N is explained in Example B.1.33.

We apply the Proposition 1.4.28 in a straightforward manner. Hence, we dispose of a concrete scheme of how to construct the right Haar weight ψ.

The first step is to define a weight ψ_1 and a projection q.

Definition 2.7.5 *We define an n.s.f. weight ψ_1 on $B(H)^+$ by $\psi_1 = \mathrm{Tr}_N$. Hence, for all $x \in B(H)^+$, we have*

$$\psi_1(x) = \mathrm{Tr}_N(x) = \sum_{k,l\in\mathbb{Z}} \mu^{2(k-l)} \langle x(e_k \otimes e_l), e_k \otimes e_l \rangle.$$

Definition 2.7.6 *Let p be the one-dimensional projection on the subspace $\mathbb{C}e_0$. We define a projection $q \in B(H)$ by $q = p \otimes 1$.*

The next Lemma 2.7.7 is crucial.

Lemma 2.7.7 *We have $q \in \hat{M} \cap \hat{M}'$.*

Proof. Define $g \in \mathrm{C}_b(E)$ by $g(s,t) = \chi_{\{1\}}(t)$. We then have $q = g(a, |b|)$. Thus, we get that $q \in M(\hat{A})$; see Theorem 2.5.21.

The property that $q \in \hat{M}'$ follows easily from Lemma 2.7.4. ∎

It is now a cushy job to produce the Haar weight ψ.

Definition 2.7.8 *We define a weight ψ on A^+ by setting $\psi(x) = \psi_1(qxq)$. Thus, for all $x \in A^+$, we have*

$$\psi(x) = \sum_{k\in\mathbb{Z}} \mu^{-2k} \langle x(e_0 \otimes e_k), e_0 \otimes e_k \rangle.$$

The next Theorem 2.7.9 is the central property in this Section 2.7.

Theorem 2.7.9 *We have that ψ is a faithful KMS-weight on A^+. Further, the weight ψ is strongly left and strongly right invariant, i.e., we have*

$$\psi\big((\omega \otimes \iota)\Phi(x)\big) = \psi\big((\iota \otimes \omega)\Phi(x)\big) = \omega(1)\psi(x).$$

for all $x \in A^+$ and $\omega \in A^*_+$.

Proof. We define $y \in A$ by $y = (\omega_{0,0} \otimes \iota)W$. It is not difficult to compute that we have $\psi(y^*y) = 1$. See Proposition 2.7.13 for an explicit calculation.

The Proposition 1.4.28 yields that ψ is a faithful KMS-weight on A^+ that is strongly right invariant. Further, we also get that ψR is strongly left invariant.

Let $x \in A^+$ and $k, l \in \mathbb{Z}$. An easy calculation (using Lemma 2.7.3) gives

$$\begin{aligned}\langle R(x)(e_k \otimes e_l), e_k \otimes e_l \rangle &= \langle x(e_{-k+2l} \otimes e_l), e_{-k+2l} \otimes e_l \rangle \\ &= \langle x(e_0 \otimes e_l), e_0 \otimes e_l \rangle = \langle x(e_k \otimes e_l), e_k \otimes e_l \rangle.\end{aligned} \tag{2.63}$$

Together with Definition 2.7.8, this Equation (2.63) yields $\psi R = \psi$. ∎

The next Remark 2.7.10 is just a trivia.

Remark 2.7.10 *In the case of the positive vector functionals $\omega_{p,q}$, it is possible to prove the (strong) left and right invariance by a direct calculation.*

We illustrate how to do this in the case of left invariance.

Let $x \in A^+$ and $p, q \in \mathbb{Z}$. Using Proposition 2.3.13, we see that

$$\begin{aligned}\psi\big((\omega_{p,q} \otimes \iota)\Phi(x)\big) &= \sum_k \mu^{-2k} \big\langle (\omega_{p,q} \otimes \iota)\Phi(x)(e_0 \otimes e_k), e_0 \otimes e_k \big\rangle \\ &= \sum_k \mu^{-2k} \big\langle (x \otimes 1)W^*(e_p \otimes e_q \otimes e_0 \otimes e_k), W^*(e_p \otimes e_q \otimes e_0 \otimes e_k) \big\rangle \\ &= \sum_{k,t} \mu^{-2k} B(t, q-k+1)^2 \langle x(e_p \otimes e_{q-t}), e_p \otimes e_{q-t} \rangle \\ &= \sum_t \Big(\sum_k \mu^{-2k} B(q-t, q-k+1)^2 \Big) \langle x(e_0 \otimes e_t), e_0 \otimes e_t \rangle\end{aligned}$$

Notice that $x(e_p \otimes e_l) = x(e_0 \otimes e_l)$ for all $l \in \mathbb{Z}$; see Lemma 2.7.3.

For all $k, t \in \mathbb{Z}$, we have that

$$\mu^{-2k} B(q-t, q-k+1)^2 = \mu^{-2t} B(q-k, q-t+1)^2.$$

This follows from the formulas in Proposition 2.2.6.

Using the above results, we get

$$\begin{aligned}\psi\big((\omega_{p,q} \otimes \iota)\Phi(x)\big) &= \sum_t \Big(\sum_k \mu^{-2t} B(q-k, q-t+1)^2 \Big) \langle x(e_0 \otimes e_t), e_0 \otimes e_t \rangle \\ &= \sum_t \mu^{-2t} \langle x(e_0 \otimes e_t), e_0 \otimes e_t \rangle = \psi(x).\end{aligned}$$

This proves the left invariance property.

We now prove several useful properties of the KMS-weight ψ. This extra information about ψ does not contain any astonishing result. Nonetheless, it helps to get a clear picture of ψ and is therefore important.

From Theorem 1.1.10, we get the following uniqueness result.

Theorem 2.7.11 *Let η be a non-zero, densely defined, lower semi-continuous weight on A^+ that is either left or right invariant.*

Then, there exists a number $r > 0$ such that $\eta = r\psi$.

The next Proposition 2.7.12 displays the action of ψ on basic elements.

Proposition 2.7.12 *Let $x = \sum c^k f_k(d) \in A$. We have*

$$\psi(x^*x) = \frac{1}{2\pi} \sum_{k,m} \mu^{-2m} \int_0^{2\pi} |f_k(\mu^{-m} e^{it})|^2 \, dt.$$

If $f_k \in \mathrm{K}(\bar{\mathbb{C}}^\mu)$ for every k, then $x \in \mathfrak{N}_\psi \cap \mathfrak{N}_\psi^*$.

Proof. Let $x = \sum c^k f_k(d) \in A$. It is straightforward to compute that

$$\psi(x^*x) = \sum_{k,m} \mu^{-2m} \big\langle |f_k|^2(\mu^{-m}s)e_0, e_0 \big\rangle.$$

The Stone-Weierstrass theorem gives that the set of polynomials on S^1 is dense in $\mathrm{C}(S^1)$. We can then use the functional calculus to show that

$$\langle g(s)e_n, e_n \rangle = \frac{1}{2\pi} \int_0^{2\pi} g(e^{it}) \, dt$$

for all $g \in \mathrm{C}(S^1)$ and $n \in \mathbb{Z}$. ■

We also have the Proposition 2.7.13 below.

Proposition 2.7.13 *Let $k, l, m, n \in \mathbb{Z}$. Denote $x = (\omega_{k,l,m,n} \otimes \iota)W$. Then, we have that $x \in \mathfrak{N}_\psi \cap \mathfrak{N}_\psi^*$. We have*

$$\begin{aligned} \psi(x^*x) &= \mu^{2(n-k)} \delta_{k,m} < +\infty, \\ \psi(xx^*) &= \mu^{-2l} \delta_{k,m} < +\infty. \end{aligned}$$

Proof. Let $k, l, m, n \in \mathbb{Z}$ and $x = (\omega_{k,l,m,n} \otimes \iota)W$. We only consider the first formula above. It is completely similar to prove the second one.
Take $q \in \mathbb{Z}$. From Corollary 2.3.14, we get that

$$\langle x^*x(e_0 \otimes e_q), e_0 \otimes e_q \rangle = B(n-l, k-l-q+1)^2 \delta_{k,m}. \tag{2.64}$$

The Proposition 2.2.6 implies that we have

$$\mu^{-2q} B(n-l, k-l-q+1)^2 = \mu^{2(n-k)} B(k-l-q, n-l+1)^2. \tag{2.65}$$

Using Definition 2.7.8 and Equations (2.64) and (2.65), we find that

$$\begin{aligned} \psi(x^*x) &= \sum_q \mu^{-2q} \langle x^*x(e_0 \otimes e_q), e_0 \otimes e_q \rangle = \sum_q \mu^{-2q} B(n-l, k-l-q+1)^2 \delta_{k,m} \\ &= \sum_q \mu^{2(n-k)} B(k-l-q, n-l+1)^2 \delta_{k,m} = \mu^{2(n-k)} \delta_{k,m}. \end{aligned}$$

The last step follows from Proposition A.9. ■

The next Remark 2.7.14 further considers the action of ψ.

Remark 2.7.14 *The Propositions 2.7.12 and 2.7.13 can be used to construct elements $y \in A$ for which we have $y \notin \mathfrak{M}_\psi$ or $y \notin \mathfrak{N}_\psi$.*

Define $f \in \mathrm{C}_0(\bar{\mathbb{C}}^\mu)$ by $f(z) = 1$ if $|z| \leq 1$ and $f(z) = \frac{1}{|z|}$ if $|z| > 1$. We get from the Proposition 2.7.12 that $f(d) \in A^+$ and $\psi(f(d)) = +\infty$.

Then, define a vector $\xi \in H$ by $\xi = e_0 \otimes \sum_{n\in\mathbb{Z}} \frac{1}{n} e_n$. From Proposition 2.7.13, it follows that $(\omega_\xi \otimes \iota)W \notin \mathfrak{N}_\varphi \cup \mathfrak{N}_{\varphi^*}$.

We now introduce a one-parameter group (σ_t). It will appear as the modular automorphism group of the KMS-weight ψ; see Proposition 2.7.17.

All the results in Definition 2.7.15 follow directly from Equation (2.66) or can be obtained by using the formulas in Proposition 2.7.16 below.

Definition 2.7.15 *For every $t \in \mathbb{R}$, we consider the *-automorphism σ_t on A defined by*

$$\sigma_t : A \to A : x \mapsto \hat{N}_0^{-it} x \hat{N}_0^{it}. \tag{2.66}$$

Then (σ_t) is a norm continuous one-parameter group on A.

It is easy to calculate the action of σ on basic elements.

Proposition 2.7.16 *Let $x = \sum c^k f_k(d) \in A$. Then, we have that x is analytic with respect to σ. For every $z \in \mathbb{C}$, we have*

$$\sigma_z(x) = \sum \mu^{-2ikz} c^k f_k(d).$$

Further, for all $z \in \mathbb{C}$, we have that

$$\Big\{ \sum_{k\in\mathbb{Z}} c^k f_k(d) \;\Big|\; f_k \in \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \text{ for every } k \text{ and } f_k \neq 0 \text{ for finitely many } k \Big\}$$

is a core for σ_z.

The Proposition 2.7.17 describes the most important (standard) objects related to the Haar weight ψ. In the proof, we use Tomita-Takesaki theory.

Proposition 2.7.17 *We have that ψ is a faithful KMS-weight on A^+ with σ as the (unique) modular automorphism group.*

The GNS-construction $(H_\psi, \pi_\psi, \Lambda_\psi)$ of ψ is given by

- $H_\psi = \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$;
- $\Lambda_\psi : \mathfrak{N}_\psi \to H_\psi : x \mapsto \Lambda_\psi(x) = \sum_{k\in\mathbb{Z}} \mu^{-k} x(e_0 \otimes e_k) \otimes e_k$;
- $\pi_\psi : A \to B(H_\psi) : x \mapsto x \otimes 1$.

The modular conjugation J_ψ and the modular operator ∇_ψ of ψ in the above GNS-construction $(H_\psi, \pi_\psi, \Lambda_\psi)$ are given by the formulas

$$\nabla_\psi = 1 \otimes m^{-2} \otimes m^2$$

and

$$J_\psi(e_p \otimes e_q \otimes e_r) = e_{-p} \otimes e_r \otimes e_q$$

where $p, q, r \in \mathbb{Z}$.

Proof. Theorem 2.7.9 yields that ψ is a faithful KMS-weight. It is easy to check that $(H_\psi, \pi_\psi, \Lambda_\psi)$ is a GNS-construction for ψ. The only thing that may ask for some extra explanation is the fact that $\Lambda_\psi(\mathfrak{N}_\psi)$ is dense in H_ψ.

Let $k, l, m \in \mathbb{Z}$. We consider the function $f \in \mathrm{K}(\bar{\mathbb{C}}^\mu)$ defined by

$$f(z) = \begin{cases} \left(\frac{z}{|z|}\right)^{k-l+m} & \text{if } |z| = \mu^{-l}; \\ 0 & \text{otherwise.} \end{cases}$$

We then define an element $x \in A$ by $x = \mu^m f_1(d) c^{l-m}$.

From Proposition 2.7.12, we get $x \in \mathfrak{N}_\psi \cap \mathfrak{N}_\psi^*$. It is straightforward to prove that $\Lambda_\psi(x) = e_k \otimes e_l \otimes e_m$. This implies that $\Lambda_\psi(\mathfrak{N}_\psi)$ is dense in H_ψ.

We define a weight ψ_{vn} on M^+ by setting $\psi_{\mathrm{vn}}(x) = \psi_1(qxq)$. It follows from Proposition 1.4.17 that ψ_{vn} is an n.s.f. weight.

Of course, the GNS-construction $(H_{\psi_{\mathrm{vn}}}, \pi_{\psi_{\mathrm{vn}}}, \Lambda_{\psi_{\mathrm{vn}}})$ of ψ_{vn} is then given by the same formulas as $(H_\psi, \pi_\psi, \Lambda_\psi)$. Starting from this GNS-construction, it is quite standard to built up the Tomita-Takesaki objects of the n.s.f. weight ψ_{vn}. In this way, we can find the given formulas for ∇_ψ and J_ψ.

If we now invoke Tomita-Takesaki theory, it is not too hard to prove that the modular automorphism group σ^{vn} of the n.s.f. weight ψ_{vn} is given by

$$\sigma_t^{\mathrm{vn}} : M \to M : x \mapsto \hat{N}_0^{-it} x \hat{N}_0^{it}.$$

We know from Definition 2.7.15 that σ is a norm continuous one-parameter group on A. Using the above results, it is not difficult anymore to prove that σ is the modular automorphism group of the faithful KMS-weight ψ. ∎

The next Proposition 2.7.18 gives some further properties of ψ. The fact that ψ is invariant with respect to the scaling group (τ_t) implies that the *scaling constant* ν of the quantum $E(2)$ group is equal to 1; see Proposition 2.8.3.

Proposition 2.7.18 *Let $t \in \mathbb{R}$. We have*

1. $\psi\sigma_t = \psi$,

2. $\psi\tau_t = \psi$,

3. $\psi R = \psi$.

Proof. Take $t \in \mathbb{R}$. The property $\psi\sigma_t = \psi$ is contained in Proposition 2.7.17. From the proof of Theorem 2.7.9, we know that $\psi R = \psi$.

Let $x \in A$ and $k, l \in \mathbb{Z}$. It is direct to check that

$$\langle \tau_t(x)(e_k \otimes e_l), e_k \otimes e_l \rangle = \langle x(e_k \otimes e_l), e_k \otimes e_l \rangle.$$

Together with Definition 2.7.8, this equation yields that $\psi\tau_t = \psi$. ∎

The Remark 2.7.19 contains some simple facts.

Remark 2.7.19 *It is possible to apply the technique from Section 1.4 with other operators than N and q. We give some concise comment on this fact.*

Let $k \in \mathbb{Z}$. We define a strictly positive operator $\hat{N}_k$ on H by $\hat{N}_k = m^k \otimes m^2$. It is easy to check that $\hat{N}_k$ satisfies $\hat{\tau}_t(x) = \hat{N}_k^{-it} x \hat{N}_k^{it}$ for $t \in \mathbb{R}$ and $x \in \hat{A}$.

Let p_k be the projection on $\mathbb{C}e_k$. We define $q_k \in \hat{M} \cap \hat{M}'$ by $q_k = p_k \otimes 1$.

For all $r, s \in \mathbb{Z}$, we define a weight $\psi_{r,s}$ on A^+ by $\psi_{r,s}(x) = \mathrm{Tr}_{\hat{N}_r}(q_s x q_s)$. We can now apply Proposition 1.4.28 to show that every weight $\psi_{r,s}$ is a faithful KMS-weight on A^+ that is strongly right invariant.

The Lemma 2.7.3 gives that $\psi_{r,s} = \mu^{2rs}\psi$ for all $r, s \in \mathbb{Z}$.

For every $k \in \mathbb{Z}$, we have that q_k commutes with $\hat{N}_0$. So, we can define a positive operator h on H by setting $h = \hat{N}_0^{-1} q_k$. We then have that $\psi(x) = \mathrm{Tr}_h(x)$ for all $x \in A^+$. However, h is not strictly positive.

We have that $q_k \in D(\hat{S})$ and $\hat{S}(q_k) = q_k$ for every $k \in \mathbb{Z}$.

The next Remark 2.7.20 shows that Woronowicz' formula in [166] cannot be used to produce a Haar weight in the case of the quantum $E(2)$ group.

Remark 2.7.20 *Theorem 1.4.26 implies that the weight ψ_1 is a faithful, lower semi-continuous weight on A^+ that is strongly right invariant.*

It is not difficult to prove that ψ_1 is not densely defined.

Let $x \in M^+$. From Lemma 2.7.3, we get

$$\psi_1(x) = \sum_{k \in \mathbb{Z}} \mu^{2k} \psi(x) \tag{2.67}$$

Because ψ is faithful, we can deduce that $\mathfrak{M}_{\psi_1} \cap A^+ = \{0\}$.

The quantum $\hat{E}(2)$ group

We now apply the results of Section 1.4 to find a formula for the Haar weights on the quantum $\hat{E}(2)$ group $(\hat{A}, \hat{\Phi})$. It turns out that the left Haar weight $\hat{\varphi}$ and the right Haar weight $\hat{\psi} = \hat{\varphi}\hat{R}$ are different from each other.

From the general theory (cf. Proposition 1.1.25), we know that there exists a unique *modular element* $\hat{\delta}$ that connects the two weights $\hat{\varphi}$ and $\hat{\psi}$ via the formula $\hat{\psi} = \hat{\varphi}_{\hat{\delta}}$. We will calculate in Proposition 2.8.4 that $\hat{\delta} = a^4$.

The Definition 2.7.21 defines the Haar weights $\hat{\varphi}$ and $\hat{\psi}$. Unlike the case of the quantum $E(2)$ group, we can employ the formulas from Section 1.4 with $q = 1$. We do not know any reason why this dissimilarity occurs.

Definition 2.7.21 *We define weights $\hat{\varphi}, \hat{\psi}$ on $\hat{A}^+$ by setting that*

$$\hat{\varphi}(x) = \mathrm{Tr}_N(x) \qquad \textit{and} \qquad \hat{\psi}(x) = \mathrm{Tr}_{\hat{N}_0}(x).$$

Thus, for all $x \in \hat{A}^+$, we have

$$\hat{\varphi}(x) = \sum_{k,l \in \mathbb{Z}} \mu^{2(k-l)} \langle x(e_k \otimes e_l), e_k \otimes e_l \rangle,$$

$$\hat{\psi}(x) = \sum_{k,l \in \mathbb{Z}} \mu^{2l} \langle x(e_k \otimes e_l), e_k \otimes e_l \rangle.$$

It is clear that we can now proceed quite rapidly. The fact that we can take $q = 1$ gives us the opportunity to dodge extra computations.

The next Theorem 2.7.22 states that $\hat{\varphi}$ is the left Haar weight.

Theorem 2.7.22 *We have that $\hat{\varphi}$ is a faithful KMS-weight on $\hat{A}^+$. Further, the weight $\hat{\varphi}$ is strongly left invariant, i.e., we have*

$$\hat{\varphi}\big((\omega \otimes \iota)\hat{\Phi}(x)\big) = \omega(1)\hat{\varphi}(x).$$

for all $x \in \hat{A}^+$ and $\omega \in \hat{A}^*_+$.

Proof. We define $y \in \hat{A}$ by $y = (\iota \otimes \omega_{0,0})W$. It is not difficult to calculate that we have $\hat{\varphi}(y^* y) = 1$. See Proposition 2.7.27 for an explicit calculation.

Because $\hat{\Phi}$ is defined in the opposite way, Proposition 1.4.28 yields that $\hat{\varphi}$ is a faithful KMS-weight on A^+ which is strongly left invariant. ∎

It is then easy to derive that $\hat{\psi}$ is the right Haar weight.

Theorem 2.7.23 *We have that $\hat{\psi}$ is a faithful KMS-weight on $\hat{A}^+$. Further, the weight $\hat{\psi}$ is strongly right invariant, i.e., we have*

$$\hat{\psi}\big((\iota \otimes \omega)\hat{\Phi}(x)\big) = \omega(1)\hat{\psi}(x).$$

for all $x \in \hat{A}^+$ and $\omega \in \hat{A}^*_+$.

Proof. We get from Theorem 2.7.23 and Proposition 1.4.28 that $\hat{\varphi}\hat{R}$ is a faithful KMS-weight which is strongly right invariant.

Let $x \in \hat{A}^+$ and $k, l \in \mathbb{Z}$. An easy calculation gives that

$$\langle \hat{R}(x)(e_k \otimes e_l), e_k \otimes e_l \rangle = \langle x(e_k \otimes e_{k-l}), e_k \otimes e_{k-l} \rangle.$$

We hence have

$$\begin{aligned} \hat{\varphi}\hat{R}(x) &= \sum_{k,l} \mu^{2(k-l)} \langle \hat{R}(x)(e_k \otimes e_l), e_k \otimes e_l \rangle = \sum_{k,l} \mu^{2(k-l)} \langle x(e_k \otimes e_{k-l}), e_k \otimes e_{k-l} \rangle \\ &= \sum_{k,l} \mu^{2l} \langle x(e_k \otimes e_l), e_k \otimes e_l \rangle = \hat{\psi}(x) = \hat{\psi}(x). \end{aligned}$$

This implies that $\hat{\varphi}\hat{R} = \hat{\psi}$. ∎

We now prove several useful properties of the KMS-weights $\hat{\varphi}$ and $\hat{\psi}$. This extra information helps to get a clear picture of their behavior.

From Theorem 1.1.10, we get the following uniqueness result.

Theorem 2.7.24 *Let η be a non-zero, densely defined, lower semi-continuous weight on $\hat{A}^+$. The following properties hold:*

- *If η is left invariant, there exist a number $r > 0$ such that $\eta = r\hat{\varphi}$;*
- *If η is right invariant, there exist a number $r > 0$ such that $\eta = r\hat{\psi}$.*

The next Proposition 2.7.25 displays the action of $\hat{\varphi}$ and $\hat{\psi}$ on basic elements. It is straightforward to prove the mentioned formulas.

Proposition 2.7.25 *Let $x = \sum u^k g_k(a, |b|) \in \hat{A}$. We have that*

$$\begin{aligned} \hat{\varphi}(x^* x) &= \sum_{k,m,n} \mu^{2(m-n)} \big| g_k(\mu^{-\frac{1}{2}m+n}, \mu^{\frac{1}{2}m}) \big|^2, \\ \hat{\psi}(x^* x) &= \sum_{k,m,n} \mu^{2n} \big| g_k(\mu^{-\frac{1}{2}m+n}, \mu^{\frac{1}{2}m}) \big|^2. \end{aligned}$$

If $g_k \in \mathrm{K}(E)$ for every k, then $x \in \mathfrak{N}_{\hat{\varphi}} \cap \mathfrak{N}_{\hat{\varphi}}^* \cap \mathfrak{N}_{\hat{\psi}} \cap \mathfrak{N}_{\hat{\psi}}^*$.

The Remark 2.7.26 is just a detail.

Remark 2.7.26 *Consider the two sets*

$$D_1 = \{(\mu^{-\frac{1}{2}k}, \mu^{-\frac{1}{2}k}) \mid k \in \mathbb{N}\} \quad \textit{and} \quad D_2 = \{(\mu^{\frac{1}{2}k}, \mu^{-\frac{1}{2}k}) \mid k \in \mathbb{N}\}.$$

Define $g_1, g_2 \in \mathrm{C}_0(E)$ by $g_1(x, y) = \frac{1}{x^2} \chi_{D_1}(x, y)$ and $g_2(x, y) = x^2 \chi_{D_2}(x, y)$.

Using the formulas in Proposition 2.7.25, we see that

- $g_1(a,|b|) \in \mathfrak{M}_{\hat{\varphi}}^+$ *with* $\hat{\psi}(g_1(a,|b|)) = +\infty$,
- $g_2(a,|b|) \in \mathfrak{M}_{\hat{\psi}}^+$ *with* $\hat{\varphi}(g_2(a,|b|)) = +\infty$.

Hence, $\mathfrak{M}_{\hat{\varphi}}^+ \not\subseteq \mathfrak{M}_{\hat{\psi}}^+$ *and* $\mathfrak{M}_{\hat{\psi}}^+ \not\subseteq \mathfrak{M}_{\hat{\varphi}}^+$.

We then have the Proposition 2.7.27 below.

Proposition 2.7.27 *Let* $k,l,m,n \in \mathbb{Z}$. *Denote* $x = (\iota \otimes \omega_{k,l,m,n})W$. *Then, we have that* $x \in \mathfrak{N}_{\hat{\varphi}} \cap \mathfrak{N}_{\hat{\varphi}}^* \cap \mathfrak{N}_{\hat{\psi}} \cap \mathfrak{N}_{\hat{\psi}}^*$. *We have*

$$\begin{aligned} \hat{\varphi}(x^*x) &= \mu^{-2(k-2l-m+2n)} < +\infty, \qquad & \hat{\psi}(x^*x) &= \mu^{2n} < +\infty, \\ \hat{\varphi}(xx^*) &= \mu^{2(2k+l-2m)} < +\infty, & \hat{\psi}(xx^*) &= \mu^{2(k+l-m)} < +\infty. \end{aligned}$$

Proof. Let $k,l,m,n \in \mathbb{Z}$ and $x = (\omega_{k,l,m,n} \otimes \iota)W$. We only consider the first formula above. It is completely similar to prove the other ones.

Take $p,q \in \mathbb{Z}$. From Corollary 2.3.14, we get that

$$\begin{aligned} &\langle x^*x(e_p \otimes e_q), e_p \otimes e_q \rangle \\ &\qquad = B(-k+l+m-n, q-k+l+m-2n+1)^2 \delta_{p-2q,-k+2l+m-2n}. \end{aligned} \tag{2.68}$$

The Proposition 2.2.6 implies that we have

$$\begin{aligned} &\mu^{2(q-k+2l+m-2n)} B(-k+l+m-n, q-k+l+l-2n+1)^2 \\ &\qquad\qquad = \mu^{2(-k+2l+m-2n)} B(q-k+l-n, -k+l+m-n+1)^2. \end{aligned} \tag{2.69}$$

Using Definition 2.7.21 and Equations (2.68) and (2.69), we find that

$$\begin{aligned} \hat{\varphi}(x^*x) &= \sum_{p,q} \mu^{2(p-q)} \langle x^*x(e_p \otimes e_q), e_p \otimes e_q \rangle \\ &= \sum_{p,q} \mu^{2(p-q)} B(-k+l+m-n, q-k+l+m-2n+1)^2 \delta_{p-2q,-k+2l+m-2n} \\ &= \sum_{q} \mu^{2(q-k+2l+m-2n)} B(-k+l+m-n, q-k+l+m-2n+1)^2 \\ &= \sum_{q} \mu^{2(-k+2l+m-2n)} B(q-k+l-n, -k+l+m-n+1)^2 \\ &= \mu^{-2(k-2l-m+2n)}. \end{aligned}$$

The last step follows from Proposition A.9. ∎

The Remark 2.7.28 is again just a detail.

Remark 2.7.28 *Define a vector* $\xi \in H$ *by* $\xi = e_0 \otimes \sum_{n\in\mathbb{Z}} \frac{1}{n} e_n$. *From Proposition 2.7.27, it follows that* $(\iota \otimes \omega_\xi)W \notin \mathfrak{N}_{\hat{\varphi}} \cup \mathfrak{N}_{\hat{\varphi}^*} \cup \mathfrak{N}_{\hat{\psi}} \cup \mathfrak{N}_{\hat{\psi}^*}$.

We now introduce two one-parameter groups $(\hat{\sigma}_t)$ and $(\hat{\sigma}'_t)$. They will appear as the modular automorphism groups of respectively the KMS-weight $\hat{\varphi}$ and the KMS-weight $\hat{\psi}$; see Propositions 2.7.32 and 2.7.33.

All the results in Definition 2.7.29 follow directly from the defining relations or can be obtained by using the formulas in Proposition 2.7.30.

Definition 2.7.29 *For every $t \in \mathbb{R}$, we consider the *-automorphisms $\hat{\sigma}_t$ and $\hat{\sigma}'_t$ on $\hat{A}$ defined by*

$$\hat{\sigma}_t : \hat{A} \to \hat{A} : x \mapsto N^{it} x N^{-it},$$
$$\hat{\sigma}'_t : \hat{A} \to \hat{A} : x \mapsto \hat{N}_0^{it} x \hat{N}_0^{-it}.$$

Then $(\hat{\sigma}_t)$ and $(\hat{\sigma}'_t)$ are norm continuous one-parameter groups on $\hat{A}$.

It is easy to calculate the action of $\hat{\sigma}$ and $\hat{\sigma}'$ on basic elements.

Proposition 2.7.30 *Let $x = \sum u^k g_k(a,|b|) \in \hat{A}$. Then, we have that x is analytic with respect to both $\hat{\sigma}$ and $\hat{\sigma}'$. For every $z \in \mathbb{C}$, we have*

$$\hat{\sigma}_z(x) = \sum \mu^{-2ikz} u^k g_k(a,|b|) \quad \text{and} \quad \hat{\sigma}'_z(x) = \sum \mu^{2ikz} u^k g_k(a,|b|).$$

Further, for all $z \in \mathbb{C}$, we have that

$$\Big\{ \sum_{k \in \mathbb{Z}} u^k g_k(a,|b|) \mid g_k \in \mathrm{C}_0(E) \text{ for every } k,\ g_k \neq 0 \text{ for finitely many } k$$
$$\text{and } g_k(s,0) = 0 \text{ for every } s \in \mathbb{R}(\mu^{\frac{1}{2}}) \text{ if } k \neq 0 \Big\}$$

is a core for both $\hat{\sigma}_z$ and $\hat{\sigma}'_z$.

We observe the Lemma 2.7.31 below.

Lemma 2.7.31 *Let $t \in \mathbb{R}$. For all $x \in \hat{A}$, we have that*

$$\hat{\sigma}_t(x) = \hat{\sigma}'_{-t}(x) = \hat{\tau}_t(x).$$

Proof. This is a direct consequence of Definitions 2.7.29 and 2.6.35 combined with Lemma 2.6.37. It can also easily be checked on the generators. ∎

The two following Propositions 2.7.32 and 2.7.33 describe the most important (standard) objects related to the Haar weights $\hat{\varphi}$ and $\hat{\psi}$.

Proposition 2.7.32 *We have that $\hat{\varphi}$ is a faithful KMS-weight on $\hat{A}^+$ with $\hat{\sigma}$ as the (unique) modular automorphism group.*

The GNS-construction $(H_{\hat{\varphi}}, \pi_{\hat{\varphi}}, \Lambda_{\hat{\varphi}})$ of $\hat{\varphi}$ is given by

- $H_{\hat{\varphi}} = \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$;

- $\Lambda_{\hat{\varphi}} : \mathfrak{N}_{\hat{\varphi}} \to H_{\hat{\varphi}} : x \mapsto \Lambda_{\hat{\varphi}}(x) = \sum_{k,l \in \mathbb{Z}} \mu^{k-l} x(e_k \otimes e_l) \otimes e_l$;
- $\pi_{\hat{\varphi}} : A \to B(H_{\hat{\varphi}}) : x \mapsto x \otimes 1$.

The modular conjugation $J_{\hat{\varphi}}$ and the modular operator $\nabla_{\hat{\varphi}}$ of $\hat{\varphi}$ in the above GNS-construction $(H_{\hat{\varphi}}, \pi_{\hat{\varphi}}, \Lambda_{\hat{\varphi}})$ are given by the formulas

$$\nabla_{\hat{\varphi}} = 1 \otimes m^{-2} \otimes m^2$$

and

$$J_{\hat{\varphi}}(e_p \otimes e_q \otimes e_r) = e_p \otimes e_r \otimes e_q$$

where $p, q, r \in \mathbb{Z}$.

Proof. Theorem 2.7.22 yields that $\hat{\varphi}$ is a faithful KMS-weight. It is easy to check that $(H_{\hat{\varphi}}, \pi_{\hat{\varphi}}, \Lambda_{\hat{\varphi}})$ is a GNS-construction for $\hat{\varphi}$.

We remark that the result of Lemma 2.7.4 is certainly helpful when proving the formula $\langle \Lambda_{\hat{\varphi}}(x), \Lambda_{\hat{\varphi}}(y) \rangle = \hat{\varphi}(y^* x)$. Further, we also give some extra explanation about the property that $\Lambda_{\hat{\varphi}}(\mathfrak{N}_{\hat{\varphi}})$ is dense in $H_{\hat{\varphi}}$.

Let $k, l, m \in \mathbb{Z}$. We define a function $g \in \mathrm{K}(E)$ by $g = \chi_{\{(\mu^{-\frac{1}{2}k+m}, \mu^{\frac{1}{2}k})\}}$.

We consider the element $y \in \hat{A}$ defined by $y = \mu^{m-k} u^{l-m} g(a, |b|)$. Then, we get from Proposition 2.7.27 that $y \in \mathfrak{N}_{\hat{\varphi}} \cap \mathfrak{N}_{\hat{\varphi}}^*$. It is straightforward to prove that $\Lambda_{\hat{\varphi}}(y) = e_k \otimes e_l \otimes e_m$. This implies that $\Lambda_{\hat{\varphi}}(\mathfrak{N}_{\hat{\varphi}})$ is dense in $H_{\hat{\varphi}}$.

Starting from the GNS-construction $(H_{\hat{\varphi}}, \pi_{\hat{\varphi}}, \Lambda_{\hat{\varphi}})$, it is quite standard to find the mentioned formulas for for $\nabla_{\hat{\varphi}}$ and $J_{\hat{\varphi}}$.

From Definition 2.7.29, we know that the one-parameter group $\hat{\sigma}$ is norm continuous on $\hat{A}$. It is then a direct consequence of Definitions 2.7.21 and 2.7.29 that $\hat{\sigma}$ is the modular automorphism group of the KMS-weight $\hat{\varphi}$. ∎

Proposition 2.7.33 *We have that $\hat{\psi}$ is a faithful KMS-weight on $\hat{A}^+$ with $\hat{\sigma}'$ as the (unique) modular automorphism group.*

The GNS-construction $(H_{\hat{\psi}}, \pi_{\hat{\psi}}, \Lambda_{\hat{\psi}})$ of $\hat{\psi}$ is given by

- $H_{\hat{\psi}} = \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$;
- $\Lambda_{\hat{\psi}} : \mathfrak{N}_{\hat{\psi}} \to H_{\hat{\psi}} : x \mapsto \Lambda_{\hat{\psi}}(x) = \sum_{k,l \in \mathbb{Z}} \mu^{l} x(e_k \otimes e_l) \otimes e_l$;
- $\pi_{\hat{\psi}} : A \to B(H_{\hat{\psi}}) : x \mapsto x \otimes 1$.

The modular conjugation $J_{\hat{\psi}}$ and the modular operator $\nabla_{\hat{\psi}}$ of $\hat{\psi}$ in the above GNS-construction $(H_{\hat{\psi}}, \pi_{\hat{\psi}}, \Lambda_{\hat{\psi}})$ are given by the formulas

$$\nabla_{\hat{\psi}} = 1 \otimes m^2 \otimes m^{-2}$$

and

$$J_{\hat{\psi}}(e_p \otimes e_q \otimes e_r) = e_{-p} \otimes e_r \otimes e_q$$

where $p, q, r \in \mathbb{Z}$.

Proof. We can almost copy the proof of Proposition 2.7.32. For $\hat{\psi}$, we have to consider the element $y' \in \hat{A}$ defined by $y' = \mu^{-m} u^{l-m} g(a, |b|)$ in order to have $\Lambda_{\hat{\psi}}(y') = e_k \otimes e_l \otimes e_m$. The other things are completely similar. ■

The next Proposition 2.7.34 gives some further properties of $\hat{\varphi}$ and $\hat{\psi}$. We make use of the above results so as to formulate a short proof.

Proposition 2.7.34 *Let $t \in \mathbb{R}$. We have*

1. $\hat{\varphi}\hat{\sigma}_t = \hat{\varphi}\hat{\sigma}'_t = \hat{\varphi}$ *and* $\hat{\psi}\hat{\sigma}'_t = \hat{\psi}\hat{\sigma}_t = \hat{\psi}$,
2. $\hat{\varphi}\hat{\tau}_t = \hat{\varphi}$ *and* $\hat{\psi}\hat{\tau}_t = \hat{\psi}$,
3. $\hat{\varphi}\hat{R} = \hat{\psi}$ *and* $\hat{\psi}\hat{R} = \hat{\varphi}$.

Proof. First notice that the properties $\hat{\varphi}\hat{\sigma}_t = \hat{\varphi}$ and $\hat{\psi}\hat{\sigma}'_t = \hat{\psi}$ are contained in the Propositions 2.7.32 and 2.7.33.

Take $x \in \hat{A}$ and $k, l \in \mathbb{Z}$. It is straightforward to check that

$$\langle \hat{\tau}_t(x)(e_k \otimes e_l), e_k \otimes e_l \rangle = \langle x(e_k \otimes e_l), e_k \otimes e_l \rangle.$$

If we combine this equation with Definition 2.7.21 and Lemma 2.7.31, we find all the mentioned formulas concerning $\hat{\tau}$, $\hat{\sigma}$ and $\hat{\sigma}'$.

We know from the proof of Theorem 2.7.23 that $\hat{\varphi}\hat{R} = \hat{\psi}$. Because $\hat{R}^2 = \iota$ (cf. Definition 2.6.33), we also get $\hat{\psi}\hat{R} = \hat{\varphi}$. ■

2.8 Features of the quantum $E(2)$ group

We contemplate in this Section 2.8 the main features of the quantum $E(2)$ group. In this study, we are of course swayed by the general theory.

Each of the subsequent subsections is used to focus on one typical aspect of the quantum groups (A, Φ) and $(\hat{A}, \hat{\Phi})$. Since we can rely on all of the elaborated results in Sections 2.4 to 2.7, it is possible to keep the treatments quite concise. The Theorem 2.8.54 then sums up the most important properties.

Before we explore the traits of the quantum $E(2)$ group and its dual, we first restate the two central Theorems 2.4.3 and 2.4.4.

The Theorem 2.8.1 is about the quantum $E(2)$ group.

Theorem 2.8.1 *The pair (A, Φ) is a locally compact quantum group.*

The Theorem 2.8.2 states the dual result.

Theorem 2.8.2 *The pair $(\hat{A}, \hat{\Phi})$ is a locally compact quantum group.*

2.8.1 The modular elements and the scaling constants

We first study the *modular elements* and the *scaling constants*. These two objects are defined in Propositions 1.1.23 and 1.1.25.

We recall that we already know that the quantum $E(2)$ group is unimodular and that the quantum $\hat{E}(2)$ group is non-unimodular. This follows from the results about the Haar weights as formulated in the Section 2.7.

It is not difficult to find concrete formulas for the modular elements. We just bring together some of the aforesaid properties of the Haar weights. Further, it is almost direct to find that the scaling constants are equal to 1.

The case of the quantum $E(2)$ group is quite trivial.

Proposition 2.8.3 *The modular element δ of (A, Φ) is the identity operator. Thus, the quantum $E(2)$ group is unimodular.*

Further, the scaling constant ν of (A, Φ) equals 1.

Proof. Proposition 2.7.18 gives that $\psi R = \psi$. Hence, we get $\delta = 1$. This immediately implies that $\nu = 1$. ∎

For the quantum $\hat{E}(2)$ group, things are more interesting.

Proposition 2.8.4 *The modular element $\hat{\delta}$ of $(\hat{A}, \hat{\Phi})$ is the strictly positive operator a^4. Thus, the quantum $\hat{E}(2)$ group is non-unimodular.*

Further, the scaling constant $\hat{\nu}$ of $(\hat{A}, \hat{\Phi})$ equals 1.

Proof. From Proposition 2.7.34, we know that $\hat{\varphi}\hat{\tau}_t = \hat{\varphi}$ for all $t \in \mathbb{R}$. This then yields that we have $\hat{\nu} = 1$.

Using standard techniques from the Tomita-Takesaki theory, it is not difficult to prove that $\hat{\psi} = \hat{\varphi}_{a^4}$. It follows from Proposition 2.7.30 that $\hat{\sigma}_t(a^4) = a^4$ for all $t \in \mathbb{R}$. We hence get that $\hat{\delta} = a^4$. ∎

From the general theory, we can find a lot of information about the modular element $\hat{\delta}$. We include the most important properties.

The next Proposition 2.8.5 ensues from Proposition 1.1.25.

Proposition 2.8.5 *We have that $\hat{\delta}$ is characterized as the only strictly positive element in $\hat{A}^\eta$ such that $\hat{\sigma}_t(\hat{\delta}) = \hat{\delta}$ for all $t \in \mathbb{R}$ and $\hat{\psi} = \hat{\varphi}_{\hat{\delta}}$.*

We have the following basic properties:

1. $\hat{\psi} = \hat{\varphi}_{\hat{\delta}}$ *and* $\hat{\varphi} = \hat{\psi}_{\hat{\delta}^{-1}}$,
2. $\hat{\Phi}(\hat{\delta}) = \hat{\delta} \otimes \hat{\delta}$,
3. $\hat{R}(\hat{\delta}) = \hat{\delta}^{-1}$ *and* $\hat{\tau}_t(\hat{\delta}) = \hat{\delta}$ *for all* $t \in \mathbb{R}$,
4. *If* $t \in \mathbb{R}$, *then* $\hat{\delta}^{it} \in D(\hat{S})$ *and* $\hat{S}(\hat{\delta}^{it}) = \hat{\delta}^{-it}$,
5. $\hat{\sigma}_t(\hat{\delta}) = \hat{\sigma}'_t(\hat{\delta}) = \hat{\delta}$ *for all* $t \in \mathbb{R}$,
6. $\hat{\sigma}'_t(a) = \hat{\delta}^{it}\hat{\sigma}_t(a)\hat{\delta}^{-it}$ *for all* $t \in \mathbb{R}$ *and* $a \in \hat{A}$.

We further mention two formulas saying how $\hat{\varphi}$ behaves under right translations and how $\hat{\psi}$ behaves under left translations.

From Proposition 1.1.26, we can infer the Proposition 2.8.6 below.

Proposition 2.8.6 *Let $x \in \hat{A}^+$. Take $\xi \in D(\hat{\delta}^{\frac{1}{2}})$ and $\eta \in D(\hat{\delta}^{-\frac{1}{2}})$. Then, we have that*

1. $\hat{\varphi}\big((\iota \otimes \omega_\xi)\hat{\Phi}(x)\big) = \hat{\varphi}(x)\langle \hat{\delta}^{\frac{1}{2}}\xi, \hat{\delta}^{\frac{1}{2}}\xi \rangle$,
2. $\hat{\psi}\big((\omega_\eta \otimes \iota)\hat{\Phi}(x)\big) = \hat{\psi}(x)\langle \hat{\delta}^{-\frac{1}{2}}\eta, \hat{\delta}^{-\frac{1}{2}}\eta \rangle$.

2.8.2 The regular representations

We now consider the left regular representation $\widetilde{W}$ and the right regular representation $\widetilde{V}$ of the quantum $E(2)$ group (A, Φ). We display formulas for both these primordial multiplicative unitaries.

From Section 1.1, we know that the (left) regular representation $\widetilde{W}$ is one of the main actors in the general theory. It is a fundamental tool appearing almost everywhere in the built-up of the Kustermans-Vaes theory.

The left regular representation $\widetilde{W}$ is used below to bridge between the concrete setting in Chapter 2 and the general theory as expounded in Section 1.1.

We start by recalling the general definitions.

Definition 2.8.7 *We denote by* $\widetilde{W}$ *and* $\widetilde{V}$ *the left, respectively right,* regular representation *of the quantum* $E(2)$ *group* (A,Φ). *This means that the unitaries* $\widetilde{W},\widetilde{V} \in B(H_\psi \otimes H_\psi)$ *are defined by the formulas*

$$\widetilde{W}^*\big(\Lambda_\psi(x)\otimes\Lambda_\psi(y)\big) = (\Lambda_\psi\otimes\Lambda_\psi)\big(\Phi(y)(x\otimes 1)\big)$$
$$\widetilde{V}\big(\Lambda_\psi(x)\otimes\Lambda_\psi(y)\big) = (\Lambda_\psi\otimes\Lambda_\psi)\big(\Phi(x)(1\otimes y)\big)$$

where $x,y \in \mathfrak{N}_\psi$.

From Proposition 1.1.16, we get the following property.

Proposition 2.8.8 *The two operators* $\widetilde{W}$ *and* $\widetilde{V}$ *are multiplicative unitaries. Moreover, we have that* $\widetilde{W}$ *and* $\widetilde{V}$ *are manageable.*

We display a formula for $\widetilde{V}$ in Proposition 2.8.9 below.

Proposition 2.8.9 *We have that*

$$\widetilde{V} = F_\mu(P \dotplus Q)\chi(R,S)$$

where P,Q,R,S *are the normal operators on* $H_\psi\otimes H_\psi$ *defined by*

$$\begin{aligned} P &= 1\otimes ms\otimes 1\otimes s^2\otimes sm^{-1}\otimes 1,\\ Q &= -m^{-1}\otimes m^2\otimes m^{-1}s^*\otimes s^2\otimes sm^{-1}\otimes 1,\\ R &= m^{-\frac{1}{2}}\otimes m\otimes m^{-1}\otimes 1\otimes 1\otimes 1,\\ S &= 1\otimes 1\otimes 1\otimes s\otimes s\otimes 1. \end{aligned}$$

For all $k,l,m,p,q,r\in\mathbb{Z}$, *we have*

$$\begin{aligned} &\widetilde{V}(e_k\otimes e_l\otimes e_m\otimes e_p\otimes e_q\otimes e_r)\\ &\quad= \sum_{v,w\in\mathbb{Z}} \mu^{m-v}B(-m+v,-q+v+1)B(w,k-l+m-q+v+1)\\ &\qquad\qquad (e_k\otimes e_{l+w}\otimes e_v\otimes e_{-k+2l+p-2v+2w}\otimes e_{-k+2l-m+q-v+w}\otimes e_r). \end{aligned}$$

Proof. Let $k,l,m,p,q,r\in\mathbb{Z}$. We define functions $f_1,f_2\in \mathrm{K}(\bar{\mathbb{C}}^\mu)$ by

$$f_1(z) = \begin{cases} \big(\frac{z}{|z|}\big)^{k-l+m} & \text{if } |z| = \mu^{-l};\\ 0 & \text{otherwise.}\end{cases} \quad\text{and}\quad f_2(z) = \begin{cases} \big(\frac{z}{|z|}\big)^{p-q+r} & \text{if } |z| = \mu^{-q};\\ 0 & \text{otherwise.}\end{cases}$$

Then, we define elements $x,y\in\hat{A}$ by

$$x = \mu^m f_1(d)c^{l-m} \qquad\text{and}\qquad y = \mu^r f_2(d)c^{q-r}.$$

From Proposition 2.7.12, we get that $x,y\in\mathfrak{N}_\psi\cap\mathfrak{N}_\psi^*$.

A straightforward calculation gives that

$$\Lambda_\psi(x) \otimes \Lambda_\psi(y) = e_k \otimes e_l \otimes e_m \otimes e_p \otimes e_q \otimes e_r .$$

We can also calculate (using Proposition 2.3.13) that we have

$$\begin{aligned}&(\Lambda_\psi \otimes \Lambda_\psi)\big(\Phi(x)(1 \otimes y)\big)\\ &\quad = \sum_{v,w \in \mathbb{Z}} \mu^{m-v} B(-m+v, -q+v+1) B(w, k-l+m-q+v+1)\\ &\qquad\qquad (e_k \otimes e_{l+w} \otimes e_v \otimes e_{-k+2l+p-2v+2w} \otimes e_{-k+2l-m+q-v+w} \otimes e_r).\end{aligned}$$

If we work as in the proof of Proposition 2.3.13, we can find that

$$\begin{aligned}&F_\mu(P) F_\mu(Q) \chi(R,S)(e_k \otimes e_l \otimes e_m \otimes e_p \otimes e_q \otimes e_r)\\ &\quad = \sum_{v,w \in \mathbb{Z}} (-1)^{m+v} B(m-v, m-q+1) B(w, k-l+m-q+v+1)\\ &\qquad\qquad (e_k \otimes e_{l+w} \otimes e_v \otimes e_{-k+2l+p-2v+2w} \otimes e_{-k+2l-m+q-v+w} \otimes e_r).\end{aligned}$$

The formulas in Proposition 2.2.6 imply that, for all $v \in \mathbb{Z}$, we have

$$(-1)^{m+v} B(m-v, m-q+1) = \mu^{m-v} B(-m+v, -q+v+1).$$

Looking careful at Definition 2.8.7 and the above formulas, we see that

$$\widetilde{V} = F_\mu(P) F_\mu(Q) \chi(R,S).$$

It is easy to check that $(P,Q) \in D_\mu$ and $\sigma(P), \sigma(Q) \subseteq \bar{\mathbb{C}}^\mu$. Proposition 2.2.12 then yields

$$F_\mu(P) F_\mu(Q) = F_\mu(P \dotplus Q).$$

This completes the proof of the proposition. ∎

Remark 2.8.10 *By reasoning similar to Remark 4.8 in [148], we can find an intuition behind the formula for $\widetilde{V}$ given above. Of course, we have to be careful as the quantum $E(2)$ group is not self-dual.*

It is not difficult to construct a non-degenerate *-homomorphism $\tilde{\gamma} : \hat{A} \to B(H_\psi)$ such that

$$\tilde{\gamma}(a) = 1 \otimes 1 \otimes m^{-1} \qquad \textit{and} \qquad \tilde{\gamma}(b) = -(1 \otimes 1 \otimes s^*).$$

Then, working as in [148, Remark 4.8], we find that

$$\widetilde{V} = (\pi_{\hat{\psi}} \otimes \pi_\psi)(W)(\tilde{\gamma} \otimes \pi_\psi)(W).$$

Calculating this, we arrive at the formula for $\widetilde{V}$ given in Proposition 2.8.9.

The next Proposition 2.8.11 portrays $\widetilde{W}$ in a clear way.

Proposition 2.8.11 *We have that*

$$\widetilde{W} = F_\mu(\tilde{P} \dotplus \tilde{Q})\chi(\tilde{R}, \tilde{S})$$

where $\tilde{P}, \tilde{Q}, \tilde{R}, \tilde{S}$ *are the normal operators on* $H_\psi \otimes H_\psi$ *defined by*

$$\begin{aligned}
\tilde{P} &= -s^2 \otimes sm^{-1} \otimes 1 \otimes s^{-2} \otimes 1 \otimes ms, \\
\tilde{Q} &= s^2 \otimes sm^{-1} \otimes 1 \otimes m^{-1}s^{-2} \otimes ms^* \otimes 1, \\
\tilde{R} &= 1 \otimes 1 \otimes 1 \otimes m^{-\frac{1}{2}} \otimes 1 \otimes 1, \\
\tilde{S} &= s \otimes s \otimes 1 \otimes 1 \otimes 1 \otimes 1.
\end{aligned}$$

For all $k, l, m, p, q, r \in \mathbb{Z}$, *we have*

$$\begin{aligned}
&\widetilde{W}(e_k \otimes e_l \otimes e_m \otimes e_p \otimes e_q \otimes e_r) \\
&\quad = \sum_{v,w \in \mathbb{Z}} \mu^{-v} B(v, -l+p+r-v-w+1) B(w, -l+q+1) \\
&\qquad\qquad (e_{k-p-2v+2w} \otimes e_{l-p-v+w} \otimes e_m \otimes e_{p+2v-2w} \otimes e_{q+2w} \otimes e_{r-v}).
\end{aligned}$$

Proof. It is possible to prove the above formula for $\widetilde{W}$ along the lines of Proposition 2.8.9 (considering $\widetilde{W}^*$ rather than $\widetilde{W}$). However, we opt to give a different proof which is based on the general theory; see Equation (2.70).

We define an anti-unitary operator $\hat{J}$ on H_ψ by the formulas

$$\hat{J}(e_k \otimes e_l \otimes e_m) = (-1)^{k+l+m}(e_{-k+2l-2m} \otimes e_l \otimes e_m)$$

where $k, l, m \in \mathbb{Z}$.

Then, we have $\hat{J}^* = \hat{J}$ and $\hat{J}^2 = 1$. It is easy to check that

$$\hat{J}\big(\Lambda_\psi(x)\big) = \Lambda_\psi\big(R(x)^*\big)$$

for all $x \in \mathfrak{N}_\psi$.

From Lemma 8.26 in [66], it ensues that

$$\widetilde{W} = (\hat{J} \otimes \hat{J})\Sigma\widetilde{V}^*\Sigma(\hat{J} \otimes \hat{J}). \tag{2.70}$$

We define a pair $(P_0, Q_0) \in D_\mu$ by

- $P_0 = m \otimes sm^{-1} \otimes m^2 \otimes 1 \otimes m^{-1}s^* \otimes 1$,
- $Q_0 = -1 \otimes 1 \otimes s^*m \otimes 1 \otimes m^{-1}s^* \otimes 1$.

By using the notations and results from Proposition 2.8.9 together with the commutation relation $ms = \mu sm$, we get (after quite some work) that

$$\widetilde{V}^* = \chi(R^{-1}, S)F_\mu(Q^*)F_\mu(P^*) = F_\mu(Q_0^*)F_\mu(P_0^*)\chi(R^{-1}, S).$$

Now, we can deduce that

$$\begin{aligned}\widetilde{W} &= (\hat{J} \otimes \hat{J})\Sigma\widetilde{V}^*\Sigma(\hat{J} \otimes \hat{J}) \\ &= (\hat{J} \otimes \hat{J})\Sigma F_\mu(Q_0^*)F_\mu(P_0^*)\chi(R^{-1}, S)\Sigma(\hat{J} \otimes \hat{J}) \\ &= F_\mu(\tilde{P})F_\mu(\tilde{Q})\chi(\tilde{R}, \tilde{S}).\end{aligned}$$

The last step in this calculation again requires some effort.

It is easy to check that $(\tilde{P}, \tilde{Q}) \in D_\mu$ and $\sigma(\tilde{P}), \sigma(\tilde{Q}) \subseteq \bar{\mathbb{C}}^\mu$. Proposition 2.2.12 then yields

$$F_\mu(\tilde{P})F_\mu(\tilde{Q}) = F_\mu(\tilde{P} \dotplus \tilde{Q}).$$

Thus, we get

$$\widetilde{W} = F_\mu(\tilde{P} \dotplus \tilde{Q})\chi(\tilde{R}, \tilde{S}).$$

We can use the techniques from the proof of Proposition 2.3.13 to find the given formula of the action of $\widetilde{W}$ on basis vectors. ■

2.8.3 Duality theory

The central objects in this Chapter 2 are the pairs (A, Φ) and $(\hat{A}, \hat{\Phi})$. We now have a look at the duality between these two quantum groups.

In the previous sections, we frequently referred to the quantum $\hat{E}(2)$ group as the dual of the quantum $E(2)$ group. However, we did not yet give a correct proof of this duality property. We prove in Proposition 2.8.21 below that $(\hat{A}, \hat{\Phi})$ is indeed the (opposite) dual quantum group of (A, Φ).

We study the duality between the two quantum groups (A, Φ) and $(\hat{A}, \hat{\Phi})$ in the Kustermans-Vaes framework. The duality theory below also has a reflection on the representation theory of the quantum $E(2)$ group. This is explicated in Chapter 4. We mention that S.L. Woronowicz and A. Van Daele already studied this kind of duality properties before; see [159, 149].

First, we prove a connection between the multiplicative unitary W and the left regular representation $\widetilde{W}$ as introduced in Definition 2.8.7.

The Definition 2.8.12 defines a few operators.

Definition 2.8.12 *We consider the normal operator X on H_ψ defined by*

$$X = -m^{-1} \otimes s^*m \otimes m^{-1}s^*.$$

We define a unitary $U_0 \in B(H_\psi)$ by setting

$$U_0(e_k \otimes e_l \otimes e_m) = (-1)^m(e_{k-2m} \otimes e_l \otimes e_m)$$

when $k, l, m \in \mathbb{Z}$.

Further, we define a unitary $\mathcal{U} \in B(H_\psi)$ by $\mathcal{U} = F_\mu(X)U_0$.

The next Definition 2.8.13 is important in this Subsection 2.8.3.

Definition 2.8.13 *Let $\boldsymbol{\alpha}, \boldsymbol{\beta} : B(H) \to B(H_\psi)$ be the two injective, normal and unital *-homomorphisms defined by*

$$\boldsymbol{\alpha}(x) = x_{12} \qquad \textit{and} \qquad \boldsymbol{\beta}(y) = \mathcal{U} y_{13} \mathcal{U}^* .$$

when $x, y \in B(H)$.

The next Proposition 2.8.14 displays how W and $\widetilde{W}$ are related.

Proposition 2.8.14 *We have that*

$$(\boldsymbol{\alpha} \otimes \boldsymbol{\beta})(\Sigma W \Sigma) = \widetilde{W}.$$

Proof. Throughout the proof, we use the notations introduced in Proposition 2.8.11. From this Proposition 2.8.11, we know that

$$\widetilde{W} = F_\mu(\tilde{P} \dotplus \tilde{Q}) \chi(\tilde{R}, \tilde{S}) \tag{2.71}$$

It is not difficult to prove that $(\tilde{P}, \tilde{Q}) \in D_\mu$ and $\tilde{P}^{-1}\tilde{Q} = 1 \otimes X$. We then get from Theorem 2.2.11 that

$$\tilde{P} \dotplus \tilde{Q} = \big(1 \otimes F_\mu(X)\big) \tilde{P} \big(1 \otimes F_\mu(X)^*\big). \tag{2.72}$$

Equations (2.71) and (2.72) together give

$$\begin{aligned} \widetilde{W} &= F_\mu(\tilde{P} \dotplus \tilde{Q}) \chi(\tilde{R}, \tilde{S}) = \big(1 \otimes F_\mu(X)\big) F_\mu(\tilde{P}) \big(1 \otimes F_\mu(X)^*\big) \chi(\tilde{R}, \tilde{S}) \\ &= \big(1 \otimes F_\mu(X)\big) F_\mu(\tilde{P}) \chi(\tilde{R}, \tilde{S}) \big(1 \otimes F_\mu(X)^*\big). \end{aligned}$$

The third equality holds because $1 \otimes X$ commutes with both $\tilde{R}$ and $\tilde{S}$.

Using basic calculations, it is straightforward to check that

1. $(1 \otimes U_0^*) \tilde{P} (1 \otimes U_0) = (cd \otimes ab)_{1246}$,
2. $(1 \otimes U_0^*) \tilde{R} (1 \otimes U_0) = (1 \otimes a)_{1246}$,
3. $(1 \otimes U_0^*) \tilde{S} (1 \otimes U_0) = (c \otimes 1)_{1246}$.

If we combine all the above results, we get

$$F_\mu(cd \otimes ab)_{1246} \chi(1 \otimes a, c \otimes 1)_{1246} = \big(1 \otimes U_0^* F_\mu(X)^*\big) \widetilde{W} \big(1 \otimes F_\mu(X) U_0\big).$$

This precisely means that

$$(1 \otimes \mathcal{U})(\Sigma W \Sigma)_{1246} (1 \otimes \mathcal{U}^*) = \widetilde{W}.$$

Looking at Definition 2.8.13, we see that this formula completes the proof. ∎

We then use the general theory to construct a dual pair of quantum groups out of the left regular representation $\widetilde{W}$.

The results in Definition 2.8.15 and Proposition 2.8.16 directly ensue from the Kustermans-Vaes theory as expounded in [66].

Definition 2.8.15 *We define two bi-C^*-algebras $(\mathcal{B}, \Phi_{\mathcal{B}})$ and $(\hat{\mathcal{B}}, \hat{\Phi}_{\hat{\mathcal{B}}})$ as follows. First, we construct $(\mathcal{B}, \Phi_{\mathcal{B}})$ by setting*

- $\mathcal{B} = [(\iota \otimes \omega)\widetilde{W} \mid \omega \in B(H_\psi)]$,
- $\Phi_{\mathcal{B}} : \mathcal{B} \to M(\mathcal{B} \otimes \mathcal{B}) : x \mapsto \widetilde{W}^*(1 \otimes x)\widetilde{W}$.

Similarly, we define $(\hat{\mathcal{B}}, \hat{\Phi}_{\hat{\mathcal{B}}})$ by

- $\hat{\mathcal{B}} = [(\omega \otimes \iota)\widetilde{W} \mid \omega \in B(H_\psi)]$,
- $\hat{\Phi}_{\hat{\mathcal{B}}} : \hat{\mathcal{B}} \to M(\hat{\mathcal{B}} \otimes \hat{\mathcal{B}}) : y \mapsto \Sigma\widetilde{W}(y \otimes 1)\widetilde{W}^*\Sigma$.

Proposition 2.8.16 *We have that $(\mathcal{B}, \Phi_{\mathcal{B}})$ and $(\hat{\mathcal{B}}, \hat{\Phi}_{\hat{\mathcal{B}}})$ are locally compact quantum groups. By definition, $(\hat{\mathcal{B}}, \hat{\Phi}_{\hat{\mathcal{B}}})$ is the dual of $(\mathcal{B}, \Phi_{\mathcal{B}})$.*

The next Proposition 2.8.17 states that the pair $(\mathcal{B}, \Phi_{\mathcal{B}})$ is isomorphic to the quantum $E(2)$ group (A, Φ). In particular, we have that $\boldsymbol{\alpha}$ is a *-isomorphism between A and $\mathcal{B}$ that respects the comultiplication.

It is clear that Proposition 2.8.14 is crucial in the proof below.

Proposition 2.8.17 *We have that $\alpha : A \to \mathcal{B}$ is a *-isomorphism such that*

$$\Phi_{\mathcal{B}}\big(\boldsymbol{\alpha}(x)\big) = (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha})\Phi(x)$$

for all $x \in A$.

Proof. We know from Definition 2.8.13 that $\boldsymbol{\alpha} : A \to B(H_\psi)$ is an injective *-homomorphism. Let $\omega \in B(H_\psi)_*$. We define $\rho \in B(H)_*$ by $\rho(x) = \omega(\boldsymbol{\beta}(x))$. Using Proposition 2.8.14, it is direct to check that

$$\boldsymbol{\alpha}\big((\rho \otimes \iota)W\big) = (\iota \otimes \omega)\widetilde{W}.$$

From this, it easily follows that $\boldsymbol{\alpha}(A) = \mathcal{B}$.

The general theory from [66] gives that $\Phi_{\mathcal{B}}$ can be characterized as the only non-degenerate *-homomorphism from $\mathcal{B}$ to $M(\mathcal{B} \otimes \mathcal{B})$ such that

$$(\iota \otimes \Phi_{\mathcal{B}})(\Sigma\widetilde{W}\Sigma) = (\Sigma\widetilde{W}\Sigma)_{12}(\Sigma\widetilde{W}\Sigma)_{13}.$$

By applying Proposition 2.8.14, we then get

$$\begin{aligned}\big(\iota\otimes(\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})^{-1}\Phi_{\mathcal{B}}\,\boldsymbol{\alpha}\big)(W) &= \big(\boldsymbol{\beta}^{-1}\otimes(\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})^{-1}\Phi_{\mathcal{B}}\big)(\boldsymbol{\beta}\otimes\boldsymbol{\alpha})(W)\\ &= (\boldsymbol{\beta}\otimes\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})^{-1}(\iota\otimes\Phi_{\mathcal{B}})(\Sigma\widetilde{W}\Sigma) = (\boldsymbol{\beta}\otimes\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})^{-1}(\Sigma\widetilde{W}\Sigma)_{12}(\Sigma\widetilde{W}\Sigma)_{13}\\ &= \big((\boldsymbol{\beta}\otimes\boldsymbol{\alpha})^{-1}(\Sigma\widetilde{W}\Sigma)\big)_{12}\big((\boldsymbol{\beta}\otimes\boldsymbol{\alpha})^{-1}(\Sigma\widetilde{W}\Sigma)\big)_{13} = W_{12}W_{13}.\end{aligned}$$

Proposition 2.5.10 now yields that

$$\Phi = (\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})^{-1}\Phi_{\mathcal{B}}\,\boldsymbol{\alpha}.$$

This ends the proof of the proposition. ∎

It is trivial to calculate the action of $\boldsymbol{\alpha}$ on the generators.

Lemma 2.8.18 *We have that*

- $\boldsymbol{\alpha}(c) = s\otimes s\otimes 1,$
- $\boldsymbol{\alpha}(d) = s\otimes m^{-1}\otimes 1.$

Moreover, we have

- $\boldsymbol{\alpha}(cd) = \boldsymbol{\alpha}(c)\boldsymbol{\alpha}(d) = \mu\boldsymbol{\alpha}(d)\boldsymbol{\alpha}(c) = s^2\otimes sm^{-1}\otimes 1.$

In the dual case, we then have the Proposition 2.8.19 below. We see that $\boldsymbol{\beta}$ is a *-isomorphism between $\hat{A}$ and $\hat{\mathcal{B}}$ that flips the comultiplication.

The pair $(\hat{\mathcal{B}},\hat{\Phi}_{\hat{\mathcal{B}}})$ is thus isomorphic to the quantum group $(\hat{A},\hat{\Phi})^{\mathrm{op}}$.

Proposition 2.8.19 *We have that $\boldsymbol{\beta}:\hat{A}\to\hat{\mathcal{B}}$ is a *-isomorphism such that*

$$\hat{\Phi}_{\hat{\mathcal{B}}}\big(\boldsymbol{\beta}(y)\big) = \dot{\sigma}(\boldsymbol{\beta}\otimes\boldsymbol{\beta})\hat{\Phi}(y)$$

for all $y\in\hat{A}$.

Proof. We know from Definition 2.8.13 that $\boldsymbol{\beta}:\hat{A}\to B(H_\psi)$ is an injective *-homomorphism. Let $\omega\in B(H_\psi)_*$. We define $\rho\in B(H)_*$ by $\rho(x)=\omega(\boldsymbol{\alpha}(x))$. Using Proposition 2.8.14, it is direct to check that

$$\boldsymbol{\beta}\big((\iota\otimes\rho)W\big) = (\omega\otimes\iota)\widetilde{W}.$$

From this, it easily follows that $\boldsymbol{\beta}(\hat{A}) = \hat{\mathcal{B}}$.

The general theory from [66] gives that $\hat{\Phi}_{\hat{\mathcal{B}}}$ can be characterized as the only non-degenerate *-homomorphism from $\hat{\mathcal{B}}$ to $M(\hat{\mathcal{B}}\otimes\hat{\mathcal{B}})$ such that

$$(\dot{\sigma}\hat{\Phi}_{\hat{\mathcal{B}}}\otimes\iota)(\Sigma\widetilde{W}\Sigma) = (\Sigma\widetilde{W}\Sigma)_{13}(\Sigma\widetilde{W}\Sigma)_{23}.$$

By applying Proposition 2.8.14, we then get

$$\begin{aligned}\big((\boldsymbol{\beta}\otimes\boldsymbol{\beta})^{-1}\dot{\sigma}\hat{\Phi}_{\hat{\mathcal{B}}}\,\boldsymbol{\beta}\otimes\iota\big)(W) &= \big((\boldsymbol{\beta}\otimes\boldsymbol{\beta})^{-1}\dot{\sigma}\hat{\Phi}_{\hat{\mathcal{B}}}\otimes\boldsymbol{\alpha}^{-1}\big)(\boldsymbol{\beta}\otimes\boldsymbol{\alpha})(W)\\ &= ((\boldsymbol{\beta}\otimes\boldsymbol{\beta})^{-1}\otimes\boldsymbol{\alpha}^{-1})(\dot{\sigma}\hat{\Phi}_{\hat{\mathcal{B}}}\otimes\iota)(\Sigma\widetilde{W}\Sigma)\\ &= \big((\boldsymbol{\beta}\otimes\boldsymbol{\beta})^{-1}\otimes\boldsymbol{\alpha}^{-1}\big)(\Sigma\widetilde{W}\Sigma)_{13}(\Sigma\widetilde{W}\Sigma)_{23}\\ &= \big((\boldsymbol{\beta}\otimes\boldsymbol{\alpha})^{-1}(\Sigma\widetilde{W}\Sigma)\big)_{13}\big((\boldsymbol{\beta}\otimes\alpha)^{-1}(\Sigma\widetilde{W}\Sigma)\big)_{23} = W_{13}W_{23}.\end{aligned}$$

Proposition 2.5.32 now yields that

$$\hat{\Phi} = (\boldsymbol{\beta}\otimes\boldsymbol{\beta})^{-1}\dot{\sigma}\hat{\Phi}_{\hat{\mathcal{B}}}\,\boldsymbol{\beta}.$$

This ends the proof of the proposition. ∎

We can use Theorem 2.2.11 to calculate the action of $\boldsymbol{\beta}$ on the generators.

Lemma 2.8.20 *We have that*

- $\boldsymbol{\beta}(a) = m^{-\frac{1}{2}}\otimes 1\otimes 1,$
- $\boldsymbol{\beta}(b) = (-m^{\frac{1}{2}}s^{-2}\otimes 1\otimes ms) \dotplus (m^{-\frac{1}{2}}s^{-2}\otimes ms^*\otimes 1).$

Moreover, we have

- $\boldsymbol{\beta}(ab) = \boldsymbol{\beta}(a)\boldsymbol{\beta}(b) = \mu\boldsymbol{\beta}(a)\boldsymbol{\beta}(b) = (-s^{-2}\otimes 1\otimes ms)\dotplus(m^{-1}s^{-2}\otimes ms^*\otimes 1).$

From Propositions 2.8.17 and 2.8.19, we can infer the Proposition 2.8.21 below. This duality result states that $(\hat{A},\hat{\Phi})$ is the opposite dual of (A,Φ).

Proposition 2.8.21 *We have* $(\hat{A},\hat{\Phi}) = (\widehat{A,\Phi})^{\mathrm{op}}$.

2.8.4 Regularity and semi-regularity

We now prove that the regular representations $\widetilde{W}$ and $\widetilde{V}$ are semi-regular, but not regular. This result is stated in Corollary 2.8.24 below.

The regularity properties of the quantum $E(2)$ group were already considered by Baaj in [1]. However, we here write things in our setting. By using the recent results from [5], we can moreover introduce some simplifications.

Thanks to Proposition 2.6 in [5], we can give the next Definition 2.8.22.

Definition 2.8.22 *We define a C^*-algebra $\mathcal{C}(\widetilde{W})$ by*

$$\mathcal{C}(\widetilde{W}) = [(\iota\otimes\omega)(\Sigma\widetilde{W}) \mid \omega\in B(H_\varphi)_*].$$

The next Proposition 2.8.23 describes the regularity properties of the multiplicative unitary $\widetilde{W}$. It clearly resembles Proposition 2.3.33.

Proposition 2.8.23 *We have that* $B_0(H_\varphi) \subsetneq \mathcal{C}(\widetilde{W}) \subsetneq B(H_\varphi)$.

Proof. Combining Proposition 2.6 in [5] with Proposition 2.5 in [134], we get that $\mathcal{C}(\widetilde{W})$ is a C*-algebra acting irreducibly on H_φ.

We adopt the notations introduced in Proposition 2.8.11. We then define a unitary operator $Z \in B(H_\varphi)$ and a normal operator Y on H_φ by

- $Z = F_\mu(\widetilde{Q})\chi(\widetilde{R},\widetilde{S})$,
- $Y = -m \otimes sm^{-1} \otimes ms$.

It is easy to check that $(\widetilde{P},\widetilde{Q}) \in D_\mu$ and $\widetilde{P}\widetilde{Q}^{-1} = 1 \otimes Y$.

From Proposition 2.8.11 and Theorem 2.2.11, it ensues that

$$\begin{aligned}\widetilde{W} &= F_\mu(\tilde{P} \dotplus \tilde{Q})\chi(\tilde{R},\tilde{S}) \\ &= \big(1 \otimes F_\mu(Y)\big)F_\mu(\widetilde{Q})\big(1 \otimes F_\mu(Y)^*\big)\chi(\widetilde{R},\widetilde{S}) \\ &= \big(1 \otimes F_\mu(Y)\big)Z\big(1 \otimes F_\mu(Y)^*\big).\end{aligned}$$

The last equality holds because $1 \otimes Y$ commutes with both $\widetilde{R}$ and $\widetilde{S}$.

Let $k,l,m,p,q,r \in \mathbb{Z}$. We use the abridged notation $\omega_{k,l,m,p,q,r}$ to denote the vector functional $\omega_{e_k\otimes e_l\otimes e_m,e_p\otimes e_q\otimes e_r}$.

We then consider the operator $(\iota \otimes \omega_{k,l,m,p,q,r})(\Sigma Z) \in B(H_\varphi)$.

It follows from a straightforward calculation that

$$\begin{aligned}&(\iota \otimes \omega_{k,l,m,p,q,r})(\Sigma Z)(e_u \otimes e_v \otimes e_w) \\ &\quad = B(k-l+q,-l+v+1)\delta_{k-p+2q,-u+2v}\delta_{w,r}(e_{u-p} \otimes e_{-k+l-q+v} \otimes e_m)\end{aligned} \tag{2.73}$$

when $u,v,w \in \mathbb{Z}$.

Combining Equation (2.73) with Proposition A.9, it is not too difficult to prove that $(\iota \otimes \omega_{k,l,m,p,q,r})(\Sigma Z) \in B_0(H_\varphi)$ if and only if $l \neq k+q$.

By simple computations, one can show that $\mathcal{C}(W) \subseteq B(H) \otimes B_0(\ell^2(\mathbb{Z}))$. We thus find that we have $\mathcal{C}(W) \subsetneq B(H_\varphi)$.

It is clear that $F_\mu(Y)$ is a unitary. We can thus complete the proof by remarking that $F_\mu(Y)\big((\iota \otimes \omega)(\Sigma Z)\big) \in \mathcal{C}(\widetilde{W})$ for all $\omega \in B(H_\varphi)_*$. ∎

The next Corollary 2.8.24 is then an almost direct consequence.

Corollary 2.8.24 *We have that* $\widetilde{W}$ *and* $\widetilde{V}$ *are semi-regular, but not regular.*

Proof. Proposition 2.8.23 yields that $\widetilde{W}$ is semi-regular, but not regular. From Equation (2.70), we get that $\widetilde{V}$ has the same regularity properties as $\widetilde{W}$. ∎

The Corollaries 2.8.25 and 2.8.26 then portray the regularity properties of (A, Φ) and $(\hat{A}, \hat{\Phi})$. They follow from Corollary 2.8.24 in a standard way.

Corollary 2.8.25 *We have that* (A, Φ) *is semi-regular, but not regular.*

Proof. This follows from Definition 1.3.4 and Corollary 2.8.24. ∎

Corollary 2.8.26 *We have that* $(\hat{A}, \hat{\Phi})$ *is semi-regular, but not regular.*

Proof. We know from Proposition 2.8.21 that $(\hat{A}, \hat{\Phi})$ is the opposite dual of (A, Φ). If we then combine Corollary 2.8.25 with Proposition 2.6 in [5], we can easily derive the formulated regularity properties of $(\hat{A}, \hat{\Phi})$. ∎

It is known that all quantum groups arising from algebraic quantum groups in the sense of [57, 70] are regular; see Remark 1.3.5.

From Corollaries 2.8.25 and 2.8.26, we thus get the following property.

Proposition 2.8.27 *The pairs* (A, Φ) *and* $(\hat{A}, \hat{\Phi})$ *do not belong to the category of quantum groups arising from algebraic quantum groups.*

2.8.5 Closed quantum subgroups

We then describe some closed quantum subgroups. We recall that the notion of a *closed quantum subgroup* is introduced in Definition 1.1.45.

It is easy to prove that both the quantum $E(2)$ group and its dual have a commutative closed quantum subgroup. Since we adopt Definition 1.1.45, it is necessary to consider the von Neumann algebra setting.

In the treatment below, we are quite laborious. The closed quantum subgroups of (A, Φ) and $(\hat{A}, \hat{\Phi})$ are defined via the operator $\chi(a \otimes 1, 1 \otimes c)$. Nevertheless, this method of working gives some extra insight into the features of W.

We again apply the Notation 2.8.28.

Notation 2.8.28 *We use* $\boldsymbol{\chi}$ *to shortly denote* $\chi(a \otimes 1, 1 \otimes c)$.

The next Proposition 2.8.29 is a consequence of Proposition 2.3.21.

Proposition 2.8.29 *We have that* $\boldsymbol{\chi}$ *is a manageable multiplicative unitary.*

Proof. By using Proposition 2.3.13, it is direct to check that $\boldsymbol{\chi}$ satisfies the pentagonal equation

$$\boldsymbol{\chi}_{12}\boldsymbol{\chi}_{13}\boldsymbol{\chi}_{23} = \boldsymbol{\chi}_{23}\boldsymbol{\chi}_{12}.$$

The Proposition 2.3.21 yields that $\boldsymbol{\chi}$ is a unitary which is adapted to W.

We get from Theorem 2.3.30 that W is manageable. Theorem 1.6 in [162] then gives that also $\boldsymbol{\chi}$ is manageable. ∎

Thanks to Proposition 2.8.29, we can apply Theorem 1.3.9. In particular, we find that the next Definition 2.8.30 is justified.

This Definition 2.8.30 then defines two bi-C*-algebras.

Definition 2.8.30 *We define bi-C*-algebras (A_0, Φ_0) and $(\hat{A}_0, \hat{\Phi}_0)$ as follows. First, we construct (A_0, Φ_0) by setting*

- $A_0 = [(\iota \otimes \omega)\boldsymbol{\chi} \mid \omega \in B(H)]$,
- $\Phi_0 : B \to M(A_0 \otimes A_0) : x \mapsto \boldsymbol{\chi}(x \otimes 1)\boldsymbol{\chi}^*$.

Similarly, we define $(\hat{A}_0, \hat{\Phi}_0)$ by

- $\hat{A}_0 = [(\omega \otimes \iota)\boldsymbol{\chi} \mid \omega \in B(H)]$,
- $\hat{\Phi}_0 : \hat{A}_0 \to M(\hat{A}_0 \otimes \hat{A}_0) : y \mapsto \Sigma\boldsymbol{\chi}^*(1 \otimes y)\boldsymbol{\chi}\Sigma$.

The next Proposition 2.8.32 gives that (A_0, Φ_0) and $(\hat{A}_0, \hat{\Phi}_0)$ are commutative locally compact quantum groups.

We recall that, for commutative quantum groups, we always use Φ to denote the standard comultiplication; see Notation 1.2.5.

Definition 2.8.31 *Let (A_1, Φ_1) and (A_2, Φ_2) be two locally compact quantum groups. We then say that (A_1, Φ_1) is* isomorphic *to (A_2, Φ_2) if there exists a *-isomorphism $\pi : A_1 \to A_2$ such that*

$$(\pi \otimes \pi)\Phi_1 = \Phi_2 \pi.$$

Proposition 2.8.32 *We have the following results:*

- *The pair (A_0, Φ_0) is isomorphic to the compact quantum group $(\mathrm{C}(S^1), \Phi)$,*
- *The pair $(\hat{A}_0, \hat{\Phi}_0)$ is isomorphic to the discrete quantum group $(\mathrm{C}_0(\mathbb{Z}), \Phi)$.*

Proof. It is easy to prove that

- $[(\omega \otimes \iota)(\boldsymbol{\chi}) \mid \omega \in B(H)_*] = [f(c) \mid f \in \mathrm{C}(S^1)] \cong \mathrm{C}(S^1)$,
- $[(\iota \otimes \omega)(\boldsymbol{\chi}) \mid \omega \in B(H)_*] = [f(a) \mid f \in \mathrm{C}_0(\mathbb{R}(\mu^{\frac{1}{2}}))] \cong \mathrm{C}_0(\mathbb{Z})$.

Further, we can use Proposition 2.3.13 to check that

- $\Phi_0(f(c)) = f(c \otimes c) = \Phi(f)(c \otimes 1, 1 \otimes c)$ for all $f \in \mathrm{C}(S^1)$,
- $\hat{\Phi}_0(g(a)) = g(a \otimes a) = \Phi(g)(a \otimes 1, 1 \otimes a)$ for all $g \in \mathrm{C}_0(\mathbb{R}(\mu^{\frac{1}{2}}))$.

This completes the proof of the proposition. ∎

By using Remark 1.2.8, we can deduce the following duality result.

Lemma 2.8.33 *We have that* $\widehat{(A_0, \Phi_0)} = (\hat{A}_0, \hat{\Phi}_0)$.

We introduce some more notations.

Notation 2.8.34 *We use* (M, Φ) *and* $(\hat{M}, \hat{\Phi})$ *to denote the von Neumann algebra versions of* (A, Φ) *and* $(\hat{A}, \hat{\Phi})$ *as introduced in Remark 1.3.14.*

It follows from Theorems 2.9.3 and 2.9.6 that the pairs (M, Φ) *and* $(\hat{M}, \hat{\Phi})$ *are von Neumann algebraic quantum groups.*

Notation 2.8.35 *We use* (M_0, Φ_0) *and* $(\hat{M}_0, \hat{\Phi}_0)$ *to denote the von Neumann algebra versions of* (A_0, Φ_0) *and* $(\hat{A}_0, \hat{\Phi}_0)$ *as introduced in Remark 1.3.14.*

It follows from Proposition 2.8.32 that the pairs (M_0, Φ_0) *and* $(\hat{M}_0, \hat{\Phi}_0)$ *are von Neumann algebraic quantum groups.*

The next Proposition 2.8.36 roughly says that $(\mathrm{C}(S^1), \Phi)$ and $(\mathrm{C}_0(\mathbb{Z}), \Phi)$ are closed quantum subgroups of respectively (A, Φ) and $(\hat{A}, \hat{\Phi})$.

Proposition 2.8.36 *We have the following results:*

- *The pair* (M_0, Φ_0) *is a closed quantum subgroup of* (M, Φ)*,*
- *The pair* $(\hat{M}_0, \hat{\Phi}_0)$ *is a closed quantum subgroup of* $(\hat{M}, \hat{\Phi})$*.*

Proof. From Theorems 2.5.3 and 2.5.24, we get $c \in M(A)$ and $a \,\eta\, \hat{A}$. It follows that $\boldsymbol{\chi} \in M(\hat{A} \otimes A)$. We then find from Definition 2.8.30 that $A_0 \subseteq M(A)$ and $\hat{A}_0 \subseteq M(\hat{A})$. This yields $M_0 \subseteq M$ and $\hat{M} \subseteq \hat{M}$.

Propositions 2.5.12 and 2.5.34 give $\Phi(c) = c \otimes c$ and $\hat{\Phi}(a) = a \otimes a$. If we combine these formulas with the results found in the proof of Proposition 2.8.32, we see that $\Phi_0(x) = \Phi(x)$ for all $x \in M_0$ and $\hat{\Phi}_0(y) = \hat{\Phi}(y)$ for all $y \in \hat{M}_0$.

We can complete the proof by remarking that $\widehat{(M_0, \Phi_0)} = (\hat{M}_0, \hat{\Phi}_0)$. This duality result follows from Lemma 2.8.33. ∎

2.8.6 An ergodic coaction

We describe a left coaction $\boldsymbol{\gamma}$ of the quantum $E(2)$ group (A, Φ). As it is common for the theory of coactions on quantum groups, we formulate our basic results in the von Neumann algebra setting. We again adopt the Notation 2.8.34.

From Proposition 2.1.18, we get an intuition for constructing a left coaction $\boldsymbol{\gamma}$ of (M, Φ). S.L. Woronowicz gave in [157] the initiative for describing $\boldsymbol{\gamma}$ on the operator algebra level, but he did not elaborate his statements. Also A. Pal considered some related results [88]. In the treatment below, we give a more meticulous approach and prove some extra properties.

We first mention the preliminary Definition 2.8.37.

Definition 2.8.37 *We define a strictly positive operator T_0 on $\ell^2(\mathbb{Z})$ by setting $T_0(e_k) = e^k e_k$ for every $k \in \mathbb{Z}$. Further, we set $T = T_0 \otimes T_0^{-2}$.*

Then T is a strictly positive operator on H.

It is direct to check the Lemma 2.8.38.

Lemma 2.8.38 *For all $t \in \mathbb{R}$, we have $\chi(a, e^{-it}) = T^{it}$.*

We consider a one-parameter group (h_t) on M.

Definition 2.8.39 *Let $t \in \mathbb{R}$. Then, we consider the *-automorphism h_t on M defined by*

$$h_t : M \to M : x \mapsto T^{-it} x T^{it}.$$

We have that (h_t) is a strongly continuous one-parameter group on M.

Proposition 2.8.40 *Let $t \in \mathbb{R}$. We have that $h_t|_A$ is characterized as the only non-degenerate *-homomorphism from A to $M(A)$ such that*

$$h_t(c) = e^{-it} c \qquad \text{and} \qquad h_t(d) = e^{it} d.$$

Let $\sum c^k f_k(d) \in A$. Then, we have

$$h_t\big(\sum c^k f_k(d)\big) = \sum e^{-itk} c^k f_k(e^{it} d).$$

Next, we define a von Neumann algebra $\mathcal{N}$.

Definition 2.8.41 *Let $\mathcal{N}$ be the von Neumann subalgebra of M defined by*

$$\mathcal{N} = \{a \in M \mid h_t(a) = a \text{ for all } t \in \mathbb{R}\}.$$

We describe the basic elements of $\mathcal{N}$ in Proposition 2.8.43 below.

Notation 2.8.42 *For every $k \in \mathbb{Z}$, we define a subset $F_k \subseteq \mathrm{C}_0(\bar{\mathbb{C}}^\mu)$ as follows. First, we define sets F'_k by*

$$F'_k = \big\{f \in \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \mid \text{there exists a function } g \in \mathrm{C}_0(\bar{\mathbb{R}}^\mu) \text{ such that } f(\mu^n z) = g(\mu^n) z^k \text{ for all } n \in \mathbb{Z} \text{ and } z \in S^1\big\}.$$

We denote $F_0 = F'_0$. For $k \neq 0$, we set

$$F_k = \{f \in F'_k \mid f(0) = 0\}.$$

Proposition 2.8.43 *Let $\sum c^k f_k(d) \in A$. Then, we have $\sum c^k f_k(d) \in \mathcal{N}$ if and only if $f_k \in F_k$ for every $k \in \mathbb{Z}$.*

Proof. Take $\sum c^k f_k(d) \in A$ and suppose that $\sum c^k f_k(d) \in \mathcal{N}$. We then derive that $f_k \in F_k$ for every $k \in \mathbb{Z}$. The converse implication is obvious.

The Proposition 2.8.40 gives that we have

$$\sum c^k f_k(d) = \sum e^{-itk} c^k f_k(e^{it} d).$$

If we now apply Proposition 4.1.5, we can deduce that

$$f_k(z) = e^{-itk} f_k(e^{it} z)$$

for all $z \in \bar{\mathbb{C}}^\mu$ and $k \in \mathbb{Z}$.

Using the Fourier transform, it is direct to show that $f_k \in F_k$ for every k. In fact, we have

$$f_k(\mu^l z) = \Big(\int f_{n,k}(\mu^l e^{it}) e^{itk}\, dt \Big) z^k$$

for all $k, l \in \mathbb{Z}$ and $z \in S^1$. ∎

From Proposition 2.8.43, it ensues that $\mathcal{N}$ is non-trivial.

Lemma 2.8.44 *We have that $\mathcal{N}$ acts non-degenerately on H.*

Proof. We again consider the functions $f_{k,n} \in \mathrm{C}_0(\bar{\mathbb{C}}^\mu)$ as defined in the proof of Proposition 2.2.14. Recall that, for $z \in \bar{\mathbb{C}}^\mu$, we have

$$f_{k,n}(z) = \frac{1}{2\pi} \int_0^{2\pi} F_\mu(\mu^n e^{it} z) e^{-ikt}\, dt.$$

From Equation (2.45), we get that

$$f_{k,n}(e^{i\theta} x) = e^{ik\theta} f_{k,n}(x)$$

for all $\theta \in [0, 2\pi]$ and $x \in \bar{\mathbb{C}}^\mu$.

The Proposition 2.8.43 yields that $c^k f_{k,n}(d) \in \mathcal{N}$ for all $k, n \in \mathbb{Z}$.

From the proof of Proposition 2.5.15, we know that $f_{0,n}(d) \to 1$ in the strict topology on $M(A)$ if $n \to +\infty$. Proposition 2.5.1 gives that A is non-degenerate. We can thus conclude that also $\mathcal{N}$ is non-degenerate. ∎

We also introduce a C*-algebra C. The Definition 2.8.45 below is justified by the commutation relations in Proposition 2.3.6.

The non-degeneracy of C follows from the proof of Lemma 2.8.44.

Definition 2.8.45 *We define a C*-algebra C by setting*

$$C = \Big[\sum_{k \in \mathbb{Z}} c^k f_k(d) \mid f_k \in F_k \textit{ for every } k \textit{ and } f_k \neq 0 \textit{ for finitely many } k \Big].$$

Then C is a non-unital C*-algebra acting non-degenerately on H.

It is clear that Proposition 2.8.43 entails the result below.

Proposition 2.8.46 *We have $C'' \subseteq \mathcal{N}$.*

The definition of C is given with a view to the following feature.

Proposition 2.8.47 *We have $cd \,\eta\, C$. Further, the C^*-algebra C is generated by the affiliated element $cd \,\eta\, C$.*

Proof. From Definition 2.8.45, it ensues that $cd \,\eta\, C$ and $(1 + (cd)^*(cd))^{-1} \in C$. It is not so difficult to prove that cd separates non-degenerate *-representations of C. Now, Proposition 1.5.54 gives that B is generated by $cd \,\eta\, C$. ∎

The next Proposition 2.8.48 is crucial.

Proposition 2.8.48 *For all $x \in \mathcal{N}$, we have $\Phi(x) \in M \bar{\otimes} \mathcal{N}$.*
Furthermore, we have that

$$\Phi|_{\mathcal{N}} : \mathcal{N} \to M \bar{\otimes} \mathcal{N} : x \mapsto \Phi(x)$$

is an injective, unital and normal *-homomorphism.

Proof. Let $t \in \mathbb{R}$. It is not difficult to check that

$$h_t(W) = F_\mu(ab \otimes cd)\chi(a \otimes 1, 1 \otimes e^{-it}c) = W(\chi(a, e^{-it}) \otimes 1).$$

We can then use Lemma 2.8.38 to prove that we have for all $x \in M$ that

$$(\iota \otimes h_t)\Phi(x) = \Phi(h_{-t}(x)). \tag{2.74}$$

Take $x \in \mathcal{N}$. From Definition 2.9.1, we get that $\Phi(x) \in M \bar{\otimes} M$. It then follows from Equation (2.74) that we have $\Phi(x) \in (M' \bar{\otimes} \mathcal{N}')'$.
We now observe that $(M' \bar{\otimes} \mathcal{N}')' = M \bar{\otimes} \mathcal{N}$. This equality is a standard property in von Neumann algebra theory. ∎

The Proposition 2.8.48 enables us to give the next Definition 2.8.49.

Definition 2.8.49 *We define $\boldsymbol{\gamma} = \Phi|_{\mathcal{N}}$. Then, we have that*

$$\boldsymbol{\gamma} : \mathcal{N} \to M \bar{\otimes} \mathcal{N} : x \mapsto \boldsymbol{\gamma}(x)$$

is an injective, unital and normal *-homomorphism.

The coassociativity of Φ implicates the Proposition 2.8.50 below.

Proposition 2.8.50 *We have $(\Phi \otimes \iota)\boldsymbol{\gamma} = (\iota \otimes \boldsymbol{\gamma})\boldsymbol{\gamma}$.*

The next Corollary 2.8.51 is an immediate consequence.

Corollary 2.8.51 *We have that γ is a left coaction of (M, Φ) on $\mathcal{N}$.*

We also give some information related to the Haar weight ψ. Hereby, we use the notation ψ to also denote the restriction of ψ to $\mathcal{N}$.

The next Proposition 2.8.52 implies that the left coaction γ is integrable.

Proposition 2.8.52 *We have that ψ is an n.s.f. weight on $\mathcal{N}$. For all $x \in \mathcal{N}$ and $\omega \in \mathcal{N}_*$, we have that*

$$\psi\big((\iota \otimes \omega)\gamma(x)\big) = \omega(1)\psi(x).$$

Proof. By virtue of Theorem 2.7.9, we see that it is sufficient to prove that ψ is semi-finite on N. Using Propositions 2.7.12 and 2.8.46, it is easy to produce a non-zero element in $\mathfrak{N}_\psi$. Arguing as in the proof of Proposition 1.3.45, we can then deduce that ψ is indeed semi-finite. ∎

From Proposition 1.3.31, we can deduce the property below.

Proposition 2.8.53 *The left coaction γ is* ergodic*, i.e., we have*

$$\{x \in \mathcal{N} \mid \gamma(x) = 1 \otimes x\} = \mathbb{C}1.$$

2.8.7 Summary

We consolidate the main properties of the quantum $E(2)$ group (A, Φ) and its dual $(\hat{A}, \hat{\Phi})$ in Theorem 2.8.54 below.

All the mentioned results are proved above.

Theorem 2.8.54 *We have that (A, Φ) and $(\hat{A}, \hat{\Phi})$ are locally compact quantum groups in the sense of Definition 1.1.6.*

We call (A, Φ) the quantum $E(2)$ group *and $(\hat{A}, \hat{\Phi})$ the* quantum $\hat{E}(2)$ group*. These quantum group are shortly referred to as $E_\mu(2)$ and $\hat{E}_\mu(2)$.*

Furthermore, the following properties hold:

- *$E_\mu(2)$ and $\hat{E}_\mu(2)$ are non-commutative and non-cocommutative,*
- *$E_\mu(2)$ and $\hat{E}_\mu(2)$ are non-compact and non-discrete,*
- *$\hat{E}_\mu(2)$ is the opposite dual of $E_\mu(2)$,*
- *$E_\mu(2)$ and $\hat{E}_\mu(2)$ are not Kac algebra's,*

- $E_\mu(2)$ *and* $\hat{E}_\mu(2)$ *do not arise from algebraic quantum groups,*
- $E_\mu(2)$ *and* $\hat{E}_\mu(2)$ *are not self-dual,*
- $E_\mu(2)$ *is unimodular;* $\hat{E}_\mu(2)$ *is non-unimodular,*
- $E_\mu(2)$ *and* $\hat{E}_\mu(2)$ *have scaling constant* 1*,*
- $E_\mu(2)$ *and* $\hat{E}_\mu(2)$ *are semi-regular, but not regular.*

2.9 The von Neumann algebra case

In this Section 2.9, we consider the quantum $E(2)$ group as a von Neumann algebraic quantum group. We are very concise in this study. We solely mention the basic results and are frugal of comment.

There is of course an equivalence between the von Neumann algebra level and the C*-algebra level. However, it is quite natural to prefer the C*-algebra setting for a description of the quantum $E(2)$ group. This choice has the advantage that we can use the notion of affiliated elements as described in Section 1.5. In the von Neumann algebra case, there is no such framework to study the action of (for instance) Φ and $\hat{\Phi}$ on the unbounded generators.

Notice that there is an overlap between several notations used below and their C*-algebraic counterpart. This does not cause any confusion.

We first look at the quantum $E(2)$ group.

Definition 2.9.1 *We consider the von Neumann algebra* $M = A''$*. Further, we consider the unital normal* **-homomorphism*

$$\Phi : M \to M \mathbin{\bar{\otimes}} M : x \mapsto W(x \otimes 1)W^*.$$

Definition 2.9.2 *We define a normal weight* ψ *on* M^+ *by setting*

$$\psi(x) = \mathrm{Tr}_N(qxq).$$

We immediately state the main Theorem 2.9.3. It is the von Neumann algebra version of Theorem 2.4.3. In the proof, we are sparing of details.

This Theorem 2.9.3 is central in the Section 2.9.

Theorem 2.9.3 *The pair* (M, Φ) *is a von Neumann algebraic quantum group. Further, we have that* ψ *is an n.s.f. weight on* M^+ *that is strongly left and strongly right invariant, i.e., we have*

$$\psi\big((\omega \mathbin{\bar{\otimes}} \iota)\Phi(x)\big) = \psi\big((\iota \mathbin{\bar{\otimes}} \omega)\Phi(x)\big) = \psi(x).$$

for all $x \in M^+$ *and* $\omega \in M_*^+$*.*

Proof. From the proof of Theorem 2.7.9, we can infer that $\psi = \psi R$ and that there exists an element $x \in M$ such that $\psi(x) = 1$.

An application of Corollary 1.4.20 now already ends the proof. ∎

The case of the quantum $\hat{E}(2)$ group can be treated in a similar way.

Definition 2.9.4 *We consider the von Neumann algebra $\hat{M} = \hat{A}''$. Further, we consider the unital normal *-homomorphism*

$$\Phi : \hat{M} \to \hat{M} \,\bar{\otimes}\, \hat{M} : x \mapsto W^*(1 \otimes x)W.$$

Definition 2.9.5 *We define two normal weight $\hat{\varphi}$ and $\hat{\psi}$ on M^+ by setting*

$$\hat{\varphi}(x) = \mathrm{Tr}_N(x) \qquad \textit{and} \qquad \hat{\psi}(x) = \mathrm{Tr}_{N_0}(x).$$

The next Theorem 2.9.6 holds.

Theorem 2.9.6 *The pair $(\hat{M}, \hat{\Phi})$ is a von Neumann algebraic quantum group. Further, we have that $\hat{\varphi}$ and $\hat{\psi}$ are n.s.f. weights on M^+ such that $\hat{\varphi}$ is strongly left invariant and $\hat{\psi}$ is strongly right invariant, i.e., we have*

- $\hat{\varphi}\big((\omega \,\bar{\otimes}\, \iota)\Phi(x)\big) = \omega(1)\hat{\varphi}(x)$,
- $\hat{\psi}\big((\iota \,\bar{\otimes}\, \omega)\Phi(x)\big) = \omega(1)\hat{\psi}(x)$ *for all $x \in M^+$ and $\omega \in M_*^+$.*

Proof. From the proof of Theorems 2.7.22 and 2.7.23, we can infer that $\hat{\psi} = \hat{\varphi}\hat{R}$ and that there exists an element $x \in \hat{M}$ such that $\hat{\varphi}(x) = 1$.

An application of Corollary 1.4.20 ends the proof of the theorem. ∎

The next Proposition 2.9.7 follows from Remark 1.3.14.

Proposition 2.9.7 *We have $W \in \hat{M} \,\bar{\otimes}\, M$.*

We state a simple result about $\hat{M}$. In comparison with the C*-algebra level, we can here be less fussy when working with the elements $u^k g(a, |b|)$.

The Proposition 2.9.8 below clearly resembles Theorem 2.5.21.

Proposition 2.9.8 *We have that*

$$\hat{M} = \Big\{ \sum_{k \in \mathbb{Z}} u^k g_k(a, |b|) \,\big|\, g_k \in \mathrm{C}_0(E) \textit{ for every } k \textit{ and } g_k \neq 0 \textit{for finitely many } k \Big\}''.$$

Proof. Theorem 2.5.24 implies that $f(b) \in \hat{M}$ for all $f \in \mathrm{C}_b(\bar{\mathbb{C}}(\mu^{\frac{1}{2}}))$. By using this property, it is easy to deduce that $u \in \hat{M}$.

If we now apply Theorem 2.5.21, the above description of $\hat{M}$ follows. ∎

Chapter 3

Amenability properties

A nowadays popular topic in the theory of locally compact quantum groups is the study of amenability properties. In this Chapter 3, we focus on the amenability results concerning the quantum $E(2)$ group.

The amenability theory for quantum groups generalizes the results around the familiar notion of amenability for locally compact groups. It is studied (among others) by E. Bédos and L. Tuset [9], P. Desmedt [22] and Y. Nakagami [84]. We refer to their articles for a full elaboration of the subject.

From Woronowicz' treatment in [159], it follows that the quantum $E(2)$ group is strongly amenable. In Section 3.2 below, we give an alternative proof of this property and we unveil some extra amenability results.

The Chapter 3 contains two parts.

Section 3.1 - Definition and equivalent notions. In this Section 3.1, we consider the general case. We introduce the definitions of the different notions of amenability and we give some related remarks.

Section 3.2 - Amenability of the quantum $E(2)$ group. We here study the specific amenability properties of the quantum groups $E(2)$ and $\hat{E}(2)$. In particular, we state simple proofs of their strong amenability.

3.1 Definition and equivalent notions

We first mention some generalities of the amenability theory of locally compact quantum groups. We state the main definitions and we look at a few basic properties. The overview is very condensed and contains no proof. As it is said above, we refer to [9, 22, 84] for more detailed treatments.

In the theory of locally compact groups, the notion of amenability gives a very important and intensively studied regularity condition. Standard references for the study of amenability of locally compact groups are [32] and [90].

Taking into account the importance of amenability in the group case, it is natural to look at generalizations of this concept to the quantum group world.

The first ones to study amenability for quantum groups were Enock and Schwartz in 1986. They gave in [28] a full treatment of amenability for Kac algebras (but with a defect in the proof of the main property; see [111]). All of their valid results were later proved to be true for locally compact quantum groups.

The amenability theory of quantum groups is of course inspired by the profusion of amenability results for locally compact groups. All amenability properties for quantum groups spring out of a corresponding result for groups.

For locally compact groups, we have a lot of equivalent characterizations of amenability. Most of these characterizations can be translated into the quantum language. However, it is not yet known if the equivalence between the different notions is still true on the level of quantum groups.

At the moment, we have two sets of equivalent characterizations. This leads to two notions of amenability; see Definitions 3.1.3 and 3.1.5.

The Notation 3.1.1 is kept fixed for the remainder of Section 3.1.

Notation 3.1.1 *Let* (A, Φ) *be a locally compact quantum group. We use* $(\hat{A}, \hat{\Phi})$ *to denote the dual quantum group. The corresponding von Neumann algebraic quantum groups are respectively denoted by* (M, Φ) *and* $(\hat{M}, \hat{\Phi})$.

Further, $\widetilde{W} \in B(K \otimes K)$ *denotes the left regular representation of* (A, Φ).

We now give all the definitions involved.

Definition 3.1.2 *Let* $m \in M^*$ *be a state. We then call* m *a* left *(resp.* right*)* invariant mean *on* (A, Φ) *if we have*

$$m\big((\omega \otimes \iota)\Phi(x)\big) = \omega(1)m(x) \qquad \textit{resp.} \qquad m\big((\iota \otimes \omega)\Phi(x)\big) = \omega(1)m(x)$$

for all $\omega \in M_*$ *and* $x \in M$.

If m *is both left and right invariant, then we call* m *an* invariant mean*.*

Definition 3.1.3 *We call* (A, Φ) *an* amenable *quantum group if and only if there exists a left invariant mean* m *on* (A, Φ).

Definition 3.1.4 *A* **-homomorphism* $\varepsilon : A \to \mathbb{C}$ *is called a* bounded counit *on* (A, Φ) *if it satisfies*

$$(\varepsilon \otimes \iota)\Phi = (\iota \otimes \varepsilon)\Phi = \iota.$$

Definition 3.1.5 *We call* (A, Φ) *a* strongly amenable *quantum group if and only if there exists a bounded counit* $\hat{\varepsilon}$ *on* $(\hat{A}, \hat{\Phi})$.

The next Lemma 3.1.6 is e.g. proved in [22, Proposition 2.2.3]. We remark that a detailed definition for $mm_0 \in M^*$ can be found in [22, Section A.1].

This Lemma 3.1.6 sums up a few basic properties.

Lemma 3.1.6 *Let R be the unitary antipode of (A, Φ). Assume that m, n are left invariant means on (A, Φ). We define $m_0 \in M^*$ by $m_0 = mR$. Further, we consider $mm_0 \in M^*$ defined by $mm_0 = (m \otimes m_0)\Phi$.*

For all $\lambda \in [0, 1]$, we have that $\lambda m + (1 - \lambda)n$ is a left invariant mean.

Further, the following properties hold:

- *We have that m_0 is a right invariant mean,*
- *We have that mm_0 is an invariant mean.*

Thus, (A, Φ) is amenable if and only if it possesses an invariant mean.

In the next two Examples 3.1.7 and 3.1.8, we display the standard amenability properties of some special types of quantum groups.

We first give information about the amenability results concerning commutative and cocommutative quantum groups. It is clear that these two cases are the motivating examples behind Definitions 3.1.3 and 3.1.5 above.

Example 3.1.7 *Let G be a locally compact group. We consider the quantum groups $(\mathrm{C}_0(G), \Phi)$ and $(\mathrm{C}^*_r(G), \hat{\Phi})$ defined in Examples 1.2.1 and 1.2.4.*

For both these quantum groups, the notions of amenability and strong amenability coincide; see Proposition 3.1.10 below.

The following properties hold:

- *$(\mathrm{C}_0(G), \Phi)$ is (strongly) amenable if and only if G is amenable;*
- *$(\mathrm{C}^*_r(G), \hat{\Phi})$ is always (strongly) amenable.*

Of course, amenability of G is considered in the usual sense.

It is important to notice that a state $m \in L^\infty(G)^*$ is a left invariant mean on $(\mathrm{C}_0(G), \Phi)$ in the sense of Definition 3.1.2 if and only if it is a topological left invariant mean in the usual sense. The counit *ε on $(\mathrm{C}_0(G), \Phi)$ is defined by the formula*

$$\varepsilon(f) = f(e)$$

where e is the identity of G.

For the reduced group C-algebra $\mathrm{C}^*_r(G)$, we consider the linear map $\hat{\varepsilon}$ which is defined by*

$$\hat{\varepsilon} : L^1(G) \subseteq \mathrm{C}^*_r(G) \to \mathbb{C} : f \mapsto \int f(s)\, ds$$

where we integrate over the left Haar measure.

It is known from the theory of locally compact groups that G is amenable if and only if it is possible to extend $\hat{\varepsilon}$ to a bounded linear map on $\mathrm{C}^*_r(G)$.

We also shed light on the situation for compact and discrete quantum groups. These special cases are studied intensively in [11, 12, 123, 10].

In the Example 3.1.8 below, we make use of Proposition 1.1.38.

Example 3.1.8 *For every compact quantum group, the Haar state gives the unique invariant mean. Thus, all compact quantum groups are amenable.*

Further, every discrete quantum group has a bounded counit.

We hence have that all compact quantum groups are strongly amenable.

The next Proposition 3.1.9 is standard. See e.g. [9] or [22].

Proposition 3.1.9 *If* (A, Φ) *is amenable, then* $\hat{A}$ *is nuclear.*

We already mentioned above that it is not known if the notions of amenability and strong amenability coincide. The possible discrepancy between these two concepts is the subject of nowadays research.

The equivalence between the two notions of amenability is proved to be true in special cases, but the general case is still an open question.

The following Proposition 3.1.10 combines the related results. It also justifies the used terminology for the notion of 'strong' amenability.

Proposition 3.1.10 *If* (A, Φ) *is strongly amenable, then it is also amenable. The converse is true in the following cases:*

- *We have that* (A, Φ) *is commutative (old result);*
- *We have that* (A, Φ) *is cocommutative (trivial);*
- *We have that* (A, Φ) *is compact (trivial);*
- *We have that* (A, Φ) *is discrete (see [12, 123, 10]).*

The next Proposition 3.1.11 states that we can deduce strong amenability out of amenability if we demand an extra condition.

It was C.-K. Ng who first gave a proof of the result below.

Proposition 3.1.11 ([85]) *The following statements are equivalent:*

1. *We have that* (A, Φ) *is strongly amenable,*
2. *There exists a left invariant mean* m *on* (A, Φ) *such that*

$$m\big(\{\, (\iota \otimes \omega)\widetilde{W} \mid \omega \in \hat{A}^*\}\big) \neq \{0\}.$$

There exist several equivalent characterizations of both amenability and strongly amenability. We refer to [9] and [22] for a complete list.

We only include the Proposition 3.1.12 and 3.1.13. The latter one appears (in the adapted form of Proposition 3.1.19) in our calculations in Section 3.2.

The Proposition 3.1.12 below concerns amenability.

Proposition 3.1.12 ([22]) *The three statements below are equivalent:*

1. *We have that (A, Φ) is amenable;*
2. *There exists a net (ω_i) of states in M_* such that $\omega\omega_i(x) - \omega(1)\omega_i(x) \to 0$ (resp., $\omega_i\omega(x) - \omega(1)\omega_i(x) \to 0$) for all $\omega \in M_*$ and $x \in M$,*
3. *There exists a net (ω_i) of states in M_* such that $\|\omega\omega_i - \omega(1)\omega_i\| \to 0$ (resp., $\|\omega_i\omega - \omega(1)\omega_i\| \to 0$) for all $\omega \in M_*$.*

The next Proposition 3.1.13 is about strong amenability.

Proposition 3.1.13 ([9, 22]) *The four statements below are equivalent:*

1. *We have that (A, Φ) is strongly amenable;*
2. *There exists a state $\theta \in B(K)^*$ such that $(\iota \otimes \theta)(\widetilde{W}) = 1$,*
3. *There exists a net (ξ_j) of unit vectors in K such that*
$$(\iota \otimes \omega_{\xi_j})(\widetilde{W}) \to 1$$
in the weak topology.
4. *There exists a net (ξ_j) of unit vectors in K such that*
$$\|\widetilde{W}(\eta \otimes \xi_j) - (\eta \otimes \xi_j)\| \to 0$$
for all $\eta \in K$.

We recall the standard Notation 3.1.14.

Notation 3.1.14 *We use R and (τ_t) to denote the unitary antipode respectively the scaling group of (A, Φ) as introduced in Section 1.1.*

The three Lemmas 3.1.15, 3.1.16 and 3.1.17 contain simple properties.

Lemma 3.1.15 ([84]) *Assume that (A, Φ) is amenable. Then, there exists an invariant mean $m \in M^*$ such that $m\tau_t = m$ for all $t \in \mathbb{R}$.*

Lemma 3.1.16 ([9]) *Let $m \in M^*$ be a left invariant mean on (A, Φ). We then have $m \in M_*$ if and only if A is unital, i.e., if (A, Φ) is compact.*

Further, we have $m|_A \neq 0$ if and only if A is unital.

Lemma 3.1.17 ([9]) *Let $\varepsilon \in A^*$ be a bounded counit on (A, Φ). Then ε is the unique state on A such that $(\varepsilon \otimes \iota)\Phi = (\iota \otimes \varepsilon)\Phi = \iota$.*

Moreover, we have that $\varepsilon R = \varepsilon$ and $\varepsilon \tau_t = \varepsilon$ for all $t \in \mathbb{R}$.

We then also mention the Proposition 3.1.18. J. Kustermans studies the theory of universal quantum groups in [61]. This appearance of a quantum group is especially useful if one wants to disentangle its representation theory.

Proposition 3.1.18 ([9, 22]) *If (A, Φ) is strongly amenable, then $(\hat{A}, \hat{\Phi})$ is a universal quantum group in the sense of Kustermans [61].*

The next Proposition 3.1.19 extends Proposition 3.1.13. We state it in a version which is adapted to the manageability theory. In this way, it is best suited to examine the amenability properties of the quantum $E(2)$ group.

One can apply the techniques from [9, 22] to prove the property below.

Proposition 3.1.19 *Let $W \in B(H \otimes H)$ be a modular multiplicative unitary such that (A, Φ) is the bi-C^*-algebra associated to W.*

Suppose that $(\xi_i)_{i \in I}$ is a net in H such that

$$(\omega_{\xi_i} \otimes \iota)(W) \rightarrow 1$$

in the weak topology.

Let $\theta \in B(H)^*$ be a weak* limit point of the set $\{\, \omega_{\xi_i} \mid i \in I\}$.

We then define $\hat{\varepsilon} \in \hat{A}^*$ by $\hat{\varepsilon} = \theta|_{\hat{A}}$ and $m \in M^*$ by $m = \theta|_M$.

The following properties hold:

- *We have that $\hat{\varepsilon}$ is a bounded counit on $(\hat{A}, \hat{\Phi})$,*
- *We have that m is a right invariant mean on (A, Φ).*

It ensues that (A, Φ) is strongly amenable.

3.2 Amenability of the quantum $E(2)$ group

We now make use of the results in Section 3.1 to give a detailed study of the amenability properties of the quantum $E(2)$ group. The main property is that both the quantum groups $E(2)$ and $\hat{E}(2)$ are strongly amenable. This result is not new. It was first mentioned by Woronowicz in [159, 149].

We approach the amenability features of the quantum $E(2)$ group in a completely different way. In our treatment, we follow the lines of the general theory and, in particular, apply Proposition 3.1.19. We hereby go into full detail. Further, we also display some new formulas concerning the invariant means.

Proposition 3.1.19 plays a crucial role in the amenability study below. It is the main actor in our calculations. We apply Proposition 3.1.19 to the manageable multiplicative unitary W as introduced in Definition 2.3.9.

By performing the procedure that is encoded in Proposition 3.1.19, we can quite easily prove some extra amenability results. For the quantum $E(2)$ group, we construct an infinite set of linearly independent invariant means and we display explicit formulas for their action on basic elements.

There is a similar result for the quantum $\hat{E}(2)$ group. In this case, we stumble across a distinction between left and right invariant means.

First, we fix the Notation 3.2.1.

Notation 3.2.1 *We adopt all notations introduced in Chapter 2. In particular, we have that (A, Φ) denotes the quantum $E(2)$ group.*

We now state the two basic properties. The Propositions 3.2.3 and 3.2.5 below allow us to apply the Proposition 3.1.19 and therefore are the key results in our amenability study of the quantum $E(2)$ group.

To prove Propositions 3.2.3 and 3.2.5, we make use of the convergence properties of the Fourier coefficients $B(k, n)$; see Proposition 2.2.5.

Definition 3.2.2 *For every $n \in \mathbb{N}$, we define a unit vector ξ_n by*

$$\xi_n = \frac{1}{\sqrt{2n+1}} \sum_{k=-n}^{n} e_k \otimes e_{k-2n}.$$

Proposition 3.2.3 *If $n \to +\infty$, we have that*

$$(\iota \otimes \omega_{\xi_n})W \to 1$$

in the weak topology.

Proof. Let $p, q, r, s \in \mathbb{Z}$. For all $n \in \mathbb{N}$, we have

$$\big\langle \big((\iota \otimes \omega_{\xi_n})W\big)(e_p \otimes e_q), e_r \otimes e_s \big\rangle = \frac{1}{2n+1} \sum_{k,l=-n}^{n} B(0, p-q-k+2n+1)\delta_{k-p+2q,l}\delta_{p,r}\delta_{q,s}.$$

This ensues directly from Definition 3.2.2 and Corollary 2.3.14.

We take $n \geq \frac{1}{2}|p-2q|$. If $p \leq 2q$, we get that

$$\big\langle \big((\iota \otimes \omega_{\xi_n})W\big)(e_p \otimes e_q), e_r \otimes e_s \big\rangle = \frac{1}{2n+1} \sum_{k=-n}^{n+p-2q} B(0, p-q-k+2n+1)\delta_{p,r}\delta_{q,s}. \quad (3.1)$$

Proposition 2.2.5 and Equation (3.1) together yield

$$\big\langle \big((\iota \otimes \omega_{\xi_n})W\big)(e_p \otimes e_q), e_r \otimes e_s \big\rangle \to \delta_{p,r}\delta_{q,s}. \quad (3.2)$$

In the case $p > 2q$, we find this result in a completely similar way.

Equation (3.2) implies that

$$\big\langle (\iota \otimes \omega_{\xi_n}) W \eta_1, \eta_2 \big\rangle \to \langle \eta_1, \eta_2 \rangle$$

for all $\eta_1, \eta_2 \in H_0$.
From this, we can conclude that

$$(\iota \otimes \omega_{\xi_n}) W \to 1$$

in the weak topology. ∎

Definition 3.2.4 *For every $n \in \mathbb{N}$, we define a unit vector ξ'_n by*

$$\xi'_n = e_{2n} \otimes e_n .$$

Proposition 3.2.5 *If $n \to +\infty$, we have that*

$$(\omega_{\xi'_n} \otimes \iota) W \to 1$$

in the weak topology.

Proof. Let $p, q, r, s \in \mathbb{Z}$. For all $n \in \mathbb{N}$, we have

$$\Big\langle \big((\omega_{\xi'_n} \otimes \iota) W \big)(e_p \otimes e_q), e_r \otimes e_s \Big\rangle = B(0, n - q + 1) \delta_{p,r} \delta_{q,s} . \tag{3.3}$$

This ensues directly from Definition 3.2.4 and Corollary 2.3.14.
Proposition 2.2.5 and Equation (3.3) together yield

$$\big\langle (\omega_{\xi'_n} \otimes \iota) W (e_p \otimes e_q), e_r \otimes e_s \big\rangle \to \delta_{p,r} \delta_{q,s} .$$

Similar as in the proof of Proposition 3.2.3, we then can derive that

$$(\omega_{\xi'_n} \otimes \iota) W \to 1$$

in the weak topology. ∎

The next Definition 3.2.6 is given with a view to Proposition 3.1.19. We keep the functionals θ and $\hat{\theta}$ fixed for the remainder of the section.

Definition 3.2.6 *Because the unit ball of $B(H)^*$ is weak* compact, we can take linear functionals $\theta, \hat{\theta} \in B(H)^*$ such that*

a) We have that θ is a weak limit point of the set $\{\, \omega_{\xi_n} \mid n \in \mathbb{N} \}$,*

b) We have that $\hat{\theta}$ is a weak limit point of the set $\{\, \omega_{\xi'_n} \mid n \in \mathbb{N} \}$.*

The quantum $E(2)$ group

We now give a detailed account of the amenability properties of the quantum $E(2)$ group (A, Φ). We prove that (A, Φ) is strongly amenable and construct both a *bounded counit* ε and an *invariant mean* m on (A, Φ).

The aforementioned Propositions 3.2.3 and 3.2.5 are the main ingredients in our amenability study. Having these properties at hand, it is just a simple task to apply Proposition 3.1.19 and deduce all the desired results.

Besides the basic amenability properties, we also include a reflection about the non-uniqueness of the invariant mean m. We prove in Proposition 3.2.28 below that (A, Φ) possesses infinitely many different invariant means. This result may be not too telling, but we are brash enough to include it.

The next Theorem 3.2.7 is the central property of this section. It ensues directly from a combination of Propositions 3.1.19 and 3.2.5.

Theorem 3.2.7 *We have that (A, Φ) is strongly amenable.*

We then mention the Proposition 3.2.8 below. It can be proved by combining Propositions 3.1.19 and 3.2.3 with Proposition 3.1.18.

The main importance of Proposition 3.2.8 lies in its consequences with regards to the representation theory of (A, Φ).

We recall that universal quantum groups are studied in [61].

Proposition 3.2.8 *The pair (A, Φ) is a universal quantum group.*

We construct a bounded counit ε on (A, Φ). The Definition 3.2.9 below is clearly inspired by the general amenability results as expounded in Section 3.1.

In the Corollary 4.2.4, we regard ε via an alternative approach.

Definition 3.2.9 *We define a state $\varepsilon \in A^*$ by $\varepsilon = \theta|_A$.*

The next Proposition 3.2.10 infers from Propositions 3.1.19 and 3.2.3.

Proposition 3.2.10 *We have that ε is a* bounded counit *on (A, Φ), i.e., we have that ε is the only *-homomorphism from A to $\mathbb{C}$ such that*

$$(\varepsilon \otimes \iota)\Phi = (\iota \otimes \varepsilon)\Phi = \iota.$$

By virtue of Lemma 3.1.17, the two following properties hold.

Lemma 3.2.11 *We have $\varepsilon R = \varepsilon$.*

Lemma 3.2.12 *For all $t \in \mathbb{R}$, we have $\varepsilon \tau_t = \varepsilon$.*

We can easily compute the action of ε on basic elements. The formulas below completely resemble the related algebraic results in Proposition 2.1.1.

The next Proposition 3.2.13 describes ε in full detail.

Proposition 3.2.13 *The counit ε is characterized as the only *-homomorphism from A to $\mathbb{C}$ such that*

$$\varepsilon(c) = 1 \qquad \textit{and} \qquad \varepsilon(d) = 0.$$

Let $\sum c^k f_k(d) \in A$. Then, we have

$$\varepsilon\Big(\sum c^k f_k(d)\Big) = \sum f_k(0).$$

Proof. The uniqueness part is a consequence of Theorem 2.5.3. We thus only have to bother about the formulas for $\varepsilon(c)$ and $\varepsilon(d)$.

We know from Proposition 2.5.12 that

$$\Phi(c) = c \otimes c \qquad \text{and} \qquad \Phi(d) = c \otimes d \dotplus d \otimes c^*. \tag{3.4}$$

If we now apply Proposition 3.2.10, we get

$$c = (\varepsilon \otimes \iota)\Phi(c) = (\varepsilon \otimes \iota)(c \otimes c) = \varepsilon(c)c.$$

This gives that $\varepsilon(c) = 1$.

It is somewhat more difficult to prove that $\varepsilon(d) = 0$. We remark that one should not be too hasty when checking formulas concerning affiliated elements.

We combine Equation (3.4) with Proposition 3.2.10. If we are careful, we can prove that

$$d = (\varepsilon \otimes \iota)\Phi(d) = (\varepsilon \otimes \iota)(c \otimes d \dotplus d \otimes c^*) = \varepsilon(c)d \dotplus \varepsilon(d)c^* = d + \varepsilon(d)c^*.$$

We hence find that $\varepsilon(d) = 0$. ∎

It is also easy to calculate the action of ε on elements $(\omega \otimes \iota)W$. This leads to the standard formulas in Proposition 3.2.14 below.

Proposition 3.2.14 *Let $\omega \in B(H)_*$. Then, we have*

$$\varepsilon\big((\omega \otimes \iota)W\big) = \omega(1).$$

Proof. For all $\omega \in B(H)_*$ and $n \in \mathbb{N}_0$, we have that

$$\omega_{\xi_n}\big((\omega \otimes \iota)W\big) = \omega\big((\iota \otimes \omega_{\xi_n})W\big).$$

An application of Proposition 3.2.3 and Definition 3.2.9 ends the proof. ∎

It is a routine verification to deduce the Corollary 3.2.15 below.

Corollary 3.2.15 *We have that $\varepsilon \in A^*$ is characterized by the formula*

$$(\iota \otimes \varepsilon)W = 1.$$

We now construct an invariant mean m on (A, Φ). It is clear that we again find inspiration in the amenability results from Section 3.1.

The next Definition 3.2.16 defines m in a standard way.

Definition 3.2.16 *We define a state $m \in M^*$ by $m = \hat{\theta}|_M$.*

The main feature of m is stated in the Proposition 3.2.17 below.

Proposition 3.2.17 *The state m is an* invariant mean *on (A, Φ), i.e., we have that*

$$m\big((\omega \otimes \iota)\Phi(x)\big) = m\big((\iota \otimes \omega)\Phi(x)\big) = \omega(1)m(x)$$

for all $\omega \in M_*$ and $x \in M$.

Proof. From Propositions 3.1.19 and 3.2.5, it ensues that m is a right invariant mean on (A, Φ). Using Equation (2.63), it is direct to prove that $mR = m$.

We can then deduce that m is also left invariant. ∎

It is easy to check the following properties.

Lemma 3.2.18 *We have $mR = m$.*

Lemma 3.2.19 *For all $t \in \mathbb{R}$, we have $m\tau_t = m$.*

The Proposition 3.2.20 displays the action of m on basic elements. It is explicated in Remark 3.2.21 below that we are somewhat sloppy with the notation.

The fact that $m|_A = 0$ is standard; see Lemma 3.1.16.

Proposition 3.2.20 *Let $\sum c^k f_k(d) \in M(A)$. We have*

$$m\Big(\sum c^k f_k(d)\Big) = \lim_{n \to +\infty} \frac{1}{2\pi} \int_0^{2\pi} f_0(\mu^{-n} e^{it})\, dt. \tag{3.5}$$

Hence, we see that $m|_A = 0$.

Proof. Let $n \in \mathbb{N}$. For every $l \in \mathbb{Z}$, we define a number $a(l)$ by setting

$$a(l) = \frac{1}{2\pi} \int_0^{2\pi} f_0(\mu^{-n} e^{it}) e^{-ilt}\, dt.$$

The polar decomposition $d = (s \otimes 1)(1 \otimes m^{-1})$ holds; see Lemma 2.3.4.

We then have that

$$\begin{aligned}\omega_{\xi'_n}\Big(\sum c^k f_k(d)\Big) &= \sum \big\langle c^k f_k(d)(e_{2n}\otimes e_n), e_{2n}\otimes e_n\big\rangle \\ &= \sum \big\langle f_k(d)(e_{2n}\otimes e_n), e_{2n-k}\otimes e_{n-k}\big\rangle = \big\langle f_0(\mu^{-n}s)e_{2n}, e_{2n}\big\rangle \\ &= \sum_{l\in\mathbb{Z}} \big\langle a(l)s^l e_{2n}, e_{2n}\big\rangle = a(0).\end{aligned}$$

It is now direct to deduce the formula in Equation (3.5). ∎

The next Remark 3.2.21 is quite important.

Remark 3.2.21 *We have to be careful with the loose notations that are used in Proposition 3.2.20. The Equation* (3.5) *actually means that the considered equality holds if the limit on the right hand side exists.*

In the case that this limit does not exist, we are in a less comfortable situation. The freedom in Definition 3.2.6 then leaves open several possibilities and we get that the action of m is not unambiguously defined.

We now digress about the non-uniqueness of the invariant mean m as introduced in Definition 3.2.16. The results below can be considered as an additional side remark to our amenability study of the quantum $E(2)$ group.

It is a foreseen property that any amenable quantum group which is non-compact possesses an infinity of invariant means. This result is known to be true in the commutative and the cocommutative case; see [90] and [36]. It is an open problem how the general case can be proved (if possible).

We perform minor changes to the construction procedure of m. In this way, we get hold on a lot of different invariant means; see Proposition 3.2.28.

Definition 3.2.22 *Let $p\in\mathbb{Z}$ and $q,n\in\mathbb{N}$ with $q\neq 0$. Then, we define a unit vector $\xi'_{n,p,q}$ by*

$$\xi'_{n,p,q} = e_{2qn+2p}\otimes e_{qn+p}.$$

Proposition 3.2.23 *Let $p\in\mathbb{Z}$ and $q\in\mathbb{N}_0$. If $n\to+\infty$, we have*

$$(\omega_{\xi'_{n,p,q}}\otimes\iota)W \to 1$$

in the weak topology.

Proof. Let $p,u,v,r,s\in\mathbb{Z}$ and $q\in\mathbb{N}_0$. For all $n\in\mathbb{N}$, we have that

$$\Big\langle\big((\omega_{\xi'_{n,p,q}}\otimes\iota)W\big)(e_u\otimes e_v), e_r\otimes e_s\Big\rangle = B(0, qn+p-v+1)\delta_{u,r}\delta_{v,s}.$$

This ensues directly from Definition 3.2.22 and Corollary 2.3.14.

The remainder of the proof is identical to the proof of Proposition 3.2.5. ∎

The next Definition 3.2.24 introduces a set of states $m_{p,q} \in M^*$.

Definition 3.2.24 *For all $p \in \mathbb{Z}$ and $q \in \mathbb{N}_0$, we consider a weak* limit point $\hat{\theta}_{p,q} \in B(H)^*$ of the set $\{\, \omega_{\xi'_{n,p,q}} \,|\, n \in \mathbb{N}\}$.*

We then define a state $m_{p,q} \in M^*$ by $m_{p,q} = \hat{\theta}_{p,q}|_M$.

It is straightforward to generalize Propositions 3.2.17 and 3.2.20. This leads to the two Propositions 3.2.25 and 3.2.26 below.

We observe that Remark 3.2.21 also alludes to Equation (3.6).

Proposition 3.2.25 *For all $p \in \mathbb{Z}$ and $q \in \mathbb{N}_0$, the state $m_{p,q} \in M^*$ is an invariant mean on (A, Φ).*

Proposition 3.2.26 *Let $\sum c^k f_k(d) \in M(A)$. We have*

$$m_{p,q}\Big(\sum c^k f_k(d)\Big) = \lim_{n\to+\infty} \frac{1}{2\pi}\int_0^{2\pi} f_0(\mu^{-qn-p}e^{it})\,dt \tag{3.6}$$

Hence, we see that $m_{p,q}|_A = 0$.

We now arrive at the Proposition 3.2.27 below.

Proposition 3.2.27 *Let $q \in \mathbb{N}_0$. The states in $\{\, m_{p,q} \,|\, p \in \mathbb{N},\, 0 \leq p < q\}$ are then linearly independent from each other.*

Proof. Let $q \in \mathbb{N}_0$. For all $p \in \{0, 1, \ldots q-1\}$, we define a subset $Y_p \subseteq \bar{\mathbb{R}}^\mu$ by setting $Y_p = \{\mu^{-qn-p} \,|\, n \in \mathbb{N}\}$.

Next, we fix an element $p_1 \in \{0, 1, \ldots q-1\}$. We define $f \in \mathrm{C}_b(\bar{\mathbb{C}}^\mu)$ by

$$f : \bar{\mathbb{C}}^\mu \to \mathbb{C} : x \mapsto \begin{cases} 1 & \text{if } |x| \in Y_{p_1}, \\ 0 & \text{otherwise.} \end{cases}$$

We get from Lemma 2.5.4 that $f(d) \in M(A)$.

From Proposition 3.2.26, it follows that

$$m_{p_1,q}(f(d)) = \lim_{n\to+\infty} \frac{1}{2\pi}\int_0^{2\pi} f(\mu^{-qn-p_1}e^{it})\,dt = 1.$$

Take $p_2 \in \{0, 1, \ldots q-1\}$ with $p_2 \neq p_1$. It is easy to check that $L_{p_1} \cap L_{p_2} = \emptyset$.

If we again apply Proposition 3.2.26, we get

$$m_{p_2,q}(f(d)) = \lim_{n\to+\infty} \frac{1}{2\pi}\int_0^{2\pi} f(\mu^{-qn-p_2}e^{it})\,dt = 0.$$

This ends the proof of the proposition. ∎

The Propositions 3.2.25 and 3.2.27 together imply that there exist infinitely many invariant means $m_{p,q}$ on (A, Φ).

We state this result in the next Proposition 3.2.28.

Proposition 3.2.28 *We have that the set* $\{ m_{p,q} \mid p \in \mathbb{Z}, q \in \mathbb{N}_0 \}$ *contains an infinite subset of linearly independent states.*

The number of different invariant means $m_{p,q}$ *on* (A, Φ) *is thus infinite.*

The quantum $\hat{E}(2)$ group

We look at the amenability properties of the quantum $\hat{E}(2)$ group $(\hat{A}, \hat{\Phi})$. It is proved in Theorem 3.2.29 that $(\hat{A}, \hat{\Phi})$ is strongly amenable. Further, we construct both a *bounded counit* $\hat{\varepsilon}$ and an *invariant mean* $\hat{m}$ on $(\hat{A}, \hat{\Phi})$.

The amenability properties of $(\hat{A}, \hat{\Phi})$ are very similar to those of (A, Φ). For this reason, we are quite succinct in the treatment below.

Similar to the case of (A, Φ), the Propositions 3.2.3 and 3.2.5 are crucial in our amenability study. They carry almost all necessary information about the amenability of $(\hat{A}, \hat{\Phi})$ and are extensively used in the proofs below.

If we combine Propositions 3.2.3 and 3.2.5 with Propositions 3.1.18 and 3.1.19, we get Theorem 3.2.29 and Proposition 3.2.30.

The next Theorem 3.2.29 states the strong amenability of $(\hat{A}, \hat{\Phi})$.

Theorem 3.2.29 *We have that* $(\hat{A}, \hat{\Phi})$ *is strongly amenable.*

The Proposition 3.2.30 below is again to be considered in the setting of [61].

Proposition 3.2.30 *The pair* $(\hat{A}, \hat{\Phi})$ *is a universal quantum group.*

We construct a bounded counit $\hat{\varepsilon}$ on $(\hat{A}, \hat{\Phi})$. The Definition 3.2.31 below is dual to Definition 3.2.9. It is based on the results in Section 3.1.

Definition 3.2.31 *We define a state* $\hat{\varepsilon} \in \hat{A}^*$ *by* $\hat{\theta}|_{\hat{A}}$.

The next Proposition 3.2.32 infers from Propositions 3.1.19 and 3.2.5.

Proposition 3.2.32 *We have that* $\hat{\varepsilon}$ *is a* bounded counit *on* $(\hat{A}, \hat{\Phi})$*, i.e., we have that* $\hat{\varepsilon}$ *is the only* **-homomorphism from* $\hat{A}$ *to* $\mathbb{C}$ *such that*

$$(\hat{\varepsilon} \otimes \iota)\hat{\Phi} = (\iota \otimes \hat{\varepsilon})\hat{\Phi} = \iota.$$

By virtue of Lemma 3.1.17, the two following properties hold.

Lemma 3.2.33 *We have* $\hat{\varepsilon}\hat{R} = \hat{\varepsilon}$.

Lemma 3.2.34 *For all* $t \in \mathbb{R}$*, we have* $\hat{\varepsilon}\hat{\tau}_t = \hat{\varepsilon}$.

We compute the action of $\hat{\varepsilon}$ on basic elements. The formulas below are in correspondence with the algebraic results in Proposition 2.1.1.

The next Proposition 3.2.35 describes $\hat{\varepsilon}$ in full detail.

Proposition 3.2.35 *The counit $\hat{\varepsilon}$ is characterized as the only *-homomorphism from $\hat{A}$ to $\mathbb{C}$ such that*

$$\hat{\varepsilon}(a) = 1 \qquad \textit{and} \qquad \hat{\varepsilon}(b) = 0.$$

Let $\sum u^k g_k(a, |b|) \in \hat{A}$. Then, we have

$$\hat{\varepsilon}\Big(\sum u^k g_k(a, |b|)\Big) = g_0(1, 0).$$

Proof. The uniqueness part is a consequence of Theorem 2.5.24. We thus only have to bother about the formulas for $\hat{\varepsilon}(a)$ and $\hat{\varepsilon}(b)$.

We know from Proposition 2.5.34 that

$$\hat{\Phi}(a) = a \otimes a \qquad \text{and} \qquad \hat{\Phi}(b) = a \otimes b \dotplus b \otimes a^{-1}. \tag{3.7}$$

If we now apply Proposition 3.2.32, we get

$$a = (\hat{\varepsilon} \otimes \iota)\hat{\Phi}(a) = (\hat{\varepsilon} \otimes \iota)(a \otimes a) = \hat{\varepsilon}(a)a.$$

This gives that $\hat{\varepsilon}(a) = 1$.

We then combine Equation (3.7) with Proposition 3.2.32. If we are careful, we can prove that

$$b = (\hat{\varepsilon} \otimes \iota)\hat{\Phi}(b) = (\hat{\varepsilon} \otimes \iota)(a \otimes b \dotplus b \otimes a^{-1}) = \hat{\varepsilon}(a)b \dotplus \hat{\varepsilon}(b)a^{-1} = b \dotplus \hat{\varepsilon}(b)a^{-1}.$$

We hence find that $\hat{\varepsilon}(b) = 0$. ∎

The Remark 3.2.36 is an inessential detail.

Remark 3.2.36 *It is easy to calculate that $\hat{\theta}(u) = 0$. We hence see that it is impossible to extend $\hat{\varepsilon}$ to a *-homomorphism on $\hat{M}$.*

We also describe the action of $\hat{\varepsilon}$ on the elements $(\iota \otimes \omega)W$.

Proposition 3.2.37 *Let $\omega \in B(H)_*$. Then, we have*

$$\hat{\varepsilon}\big((\iota \otimes \omega)W\big) = \omega(1).$$

Proof. For all $\omega \in B(H)_*$ and $n \in \mathbb{N}_0$, we have that

$$\omega_{\xi'_n}\big((\iota \otimes \omega)W\big) = \omega\big((\omega_{\xi'_n} \otimes \iota)W\big).$$

An application of Proposition 3.2.5 and Definition 3.2.31 ends the proof. ∎

It is a routine verification to deduce the Corollary 3.2.38 below.

Corollary 3.2.38 *We have that $\hat{\varepsilon} \in \hat{A}^*$ is characterized by the formula*

$$(\hat{\varepsilon} \otimes \iota)W = 1.$$

We then construct a left invariant mean $\hat{m}$ on $(\hat{A}, \hat{\Phi})$. The duality between the two Definitions 3.2.39 and 3.2.16 is obvious.

Definition 3.2.39 *We define a state $\hat{m} \in \hat{M}^*$ by $\hat{m} = \theta|_{\hat{M}}$.*

The main feature of $\hat{m}$ is stated in the Proposition 3.2.40 below.

Proposition 3.2.40 *The state $\hat{m}$ is a* left invariant mean *on $(\hat{A}, \hat{\Phi})$, i.e., we have that*

$$\hat{m}\big((\omega \otimes \iota)\hat{\Phi}(x)\big) = \omega(1)\hat{m}(x)$$

for all $\omega \in \hat{M}_*$ and $x \in \hat{M}$.

Proof. From Propositions 3.1.19 and 3.2.3, it ensues that $\hat{m}$ is a right invariant mean on $(\widehat{A, \Phi})$. The Proposition 2.8.21 gives that $(\hat{A}, \hat{\Phi}) = (\widehat{A, \Phi})^{\mathrm{op}}$. ∎

We use Lemma 3.1.6 to deduce some allied results.

Definition 3.2.41 *We define a state $\hat{m}_0 \in \hat{M}^*$ by $\hat{m}_0 = \hat{m}\hat{R}$.*

Proposition 3.2.42 *The properties below hold:*

- *We have that $\hat{m}_0$ is a right invariant mean on $(\hat{A}, \hat{\Phi})$,*
- *We have that $\hat{m}\hat{m}_0$ is an invariant mean on $(\hat{A}, \hat{\Phi})$.*

The next Remark 3.2.43 is just a trivia.

Remark 3.2.43 *It is possible to also write $\hat{m}_0$ as a limit of vector functionals. Define for every $n \in \mathbb{Z}$ a vector ξ''_n by*

$$\xi''_n = \frac{1}{\sqrt{2n+1}} \sum_{k=-n}^{n} e_k \otimes e_{2n}.$$

We then have $\omega_{\xi_n}\hat{R} = \omega_{\xi''_n}$ for every n.

It is easy to check the Lemma 3.2.44.

Lemma 3.2.44 *For all $t \in \mathbb{R}$, we have $\hat{m}\hat{\tau}_t = \hat{m}$.*

The next Proposition 3.2.45 displays the action of $\hat{m}$ and $\hat{m}_0$ on basic elements. We again denote the formulas in the spirit of Remark 3.2.21.

Proposition 3.2.45 *Let $\sum u^k g_k(a,|b|) \in M(\hat{A})$. We have*

$$\hat{m}\Big(\sum u^k g_k(a,|b|)\Big) = \lim_{n\to+\infty} \frac{1}{2n+1} \sum_{r=-n}^{n} g_0(\mu^{\frac{1}{2}r-2n}, \mu^{\frac{1}{2}r}),$$

$$\hat{m}_0\Big(\sum u^k g_k(a,|b|)\Big) = \lim_{n\to+\infty} \frac{1}{2n+1} \sum_{r=-n}^{n} g_0(\mu^{-\frac{1}{2}r+2n}, \mu^{\frac{1}{2}r}).$$

Hence, we see that $\hat{m}|_{\hat{A}} = 0$ and $\hat{m}_0|_{\hat{A}} = 0$.

Proof. Let $n \in \mathbb{N}$. We have that

$$\begin{aligned}
\omega_{\xi_n}\Big(\sum_k u^k g_k(a,|b|)\Big) &= \sum_k \frac{1}{2n+1} \sum_{r,s=-n}^{n} \big\langle u^k g_k(a,|b|)(e_r \otimes e_{r-2n}), (e_s \otimes e_{s-2n})\big\rangle \\
&= \sum_k \frac{1}{2n+1} \sum_{r,s=-n}^{n} g_k(\mu^{\frac{1}{2}r-2n}, \mu^{\frac{1}{2}r-n}) \langle e_r \otimes e_{r-2n+k}, e_s \otimes e_{s-2n}\rangle \\
&= \frac{1}{2n+1} \sum_{r=-n}^{n} g_0(\mu^{\frac{1}{2}r-2n}, \mu^{\frac{1}{2}r-n}).
\end{aligned}$$

This gives the action of $\hat{m}$. The action of $\hat{m}_0$ can be computed analogously. ■

We alter the basic construction of $\hat{m}$ to prove that $(\hat{A}, \hat{\Phi})$ possesses infinitely many different left invariant means; see Proposition 3.2.52.

The Definition 3.2.46 below is the start of this process.

Definition 3.2.46 *Let $p, r, s \in \mathbb{Z}$ and $q, n \in \mathbb{N}$ with $q \geq 2$. Then, we define a unit vector $\xi_{n,p,q,r,s}$ by*

$$\xi_{n,p,q,r,s} = \frac{1}{\sqrt{2n+1}} \sum_{k=-n}^{n} e_{k-sn+r} \otimes e_{k-qn+p}.$$

Proposition 3.2.47 *Let $p, q, r, s \in \mathbb{Z}$ with $q \geq 2$. If $n \to +\infty$, we have*

$$(\omega_{\xi'_{n,p,q,r,s}} \otimes \iota)W \to 1$$

in the weak topology.

Proof. Let $p, q, r, s, u, v, w, x \in \mathbb{Z}$ with $q \geq 2$. For all $n \in \mathbb{N}$, we have that

$$\begin{aligned}&\big\langle \big((\iota \otimes \omega_{\xi_{n,p,q,r,s}})W\big)(e_u \otimes e_v), e_w \otimes e_x \big\rangle \\ &\qquad = \frac{1}{2n+1} \sum_{k,l=-n}^{n} B(0, u-v-k+qn-p+1)\delta_{k-u+2v,l}\delta_{u,w}\delta_{v,x}.\end{aligned}$$

This ensues directly from Definition 3.2.46 and Corollary 2.3.14.

The remainder of the proof is identical to the proof of Proposition 3.2.3. ∎

The next Definition 3.2.48 introduces a set of states $\hat{m}_{p,q,r,s} \in \hat{M}^*$.

Definition 3.2.48 *For all $p, q, r, s \in \mathbb{Z}$ with $q \geq 2$, we consider a weak* limit point $\theta_{p,q,r,s} \in B(H)^*$ of the set $\{\, \omega_{\xi'_{n,p,q}} \,|\, n \in \mathbb{N}\}$.*

We then define a state $\hat{m}_{p,q,r,s} \in \hat{M}^*$ by $\hat{m}_{p,q,r,s} = \theta_{p,q,r,s}|_{\hat{M}}$.

It is straightforward to generalize Propositions 3.2.40 and 3.2.45. This leads to the two Propositions 3.2.49 and 3.2.50 below.

We adhere to Remark 3.2.21 for the notations in Proposition 3.2.50.

Proposition 3.2.49 *For all $p, q, r, s \in \mathbb{Z}$ with $q \geq 2$, the state $\hat{m}_{p,q,r,s} \in \hat{M}^*$ is a left invariant mean on $(\hat{A}, \hat{\Phi})$.*

Proposition 3.2.50 *Let $\sum u^k g_k(a, |b|) \in M(\hat{A})$. We then have*

$$\hat{m}_{p,q,r,s}\Big(\sum u^k g_k(a,|b|)\Big) = \lim_{n \to +\infty} \frac{1}{2n+1} \sum_{w=-n}^{n} g_0(\mu^{\frac{1}{2}(w+sn-r-2qn+2p)}, \mu^{\frac{1}{2}(w-sn+r)}).$$

Hence, we see that $\hat{m}_{p,q,r,s}|_{\hat{A}} = 0$.

We now arrive at the Proposition 3.2.51 below.

Proposition 3.2.51 *Let $p, q, p_i, q_i, r_i \in \mathbb{N}$ with $q, q_i \geq 2$ for $i = 1, 2$. We then have the following properties:*

- *If $p_1 - r_1 \neq p_2 - r_2$, then $\hat{m}_{p_1,q_1,r_1,q_1} \neq \hat{m}_{p_2,q_2,r_2,q_2}$.*
- *The states in $\{\hat{m}_{p,q,s,q} \,|\, s \in \mathbb{N}\}$ are linearly independent from each other.*

Proof. For all $k, l \in \mathbb{Z}$, we set $X_{k,l} = \{\, (x, y) \in E \mid x^{-1}y = \mu^{-k+l}\}$. We then define $h_{k,l} \in \mathrm{C}_b(E)$ by setting $h_{k,l} = \chi_{X_{k,l}}$.

We get from Lemma 2.5.23 that $h_{k,l}(a, |b|) \in M(\hat{A})$ for all $k, l \in \mathbb{Z}$.

Fix $p, q, p_i, q_i, r_i \in \mathbb{N}$ with $q, q_i \geq 2$ for $i = 1, 2$ and suppose that $p_1 - r_1 \neq p_2 - r_2$. It is immediately clear that $X_{p_1,r_1} \cap X_{p_2,r_2} = \emptyset$.

From Proposition 3.2.50, we get that

$$\hat{m}_{p_1,q_1,r_1,q_1}\big(h_{p_1,r_1}(a,|b|)\big) = 1 \quad \text{and} \quad \hat{m}_{p_2,q_2,r_2,q_2}\big(h_{p_1,r_1}(a,|b|)\big) = 0. \tag{3.8}$$

It ensues that $\hat{m}_{p_1,q_1,r_1,q_1} \neq \hat{m}_{p_2,q_2,r_2,q_2}$.

Take $s \in \mathbb{N}$. From Equation (3.8), we find that $\hat{m}_{p,q,s,q}(h_{p,s}(a,|b|)) = 1$ while we have $\hat{m}_{p,q,t,q}(h_{p,s}(a,|b|)) = 0$ for all $t \in \mathbb{N}$ with $s \neq t$. ■

We then acquire the next property.

Proposition 3.2.52 *We have that* $\{\, \hat{m}_{p,q,r,s} \mid p,q,r,s \in \mathbb{Z},\ q \geq 2\}$ *contains an infinite subset of linearly independent states.*

The number of different left invariant means $\hat{m}_{p,q,r,s}$ *is thus infinite.*

Chapter 4

Representation theory

In this Chapter 4, we compile four studies of subjects that are all in some way related to the representation theory of the quantum $E(2)$ group. These four studies together yield an almost complete deciphering of all representations and corepresentations of both the quantum $E(2)$ group and its dual.

The results stated below are not new and are mainly included for the sake of completeness. For this reason, we are quite frugal of comment. We can e.g. refer to [159, 157, 149] for more detailed treatments.

First, we fix the Notation 4.0.1.

Notation 4.0.1 *We adopt all notations introduced in Chapter 2 and Section 3.2. In particular, we have that (A, Φ) denotes the quantum $E(2)$ group.*

We then give a short overview of the sections.

Section 4.1 - Crossed products. We mentioned in Section 2.5 that we can describe the C*-algebras A and $\hat{A}$ in terms of crossed products. We here fill in the details of these constructions.

Section 4.2 - Universal pairs of operators. The Section 4.2 is then about two kinds of universal pairs of operators. These pairs appear naturally in the study of the quantum $E(2)$ group and its dual.

Section 4.3 - Irreducible *-representations. We here display formulas for all irreducible *-representations of the C*-algebras A and $\hat{A}$.

Section 4.4 - Corepresentations. In this last Section 2.4, we describe all corepresentations of the quantum groups (A, Φ) and $(\hat{A}, \hat{\Phi})$.

4.1 Crossed products

In the Propositions 2.5.7 and 2.5.29, we have already mentioned how the two C*-algebras A and $\hat{A}$ can be described in terms of crossed products. We now give some more refined information about these results. Further, we also write the von Neumann algebras M and $\hat{M}$ as crossed products.

The main results of this Section 4.1 are Propositions 4.1.5 and 4.1.16. They are e.g. mentioned by Woronowicz in [159, 149].

First, we consider the case of the C*-algebra A.

Definition 4.1.1 *We define an action $\alpha : \mathbb{Z} \to \mathrm{Aut}(\mathrm{C}_0(\bar{\mathbb{C}}^\mu))$ by*

$$\alpha_n : \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \to \mathrm{C}_0(\bar{\mathbb{C}}^\mu) : f \mapsto f(\mu^n \cdot).$$

It is easy to check the two Lemmas 4.1.2 and 4.1.3.

Lemma 4.1.2 *We have that $(\mathrm{C}_0(\bar{\mathbb{C}}^\mu), \mathbb{Z}, \alpha)$ is a C*-dynamical system.*

Lemma 4.1.3 *We define a faithful, non-degenerate *-homomorphism π and a strongly continuous representation U by setting*

- $\pi : \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \to B(H) : f \mapsto f(d)$,
- $U : \mathbb{Z} \to B(H) : k \mapsto c^k$.

Then, we have that (π, U) is a covariant representation of $(\mathrm{C}_0(\bar{\mathbb{C}}^\mu), \mathbb{Z}, \alpha)$.

The Definition 4.1.4 below is allowed by the theory of crossed products.

Definition 4.1.4 *Let $\tilde{\pi}$ be the non-degenerate *-representation of $\mathrm{C}_0(\bar{\mathbb{C}}^\mu) \ltimes_\alpha \mathbb{Z}$ that corresponds to the covariant representation (π, U).*

The next Proposition 4.1.5 then states the central property.

Proposition 4.1.5 *We have $A \cong \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \ltimes_\alpha \mathbb{Z}$.*
In particular, we have that $\tilde{\pi} : \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \ltimes_\alpha \mathbb{Z} \to A$ is a *-isomorphism.
For every $f \in K\big(\mathbb{Z}, \mathrm{C}_0(\bar{\mathbb{C}}^\mu)\big)$, we have

$$\tilde{\pi}(f) = \sum c^k f(k)(\mu^{-k} d).$$

Proof. It is direct to check that

$$\tilde{\pi}\big(\mathrm{K}(\mathbb{Z}, \mathrm{C}_0(\bar{\mathbb{C}}^\mu))\big) = \Big\{ \sum_{k\in\mathbb{Z}} c^k f_k(d) \;\Big|\; f_k \in \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \text{ for every } k \text{ and } f_k \neq 0 \text{ for finitely many } k \Big\}.$$

By using Theorem 2.5.2, we then get $\tilde{\pi}\big(\mathrm{C}_0(\bar{\mathbb{C}}^\mu) \ltimes_\alpha \mathbb{Z}\big) = A$.

We now apply the results from [92, Section 7.7]. Because $\mathbb{Z}$ is amenable, there is a unique faithful *-representation

$$\pi' \ltimes \lambda : \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \ltimes_\alpha \mathbb{Z} \to B\big(\ell^2(\mathbb{Z}, \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}))\big)$$

such that, for all $y \in K\big(\mathbb{Z}, \mathrm{C}_0(\bar{\mathbb{C}}^\mu)\big)$, we have

$$\big(((\pi' \ltimes \lambda)(y))\xi\big)(t) = \sum_{s\in\mathbb{Z}} \pi\big(\alpha_{-t}(y(s))\big)\xi(t-s)$$

where $\xi \in \ell^2\big(\mathbb{Z}, \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})\big)$ and $t \in \mathbb{Z}$.

It is easy to prove that $\pi' \ltimes \lambda$ is unitarily equivalent to the *-representation

$$\theta : \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \ltimes_\alpha \mathbb{Z} \to B\big(\ell^2(\mathbb{Z}, \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}))\big) : x \mapsto \tilde{\pi}(x)_{13}.$$

Hence, we get that also $\tilde{\pi}$ is faithful.

We conclude that $\tilde{\pi} : \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \ltimes_\alpha \mathbb{Z} \to A$ is a *-isomorphism. ∎

We can thus use the machinery of crossed products when studying aspects of the C*-algebra A. This has some useful consequences.

If we apply the results in [16], we can deduce the next property.

Corollary 4.1.6 *We have that A is a type I C*-algebra. In particular, we find that the C*-algebra A is nuclear.*

We use the above results to also write M as a crossed product.

Definition 4.1.7 *Let ν be the Haar measure on $\mathbb{C}^\mu$. We then define a regular Borel measure ν_1 on $\bar{\mathbb{C}}^\mu$ by setting*

$$\nu_1(X) = \sum_{k\in\mathbb{Z}} \frac{1}{2^k}\, \nu(X \cap S_k)$$

for every subset $X \subseteq \bar{\mathbb{C}}^\mu$.

Notation 4.1.8 *We use $L^\infty(\bar{\mathbb{C}}^\mu)$ to shortly denote $L^\infty(\bar{\mathbb{C}}^\mu, \nu_1)$. We then have that $L^\infty(\bar{\mathbb{C}}^\mu)$ is an abelian von Neumann algebra.*

Definition 4.1.9 *We define an action $\alpha'' : \mathbb{Z} \to \mathrm{Aut}(L^\infty(\bar{\mathbb{C}}^\mu))$ by*

$$\alpha''_n : L^\infty(\bar{\mathbb{C}}^\mu) \to L^\infty(\bar{\mathbb{C}}^\mu) : f \mapsto f(\mu^n \cdot).$$

It is direct to check the Lemma 4.1.10.

Lemma 4.1.10 *We have that $(L^\infty(\bar{\mathbb{C}}^\mu), \mathbb{Z}, \alpha'')$ is a W*-dynamical system.*

We then arrive at the Proposition 4.1.11 below.

Proposition 4.1.11 *We have* $M \cong L^\infty(\bar{\mathbb{C}}^\mu) \ltimes_{\alpha''} \mathbb{Z}$

Proof. It is easy to check that $\mathbb{Z}$ is a central shift in $(\mathrm{C}_0(\bar{\mathbb{C}}^\mu), \mathbb{Z}, \alpha)$ in the sense of [16]. Since $\mathbb{Z}$ is discrete, we can apply the results in [16] to get that

$$\big(\mathrm{C}_0(\bar{\mathbb{C}}^\mu) \ltimes_\alpha \mathbb{Z}\big)'' = L^\infty(\bar{\mathbb{C}}^\mu) \ltimes_{\alpha''} \mathbb{Z}.$$

An application of Proposition 4.1.5 ends the proof. ∎

It is also possible to describe $\hat{A}$ and $\hat{M}$ in terms of crossed products. The results concerning $\hat{A}$ are somewhat less elegant than those about A.

We can repeat the techniques above to get the desired properties.

Definition 4.1.12 *We define an action* $\beta : \mathbb{Z} \to \mathrm{Aut}(\mathrm{C}_0(E))$ *by*

$$\beta_n : \mathrm{C}_0(E) \to \mathrm{C}_0(E) : g \mapsto g(\mu^{-n}\cdot, \cdot).$$

It is easy to check the two Lemmas 4.1.13 and 4.1.14.

Lemma 4.1.13 *We have that* $(\mathrm{C}_0(E), \mathbb{Z}, \beta)$ *is a C*-dynamical system.*

Lemma 4.1.14 *We define a faithful, non-degenerate* **-homomorphism* $\hat{\pi}$ *and a strongly continuous representation* $\hat{U}$ *by setting*

- $\hat{\pi} : \mathrm{C}_0(E) \to B(H) : g \mapsto g(a, |b|)$,
- $\hat{U} : \mathbb{Z} \to B(H) : k \mapsto u^k$.

Then, we have that $(\hat{\pi}, \hat{U})$ *is a covariant representation of* $(\mathrm{C}_0(E), \mathbb{Z}, \beta)$.

The Definition 4.1.15 is allowed by the theory of crossed products.

Definition 4.1.15 *Let* $\breve{\pi}$ *be the non-degenerate* **-representation of* $\mathrm{C}_0(E) \ltimes_\beta \mathbb{Z}$ *that corresponds to the covariant representation* $(\hat{\pi}, \hat{U})$.

The Proposition 4.1.5 states the central property. It turns out that $\hat{A}$ is not a crossed product itself, but it can be written as a C*-subalgebra of one.

We recall that the C*-algebras A_{ex} and B are introduced in Definition 2.5.28.

Proposition 4.1.16 *We have* $\hat{A} \cong B$ *and* $\hat{A}_{ex} \cong \mathrm{C}_0(E) \ltimes_\beta \mathbb{Z}$.

In particular, $\breve{\pi} : \mathrm{C}_0(E) \ltimes_\beta \mathbb{Z} \to \hat{A}_{ex}$ *and* $\breve{\pi}|_B : B \to \hat{A}$ *are* **-isomorphisms.*

For every $g \in K\big(\mathbb{Z}, \mathrm{C}_0(E)\big)$, *we have*

$$\breve{\pi}(f) = \sum u^k g(k)(\mu^k a, |b|).$$

Proof. It is direct to check that

$$\check{\pi}\big(\mathrm{K}(\mathbb{Z}, \mathrm{C}_0(E))\big) = \Big\{ \sum_{k\in\mathbb{Z}} u^k g_k(a, |b|) \;\Big|\; g_k \in \mathrm{C}_0(E) \text{ for every } k \text{ and } g_k \neq 0 \text{ for finitely many } k \Big\}.$$

By using Theorem 2.5.21, we then get $\check{\pi}\big(\mathrm{C}_0(E) \ltimes_\beta \mathbb{Z}\big) = \hat{A}_{\mathrm{ex}}$.

We now apply the results from [92, Section 7.7]. Because $\mathbb{Z}$ is amenable, there is a unique faithful *-representation

$$\check{\pi}' \ltimes \lambda : \mathrm{C}_0(E) \ltimes_\beta \mathbb{Z} \to B\big(\ell^2(\mathbb{Z}, \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}))\big)$$

such that, for all $y \in K\big(\mathbb{Z}, \mathrm{C}_0(E)\big)$, we have

$$\big(((\check{\pi}' \ltimes \lambda)(y))\xi\big)(t) = \sum_{s\in\mathbb{Z}} \check{\pi}\big(\beta_{-t}(y(s))\big)\xi(t-s)$$

where $\xi \in \ell^2\big(\mathbb{Z}, \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})\big)$ and $t \in \mathbb{Z}$.

It is easy to prove that $\check{\pi}' \ltimes \lambda$ is unitarily equivalent to the *-representation

$$\hat{\theta} : \mathrm{C}_0(E) \ltimes_\beta \mathbb{Z} \to B\big(\ell^2(\mathbb{Z}, \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}))\big) : x \mapsto \check{\pi}(x)_{13}.$$

Hence, we get that also $\check{\pi}$ is faithful.

We conclude that $\check{\pi} : \mathrm{C}_0(E) \ltimes_\beta \mathbb{Z} \to \hat{A}_{\mathrm{ex}}$ is a *-isomorphism. Further, it is immediately clear that $\check{\pi}(\hat{A}) = B$. ■

If we apply the results in [16], we can deduce the next property.

Corollary 4.1.17 *We have that $\hat{A}$ is a type I C*-algebra. In particular, we find that the C*-algebra $\hat{A}$ is nuclear.*

We use the above results to also write $\hat{M}$ as a crossed product.

Definition 4.1.18 *Let $\hat{\nu}$ be the Haar measure on $\mathbb{R}(\mu^{\frac{1}{2}}) \times \mathbb{R}(\mu^{\frac{1}{2}})$. For every $k \in \mathbb{Z}$, we define a set V_k by $V_k = \{\, (\mu^{\frac{1}{2}p}, \mu^{\frac{1}{2}k}) \in E \mid p \in \mathbb{Z}\}$.*

We define a regular Borel measure $\hat{\nu}_1$ on E by setting

$$\hat{\nu}_1(X) = \sum_{k\in\mathbb{Z}} \frac{1}{2^k}\, \hat{\nu}(X \cap V_k)$$

for all subsets $X \subseteq E$.

Notation 4.1.19 *We use $L^\infty(E)$ to shortly denote $L^\infty(\bar{\mathbb{C}}^\mu, \hat{\nu}_1)$. We then have that $L^\infty(E)$ is an abelian von Neumann algebra.*

Definition 4.1.20 *We define an action* $\beta'' : \mathbb{Z} \to \mathrm{Aut}(L^\infty(E))$ *by*

$$\beta''_n : L^\infty(E) \to L^\infty(E) : g \mapsto g(\mu^{-n}\cdot, \cdot).$$

It is direct to check the Lemma 4.1.21.

Lemma 4.1.21 *We have that* $(L^\infty(E), \mathbb{Z}, \beta'')$ *is a* W^**-dynamical system.*

We then arrive at the Proposition 4.1.22 below.

Proposition 4.1.22 *We have* $\hat{M} \cong L^\infty(E) \ltimes_{\beta''} \mathbb{Z}$

Proof. It is easy to check that $\mathbb{Z}$ is a central shift in $(\mathrm{C}_0(E), \mathbb{Z}, \beta)$ in the sense of [16]. Since $\mathbb{Z}$ is discrete, we can apply the results in [16] to get that

$$\big(\mathrm{C}_0(E) \ltimes_\beta \mathbb{Z}\big)'' = L^\infty(E) \ltimes_{\beta''} \mathbb{Z}.$$

From Proposition 2.9.8, it follows that $\hat{M} = \hat{A}''_{\mathrm{ex}}$. An application of Proposition 4.1.16 then ends the proof. ■

4.2 Universal pairs of operators

The results in Chapter 2 demonstrate that the operators c, d are very important in our treatment of the quantum $E(2)$ group (A, Φ). In this Section 4.2, we show that the pair (c, d) is some kind of universal representative of a set of typical pairs of operators. These pairs will be called $E(2)$*-pairs of operators*.

It is no surprise that also the pair (a, b) has universality properties. These are then related to the quantum $\hat{E}(2)$ group $(\hat{A}, \hat{\Phi})$.

We did not make use of the universality properties below in our construction of the quantum groups (A, Φ) and $(\hat{A}, \hat{\Phi})$ as displayed in Section 2.4. This is very different from the working method in [159] where these universality results constitute one of the pillars in Woronowicz' original constructions.

In this Section 4.2, we compile results from [159]. We do not give proofs nor amplify on the details. For more specifics, we refer to [159, 168].

We start by giving the basic Definition 4.2.1 below.

Definition 4.2.1 *Let* $\mathbf{c}, \mathbf{d}$ *be linear operators on a Hilbert space* K.

We call $(\mathbf{c}, \mathbf{d})$ *an* $E(2)$-pair of operators *if all four conditions below are satisfied:*

1. $\mathbf{c}$ *is unitary,*
2. $\mathbf{d}$ *is normal,*
3. $\mathbf{c}\mathbf{d}\mathbf{c}^* = \mu\mathbf{d}$,
4. $\sigma(\mathbf{d}) \subseteq \bar{\mathbb{C}}^\mu$.

The Lemma 4.2.2 ensues from the results in Section 2.3.

Lemma 4.2.2 *We have that (c, d) is an $E(2)$-pair of operators.*

The next Proposition 4.2.3 states the central universality property.

Proposition 4.2.3 *Let $\mathbf{A} \subseteq B(K)$ be a non-degenerate C*-algebra. We then consider a non-degenerate *-representation $\pi : A \to B(K)$.*

The properties below hold:

1. *We have that $\big(\pi(c), \pi(d)\big)$ is an $E(2)$-pair of operators,*
2. *We have $\pi(c), \pi(d)\, \eta\, \mathbf{A}$ if and only if we can consider π as a non-degenerate *-homomorphism from A to $M(\mathbf{A})$.*

Further, also the converse result holds:

3. *If $(\mathbf{c}, \mathbf{d})$ is an $E(2)$-pair of operators, there exists a unique non-degenerate *-representation $\theta : A \to B(L)$ such that $\mathbf{c} = \theta(c)$ and $\mathbf{d} = \theta(d)$.*

From Proposition 4.2.3, we can directly deduce the result below.

Corollary 4.2.4 *For all $z \in S^1$, there is a unique *-homomorphism $\omega_z : A \to \mathbb{C}$ such that*

$$\omega_z(c) = z \qquad \text{and} \qquad \omega_z(d) = 0.$$

It is clear that $\omega_1 = \varepsilon$.

Let $z \in S^1$. We have that

1. $(\omega_z \otimes \iota)\Phi(c) = (\iota \otimes \omega_z)\Phi(c) = zc$,
2. $(\omega_z \otimes \iota)\Phi(d) = zd$ *and* $(\iota \otimes \omega_z)\Phi(d) = \overline{z}d$.

In particular, we get $(\omega_{-1} \otimes \iota)\Phi(x) = (\iota \otimes \omega_{-1})\Phi(x)$ for all $x \in A$.

For $z_1, z_2 \in S^1$, we have $(\omega_{z_1} \otimes \omega_{z_2})\Phi = \omega_{z_1 z_2}$.

We now explain how Woronowicz in [159] uses the above universality properties to construct the comultiplication Φ on the C*-algebra A.

The Remark 4.2.5 is explained in detail in Section 3 of [157].

Remark 4.2.5 *The Lemma 4.2.2 gives that (c, d) is an $E(2)$-pair. We then get from Theorem 2.2.11 that also $(c \otimes c,\, c \otimes d \dotplus d \otimes c^*)$ is an $E(2)$-pair.*

Theorem 2.5.3 yields $c \in M(A)$ and $d\, \eta\, A$. We thus have $c \otimes c \in M(A \otimes A)$. By using Theorem 2.2.11, it is easy to deduce that $(c \otimes d \dotplus d \otimes c^*)\, \eta\, A$.

From Proposition 4.2.3, we then get that there exists a unique non-degenerate *-homomorphism $\Phi : A \to M(A \otimes A)$ such that

$$\Phi(c) = c \otimes c \qquad \text{and} \qquad \Phi(d) = c \otimes d \dotplus d \otimes c^*.$$

By working as in [157, Theorem 3.4], one can show that Φ is coassociative.

We also comment on the spectral condition on $\mathbf{d}$ in Definition 4.2.1. We already mentioned in Section 1.2 that such a condition typically appears in the study of examples of locally compact quantum groups.

For the quantum $E(2)$ group (A, Φ), the spectral condition on d is needed in order to ensure the existence of the comultiplication.

The Remark 4.2.6 explains the meaning of the conditions on $\sigma(\mathbf{d})$.

Remark 4.2.6 *Let* $\mathbf{c}, \mathbf{d}$ *be linear operators on a Hilbert space* K*. Suppose that* $\mathbf{c}$ *is unitary and that* $\mathbf{d}$ *is normal. Further, we assume that* $\mathbf{c}\mathbf{d}\mathbf{c}^* = \mu\mathbf{d}$*.*

We then define a pair $(R, S) \in D_\mu$ *by*

$$R = \mathbf{d} \otimes \mathbf{c}^* \qquad \textit{and} \qquad S = \mathbf{c} \otimes \mathbf{d}.$$

From Lemma 2.2.9, we get that $R \dotplus S$ *is normal if and only if* $\sigma(R^{-1}S) \subseteq \bar{\mathbb{C}}^\mu$*.*

It is not difficult to show that the condition $\sigma(R^{-1}S) \subseteq \bar{\mathbb{C}}^\mu$ *is fulfilled if and only if there exists a number* $x \in \mathbb{C}_0$ *such that* $\sigma(d) \subseteq x\bar{\mathbb{C}}^\mu$*.*

In this case, we have $\sigma(R \dotplus S) \subseteq \bar{\mathbb{C}}^\mu$ *if and only if* $x \in \bar{\mathbb{C}}^\mu$ *and hence* $\sigma(d) \subseteq \bar{\mathbb{C}}^\mu$*.*

We now switch our attention to the pair (a, b). The universality properties of (a, b) can be handled quite similarly to those of (c, d).

First, we give the basic Definition 4.2.7 below.

Definition 4.2.7 *Let* $\mathbf{a}, \mathbf{b}$ *be linear operators on a Hilbert space* K*.*

We call $(\mathbf{a}, \mathbf{b})$ *an* $\hat{E}(2)$-pair of operators *if all five conditions below are satisfied:*

1. $\mathbf{a}$ *is strictly positive,*
2. $\mathbf{b}$ *is normal,*
3. $\mathbf{a}$ *and* $|\mathbf{b}|$ *strongly commute,*
4. $\mathbf{a}^{it}\mathbf{b}\mathbf{a}^{-it} = \mu^{it}\mathbf{b}$,
5. $\sigma(\mathbf{a}, |\mathbf{b}|) \subseteq \bar{E}$.

The small Remark 4.2.8 alludes to Definition 4.2.7.

Remark 4.2.8 *Let* $\mathbf{a}, \mathbf{b}$ *be linear operators on a Hilbert space* K*. We then use* $\mathbf{b} = \mathbf{u}|\mathbf{b}|$ *to denote the polar decomposition of* $\mathbf{b}$*.*

We have that Condition 4 in Definition 4.2.7 can be replaced by either one of the following conditions:

4'. $\mathbf{a}\mathbf{u} \supseteq \mu\mathbf{u}\mathbf{a}$,

4". *We have* $\mathbf{a}\mathbf{u} = \mu\mathbf{u}\mathbf{a}$ *as operators on* $\{\ker \mathbf{b}\}^\perp$*.*

If $(\mathbf{a}, \mathbf{b})$ *is an* $\hat{E}(2)$*-pair, then* $\mathbf{ab}$ *and* $\mathbf{ba}$ *are well-defined. It is easy to check that in this case*

- $\mathbf{ab} \supseteq \mu\mathbf{ba}$,
- *We have* $\mathbf{ab} = \mu\mathbf{ba}$ *as operators on* $\{\ker \mathbf{b}\}^{\perp}$.

We refer to [142] for a study about the spectral condition on $\mathbf{a}$ *and* $\mathbf{b}$.

The next Lemma 4.2.9 ensues from the results in Section 2.3.

Lemma 4.2.9 *We have that* (a, b) *is an* $\hat{E}(2)$*-pair of operators.*

The next Proposition 4.2.10 states the central universality property.

Proposition 4.2.10 *Let* $\hat{\mathbf{A}} \subseteq B(K)$ *be a non-degenerate C*-algebra. We then consider a non-degenerate* **-representation* $\pi : \hat{A} \to B(K)$.

The properties below hold:

1. *We have that* $\big(\pi(a), \pi(b)\big)$ *is an* $\hat{E}(2)$*-pair of operators,*
2. *We have* $\pi(a)$, $\pi(a^{-1})$, $\pi(b)\,\eta\,\hat{\mathbf{A}}$ *if and only if we can consider* π *as a non-degenerate* **-homomorphism from* $\hat{A}$ *to* $M(\hat{\mathbf{A}})$.

Further, also the converse result holds:

3. *If* $(\mathbf{a}, \mathbf{b})$ *is an* $\hat{E}(2)$*-pair of operators, there exists a unique non-degenerate* **-representation* $\theta : \hat{A} \to B(L)$ *such that* $\mathbf{a} = \theta(a)$ *and* $\mathbf{b} = \theta(b)$.

From Proposition 4.2.10, we can directly deduce the result below.

Corollary 4.2.11 *For all* $k \in \mathbb{Z}$*, there is a unique* **-homomorphism* $\hat{\omega}_k : \hat{A} \to \mathbb{C}$ *such that*

$$\hat{\omega}_k(a) = \mu^{\frac{1}{2}k} \qquad \textit{and} \qquad \hat{\omega}_k(b) = 0.$$

It is clear that $\hat{\omega}_0 = \hat{\varepsilon}$.

Let $k \in \mathbb{Z}$*. We have that*

1. $(\hat{\omega}_k \otimes \iota)\Phi(c) = (\iota \otimes \hat{\omega}_k)\hat{\Phi}(a) = \mu^{\frac{1}{2}k}a$,
2. $(\hat{\omega}_k \otimes \iota)\hat{\Phi}(b) = \mu^{\frac{1}{2}k}b$ *and* $(\iota \otimes \hat{\omega}_k)\hat{\Phi}(b) = \mu^{-\frac{1}{2}k}b$.

For $k_1, k_2 \in \mathbb{Z}$*, we have* $(\hat{\omega}_{k_1} \otimes \hat{\omega}_{k_2})\hat{\Phi} = \hat{\omega}_{k_1+k_2}$.

4.3 Irreducible *-representations

This Section 4.3 is very short. We describe all irreducible *-representations of the C*-algebras A and $\hat{A}$. The properties below appeared before in [88]. They can easily be checked by using some simple computations.

We have already studied one-dimensional *-representations of both A and $\hat{A}$ in the Corollaries 4.2.4 and 4.2.11. It is not so difficult to also find formulas for the other irreducible *-representations on these two C*-algebras.

First, we display all irreducible *-representations of A.

Proposition 4.3.1 *We have the following results:*

- *For all $z \in \mathbb{C}$, we can define a one-dimensional irreducible *-representation $\omega_z : A \to \mathbb{C}$ by setting*

$$\omega_z(c) = z \qquad \textit{and} \qquad \omega_z(d) = 0.$$

- *For all $z \in \mathbb{C}$, we can define an infinite-dimensional irreducible *-representation $\pi_z : A \to B(\ell^2(\mathbb{Z}))$ by setting*

$$\pi_z(c) = s^* \qquad \textit{and} \qquad \pi_z(d) = zm.$$

Every irreducible *-representation $\pi : A \to B(K)$ on a Hilbert space K is either one-dimensional or infinite-dimensional. We have the next classification:

- *In the case that π is one-dimensional, there exists a unique $z \in S^1$ such that π is unitarily equivalent to ω_z.*
- *In the case that π is infinite-dimensional, there exists a unique $z \in S^1$ such that π is unitarily equivalent to π_z.*

We then consider the case of the C*-algebra $\hat{A}$.

Proposition 4.3.2 *We have the following results:*

- *For all $k \in \mathbb{Z}$, we can define a one-dimensional irreducible *-representation $\hat{\omega}_k : \hat{A} \to \mathbb{C}$ by setting*

$$\hat{\omega}_k(a) = \mu^{\frac{1}{2}k} \qquad \textit{and} \qquad \hat{\omega}_k(d) = 0.$$

- *For all $z \in \mathbb{C}$, we can define an infinite-dimensional irreducible *-representation $\hat{\pi}_k : \hat{A} \to B(\ell^2(\mathbb{Z}))$ by setting*

$$\hat{\pi}_k(a) = m \qquad \textit{and} \qquad \hat{\pi}_k(b) = \mu^k s.$$

- *For all $z \in \mathbb{C}$, we can define an infinite-dimensional irreducible *-representation $\hat{\pi}'_k : \hat{A} \to B(\ell^2(\mathbb{Z}))$ by setting*

$$\hat{\pi}'_k(a) = \mu^{\frac{1}{2}} m \qquad \textit{and} \qquad \hat{\pi}'_k(b) = \mu^k s.$$

Every irreducible *-representation $\hat{\pi} : \hat{A} \to B(K)$ on a Hilbert space K is either one-dimensional or infinite-dimensional. We have the next classification:

- *In the case that $\hat{\pi}$ is one-dimensional, there exists a unique $k \in \mathbb{Z}$ such that $\hat{\pi}$ is unitarily equivalent to $\hat{\omega}_k$.*
- *In the case that $\hat{\pi}$ is infinite-dimensional, there exists a unique $k \in \mathbb{Z}$ such that $\hat{\pi}$ is unitarily equivalent to either $\hat{\pi}_k$ or $\hat{\pi}'_k$.*

4.4 Corepresentations

We now study the (unitary) corepresentations of the quantum $E(2)$ group (A, Φ) and its dual. In the Propositions 4.4.2 and 4.4.5, we give explicit formulas for all corepresentations of (A, Φ) and $(\hat{A}, \hat{\Phi})$. We state these properties with very little comment and refer to [159, 149] for more information and proofs.

From the amenability results in Section 3.2, we know that both the quantum groups (A, Φ) and $(\hat{A}, \hat{\Phi})$ are universal in the sense of [61]. It is then a direct consequence that their corepresentation theory is quite accessible.

The corepresentations of (A, Φ) are treated by Woronowicz in [159]. The case of $(\hat{A}, \hat{\Phi})$ is considered in [149]. We further mention that e.g. E. Koelink [53] and A. Pal [88] studied the relation between the matrix coefficients of irreducible corepresentations and Hahn-Exton q-Bessel functions.

We state the Definition 4.4.1 of a *corepresentation* U on (A, Φ). This Definition 4.4.1 is not the standard one, but yields an equivalent notion.

Definition 4.4.1 *Let $U \in M(B_0(K) \otimes A)$ be a unitary operator. We call U a* corepresentation *of (A, Φ) if we have*

$$(\iota \otimes \Phi)(U) = U_{12} U_{13}.$$

The Proposition 4.4.2 describes all corepresentations of (A, Φ). The concept of an $\hat{E}(2)$-pair of operators (cf. Section 4.3) hereby plays a crucial role.

It is clear that W is a typical corepresentation of (A, Φ).

Proposition 4.4.2 ([159]) *Let $\hat{\mathbf{A}} \subseteq B(K)$ be a non-degenerate C^*-algebra. We have the following properties:*

- *Let $(\mathbf{a}, \mathbf{b})$ be an $\hat{E}(2)$-pair of operators acting on K. Then, we consider $U \in M(B_0(K) \otimes A)$ defined by*

  $$U = F_\mu(\mathbf{ab} \otimes cd)\chi(\mathbf{a} \otimes 1, 1 \otimes c).$$

  We have that U is a corepresentation of (A, Φ).

- *Let $U \in M(B_0(K) \otimes A)$ be a corepresentation of (A, Φ). There exists a unique $\hat{E}(2)$-pair $\big(\mathbf{a}, \mathbf{b}\big)$ of operators acting on K such that*

$$U = F_\mu(\mathbf{ab} \otimes cd)\chi(\mathbf{a} \otimes 1, 1 \otimes c).$$

Moreover, we have $U \in M(\hat{\mathbf{A}} \otimes A)$ if and only if $\mathbf{a}, \mathbf{a}^{-1}, \mathbf{b}\, \eta\, \hat{\mathbf{A}}$.

The next Proposition 4.4.3 directly ensues from Propositions 4.2.3 and 4.4.2.

Proposition 4.4.3 *Let $\hat{\mathbf{A}} \subseteq B(K)$ be a non-degenerate C^*-algebra. The results below hold:*

- *Let $\pi : \hat{A} \to M(\hat{\mathbf{A}})$ be a non-degenerate *-homomorphism. We consider $\mathbf{W} \in M(\hat{\mathbf{A}} \otimes A)$ defined by*

$$\mathbf{W} = (\pi \otimes \iota)(W).$$

Then, we have that $\mathbf{W}$ is a corepresentation of (A, Φ).

- *Let $\mathbf{W} \in M(\hat{\mathbf{A}} \otimes A)$ be a corepresentation of (A, Φ). There exists a unique non-degenerate *-homomorphism $\pi : \hat{A} \to M(\hat{\mathbf{A}})$ such that*

$$\mathbf{W} = (\pi \otimes \iota)(W).$$

We describe the corepresentations of $(\hat{A}, \hat{\Phi})$. It is of course no surprise that there is a complete resemblance with the properties concerning (A, Φ).

We confine ourselves to only formulate the results.

Definition 4.4.4 *Let $\hat{U} \in M(\hat{A} \otimes B_0(K))$ be a unitary operator. We call $\hat{U}$ a* corepresentation *of $(\hat{A}, \hat{\Phi})$ if we have*

$$(\hat{\Phi} \otimes \iota)\hat{U} = \hat{U}_{13}\hat{U}_{23}.$$

Proposition 4.4.5 ([149]) *Let $\mathbf{A} \subseteq B(K)$ be a non-degenerate C^*-algebra. We have the following results:*

- *Let $\big(\mathbf{c}, \mathbf{d}\big)$ be an $E(2)$-pair of operators acting on K. Then, we consider $\hat{U} \in M(\hat{A} \otimes B_0(K))$ by*

$$\hat{U} = F_\mu(ab \otimes \mathbf{cd})\chi(a \otimes 1, 1 \otimes \mathbf{c}).$$

We have that $\hat{U}$ is a corepresentation of $(\hat{A}, \hat{\Phi})$.

- *Let $\hat{U} \in M(\hat{A} \otimes B_0(K))$ be a corepresentation of $(\hat{A}, \hat{\Phi})$. There exists a unique $E(2)$-pair $\big(\mathbf{c}, \mathbf{d}\big)$ of operators acting on K such that*

$$\hat{U} = F_\mu(ab \otimes \mathbf{cd})\chi(a \otimes 1, 1 \otimes \mathbf{c}).$$

Moreover, we have $\hat{U} \in M(\hat{A} \otimes \mathbf{A})$ if and only if $\mathbf{c}, \mathbf{d}\, \eta\, \mathbf{A}$.

Proposition 4.4.6 *Let $\mathbf{A} \subseteq B(K)$ be a non-degenerate C^*-algebra. The results below hold:*

- *Let $\pi : A \to M(\mathbf{A})$ be a non-degenerate *-homomorphism. We consider $\mathbf{W} \in M(\hat{A} \otimes \mathbf{A})$ defined by*

$$\mathbf{W} = (\iota \otimes \pi)(W).$$

  Then, we have that $\mathbf{W}$ is a corepresentation of $(\hat{A}, \hat{\Phi})$.

- *Let $\mathbf{W} \in M(\hat{A} \otimes \mathbf{A})$ be a corepresentation of $(\hat{A}, \hat{\Phi})$. There exists a unique non-degenerate *-homomorphism $\pi : A \to M(\mathbf{A})$ such that*

$$\mathbf{W} = (\iota \otimes \pi)(W).$$

Chapter 5

Conclusions and open problems

In this final Chapter 5, we bethink the fruition of our work. If we look at this Ph.D.-thesis, it seems that we have found what we were really looking for. Another example in our life, to construct and to adore!

We use this brief Chapter 5 also to point the reader towards questions that we look upon as interesting to further contemplate about. Of course, we restrict ourselves to only mention those problems which seem feasible to unveil.

The thesis encompasses two main parts in each of which we make a contribution to a specific branch of the quantum group theory. We first have our study of the manageability theory. This is expounded in Sections 1.3 and 1.4. Further, in Chapters 2 to 4, we elaborate all features of the quantum $E(2)$ group.

Both these two parts form a complete whole. We have tried to streamline the treatments as much as possible. There are nevertheless still some open problems. We below mention a few ones that we have encountered. They may serve as an incentive for further research. The outcome of these problems is most of the times imponderable and might lead to some new insights.

It need not be said that we hope that the problems in the list below will be answered in the near future. We would be very pleased if some of our results will open a gate to new research on the quantum $E(2)$ group.

Some thoughts on Sections 1.3 and 1.4

In Sections 1.3 and 1.4, one can find a quite complete survey of the theory of manageable multiplicative unitaries. We here first give an overview of the main properties and then prove some new results. In particular, we summon up interesting results concerning the existence of Haar weights.

We think that the results of the Sections 1.3 and 1.4 constitute a solid framework that can be useful in the construction of quantum group examples.

We account the following topics as the two most intriguing open problems:

- We wonder if any manageable multiplicative unitary W leads up to a locally compact quantum group. Thus, if (A, Φ) is the bi-C*-algebra associated to W, does there always exists a (left) Haar weight ψ on (A, Φ)?

- A stronger question is whether it is always possible to find a Haar weight ψ on (A, Φ) that is of the form ψ_q as displayed in Definition 1.4.5.

Some thoughts on Chapters 2, 3 and 4

The extensive Chapter 2 together with the Chapters 3 and 4 constitutes a study of the quantum $E(2)$ group. We assess our treatment as very complete. It might be used as a reference work in future research.

In Chapter 2, we build up the quantum $E(2)$ group by using the techniques that are introduced in Sections 1.3 and 1.4. Further, we examine the basic features of this example with special attention to the Haar weights.

In the Chapters 3 and 4, we respectively gave a study about the amenability properties and the representation theory of the quantum $E(2)$ group.

The following issues naturally cropped up in our research:

- It is expected that the quantum $E(2)$ group (A, Φ) is simple, i.e., that we cannot describe (A, Φ) as an extension in the sense of [131].

  We observe that this result is equivalent to the property that (A, Φ) cannot be written as a cocycle bicrossed product.

- It is probably useful to involve the quantum $E(2)$ group (A, Φ) in the research about quantum homogeneous spaces. The obvious first thing to do is then to determine all closed quantum subgroups of (A, Φ) together with the corresponding quantum homogeneous spaces.

  There has been very recent research on quantum homogeneous spaces and related matters; see e.g. [130]. This theory is however only elaborated for *regular* quantum groups. The non-regular case is still problematic.

  Because the quantum $E(2)$ group (A, Φ) is the simplest known non-regular quantum group (besides the bicrossed products), it should be interesting to look at which parts of the theory of quantum homogeneous spaces for regular quantum groups is still applicable for (A, Φ).

  In the most optimal case, this research may have some solutions in store for a generalization of the present theory.

- We wonder if also the quantum $\hat{E}(2)$ group has some kind of *semigroup behavior*, i.e., if there is a dual version of Proposition 2.5.15.
- In Chapter 2, we use a deformation parameter μ with $0 < \mu < 1$. Maybe it is possible to adapt the construction procedure of the quantum $E(2)$ group in such a way that also other values of μ are allowed.

  This question can be seen to be in the line of recent research on e.g. the quantum $az+b$ group; see [165, 113].
- As the final and inevitable thought, there is the open question if it is possible to construct a 'quantum $E(3)$ group'.

  Even more ambitious is then to search for a 'quantum $E(n)$ group'.

Appendix A

The function F_μ and its Fourier transform

In this Appendix A, we give a detailed study of the function F_μ and its Fourier coefficients. Most of the basic results are already formulated (without proof) in the Section 2.2. Since the function F_μ plays a crucial role in the construction of the quantum $E(2)$ group, we think that it is useful to spend a few pages on a survey of its most important basic properties.

The function F_μ was first introduced and studied by S.L. Woronowicz in [158]. Its main importance with respect to the quantum $E(2)$ group is the fact that it has a quantum exponential behavior. The most crucial properties of F_μ are stated in the Theorems 2.2.11 and 2.2.12. The proofs of these results are very advanced and we therefore do not include them in this appendix.

Although the treatment of the function F_μ is inevitably somewhat technical, the calculations in the proofs below are certainly not intricate. There is only required a little knowledge of complex analysis.

The emphasis of this Appendix A lies in the study of the numbers $B(k,n)$ as defined in Definition A.4. The formulated properties of these Fourier coefficients are not new. They were already found by S. Baaj in [1]. We nevertheless include a proof of these results to translate notations into our settings.

The Definition 2.2.1 of the function F_μ is restated in Definition A.1 below.

Definition A.1 ([158]) *We consider the function F_μ on $\mathbb{C}$ that is defined in the following way. For all $z \in \mathbb{C} \setminus \{-\mu^{-2k} \mid k \in \mathbb{N}\}$, we set*

$$F_\mu(z) = \prod_{k=0}^{\infty} \frac{1 + \mu^{2k}\overline{z}}{1 + \mu^{2k} z}$$

Further, we set $F_\mu(z) = -1$ when $z \in \{-\mu^{-2k} \mid k \in \mathbb{N}\}$.

We start by proving the basic properties of the function F_μ. These results are already mentioned in Section 2.2. They are all due to Woronowicz [158].

First, we have the next Lemma A.2.

Lemma A.2 ([158]) *It is clear that F_μ is not continuous on $\mathbb{C}$. But we have that the restriction of F_μ to $\bar{\mathbb{C}}^\mu$ is continuous.*

We also use the notation F_μ to denote this restriction.

For all $z \in \mathbb{C}$, we have $F_\mu(\overline{z}) = \overline{F_\mu(z)}$ and $|F_\mu(z)| = 1$. Hence, $F_\mu \in \mathrm{C}_b(\bar{\mathbb{C}}^\mu)$.

Proof. It is clear that $\overline{F_\mu(z)} = F_\mu(\overline{z})$ and $|F_\mu(z)| = 1$ for all $z \in \mathbb{C}$. To study the continuity of F_μ, we consider the function g_μ on $\mathbb{C}$ defined by

$$g_\mu(z) = \prod_{k=0}^{\infty}(1 + \mu^{2k} z).$$

We have that g_μ is analytic on $\mathbb{C}$. The set of its zeros is equal to $\{-\mu^{-2k} \mid k \in \mathbb{N}\}$. It is standard to prove that all these zeros are single zeros. Moreover, we have that the product $\prod_{k=0}^{\infty}(1 + \mu^{2k} z)$ converges uniformly on compact subsets.

For all $z \in \mathbb{C} \setminus \{-\mu^{-2k} \mid k \in \mathbb{N}\}$, we have

$$F_\mu(z) = \frac{g_\mu(\overline{z})}{g_\mu(z)}.$$

From this, it immediately follows that F_μ is continuous on $\mathbb{C} \setminus \{-\mu^{-2k} \mid k \in \mathbb{N}\}$.

Take $k \in \mathbb{N}$. It is not difficult to verify that

$$\lim_{\theta \to \pi} F_\mu(\mu^{-2k} e^{i\theta}) = -1.$$

This proves that F_μ is continuous on $\bar{\mathbb{C}}^\mu$. ∎

We compute the asymptotical behavior of F_μ in the Lemma A.3 below.

Lemma A.3 *If $z \in \bar{\mathbb{C}}^\mu$, then we have that $F_\mu(\mu^2 z) F_\mu(z^{-1}) = \left(\frac{z}{|z|}\right)^{\log_\mu(|z|)+1}$. Thus, for $z \in \bar{\mathbb{C}}^\mu$ large enough, we have*

$$F_\mu(z) \approx \left(\frac{z}{|z|}\right)^{\log_\mu(|z|)-1}.$$

Proof. First assume that $z = \mu^q e^{-i\theta}$ with $q \in \mathbb{Z}$ and $\theta \in [0, 2\pi]$ such that $\theta \neq \pi$. It follows from a straightforward calculation that

$$\begin{aligned} F_\mu(\mu^2 z) F_\mu(z^{-1}) &= \prod_{k=0}^{q} \frac{1 + \mu^{2k-q} e^{i\theta}}{1 + \mu^{2k-q} e^{-i\theta}} \\ &= e^{i(q+1)\theta} \prod_{k=0}^{q} \mu^{q-2k} = e^{i(q+1)\theta} = \left(\frac{z}{|z|}\right)^{\log_\mu(|z|)+1}. \end{aligned}$$

By continuity of F_μ, we can extend this result to all elements in $\bar{\mathbb{C}}^\mu$. ∎

Next, we recall the Definition 2.2.4 of the Fourier coefficients $B(k,n)$.

Definition A.4 ([1, 2]) *Let $k,n \in \mathbb{Z}$. Then, we define a number $B(k,n)$ by*

$$B(k,n) = \frac{1}{2\pi}\int_0^{2\pi} F_\mu(\mu^n e^{it})e^{-ikt}\,dt.$$

We thus have by definition that $B(k,n)$ is the k^{th} Fourier coefficient of the continuous function $S^1 \to \mathbb{C} : z \mapsto F_\mu(\mu^n z)$.

We then also introduce functions $\hat{F}_p$. It is clear from their Definition A.5 that they are closely related to the function F_μ (cf. Lemma A.7). We use them below to prove some important relations between the numbers $B(k,n)$.

Definition A.5 ([157, 1]) *Let $p \in \mathbb{Z}$. We then define a function $\hat{F}_p$ on $\mathbb{C}_0$ in the following way. For $z \in \mathbb{C}_0$, we set*

$$\hat{F}_p(z) = \prod_{k=0}^{\infty} \frac{1+\mu^{2k+p}z^{-1}}{1+\mu^{2k+p}z}.$$

The basic properties of the functions $\hat{F}_p$ are stated in Lemma A.6.

Lemma A.6 ([157, 1]) *Let $p \in \mathbb{Z}$. We have that $\hat{F}_p$ is a meromorphic function. The function $\hat{F}_p$ is analytic on the open annulus $\{z \in \mathbb{C} \mid 0 < |z| < \mu^{-1}\}$. Further, we have the following properties:*

- *If $p > 0$, then $\hat{F}_p$ has single zeros in $\{-\mu^{p+2k} \mid k \in \mathbb{N}\}$ and single poles in $\{-\mu^{-p-2k} \mid k \in \mathbb{N}\}$. We have $\hat{F}_p(1) = \hat{F}_p(-1) = 1$.*
- *If $p \leq 0$, then $\hat{F}_p$ has single zeros in $\{-\mu^{-p+2+2k} \mid k \in \mathbb{N}\}$ and single poles in $\{-\mu^{p-2-2k} \mid k \in \mathbb{N}\}$. We have $\hat{F}_p(1) = 1$ and $\hat{F}_p(-1) = (-1)^{p+1}$.*

Proof. Let $p \in \mathbb{Z}$. We consider the function $\hat{g}_p$ on $\mathbb{C}$ defined by

$$\hat{g}_p(z) = \prod_{k=0}^{\infty}(1+\mu^{2k+p}z).$$

Then, $\hat{g}_p$ is analytic on $\mathbb{C}$. The set of its zeros is equal to $\{-\mu^{-p-2k} \mid k \in \mathbb{N}\}$. It is standard to prove that all these zeros are single zeros. Moreover, we have that the product $\prod_{k=0}^{\infty}(1+\mu^{2k+p}z)$ converges uniformly on compact subsets.

For every $z \in \mathbb{C}_0$, we have by Definition A.5 that

$$\hat{F}_p(z) = \frac{\hat{g}_p(z^{-1})}{\hat{g}_p(z)}.$$

This gives the meromorphic properties of the function $\hat{F}_p$.

We observe that the small difference between the cases $p > 0$ and $p \leq 0$ is due to the fact that

- $\{-\mu^{p+2k} \,|\, k \in \mathbb{N}\} \cap \{-\mu^{-p-2k} \,|\, k \in \mathbb{N}\} = \emptyset$ if $p > 0$,
- $\{-\mu^{p+2k} \,|\, k \in \mathbb{N}\} \cap \{-\mu^{-p-2k} \,|\, k \in \mathbb{N}\} = \{-\mu^{-p-2l} \,|\, l \in \mathbb{N}, l \leq -p\}$ if $p \leq 0$.

In both cases, we see that $\hat{F}_p$ is analytic the open annulus $\{z \in \mathbb{C} \,|\, 0 < |z| < \mu^{-1}\}$. It is straightforward to calculate $\hat{F}_p(1)$ and $\hat{F}_p(-1)$. ∎

The functions $\hat{F}_p$ are related to F_μ by the following property.

Lemma A.7 ([157, 1]) *Let $p \in \mathbb{Z}$ and $\theta \in [0, 2\pi]$. We then have that*

$$F_\mu(\mu^p e^{i\theta}) = \hat{F}_p(e^{i\theta}).$$

Thus, the function F_μ is analytic on every circle S_k with $k \in \mathbb{Z}$.

Proof. Remember that $S_k = \{z \in \bar{\mathbb{C}}^\mu \,|\, |z| = \mu^k\}$ for $k \in \mathbb{Z}$; see Definition 2.6.23. Take $p \in \mathbb{Z}$ and $\theta \in [0, 2\pi]$ with $\theta \neq \pi$. Then, we have

$$F_\mu(\mu^p e^{i\theta}) = \prod_{k=0}^{\infty} \frac{1 + \mu^{2k+p} e^{-i\theta}}{1 + \mu^{2k+p} e^{i\theta}} = \hat{F}_p(e^{i\theta}).$$

The case $\theta = \pi$ follows by continuity or can be checked in a direct way. ∎

The fact the F_μ is analytic on the circles S_k makes it possible to derive some useful results about the Fourier coefficients.

The next Lemma A.8 is a direct consequence of Lemma A.7.

Lemma A.8 ([1]) *Let $n \in \mathbb{Z}$. For every $l \in \mathbb{Z}$, we have*

$$\lim_{k \to +\infty} k^l B(k, n) = \lim_{k \to -\infty} k^l B(k, n) = 0.$$

This implicates that

$$\sum_{k \in \mathbb{Z}} |B(k, n)| < +\infty.$$

It follows, for all $z \in S^1$, that we have

$$F_\mu(\mu^n z) = \sum_{k \in \mathbb{Z}} B(k, n) z^k. \tag{A.1}$$

Further, the series in Equation (A.1) has uniform convergence on S^1.

For the rest of this Appendix A, we focus on the Fourier coefficients $B(k,n)$. Below, we formulate and prove all related properties that are used throughout this thesis. We can hereby restrict ourselves to basic results.

We mention that more advanced formulas involving the numbers $B(k,n)$ can be found in Appendix A of [1]. We do not include these results in this Appendix A as they are nowhere needed in our calculations.

The next Proposition A.9 is simple, but very useful.

Proposition A.9 ([1, 2]) *We have the following properties:*

1. $B(k,n) \in \mathbb{R}$ *and* $|B(k,n)| \leq 1$ *for all* $k,n \in \mathbb{Z}$,
2. $\sum_{k\in\mathbb{Z}} |B(k,n)|^2 = 1$ *for all* $n \in \mathbb{Z}$,
3. $\lim_{n\to+\infty} B(k,n) = \delta_{k,0}$ *uniformly in* k,
4. $\lim_{n\to-\infty} B(k,n) = 0$.

Proof. Take $n \in \mathbb{Z}$. By Definition A.4, we have that $B(k,n)$ is the k^{th} Fourier coefficient of the function $S^1 \to \mathbb{C} : z \mapsto F_\mu(\mu^n z)$.

Because $|F_\mu(\mu^n z)| = 1$ for every $z \in S^1$, we get that $\sum_{k\in\mathbb{Z}} |B(k,n)|^2 = 1$. This also gives that $|B(k,n)| \leq 1$ for all $k \in \mathbb{Z}$. The fact that we have $\overline{F_\mu(z)} = F_\mu(\overline{z})$ for all $z \in \mathbb{C}$ implies that $B(k,n) \in \mathbb{R}$ for all $k \in \mathbb{Z}$.

Independently of k, we have that

$$\lim_{n\to+\infty} B(k,n) = \frac{1}{2\pi}\int_0^{2\pi} \lim_{n\to+\infty} F_\mu(\mu^n e^{i\theta})e^{-ik\theta}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta}\, d\theta = \delta_{k,0}.$$

Using the asymptotical behavior of the function F_μ (see Lemma A.3), it is not difficult to show that we have that $\lim_{n\to-\infty} B(k,n) = 0$. ∎

The Lemma A.10 and its Corollary A.11 are also important.

Lemma A.10 ([157, 1]) *Let* $m \in \mathbb{Z}$. *For all* $z \in \mathbb{C}_0$, *we then have that*

1. $\hat{F}_{m+1}(\mu z) = (1 + \mu^m z)\hat{F}_m(z)$,
2. $\hat{F}_m(\mu z) = (1 + \mu^{m-1} z^{-1})\hat{F}_{m+1}(z)$

where equality also can mean that we have single poles at both sides.

Proof. Take $m \in \mathbb{Z}$. Lemma A.6 shows that z is a pole of $\hat{F}_m$ if and only if μz is a pole of $\hat{F}_{m+1}$ or $z = -\mu^m$. If $m < 0$, this is the case if $z \in \{-\mu^{m-2-2k} \mid k \in \mathbb{N}\}$; if $m \geq 0$, this is the case if $z \in \{-\mu^{-m-2k} \mid k \in \mathbb{N}\}$.

In the case that z is not a pole of $\hat{F}_m$, we can make the straightforward computation

$$\hat{F}_{m+1}(\mu z) = \prod_{k=0}^{\infty} \frac{1+\mu^{2k+m}z^{-1}}{1+\mu^{2k+m+2}z} = (1+\mu^m z)\prod_{k=0}^{\infty} \frac{1+\mu^{2k+m}z^{-1}}{1+\mu^{2k+m}z} = \hat{F}_m(z)$$

unless $m \leq 0$ and $z \in \{-\mu^{-m-2l} \mid 0 \leq l \leq -m\}$. We can include these cases via a trivial continuity argument.

The proof of the second result is completely similar. The main calculation is then given by

$$\hat{F}_m(\mu z) = \prod_{k=0}^{\infty} \frac{1+\mu^{2k+m-1}z^{-1}}{1+\mu^{2k+m+1}z} = (1+\mu^{m-1}z^{-1})\prod_{k=0}^{\infty} \frac{1+\mu^{2k+m+1}z^{-1}}{1+\mu^{2k+m+1}z} = \hat{F}_{m+1}(z)$$

which is correct for almost all $z \in \mathbb{C}_0$. ∎

Corollary A.11 ([1, 2]) *Let $k, n \in \mathbb{Z}$. We then have that*

1. $B(k, n+1) = B(k, n)\mu^{-k} + B(k-1, n)\mu^{n-k}$,
2. $B(k, n-1) = B(k, n)\mu^{-k} + B(k+1, n)\mu^{n-k-2}$.

Proof. We only prove the first equality. The proof of the second equality is completely similar. The only difference is that we then have to use the second equality from Lemma A.10 instead of the first one.

First, we define two closed paths γ_1 and γ_2 in the following way:

$$\gamma_1 : [0, 2\pi] \to \mathbb{C} : \theta \mapsto e^{i\theta} \qquad \text{and} \qquad \gamma_2 : [0, 2\pi] \to \mathbb{C} : \theta \mapsto \mu e^{i\theta}.$$

We have that

$$\begin{aligned}
B(k, n+1) &= \frac{1}{2\pi}\int_0^{2\pi} F_\mu(\mu^{n+1}e^{it})e^{-ikt}\,dt = \frac{1}{2\pi i}\int_{\gamma_1} \hat{F}_{n+1}(z)z^{-k-1}\,dz \\
&= \frac{1}{2\pi i}\int_{\gamma_2} \hat{F}_{n+1}(z)z^{-k-1}\,dz = \frac{1}{2\pi}\int_0^{2\pi} \hat{F}_{n+1}(\mu e^{it})\mu^{-k}e^{-ikt}\,dt \\
&\overset{(1)}{=} \frac{1}{2\pi}\int_0^{2\pi} (1+\mu^n e^{it})\hat{F}_n(e^{it})\mu^{-k}e^{-ikt}\,dt \\
&= \Big(\frac{1}{2\pi}\int_0^{2\pi} F_\mu(\mu^n e^{it})e^{-ikt}\,dt\Big)\mu^{-k} \\
&\qquad\qquad + \Big(\frac{1}{2\pi}\int_0^{2\pi} F_\mu(\mu^n e^{it})e^{-i(k-1)t}\,dt\Big)\mu^{n-k} \\
&= B(k, n)\mu^{-k} + B(k-1, n)\mu^{n-k}.
\end{aligned}$$

The Equality (1) follows from the results in Lemma A.10. ∎

The next Proposition A.12 states the most important property of the Fourier coefficients $B(k,n)$ (cf. Remark 2.2.7). Its proof is based on the two equalities in Corollary A.11. The used technique is due to S. Baaj [1].

This Proposition A.12 is e.g. crucial in the Section 2.3.

Proposition A.12 ([1, 2]) *Let $k,n \in \mathbb{Z}$. We have that*

$$\begin{aligned} B(k,n) &= (-\mu)^k B(-k,n-k) \\ &= B(k-n+1,-n+2) = (-\mu)^{k-n+1} B(n-1,k+1). \end{aligned}$$

Proof. Let $\mathbf{F}$ be the set of all real double sequences $\big(\mathcal{B}(k,n)\big)_{(k,n)\in\mathbb{Z}\times\mathbb{Z}}$ satisfying the two equalities

1. $\mathcal{B}(k,n+1) = \mathcal{B}(k,n)\mu^{-k} + \mathcal{B}(k-1,n)\mu^{n-k}$,
2. $\mathcal{B}(k,n-1) = \mathcal{B}(k,n)\mu^{-k} + \mathcal{B}(k+1,n)\mu^{n-k-2}$.

It is clear that $\mathbf{F}$ is a real vector space.

From the Corollary A.11, we directly get that $\big(B(k,n)\big) \in F$.

Take $\mathcal{B} \in \mathbf{F}$. We define double sequences $\mathbf{u}, \mathbf{v}, \mathbf{w}$ by

1. $\mathbf{u}(k,n) = (-\mu)^k \mathcal{B}(-k,n-k)$,
2. $\mathbf{v}(k,n) = \mathcal{B}(k-n+1,-n+2)$,
3. $\mathbf{w}(k,n) = (-\mu)^{k-n+1}\mathcal{B}(n-1,k+1)$.

It is not difficult to check that $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{F}$.

By the definition of the double sequences $\mathbf{u}$ and $\mathbf{v}$ we immediately get that we have $\mathbf{u}(0,n) = \mathcal{B}(0,n)$ and $\mathbf{v}(k,1) = \mathcal{B}(k,1)$ for all $k,n \in \mathbb{Z}$. Using the structure of the vector space $\mathbf{F}$, we see that this implies that $\mathbf{u} = \mathbf{v} = \mathcal{B}$.

Let $k,n \in \mathbb{Z}$. Using the equality $\mathbf{u} = \mathbf{v} = \mathcal{B}$, we see that

$$\begin{aligned} \mathbf{w}(k,n) &= (-\mu)^{k-n+1}\mathcal{B}(n-1,k+1) = (-\mu)^{k-n+1}\mathbf{v}(-k+n-1,-k+1) \\ &= (-\mu)^{k-n+1}\mathcal{B}(-k+n-1,-k+1) = u(k-n+1,-n+2) \\ &= \mathcal{B}(k-n+1,-n+2) = \mathbf{v}(k,n) = \mathcal{B}(k,n). \end{aligned}$$

Hence, we can conclude that $\mathbf{u} = \mathbf{v} = \mathbf{w} = \mathcal{B}$. ■

The Lemma A.13 is a standard result for Fourier coefficients.

Lemma A.13 ([1]) *Let $p,n \in \mathbb{Z}$. We have that*

$$\sum_{k\in\mathbb{Z}} B(k,n)B(k+p,n) = \delta_{p,0}.$$

Proof. By Definition A.4, we have that $B(k,n)$ is the k^{th} Fourier coefficient of the function $S^1 \to \mathbb{C} : z \mapsto F_\mu(\mu^n z)$. We also have that $B(k+p,n)$ is the k^{th} Fourier coefficient of the function $S^1 \to \mathbb{C} : z \mapsto F_\mu(\mu^n z) z^{-p}$.

Using Parseval's identity (and remembering that $B(k+p,n) \in \mathbb{R}$ for all $k \in \mathbb{Z}$), we get that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} B(k,n)B(k+p,n) &= \frac{1}{2\pi}\int_0^{2\pi} F_\mu(\mu^n e^{i\theta})\overline{F_\mu(\mu^n e^{i\theta})e^{-ip\theta}}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{ip\theta}\, d\theta = \delta_{p,0}.\end{aligned}$$

In this calculation, we also use that $|F_\mu(z)| = 1$ for all $z \in \bar{\mathbb{C}}^\mu$. ∎

The next Lemma A.14 is used in the proof of Proposition 2.5.15.

Lemma A.14 *Let $l,n,s \in \mathbb{Z}$. We have that*

$$\sum_{t\in\mathbb{Z}} B(t,l-n+1)B(0,-l+t)B(s+t,l-n+1) = B(-s,-l-s)B(s,-n).$$

Proof. In this proof, we use the operators c,d as introduced in Definition 2.3.3. We also use the unit vectors e_k as introduced in Notation 2.3.1.

First, we define normal operators R and S by

$$R = d \otimes c^* \qquad \text{and} \qquad S = c \otimes d.$$

It is easy to check that $(R,S) \in D_\mu$ and $\sigma(R), \sigma(S) \subseteq \bar{\mathbb{C}}^\mu$. Hence, Proposition 2.2.12 gives that $F_\mu(R \dotplus S) = F_\mu(R)F_\mu(S)$. Together with Lemma 2.3.15, this implies that

$$W(F_\mu(d) \otimes 1)W^* = F_\mu(d \otimes c^*)F_\mu(c \otimes d).$$

Let $k,l,m,n,s \in \mathbb{Z}$. We can use techniques similar to the ones used in the proof of Proposition 2.3.13 to calculate that

$$\begin{aligned}&\big\langle W(F_\mu(d)\otimes 1)W^*(e_k \otimes e_l \otimes e_m \otimes e_n), e_k \otimes e_{l+s} \otimes e_{m+2s} \otimes e_{n+s}\big\rangle \\ &\qquad = \sum_{t\in\mathbb{Z}} B(t,l-n+1)B(0,-l+t)B(s+t,l-n+1)\end{aligned}$$

and also that

$$\begin{aligned}&\big\langle F_\mu(d\otimes c^*)F_\mu(c\otimes d)(e_k \otimes e_l \otimes e_m \otimes e_n), e_k \otimes e_{l+s} \otimes e_{m+2s} \otimes e_{n+s}\big\rangle \\ &\qquad = B(-s,-l-s)B(s,-n).\end{aligned}$$

This ends the proof of the lemma. ∎

Appendix B

Technical results

We use this Appendix B to collect some information about three subjects that are unremittingly used in calculations throughout the thesis:

- Weight theory,
- One-parameter groups,
- Functional calculus.

Further, in Section B.4, we also include two technical lemmas.

We are very concise in the treatments. We restrict ourselves to only mention results that are used in calculations in either of the Chapters 1 to 4. It should nevertheless be remarked that acquaintance with the three handled subjects is a prerequisite for understanding all technical details of the thesis.

This Appendix B is thus used as a technical reference. We are sparse with details and we therefore, in each case, include a list of more elaborated works.

B.1 Weight theory

In this Section B.1, we collect some notations and results concerning the theory of weights on C*-algebras and von Neumann algebras. The mentioned definitions and properties are standard. We refer to [18, 91, 58, 64] for more details.

We also mention [125, Chapter 4] as a quite complete overview.

First, we give the Definition B.1.1 of a *weight* on a C*-algebra A. This will then be the central object for the whole Section B.1.

Definition B.1.1 *Let A be a C*-algebra. Then a mapping $\varphi : A^+ \to [0, +\infty]$ is called a* weight *if the following two conditions are satisfied:*

1. *We have $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in A^+$,*
2. *We have $\varphi(rx) = r\varphi(x)$ for all $r \in \mathbb{R}^+$ and $x \in A^+$.*

The following Notation B.1.2 and Lemma B.1.3 are standard.

Notation B.1.2 *Let φ be a weight on a C*-algebra A. We then denote*

- $\mathfrak{M}_\varphi^+ = \{a \in A^+ \mid \varphi(a) < +\infty\}$,
- $\mathfrak{M}_\varphi = \operatorname{span} \mathfrak{M}_\varphi^+$,
- $\mathfrak{N}_\varphi = \{a \in A \mid \varphi(a^*a) < +\infty\}$.

Lemma B.1.3 *Let φ be a weight on a C*-algebra A. The properties below hold:*

- *We have that $\mathfrak{M}_\varphi^+$ is a hereditary cone in A^+,*
- *We have that $\mathfrak{N}_\varphi$ is a left ideal in $M(A)$,*
- *We have $\mathfrak{M}_\varphi = \mathfrak{N}_\varphi^* \mathfrak{N}_\varphi$,*
- *We have that $\mathfrak{M}_\varphi$ is a *-subalgebra of A such that $\mathfrak{M}_\varphi \cap A^+ = \mathfrak{M}_\varphi^+$.*

We have a *GNS-construction* for weights. It is very similar to the well-known GNS-construction for continuous linear functionals.

The Proposition B.1.4 is a basic result in weight theory.

Proposition B.1.4 *Let φ be a weight on a C*-algebra A. There exists a unique (up to a unitary transformation) triple $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ such that:*

- *H_φ is a Hilbert space,*
- *Λ_φ is a linear map from $\mathfrak{N}_\varphi$ into H_φ satisfying the two conditions*
  1. *$\Lambda_\varphi(\mathfrak{N}_\varphi)$ is dense in H_φ,*
  2. *For all $a, b \in \mathfrak{N}_\varphi$, we have $\langle \Lambda_\varphi(a), \Lambda_\varphi(b) \rangle = \varphi(b^*a)$,*
- *$\pi_\varphi : A \to B(H_\varphi)$ is a *-representation such that $\pi_\varphi(a)\Lambda_\varphi(b) = \Lambda_\varphi(ab)$ for every $a \in A$ and $b \in \mathfrak{N}_\varphi$.*

The triple $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ is called the GNS-construction *for φ.*

Weights on C*-algebras

We first concentrate on the basic properties of weights on C*-algebras. It should be observed that there are some very crucial results that are specifically adapted to von Neumann algebra case. These are handled further below.

The next Notation B.1.5 is used in the whole Section B.1.

Notation B.1.5 *We fix a C*-algebra $A \subseteq B(K)$ and a weight $\varphi : A^+ \to [0, +\infty]$.*

We start by giving some terminology. We can e.g. refer to [18, 91, 93] for proofs of the (non-trivial) equivalences that appear below.

The three Terminologies B.1.6, B.1.7 and B.1.8 are standard.

Terminology B.1.6 *We say that* φ *is* lower semi-continuous *when one of the following equivalent conditions is satisfied:*

- *We have for every* $\lambda \in \mathbb{R}^+$ *that the set* $\{ a \in A^+ \mid \varphi(a) \leq \lambda \}$ *is closed,*
- *For every net* (x_i) *in* A^+ *and* $x \in A^+$ *such that* $x_i \to x$*, we have*
$$\varphi(x) \leq \liminf \varphi(x_i),$$
- *There exists a family* (ω_i) *in* A^*_+ *such that we have*
$$\varphi(x) = \sum_i \omega_i(x)$$
for all $x \in A^+$*.*

Terminology B.1.7 *We say that* φ *is* densely defined *when one of the following equivalent conditions is satisfied:*

- $\mathfrak{M}_\varphi^+$ *is dense in* A^+*,*
- $\mathfrak{M}_\varphi$ *is dense in* A*,*
- $\mathfrak{N}_\varphi$ *is dense in* A*.*

Terminology B.1.8 *We say that* φ *is* faithful *if the following condition holds:*

- *If* $a \in A^+$ *and* $\varphi(a) = 0$*, then we have* $a = 0$*.*

We then concentrate on the properties of weights that are lower semi-continuous. The condition of lower semi-continuity is a very natural one to impose.

It is clear that the next Proposition B.1.9 is a simple result.

Proposition B.1.9 *Suppose that* φ *is lower semi-continuous. Let* (x_i) *be a net in* A^+ *and* $x \in A^+$ *such that* $x_i \leq x$ *for all* i *and* $x_i \to x$*.*
We then have
$$\varphi(x_i) \to \varphi(x).$$

The main property in the theory of lower semi-continuous weights is stated in the Proposition B.1.12. It shows that it is possible to approximate any lower semi-continuous weight φ from below by continuous linear functionals.

The Notation B.1.10 introduces two important sets.

Notation B.1.10 *We make use of the two following notations:*

- $\mathcal{F}_\varphi = \big\{ \omega \in A_*^+ \mid \omega(x) \leq \varphi(x) \text{ for } x \in A^+ \big\}$,
- $\mathcal{G}_\varphi = \big\{ \alpha\omega \mid \omega \in \mathcal{F}_\varphi, \alpha \in \,]0,1[\, \big\}$.

The Lemma B.1.11 below allows us to use $\mathcal{G}_\varphi$ as an index set for a net.

Lemma B.1.11 *The set $\mathcal{G}_\varphi$ is a directed subset of $\mathcal{F}_\varphi$.*

The next Proposition B.1.12 is thus crucial. We should certainly remark that it was first proved by F. Combes in his pioneering article [18].

Proposition B.1.12 *Suppose that φ is lower semi-continuous. For all $x \in A^+$, we then have*

- $\varphi(x) = \sup\{\, \omega(x) \mid \omega \in \mathcal{F}_\varphi \}$,
- $\varphi(x) = \lim_{\omega \in \mathcal{G}_\varphi} \omega(x)$.

The following Proposition B.1.13 is also very useful.

Proposition B.1.13 *Let $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ be the GNS-construction of φ. Then, we have that*

- *The mapping $\Lambda_\varphi : \mathfrak{N}_\varphi \to H_\varphi$ is closed,*
- *The *-representation $\pi_\varphi : A \to B(H_\varphi)$ is non-degenerate,*
- *If φ is faithful, then Λ_φ is injective and π_φ is faithful.*

The next Terminology B.1.14 is used in Definition 1.1.6.

Terminology B.1.14 *Let $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ be the GNS-construction of φ.*

We call a vector $v \in H_\varphi$ right bounded *with respect to $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ if there exists a number $M \geq 0$ such that $\|\pi_\varphi(x)v\| \leq M\|\Lambda_\varphi(x)\|$ for all $x \in \mathfrak{N}_\varphi$.*

Further, we call φ approximately KMS *if the conditions below are satisfied:*

- *We have that φ is densely defined and lower semi-continuous,*
- *The subspace of the vectors $v \in H_\varphi$ that are right bounded with respect to $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ is dense in H_φ.*

Extensions to the multiplier algebra

Every lower semi-continuous weight φ on the C*-algebra A has a natural extension to a weight on the multiplier algebra $M(A)$. We only very shortly describe the standard way of how this extension is constructed.

We recall that, for any $\omega \in A^*$, we also use ω to denote its strictly continuous extension to $M(A)$. This is explained in Notation 1.5.24.

The Definition B.1.15 states the extension procedure.

Definition B.1.15 *Suppose that φ is lower semi-continuous. We can define a mapping $\bar{\varphi} : M(A)^+ \to [0, +\infty]$ by setting*

$$\bar{\varphi}(x) = \sup\{\, \omega(x) \mid \omega \in \mathcal{F}_\varphi \}.$$

From Proposition B.1.12, we can deduce the following result.

Proposition B.1.16 *Suppose that φ is lower semi-continuous. Then $\bar{\varphi}$ is a weight on $M(A)$ that is an extension of φ.*

For every $x \in M(A)^+$, we always put $\varphi(x) = \bar{\varphi}(x)$.

KMS-weights on a C*-algebra

We now point our attention to the special class of KMS-weights. It is direct to observe that the KMS-property is a strong condition. It is therefore no surprise that the theory of KMS-weights clusters several very useful properties.

It should be observed that we below already make use of the technique of analytic continuation of one-parameter groups. We refer to the Section B.2 for a short survey of the main results regarding this subject.

The next Definition B.1.17 describes the KMS-property.

Definition B.1.17 *We say that φ is a* KMS-weight *on A if the two following conditions are satisfied:*

- *φ is densely defined and lower semi-continuous,*
- *There exists a norm continuous one-parameter group σ on A satisfying the conditions below:*
  1. *φ is invariant under σ, i.e., we have $\varphi\sigma_t = \varphi$ for every $t \in \mathbb{R}$,*
  2. *For every $a \in D(\sigma_{\frac{i}{2}})$, we have $\varphi(a^*a) = \varphi(\sigma_{\frac{i}{2}}(a)\sigma_{\frac{i}{2}}(a)^*)$.*

The one-parameter group σ is called a modular automorphism group *for φ.*

We have the Proposition B.1.18 below.

Proposition B.1.18 *Let φ be a faithful KMS-weight. The one-parameter group σ appearing in Definition B.1.17 is then unique.*

In this case, we call σ the *modular automorphism group of φ.*

There are several ways to express the KMS-condition for the weight φ. We list three of them in the Proposition B.1.20 below.

We first give the Notation B.1.19.

Notation B.1.19 *Let $z \in \mathbb{C}$. Then, we use the notation*

$$\mathcal{S}(z) = \big\{\, y \in \mathbb{C} \mid \operatorname{Im} y \in [0, \operatorname{Im} z] \big\}.$$

We denote the interior of $\mathcal{S}(z)$ by $\mathcal{S}(z)^\circ$.

The next Proposition B.1.20 is then about the KMS-condition.

Proposition B.1.20 *Let φ be both densely defined and lower semi-continuous. We take a GNS-construction $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ for φ. Let σ be a norm continuous one-parameter group on A such that $\varphi\sigma_t = \varphi$ for all $t \in \mathbb{R}$.*

The following conditions are equivalent:

1. *The weight ψ is a KMS-weight with σ as a modular automorphism group, i.e., for every $a \in D(\sigma_{\frac{i}{2}})$, we have $\varphi(a^*a) = \varphi(\sigma_{\frac{i}{2}}(a)\sigma_{\frac{i}{2}}(a)^*)$,*
2. *There exists a non-degenerate *-anti-homomorphism $\theta : A \to B(H_\varphi)$ such that, for all $x \in \mathfrak{N}_\varphi$ and $a \in D(\sigma_{\frac{i}{2}})$, we have that xa belongs to $\mathfrak{N}_\varphi$ and $\Lambda_\varphi(xa) = \theta(\sigma_{\frac{i}{2}}(a))\Lambda_\varphi(x)$,*
3. *For all $a, b \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$, there exists a function $f : \mathcal{S}(i) \to \mathbb{C}$ such that*
   - *f is continuous and bounded on $\mathcal{S}(i)$,*
   - *f is analytic on the interior $\mathcal{S}(i)^\circ$ of $\mathcal{S}(i)$,*
   - *$f(t) = \varphi(\sigma_t(b)a)$ and $f(t+i) = \varphi(a\sigma_t(b))$ for all $t \in \mathbb{R}$.*

We also state the Proposition B.1.21 below.

Proposition B.1.21 *Suppose that φ is a KMS-weight. Further, we then take a GNS-construction $(H_\varphi, \Lambda_\varphi, \pi_\varphi)$ for φ.*

We have the following result:

- *There exists a unique anti-unitary operator J_φ on H_φ such that*
$$J_\varphi\Lambda_\varphi(x) = \Lambda_\varphi(\sigma_{\frac{i}{2}}(x)^*)$$
  for all $x \in \mathfrak{N}_\varphi \cap D(\sigma_{\frac{i}{2}})$.
- *There exists a unique strictly positive operator ∇_φ on H_φ such that*
$$\nabla_\varphi^{it}\Lambda_\varphi(a) = \Lambda_\varphi(\sigma_t(a))$$
  for all $a \in \mathfrak{N}_\varphi$ and $t \in \mathbb{R}$.

We call J_φ the modular conjugation *of φ and ∇_φ the* modular operator *of φ.*

The results below hold:

- *If $x \in \mathfrak{N}_\varphi$ and $y \in D(\sigma_{\frac{i}{2}})$, then $xy \in \mathfrak{N}_\varphi$ and*
$$\Lambda_\varphi(xy) = J_\varphi\pi_\varphi(\sigma_{\frac{i}{2}}(y))^* J_\varphi\Lambda_\varphi(x),$$
- *If $a \in D(\sigma_{-i})$ and $x \in \mathfrak{M}_\varphi$, then ax and $x\sigma_{-i}(a)$ belong to $\mathfrak{M}_\varphi$ and $\varphi(ax) = \varphi(x\sigma_{-i}(a))$,*
- *If $x \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$ and $a \in \mathfrak{N}_\varphi^* \cap D(\sigma_{-i})$ such that $\sigma_{-i}(a) \in \mathfrak{N}_\varphi$, then we have $\varphi(ax) = \varphi(x\sigma_{-i}(a))$.*

Absolutely continuous KMS-weights

We now look at a typical construction procedure for KMS-weights. Starting from a KMS-weight φ and a strictly positive operator $\delta \,\eta\, A$ (having some properties), there is constructed a 'deformed' KMS-weight φ_δ.

First, we give the Terminology B.1.22. We remark that we also here use the convention made in the Introduction about products of operators.

Terminology B.1.22 *Let $a \in M(A)$ and $T \,\eta\, A$. We then say that a is a* left multiplier *of T if and only if we have that $aT \in M(A)$.*

The Proposition B.1.23 states the mentioned construction procedure. It should be observed that, on an informal level, we have $\varphi_\delta = \varphi(\delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}})$.

For an elaboration of Proposition B.1.23, we refer to [58, Section 8].

Proposition B.1.23 *Suppose that φ be a KMS-weight. We then take a modular automorphism group σ of φ.*

Let $\delta \,\eta\, A$ be a strictly positive operator such that there exists a number $\lambda > 0$ satisfying $\sigma_t(\delta) = \lambda^t \delta$ for all $t \in \mathbb{R}$.

We consider the subspace $\mathfrak{N}_0$ of A defined by

$$\mathfrak{N}_0 = \{\, a \in A \mid a \text{ is a left multiplier of } \delta^{\frac{1}{2}} \text{ and } a\delta^{\frac{1}{2}} \text{ belongs to } \mathfrak{N}_\varphi \}.$$

Then $\mathfrak{N}_0$ is a dense left ideal of A. Further, the mapping $\mathfrak{N}_0 \to H_\varphi : a \mapsto \Lambda(a\delta^{\frac{1}{2}})$ is closable. We use Λ_δ to denote its closure.

There is a unique KMS-weight φ_δ on A with $(H_\varphi, \pi_\varphi, \Lambda_\delta)$ as a GNS-construction.

The next Lemma B.1.24 collects a few basic features.

Lemma B.1.24 *Suppose that φ is a KMS-weight. Further, let $\delta \,\eta\, A$ and $\lambda > 0$ be as in the formulation of Proposition B.1.23.*

The properties below hold:

- *We have that φ_δ is faithful if and only if φ is faithful,*
- *For the modular automorphism group σ' of φ_δ, we have that*

  $$\sigma'_t(a) = \delta^{it}\sigma_t(a)\delta^{-it}$$

  for all $a \in A$ and $t \in \mathbb{R}$,
- *We have $\sigma'_t(\delta) = \lambda^t \delta$ for all $t \in \mathbb{R}$,*
- *For every $t \in \mathbb{R}$, we have $\varphi\sigma'_t = \lambda^t \varphi$ and $\varphi_\delta \sigma_t = \lambda^{-t} \varphi_\delta$.*

Weights on von Neumann algebras

It is aforementioned that weight theory on von Neumann algebras has some typical features. We mention a few of them that are used in the thesis.

The next Notation B.1.25 is applied for the rest of Section B.1.

Notation B.1.25 *Let ψ be a weight on a von Neumann algebra M.*

We first give some terminology. By the 'weak topologies' on M, we mean the weak, σ-weak, strong, strong*, σ-strong and σ-strong* topologies.

It should be remarked that the equivalences appearing in Terminology B.1.26 are quite delicate to prove. We refer to [120, 33] for a full treatment.

The three Terminologies B.1.26, B.1.27 and B.1.28 are again standard.

Terminology B.1.26 *We say that ψ is* normal *when one of the following equivalent conditions is satisfied:*

- *For every $\lambda \in \mathbb{R}^+$, the set $\{a \in M^+ \mid \psi(a) \leq \lambda\}$ is σ-weakly closed,*
- *For every net (x_i) in M^+ and $x \in M^+$ such that $x_i \to x$ in the strong topology, we have*
$$\psi(x) = \lim_i \psi(x_i),$$
- *There exists a family (ω_i) in M_*^+ such that we have*
$$\psi(x) = \sum_i \omega_i(x)$$
for all $x \in M^+$,
- *For every net (x_i) in M^+ and $x \in M^+$ such that $x_i \to x$ in the σ-weak topology, we have*
$$\psi(x) \leq \liminf \psi(x_i).$$

Terminology B.1.27 *We say that ψ is* semi-finite *when one of the following equivalent conditions is satisfied:*

- *$\mathfrak{M}_\psi^+$ is dense in M^+ in any of the weak topologies,*
- *$\mathfrak{M}_\psi$ is dense in M in any of the weak topologies,*
- *$\mathfrak{N}_\psi$ is dense in M in any of the weak topologies.*

Terminology B.1.28 *We say that ψ is* faithful *if the following condition holds:*

- *If $a \in M^+$ and $\psi(a) = 0$, then we have $a = 0$.*

The next Terminology B.1.29 explains an abbreviated designation.

Terminology B.1.29 *Let ψ be normal, semi-finite and faithful. It is standard to then call ψ an* n.s.f. weight.

We mention the Proposition B.1.30 below.

Proposition B.1.30 *Let $(H_\psi, \pi_\psi, \Lambda_\psi)$ be the GNS-construction of ψ. Then, we have that*

- *The mapping $\Lambda_\psi : \mathfrak{N}_\psi \to H_\psi$ is σ-weak - weak closed,*
- *The *-representation $\pi_\psi : M \to B(H_\psi)$ is normal and unital.*
- *If ψ is faithful, then Λ_ψ is injective and π_ψ is faithful.*

It should be observed that the von Neumann algebra setting is very appropriate to work with weights. Especially for n.s.f. weights, there are a lot of profound results. This is mainly manifested in the *Tomita-Takesaki theory.*

In the next Theorem B.1.31, we compile some crucial properties of the powerful Tomita-Takesaki theory. We refer to [115, 120] for a full treatment.

Theorem B.1.31 *Let ψ be an n.s.f. weight. There exists a unique strongly continuous one-parameter group (σ_t) on M such that*

1. *$\psi\sigma_t = \psi$ for all $t \in \mathbb{R}$,*
2. *For every $x \in D(\sigma_{\frac{i}{2}})$, we have $\varphi(x^*x) = \varphi(\sigma_{\frac{i}{2}}(x)\sigma_{\frac{i}{2}}(x)^*)$.*

We call (σ_t) the modular automorphism group *of ψ.*

Let $(H_\psi, \pi_\psi, \Lambda_\psi)$ be a GNS-construction for ψ. Then, there is a unique closed anti-linear operator T on H_ψ satisfying

1. *$T\Lambda_\psi(x) = \Lambda_\psi(x^*)$ for all $x \in \mathfrak{N}_\psi \cap \mathfrak{N}_\psi^*$,*
2. *$\Lambda_\psi(\mathfrak{N}_\psi \cap \mathfrak{N}_\psi^*)$ is a core for T.*

Further, let $T = J_\psi \nabla_\psi^{\frac{1}{2}}$ be the polar decomposition of T. Then J_ψ is an anti-unitary operator on H_ψ and ∇_ψ is a strictly positive operator on H_ψ.

We call J_ψ the modular conjugation *of ψ and ∇_ψ the* modular operator *of ψ.*

The following properties hold:

1. *For all $x \in \mathfrak{N}_\psi$ and $t \in \mathbb{R}$, we have $\Lambda_\psi(\sigma_t(x)) = \nabla_\psi^{it}\Lambda_\psi(x)$,*
2. *For all $x \in \mathfrak{N}_\psi \cap D(\sigma_{\frac{i}{2}})$, we have $J_\psi\Lambda_\psi(x) = \Lambda_\psi(\sigma_{\frac{i}{2}}(x)^*)$,*

3. *If $x \in \mathfrak{N}_\psi$ and $y \in D(\sigma_{\frac{i}{2}})$, then $xy \in \mathfrak{N}_\psi$ and*

$$\Lambda_\psi(xy) = J_\psi \pi_\psi(\sigma_{\frac{i}{2}}(y))^* J_\psi \Lambda_\psi(x),$$

4. *If $a \in D(\sigma_{-i})$ and $x \in \mathfrak{M}_\psi$, then ax and $x\sigma_{-i}(a)$ belong to $\mathfrak{M}_\psi$ and $\psi(ax) = \psi(x\sigma_{-i}(a))$,*

5. *If $x \in \mathfrak{N}_\psi \cap \mathfrak{N}_\psi^*$ and $a \in \mathfrak{N}_\psi^* \cap D(\sigma_{-i})$ such that $\sigma_{-i}(a) \in \mathfrak{N}_\psi$, then we have $\psi(ax) = \psi(x\sigma_{-i}(a))$.*

The Radon-Nikodym theorem

We above mentioned how a KMS-weight can be deformed by a strictly positive operator. It is possible to apply a similar method to deform n.s.f. weights.

The next Proposition B.1.32 collects the basics of the deformation procedure.

Proposition B.1.32 *Let ψ be an n.s.f. weight. We then use σ to denote the modular automorphism group of ψ.*

Let δ be a strictly positive operator that is affiliated to M in the von Neumann algebra sense. Suppose that $\lambda > 0$ satisfies $\sigma_t(\delta) = \lambda^t \delta$ for all $t \in \mathbb{R}$.

We consider the subspace $\mathfrak{N}_0$ of M defined by

$$\mathfrak{N}_0 = \{\, a \in M \mid \textit{The operator } a\delta^{\frac{1}{2}} \textit{ belongs to } \mathfrak{N}_\psi \}.$$

Then $\mathfrak{N}_0$ is a σ-strongly* dense left ideal of M.

Further, we have that the mapping $\mathfrak{N}_0 \to H_\psi : a \mapsto \Lambda(a\delta^{\frac{1}{2}})$ is σ-strong* - norm closable. We use Λ_δ to denote its closure.

There is a unique n.s.f. weight ψ_δ on M with $(H_\psi, \pi_\psi, \Lambda_\delta)$ as a GNS-construction.

The above Proposition B.1.32 provides us with a useful tool. It gives a fairly easy way to construct new n.s.f. weights out of a given one. We refer to [126] for a full treatment and for further generalizations.

We then also describe the important Example B.1.33.

Example B.1.33 *Let (e_i) be an orthonormal basis of a Hilbert space H.*

We consider the canonical trace Tr *on $B(H)^+$. Thus, we have that* Tr *is the n.s.f. weight on $B(H)^+$ defined by*

$$\mathrm{Tr}(x) = \sum_i \langle x e_i, e_i \rangle.$$

Now, let h be a strictly positive operator.

We then consider the weight Tr_h as introduced in Proposition B.1.32.

Let $\mathfrak{N}_0$ *be the subspace of* M *defined by*

$$\mathfrak{N}_0 = \{\, a \in M \mid \textit{The operator } ah^{\frac{1}{2}} \textit{ belongs to } \mathfrak{N}_\varphi \}.$$

Then $\mathfrak{N}_0$ *is a* σ*-strongly* dense left ideal of* M *satisfying* $\mathfrak{N}_0 \subseteq \mathfrak{N}_{\mathrm{Tr}_h}$.

For all $x, y \in \mathfrak{N}_0$, *we have*

$$\mathrm{Tr}_h(y^* x) = \sum_i \langle x h^{\frac{1}{2}} e_i, y h^{\frac{1}{2}} e_i \rangle.$$

Suppose that every e_i *is an eigenvector of* $h^{\frac{1}{2}}$ *with eigenvalue* a_i. *We then have for all* $x \in B(H)^+$ *that*

$$\mathrm{Tr}_h(x) = \sum_i a_i^2 \langle x e_i, e_i \rangle.$$

The next Lemma B.1.34 collects a few basic features.

Lemma B.1.34 *Suppose that* φ *is an n.s.f. weight. Further, let* δ *and* λ *be as in the formulation of Proposition B.1.32.*

The properties below hold:

- *For the modular automorphism group* σ' *of* φ_δ, *we have that*

  $$\sigma'_t(a) = \delta^{it} \sigma_t(a) \delta^{-it}$$

  for all $a \in A$ *and* $t \in \mathbb{R}$,
- *We have* $\sigma'_t(\delta) = \lambda^t \delta$ *for all* $t \in \mathbb{R}$,
- *For every* $t \in \mathbb{R}$, *we have* $\varphi \sigma'_t = \lambda^t \varphi$ *and* $\varphi_\delta \sigma_t = \lambda^{-t} \varphi_\delta$.

We also mention the following *Radon-Nikodym theorem.* This Theorem B.1.35 is one of the major advantages of n.s.f. weights on von Neumann algebras over KMS-weights on C*-algebras. We refer to [126] for a proof.

Theorem B.1.35 *Let* φ *and* ψ *be two n.s.f. weights on* M *with respectively* σ *and* σ' *as the automorphism group. Take a number* $\lambda > 0$.

The following statements are equivalent:

1. $\varphi \sigma'_t = \lambda^t \varphi$ *for all* $t \in \mathbb{R}$,
2. $\psi \sigma_t = \lambda^{-t} \psi$ *for all* $t \in \mathbb{R}$,
3. *There exists a strictly positive operator* δ *affiliated with* M *such that we have* $\sigma_t(\delta) = \lambda^t \delta$ *for all* $t \in \mathbb{R}$ *and* $\psi = \varphi_\delta$.

B.2 One-parameter groups

We use this Section B.2 to include some useful results on one-parameter groups on C*-algebras and von Neumann algebras. The treatment is quite terse and as such only contains standard definitions and properties.

We refer to [60] for a full study of the theory of one-parameter groups.

The Notation B.2.1 is already mentioned in the Introduction.

Notation B.2.1 *Let A be a C*-algebra. We then use* $\mathrm{Aut}(A)$ *to denote the set of all *-automorphisms on A.*

The Notation B.2.2 is the same as Notation B.1.19.

Notation B.2.2 *Let $z \in \mathbb{C}$. Then, we use the notation*

$$\mathcal{S}(z) = \big\{\, y \in \mathbb{C} \mid \operatorname{Im} y \in [0, \operatorname{Im} z] \big\}.$$

We denote the interior of $\mathcal{S}(z)$ by $\mathcal{S}(z)^{\circ}$.

We formulate the Definition B.2.3 of a *one-parameter group* on a C*-algebra. This gives the central object for the whole Section B.2.

Definition B.2.3 *Let A be a C*-algebra. Consider a mapping*

$$\alpha : \mathbb{R} \to \mathrm{Aut}(A) : t \mapsto \alpha_t$$

such that

1. *We have $\alpha_s \alpha_t = \alpha_{s+t}$ for all $s, t \in \mathbb{R}$,*
2. *We have $\alpha_0 = \iota$.*
3. *For all $x \in A$, the function $\mathbb{R} \to A : t \mapsto \alpha_t(x)$ is norm continuous.*

Then α is called a norm continuous one-parameter group *on A.*

One-parameter groups on C*-algebras

The main focus of this Section B.2 is on the specific properties of one-parameter groups on C*-algebras. We observe that all results are taken from [60].

We keep the Notation B.2.4 fixed throughout Section B.2.

Notation B.2.4 *We consider a C*-algebra $A \subseteq B(K)$. Further, we let α be a norm continuous one-parameter group on A.*

There is a standard way to define, for every $z \in \mathbb{C}$, a closed densely defined linear operator α_z on A. The related machinery forms an effective tool. It takes shape in the technique of *analytic continuation*.

The next Definition B.2.5 is the basis for this analytic continuation.

Definition B.2.5 *Take $z \in \mathbb{C}$. The mapping $\alpha_z : A \to A$ is defined as follows.*

We use $D(\alpha_z)$ to denote the domain of α_z.

By definition, we have for $a \in A$ that $a \in D(\alpha_z)$ if and only if there exists a function $f : \mathcal{S}(z) \to A$ satisfying

1. *f is continuous on $\mathcal{S}(z)$,*
2. *f is analytic on $\mathcal{S}(z)^\circ$,*
3. *We have $\alpha_t(a) = f(t)$ for every $t \in \mathbb{R}$.*

If such a function f exists, it is unique and we define $\alpha_z(a) = f(z)$.

The Terminology B.2.6 below is standard.

Terminology B.2.6 *Take $x \in A$. We then call x* analytic *with respect to α if we have that $x \in D(\alpha_z)$ for all $z \in \mathbb{C}$.*

The Lemmas B.2.7, B.2.8 and B.2.9 cluster some basic properties of α_z.

Lemma B.2.7 *Let $z \in \mathbb{C}$. The features below hold:*

- *We have that $\alpha_z : D(\alpha_z) \to A$ is a closed linear map,*
- *We have that α_z is injective and $\alpha_z^{-1} = \alpha_{-z}$,*
- *The mapping α_z is densely defined and has dense range.*

Lemma B.2.8 *Take $z \in \mathbb{C}$. Then, the results below hold:*

- *For all $y \in \mathcal{S}(z)$, we have $D(\alpha_z) \subseteq D(\alpha_y)$,*
- *If $y \in \mathbb{C}$ and $\operatorname{Im} y = \operatorname{Im} z$, then $D(\alpha_z) = D(\alpha_y)$.*

Lemma B.2.9 *The following results hold:*

- *For all $t \in \mathbb{R}$ and $z \in \mathbb{C}$, we have $\alpha_z \alpha_t = \alpha_t \alpha_z = \alpha_{z+t}$,*
- *For all $y, z \in \mathbb{C}$, we have $\alpha_y \alpha_z \subseteq \alpha_{y+z}$,*
- *If $y, z \in \mathbb{C}$ and $(\operatorname{Im} y)(\operatorname{Im} z) \geq 0$, then $\alpha_y \alpha_z = \alpha_{y+z}$.*

From [60, Corollary 1.22], we get the useful Proposition B.2.10 below.

Proposition B.2.10 *Take $z \in \mathbb{C}$. Let C be a dense subspace of A such that $C \subseteq D(\alpha_z)$ and $\alpha_t(C) \subseteq C$ for all $t \in \mathbb{R}$.*

Then C is a core for the closed linear mapping α_z.

Extensions to the multiplier algebra

The Lemma 1.5.19 gives that every *-automorphism α_t has a unique extension to a strictly continuous *-automorphisms of $M(A)$. We below describe how to apply the technique of analytic continuation to these extensions.

Besides the basic reference [60], we also mention [20] for details and proofs.

We first state the Lemmas B.2.11 and B.2.12.

Lemma B.2.11 *If $s, t \in \mathbb{R}$ and $a \in M(A)$, then $\alpha_s(\alpha_t(a)) = \alpha_{s+t}(a)$.*

Lemma B.2.12 *Let $a \in M(A)$. We have that the function*

$$\mathbb{R} \to M(A) : t \mapsto \alpha_t(a)$$

is strictly continuous.

The next Definition B.2.13 turns out to be viable and useful. It provides a very natural way to apply the analytic continuation on the level of $M(A)$.

It should be noticed that Lemma B.2.12 guarantees consistency between Definition B.2.13 and Proposition 1.5.12. There is no contradiction.

Definition B.2.13 *Let $z \in \mathbb{C}$. Then, $\bar{\alpha}_z : M(A) \to M(A)$ is defined as follows.*

We use $D(\bar{\alpha}_z)$ to denote the domain of $\bar{\alpha}_z$.

By definition, we have for $a \in M(A)$ that $a \in D(\bar{\alpha}_z)$ if and only if there exists a function $f : \mathcal{S}(z) \to M(A)$ satisfying

1. *f is strictly continuous on $\mathcal{S}(z)$,*
2. *f is analytic on $\mathcal{S}(z)^\circ$,*
3. *We have $\alpha_t(a) = f(t)$ for every $t \in \mathbb{R}$.*

If such a function f exists, it is unique and we define $\bar{\alpha}_z(a) = f(z)$.

The Terminology B.2.14 below is standard.

Terminology B.2.14 *Take $x \in M(A)$. We then call x* strictly analytic *with respect to α if we have that $x \in D(\bar{\alpha}_z)$ for all $z \in \mathbb{C}$.*

It is clear that the next Proposition B.2.15 is very crucial.

Proposition B.2.15 *Let $z \in \mathbb{C}$. Then $\bar{\alpha}_z$ is the strict closure of α_z.*

For all $a \in D(\alpha_z)$, we thus have that $\bar{\alpha}_z(a) = \alpha_z(a)$.

It is sometimes useful to have the next Lemma B.2.16 at hand.

Lemma B.2.16 *Take $z \in \mathbb{C}$ and $a \in A$. Then $a \in D(\alpha_z)$ if and only if we have that $a \in D(\bar{\alpha}_z)$ and $\bar{\alpha}_z(a) \in A$.*

The next Lemma B.2.17 gathers some more simple properties.

Lemma B.2.17 *Let $z \in \mathbb{C}$. The following results hold:*

- *We have that $\bar{\alpha}_z : D(\bar{\alpha}_z) \to M(A)$ is a strictly closed linear map,*
- *We have that $\bar{\alpha}_z$ is injective and $\bar{\alpha}_z^{-1} = \alpha_{-z}$,*
- *The mapping $\bar{\alpha}_z$ has strictly dense domain and strictly dense range.*
- *We have that $D(\alpha_z)$ is a strict core for $\bar{\alpha}_z$.*

The next Notation B.2.18 is introduced for its convenience.

Notation B.2.18 *Let $z \in \mathbb{C}$. We put $\alpha_z(a) = \bar{\alpha}_z(a)$ for all $a \in D(\bar{\alpha}_z)$.*

Implemented one-parameter groups

We now study the special case where all information about the one-parameter group α is contained in a strictly positive operator $T \,\eta\, A$. It is no surprise that this case is well-behaved and thus very handy.

We immediately state the regarding Definition B.2.19.

Definition B.2.19 *Let $T \,\eta\, A$ be a strictly positive operator on K.*

We say that the one-parameter group α is implemented *by T if we have*

$$\alpha_t(x) = T^{-it} x T^{it}$$

for all $x \in A$ and $t \in \mathbb{R}$.

We introduce the Terminology B.2.20. One again should be taking notice of the convention made in the Introduction about products of operators.

Terminology B.2.20 *Let $T_1, T_2 \,\eta\, A$ be closed operators on K.*

Then x is called a middle multiplier *of T_1, T_2 if we have $T_1 x T_2 \in M(A)$.*

The next Proposition B.2.21 gives a very efficient property.

Proposition B.2.21 *Suppose that the one-parameter group α is implemented by a strictly positive operator $T \,\eta\, A$.*

Take $z \in \mathbb{C}$ and $x \in M(A)$. The properties below hold:

- *We have $x \in D(\bar{\alpha}_z)$ if and only if x is a middle multiplier of T^{-iz}, T^{iz} and $T^{-iz}xT^{iz} \in M(A)$,*
- *If $x \in D(\bar{\alpha}_z)$, then we have $\alpha_z(x) = T^{-iz}xT^{iz}$.*

Further, if $x \in A$, also the following holds:

- *We have $x \in D(\alpha_z)$ if and only if x is a middle multiplier of T^{-iz}, T^{iz} and $T^{-iz}xT^{iz} \in A$,*

One-parameter groups on von Neumann algebras

In the thesis, we also make use of strongly continuous one-parameter groups on von Neumann algebras. For a full treatment, we refer to [17, 116].

In the von Neumann algebra case, the same definitions as above are applied. The only thing that changes is that the norm topology is replaced by the strong topology. For this reason, we are very concise in the treatment.

The next Notation B.2.22 is already mentioned in the Introduction.

Notation B.2.22 *Let M be a von Neumann algebra. We then use* $\mathrm{Aut}(M)$ *to denote the set of all normal *-automorphisms on M.*

We always apply the Definition B.2.23 below.

Definition B.2.23 *Let M be a von Neumann algebra. Consider a mapping*

$$\beta : \mathbb{R} \to \mathrm{Aut}(M) : t \mapsto \beta_t$$

such that

1. *We have $\beta_s \beta_t = \beta_{s+t}$ for all $s, t \in \mathbb{R}$,*
2. *We have $\beta_0 = \iota$.*
3. *For all $x \in M$, the function $\mathbb{R} \to M : t \mapsto \beta_t(x)$ is strongly continuous.*

Then β is called a strongly continuous one-parameter group *on M.*

The Notation B.2.24 is used in the rest of the Section B.3.

Notation B.2.24 *We consider a con Neumann algebra M. Further, we let β be a strongly continuous one-parameter group on M.*

The next Definition B.2.25 is the basis for the analytic continuation in the case of strongly continuous one-parameter groups on a von Neumann algebra.

Definition B.2.25 *Let $z \in \mathbb{C}$. The mapping $\beta_z : M \to M$ is defined as follows. We use $D(\beta_z)$ to denote the domain of β_z.*

By definition, we have for $x \in M$ that $x \in D(\beta_z)$ if and only if there exists a function $f : \mathcal{S}(z) \to M$ satisfying

1. *f is strongly continuous on $\mathcal{S}(z)$,*
2. *f is analytic on $\mathcal{S}(z)^\circ$,*
3. *We have $\beta_t(a) = f(t)$ for every $t \in \mathbb{R}$.*

If such a function f exists, it is unique and we define $\beta_z(a) = f(z)$.

The Terminology B.2.26 below is standard.

Terminology B.2.26 *Take $x \in M$. We then call x* analytic *with respect to β if we have that $x \in D(\beta_z)$ for all $z \in \mathbb{C}$.*

The next Lemma B.2.27 describes some basic analytic elements.

Lemma B.2.27 *Take $r > 0$ and $z \in \mathbb{C}$. For every $x \in M$, we consider the element $x(r, z) \in M$ defined by*

$$x(r,z) = \frac{r}{\sqrt{\pi}} \int \exp\big(- r^2(t-z)^2\big)\beta_t(x)\, dt$$

where the integral is taken in the strong topology.

Let $x \in M$. Then the element $x(r, z)$ is analytic with respect for β.

For all $y \in \mathbb{C}$, we have

$$\beta_y\big(x(r,z)\big) = x(r, z+y).$$

Moreover, if $x \in D(\beta_y)$, the next equation holds:

$$\beta_y\big(x(r,z)\big) = x(r, z+y) = \beta_y(x)(r,z).$$

B.3 Functional calculus

In this Section B.3, we collect useful results about the functional calculus for normal operators. We focus on some particular properties that are continually used throughout the thesis; mainly in Chapter 2 and Section 3.2.

We use [59] as the standard reference for all results below. The basic properties of the functional calculus can e.g. also be found in [19, 105, 116].

First, we fix the Notation B.3.1.

Notation B.3.1 *Let $A \subseteq B(K)$ be a non-degenerate C^*-algebra. Further, we consider normal operators $P, Q, T \,\eta\, A$.*

We below apply the Borel calculus of P, Q, T.

For the rest of the Section B.3, we examine the *functional calculus* of the normal operator T. We hereby only focus on the elements $f(T)$ where f is continuous. In this case, we can make full use of the theory of affiliated elements.

The next Theorem B.3.2 is proved in [59, Proposition 6.21]. We remark that it is in [157] that the *-homomorphism ψ_T makes its first appearance. This Theorem B.3.2 is central in the Section B.3.

Theorem B.3.2 *There exists a unique non-degenerate *-homomorphism*

$$\psi_T : \mathrm{C}_0(\sigma(T)) \to M(A)$$

such that $\psi_T(\iota|_{\sigma(T)})$. Moreover, ψ_T is injective.

For every $f \in \mathrm{C}(\sigma(T))$, we then have that $\psi_T(f) = f(T)$.

We describe a technique that makes it possible to consider elements $f(T)$ where the function f is not defined on the whole of $\sigma(T)$. This can sometimes come in very handy. For an elaboration of the details, we refer to [59].

It is needed to first introduce the Terminology B.3.3.

Terminology B.3.3 *Let G be a subset of $\mathbb{C}$. Then G is called* almost closed *if there exists a finite subset G_0 of $\mathbb{C}$ such that $G \cup G_0$ is closed.*

We then call G compatible *with T if G is almost closed and there exists a finite subset D of $\mathbb{C}$ such that*

1. *The element $T - \lambda 1$ is invertible for every $\lambda \in D$,*
2. *$\sigma(T) \setminus D \subseteq G$.*

We can now formulate the Proposition B.3.4 below.

Proposition B.3.4 *Let $G \subseteq \mathbb{C}$ be compatible with T. There is then a unique non-degenerate *-homomorphism $\pi : \mathrm{C}_0(G) \to M(A)$ such that $\pi(\iota_G) = T$.*

We have $\pi(f|_G) = f(T)$ for every $f \in \mathrm{C}(\sigma(T))$.

For every $f \in \mathrm{C}(G)$, we apply the notation $f(T) = \pi(f)$.

The two Lemmas B.3.5 and B.3.6 are quite simple.

Lemma B.3.5 *Let $G_1, G_2 \subseteq \mathbb{C}$ be compatible with T and suppose that $G_1 \subseteq G_2$. We then have for every $f \in \mathrm{C}(G_2)$ that $f(T) = f|_{G_1}(T)$.*

Lemma B.3.6 *We have the following properties:*

- *The sets* $\mathbb{C}$ *and* $\sigma(T)$ *are always compatible with* T,
- *If* T *is invertible, then* $\mathbb{C}_0$ *and* $\sigma(T) \setminus \{0\}$ *are compatible with* T,
- *If* T *is strictly positive, then* $\mathbb{R}_0^+$ *is compatible with* T.

We further mention the following Propositions B.3.7 and B.3.8. They contain results that have a great merit in calculations.

Proposition B.3.7 *Let* $G \subseteq \mathbb{C}$ *be compatible with* T.

Let B *be a* C^**-algebra and* $\pi : A \to M(B)$ *a non-degenerate* **-representation.*

We have $\sigma(\pi(T)) \subseteq \sigma(T)$. *If* π *is injective, then* $\sigma(\pi(T)) = \sigma(T)$.

The properties below hold:

- *We have that* G *is compatible with* $\pi(T)$,
- *For all* $f \in \mathrm{C}(G)$, *we have* $f(\pi(T)) = \pi(f(T))$.

Proposition B.3.8 *Let* $G \subseteq \mathbb{C}$ *be compatible with* T.

Consider $f \in \mathrm{C}(G)$ *and suppose that* $F \subseteq \mathbb{C}$ *satisfies* $f(G) \subseteq F$.

The properties below hold:

- *The set* F *is compatible with* $f(T)$,
- *We have* $(g \circ f)(T) = g(f(T))$ *for every* $g \in \mathrm{C}(F)$.

Further, the spectrum $\sigma(f(T))$ *coincides with the closure of* $f(G \cap \sigma(T))$.

We then also state a version of the Fuglede-Putnam theorem.

Proposition B.3.9 *Let* $u \in B(K)$ *be a unitary operator. Let* G *be a subset of* $\mathbb{C}$ *that is compatible with both* S *and* T.

If $uS \subseteq Tu$, *then we have* $uf(S) = f(T)u$ *for every* $f \in \mathrm{C}(G)$.

The above Proposition B.3.4 is well-suited as a framework to study powers of operators in. We focus on complex powers of strictly positive operators.

We make use of the Notation B.3.10 below.

Notation B.3.10 *Suppose that* T *is strictly positive. Take* $z \in \mathbb{C}$ *and consider* $f \in \mathrm{C}(\mathbb{R}_0^+)$ *defined by* $f(t) = t^z$. *Then, we put* $T^z = f(T)$.

We have that $T^z \,\eta\, A$ *is an invertible normal operator.*

In the following Proposition B.3.11, we rally the most useful calculation rules concerning the powers of strictly positive operators.

We refer to [59, Proposition 7.11] for additional related results.

Proposition B.3.11 *Suppose that T is strictly positive.*

Take $y, z \in \mathbb{C}$, $s \in \mathbb{R}$ and $n \in \mathbb{N}$. Then, the properties below hold:

1. *We have that T^s is a strictly positive operator,*
2. *We have that T^{is} is a unitary operator,*
3. *We have $(T^s)^z = T^{sz}$ and $(T^z)^n = T^{nz}$,*
4. *We have that $(T^z)^* = T^{\overline{z}}$,*
5. *If $0 \leq \operatorname{Im} y \leq \operatorname{Im} z$, then $D(T^z) \subseteq D(T^y)$,*
6. *We have that $T^{z+is} = T^z T^{is} = T^{is} T^z$.*

The functional calculus plays a crucial role in the study of commuting normal operators. We below mention a few basic results about this matter.

First, we give the Definition B.3.12.

Definition B.3.12 *We say that P (strongly) commutes with Q if and only if we have for all $f, g \in \mathrm{C}_0(\mathbb{C})$ that $f(P)g(Q) = g(Q)f(P)$.*

It is not so difficult to prove the Lemma B.3.13 below.

Lemma B.3.13 *Let $G \subseteq \mathbb{C}$ be compatible with both P and Q. Then P commutes with Q if and only if $f(P)g(Q) = g(Q)f(P)$ for all $f, g \in \mathrm{C}_0(G)$.*

Both Proposition B.3.14 and Lemma B.3.14 allude to Lemma B.3.13.

Proposition B.3.14 *Suppose that P and Q commute. Let F, G be subsets of $\mathbb{C}$ such that F is compatible with P and G is compatible with Q.*

The following properties hold:

1. *We have for $f \in \mathrm{C}_b(F)$ and $g \in \mathrm{C}_b(G)$ that $f(P)g(Q) = g(Q)f(P)$,*
2. *We have for $f \in \mathrm{C}_b(F)$ and $g \in \mathrm{C}(G)$ that $f(P)g(Q) \subseteq g(Q)f(P)$,*
3. *We have for $f \in \mathrm{C}(F)$ and $g \in \mathrm{C}_b(G)$ that $g(Q)f(P) \subseteq f(P)g(Q)$.*

Lemma B.3.15 *Let $F \subseteq \mathbb{C}$ be compatible with P. Then P and Q commute if and only if we have that $f(P)Q \subseteq Qf(P)$ for every $f \in \mathrm{C}_0(F)$.*

The three Lemmas B.3.16, B.3.17 and B.3.18 treat a few special cases.

Lemma B.3.16 *We have the following properties:*

- *If $P \in M(A)$, then P and Q commute if and only if $PQ \subseteq QP$,*
- *If $P, Q \in M(A)$, then P and Q commute if and only if $PQ = QP$.*

Lemma B.3.17 *Suppose that P is strictly positive. Then P and Q commute if and only if $P^{it}Q \subseteq QP^{it}$ for all $t \in \mathbb{R}$.*

Lemma B.3.18 *Suppose that P, Q are strictly positive. Then P and Q commute if and only if $P^{is}Q^{it} = Q^{it}P^{is}$ for all $s, t \in \mathbb{R}$.*

We also mention the Lemma B.3.19 below.

Lemma B.3.19 *Suppose that P and Q commute. We then have*

$$P \dotplus Q \; \eta \; A \qquad \text{and} \qquad PQ \; \eta \; A.$$

It is not a big surprise that the functional calculi of two commuting normal operators P and Q leads to a *joint functional calculus.*

The next Proposition B.3.20 displays some basic information about this joint functional calculus. For the details, we can again refer to [59].

Proposition B.3.20 *Suppose that P and Q commute. Let F, G be subsets of $\mathbb{C}$ such that F is compatible with P and G is compatible with Q.*

There exists a unique non-degenerate *-homomorphism

$$\pi : \mathrm{C}(F \times G) \to M(A)$$

such that

- *For all $f \in \mathrm{C}(F)$, we have $\pi(f \otimes 1) = f(P)$,*
- *For all $f \in \mathrm{C}(F)$, we have $\pi(1 \otimes g) = g(Q)$.*

For every $h \in \mathrm{C}(F \times G)$, we use the notation $h(P, Q) = \pi(h)$.

B.4 Two technical lemmas

We here state two lemmas that are used in the thesis. They are well-known and only included for the sake of completeness.

The first Lemma B.4.1 is e.g. proved in [125, Lemma 4.6.10].

Lemma B.4.1 *Let $A \subseteq B(H)$ and $B \subseteq B(K)$ be non-degenerate C^*-algebras. Take $\omega \in A^*_+$ and $x \in M(A \otimes B)$. We then have*

- $(\omega \otimes \iota)(x)^*(\omega \otimes \iota)(x) \leq \|\omega\|(\omega \otimes \iota)(x^*x)$,
- $(\iota \otimes \omega)(x)^*(\iota \otimes \omega)(x) \leq \|\omega\|(\iota \otimes \omega)(x^*x)$.

The next Lemma B.4.2 is e.g. proved in [125, Lemma A.3].

Lemma B.4.2 *Let $A \subseteq B(H)$ and $B \subseteq B(K)$ be non-degenerate C^*-algebras. Take orthonormal bases (e_i) and (f_j) of respectively H and K.*

Let $x, y \in M(A \otimes B)$, $r, s \in H$ and $v, w \in K$.

The properties below hold:

- *We have that the sum*

$$\sum_i (\omega_{s,e_i} \otimes \iota)(y)^* (\omega_{r,e_i} \otimes \iota)(x)$$

  is bounded and converges to $(\omega_{r,s} \otimes \iota)(y^* x)$ in the strict topology,

- *We have that the sum*

$$\sum_i (\iota \otimes \omega_{w,f_j})(y)^* (\iota \otimes \omega_{v,f_j})(x)$$

  is bounded and converges to $(\iota \otimes \omega_{v,w})(y^* x)$ in the strict topology.

Bibliography

[1] S. Baaj, *Représentation régulière du groupe quantique des déplacements de Woronowicz*. Recent advances in operator algebras (Orléans, 1992), Astérisque No. **232** (1995), 11-48.

[2] S. Baaj, *Représentation régulière du groupe quantique $E_\mu(2)$ de Woronowicz*. C.R. Acad. Sci. Paris Sér. I Math. **314** (1992), no. 13, 1021-1026.

[3] S. Baaj and P. Julg, *Théorie bivariante de Kasparov et opérateurs non bornés dans les C^*-modules hilbertiens*. C.R. Acad. Sci. Paris Sér. I Math. **296** (1983), no. 21, 875-878; see also S. Baaj, *Multiplicateurs non bornés*, Thèse 3ème cycle, Université Paris 6 (1980).

[4] S. Baaj and G. Skandalis, *Unitaires multiplicatives et dualité pour les produits croisés de C^*-algèbres*. Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 4, 425-488.

[5] S. Baaj, G. Skandalis and S. Vaes, *Non-semi-regular quantum groups coming from number theory*. Comm. Math. Phys. **235** (2003), no. 1, 139-167.

[6] S. Baaj, G. Skandalis and S. Vaes, *Measurable Kac cohomology for bicrossed products*. Trans. Amer. Math. Soc. **357** (2005), no. 4, 1497-1524.

[7] S. Baaj and S. Vaes, *Double crossed products of locally compact quantum groups*. J. Inst. Math. Jussieu **4** (2005), no. 1, 135-173.

[8] T. Banica, *Le groupe quantique compact libre* $\mathrm{U}(n)$. Comm. Math. Phys. **190** (1997), no. 1, 143-172.

[9] E. Bédos and L. Tuset, *Amenability and co-amenability for locally compact quantum groups*. Internat. J. Math. **14** (2003), no. 8, 865-884.

[10] E. Bédos, R. Conti and L. Tuset, *On amenability and co-amenability of algebraic quantum groups and their corepresentations*. Canad. J. Math., **57** (2005), no. 1, 17-60.

[11] E. Bédos, G.J. Murphy and L. Tuset, *Co-amenability of compact quantum groups*. J. Geom. Phys. **40** (2001), no. 2, 130-153.

[12] E. Blanchard and S. Vaes, *A remark on amenability of discrete quantum groups*. Unpublished notes (2001).

[13] K. Brągiel, *The twisted $SU_\mu(3)$ group. Irreducible *-representations of the C^*-algebra $C(S_\mu U(3))$*. Lett. Math. Phys. **17** (1989), no. 1, 37-44.

[14] K. Brągiel, *The twisted $SU_\mu(n)$ group. On the C^*-algebra $C(S_\mu U(n))$*. Lett. Math. Phys. **20** (1990), no. 3, 251-257.

[15] R.C. Busby, *Double centralizers and extensions of C^*-algebras*. Trans. Amer. Math. Soc. **132** (1968), 79-99.

[16] C.-H. Chu, *Shift automorphism groups of C^*-algebras*. Yokohama Math. J. **32** (1984), no. 1-2, 31-37.

[17] I. Ciorănescu, and L. Zsidó, *Analytic generators for one-parameter groups*. Tôhoku Math. J. (2) **28** (1976), no. 3, 327-362.

[18] F. Combes, *Poids sur une C^*-algèbre*. J. Math. Pures Appl. (9) **47** (1968), 57-100.

[19] J.B. Conway, *A course in functional analysis*. Second edition. Graduate Texts in Mathematics, 96. Springer-Verlag, New York, 1990.

[20] C. D'Antoni and L. Zsidó, *Groups of linear isometries on multiplier C^*-algebras*. Pacific J. Math. **193** (2000), no. 2, 279-306.

[21] J. De Cannière, *Produit croisé d'une algèbre de Kac par un groupe localement compact*. Bull. Soc. Math. France **107** (1979), no. 4, 337-372.

[22] P. Desmedt, *Aspects of the theory of locally compact quantum groups. Amenability – Plancherel measure*. Ph.D.-Thesis KU Leuven (2003).

[23] P. Desmedt, J. Quaegebeur and S. Vaes, *Amenability and the bicrossed product construction*. Illinois J. Math. **46** (2002), no. 4, 1259-1277.

[24] V.G. Drinfel'd, *Hopf algebras and the quantum Yang-Baxter equation*. Dokl. Akad. Nauk SSSR **283** (1985), no. 5, 1060-1064.

[25] E.G. Effros and Z.-J. Ruan, *Discrete quantum groups. I. The Haar measure*. Internat. J. Math. **5** (1994), no. 5, 681-723.

[26] M. Enock and J.-M. Schwartz, *Une dualité dans les algèbres de von Neumann*. C. R. Acad. Sci. Paris Sér. A-B **277** (1973), 683-685.

[27] M. Enock and J.-M. Schwartz, *Une dualité dans les algèbres de von Neumann*. Bull. Soc. Math. France Mém., No. **44** (1975), 1-144.

[28] M. Enock and J.-M. Schwartz, *Algèbres de Kac moyennables*. Pacific J. Math. **125** (1986), no. 2, 363-379.

[29] M. Enock and J.-M. Schwartz, *Kac algebras and duality of locally compact groups.* Springer-Verlag, Berlin, 1992.

[30] M. Enock and L. Vaĭnerman, *Deformation of a Kac algebra by an abelian subgroup.* Comm. Math. Phys. **178** (1996), no. 3, 571-596.

[31] M. Enock and J.-M. Vallin, *C^*-algèbres de Kac et algèbres de Kac.* Proc. London Math. Soc. (3) **66** (1993), no. 3, 619-650.

[32] F.P. Greenleaf, *Invariant means on topological groups and their applications.* Van Nostrand Mathematical Studies, No. 16, Van Nostrand Reinhold Co., New York-Toronto, Ont.-London, 1969.

[33] U. Haagerup, *Normal weights on W^*-algebras*, J. Funct. Anal. **19** (1975), 302-317.

[34] J. Hollevoet, *Lokaal compacte quantum-semigroepen: representaties en Pontryagin-dualiteit.* Ph.D.-thesis KU Leuven (1994).

[35] J. Hollevoet, *Pontryagin duality for a class of locally compact quantum groups.* Math. Nachr. **176** (1995), 93-110.

[36] Z. Hu, *On the set of topologically invariant means on the von Neumann algebra $VN(G)$*, Illinois J. Math., **39** (1995), no. 3, 463-490.

[37] T. Ichinose, *On the spectra of tensor products of linear operators in Banach spaces.* J. Reine Angew. Math. **244** (1970), 119-153.

[38] M. Izumi and H. Kosaki, *Kac algebras arizing from composition of subfactors: general theory and classification.* Mem. Amer. Math. Soc. **158** (2002), no. 750.

[39] A. Jacobs, *Faithful Haar weights and the Baaj-Skandalis condition*, Unpublished notes (2000). `http://www.wis.kuleuven.ac.be/analyse/arnoud/homepage.html`

[40] M. Jimbo, *A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation.* Lett. Math. Phys. **10** (1985), no. 1, 63-69.

[41] G.I. Kac, *Ring groups and the duality principle I & II.* Trudy Moskov. Mat. Obšč. **12** (1963), 259-301; **13** (1965), 84-113.

[42] G.I. Kac, *Group extensions which are ring groups.* Mat. Sb. (N.S.) **76 (118)** (1968), 473-496.

[43] G.I. Kac and V.G. Paljutkin, *An example of a ring group generated by Lie groups.* Ukrain. Mat. Ž. **16** (1964), 99-105.

[44] G.I. Kac and V.G. Paljutkin, *An example of a ring group of order eight.* Soviet Math. Surveys **20**, no. 5 (1965), 268-269.

[45] G.I. Kac and V.G. Paljutkin, *Finite ring groups.* Trudy Moskov. Mat. Obšč. **15** (1966), 224-261.

[46] B.-J. Kahng, *Non-compact quantum groups arising from Heisenberg type Lie bialgebras.* J. Operator Theory **44** (2000), no. 2, 303-334.

[47] B.-J. Kahng, **-representations of a quantum Heisenberg group algebra.* Houston J. Math. **28** (2002), no. 3, 529-552.

[48] B.-J. Kahng, *Haar measure on a locally compact quantum group.* J. Ramanujan Math. Soc. **18** (2003), no. 4, 384-414.

[49] B.-J. Kahng, *Construction of a quantum Heisenberg group.* Preprint University of Kansas (2003). #OA/0307126

[50] B.-J. Kahng, *Quantum double construction in the C^*-algebra setting of certain Heisenberg-type quantum groups.* Preprint University of Nevada (2004). #OA/0405263

[51] B.-J. Kahng, *Quantizations of some Poisson-Lie groups: The bicrossed product construction.* Preprint University of Nevada (2004). #OA/0410029

[52] E. Kirchberg, *Lecture on the conference 'Invariants in operator algebras'*, Copenhagen, August 1992.

[53] E. Koelink, *The quantum group of plane motions and the Hahn-Exton q-Bessel function.* Duke Math. J. **76** (1994), no. 2, 483-508; see also E. Koelink, *On quantum groups and q-special functions.* PhD-thesis, University of Leiden (1991).

[54] E. Koelink and J. Kustermans, *A locally compact quantum group analogue of the normalizer of $SU(1,1)$ in $\mathrm{SL}(2,\mathbb{C})$.* Comm. Math. Phys. **233** (2003), no. 2, 231-296.

[55] E. Koelink and J. Kustermans, *Quantum $\widetilde{SU}(1,1)$ and its Pontryagin dual.* Locally Compact Quantum Groups and Groupoids (Strasbourg, 2002), 49-77, IRMA Lect. Math. Theor. Phys., 2, de Gruyter, Berlin, 2003.

[56] E. Koelink and J. Kustermans, *The dual of the quantum group analogue of the normalizer of $SU(1,1)$ in $\mathrm{SL}(2,\mathbb{C})$ and the decomposition of its regular representation.* Preprint TU Delft and KU Leuven (2003).

[57] J. Kustermans, *C^*-algebraic quantum groups arising from algebraic quantum groups.* Ph.D.-thesis, KU Leuven (1997).

[58] J. Kustermans, *KMS-weights on C^*-algebras.* Preprint Odense Universitet (1997). #funct-an/9704008

[59] J. Kustermans, *The functional calculus of regular operators on Hilbert C^*-modules revisited.* Preprint Odense Universitet (1997). #funct-an/9706007

[60] J. Kustermans, *One-parameter representations on C^*-algebras.* Preprint Odense Universitet (1997). #funct-an/9707009

[61] J. Kustermans, *Locally compact quantum groups in the universal setting.* Internat. J. Math. **12** (2001), no. 3, 289-338

[62] J. Kustermans, *The analytic structure of algebraic quantum groups.* J. Algebra **259** (2003), no. 2, 415-450

[63] J. Kustermans, *Locally compact quantum groups.* School on Quantum Independent Increment Processes (Greifswald, 2003), Lecture notes.

[64] J. Kustermans and S. Vaes, *Weight theory for C^*-algebraic quantum groups.* preprint KU Leuven (1999). #OA/9901063

[65] J. Kustermans and S. Vaes, *A simple definition for locally compact quantum groups.* C.R. Acad. Sci. Paris Sér. I Math. **328** (1999), no. 10, 871-876.

[66] J. Kustermans and S. Vaes, *Locally compact quantum groups.* Ann. Scient École Norm. Sup. (4) **33** (2000), no. 6, 837-934.

[67] J. Kustermans and S. Vaes, *The operator algebra approach to quantum groups.* Proc. Natl. Acad. Sci. USA **97** (2000), no. 2, 547-552.

[68] J. Kustermans and S. Vaes, *Locally compact quantum groups in the von Neumann algebraic setting.* Math. Scand. **92** (2003), no. 1, 68-92.

[69] J. Kustermans, S. Vaes, L. Vaĭnerman, A. Van Daele and S.L. Woronowicz, *Locally compact quantum groups.* Lecture Notes School/Conference on Noncommutative Geometry and Quantum groups, Warsaw, 2001, Banach Centre Publications, to appear.

[70] J. Kustermans and A. Van Daele, *C^*-algebraic quantum groups arising from algebraic quantum groups.* Internat. J. Math. **8** (1997), no. 8, 1067-1139.

[71] E.C. Lance, *An explicit description of the fundamental unitary for $SU(2)_q$.* Comm. Math. Phys. **164** (1994), no. 1, 1-15.

[72] E.C. Lance, *Hilbert C^*-modules. A toolkit for operator algebraists.* London Math. Soc. Lect. Note Series **210**, Cambridge University Press, Cambridge, 1995.

[73] E.C. Lance, *The compact quantum group* $\mathrm{SO}(3)_q$. J. Operator Theory **40** (1998), no. 2, 295-307.

[74] M.B. Landstad, *Quantizations arising from abelian subgroups.* Internat. J. Math. **5** (1994), no. 6, 897-936.

[75] A. Maes, *The multiplicative unitary as a basis for duality. Quantum group frames and their crossed products with groups.* Ph.D.-thesis KU Leuven (2000).

[76] A. Maes and A. Van Daele, *Notes on compact quantum groups.* Nieuw Arch. Wisk. (4) **16** (1998), no. 1-2, 73-112.

[77] A. Maes and A. Van Daele, *The multiplicative unitary as a basis for duality.* preprint KU Leuven (2002). #OA/0205284

[78] S. Majid, *Hopf-von Neumann algebra bicrossproducts, Kac algebra bicrossproducts, and the classical Yang-Baxter equations.* J. Funct. Anal. **95** (1991), no. 2, 291-319.

[79] T. Masuda and Y. Nakagami, *A von Neumann algebra framework for the duality of the quantum groups.* Publ. Res. Inst. Math. Sci. **30** (1994), no. 5, 799-850.

[80] T. Masuda, Y. Nakagami and S.L. Woronowicz, *A C^*-algebraic framework for quantum groups.* Internat. J. Math. **14** (2003), no. 9, 903-1001.

[81] G.J. Murphy, *C^*-algebras and operator theory.* Academic Press, Inc., Boston, MA, 1990.

[82] G.J. Murphy and L. Tuset, *Aspects of compact quantum group theory.* Proc. Amer. Math. Soc. **132** (2004), no. 10, 3055-3067.

[83] G. Nagy, *On the Haar measure of the quantum* $\mathrm{SU}(n)$ *group.* Comm. Math. Phys. **153** (1993), no. 2, 217-228.

[84] Y. Nakagami *Amenability for weighted Hopf C^*-algebras.* A garden of quanta, 463-477, World Sci. Publishing, River Edge, NJ, 2003.

[85] C.-K. Ng, *Amenability of Hopf C^*-algebras.* Operator theoretical methods (Timişoara 1998), 269-284, Theta Found., Bucharest, 2000.

[86] D. Nikshych, *K_0-rings and twisting of finite dimensional Hopf algebras,* Comm. Algebra **26** (1998), no. 1, 321-342; no. 4, p. 1347; no. 6, p. 2010.

[87] A. Pal, *A counterexample on idempotent states on a compact quantum group.* Lett. Math. Phys. **37** (1996), no. 1, 75-77.

[88] A. Pal, *On some quantum groups and their representations.* Ph.D.-thesis, Indian Statistical Institute (1996).

[89] A. Pal, *Haar measure on $E_q(2)$.* Pacific J. Math. **176** (1996), no. 1, 217-233.

[90] A.L.T. Paterson, *Amenability.* Mathematical Surveys and Monographs, 29. American Mathematical Society, Providenc, RI, 1988.

[91] G.K. Pedersen, *Measure theory for C^*-algebras.* Math. Scand. **19** (1966), 131-145.

[92] G.K. Pedersen, *C^*-algebras and their automorphism groups.* London Mathematical Society Monographs, 14. Academic Press, Inc., London-New York, 1979.

[93] G.K. Pedersen and M. Takesaki, *The Radon-Nikodym theorem for von Neumann algebras.* Acta Math. **130** (1973), 53-87.

[94] P. Podleś and S.L. Woronowicz, *Quantum deformation of Lorentz group.* Comm. Math. Phys. **130** (1990), no. 2, 381-431.

[95] W. Pusz, *Irreducible unitary representations of quantum Lorentz group.* Comm. Math. Phys. **152** (1993), no. 3, 591-626.

[96] W. Pusz, *Quantum* $\mathrm{GL}(2,\mathbb{C})$ *group as double group over 'az+b' quantum group.* Rep. Math. Phys. **49** (2002), no. 1, 113-122.

[97] W. Pusz and P. Sołtan, *Functional form of unitary representations of the quantum 'az+b' group.* Rep. Math. Phys. **52** (2003), no. 2, 309-319.

[98] W. Pusz and S.L. Woronowicz, *Unitary representations of quantum Lorentz group.* Noncompact Lie groups and some of their applications (San Antonio, TX, 1993), 453-472, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 429, Kluwer Acad. Publ., Dordrecht, 1994.

[99] W. Pusz and S.L. Woronowicz, *Representations of quantum Lorentz group on Gelfand spaces.* Rev Math. Phys. **12** (2000), no. 12, 1551-1625.

[100] W. Pusz and S.L. Woronowicz, *A quantum* $\mathrm{GL}(2,\mathbb{C})$ *group at roots of unity.* Rep. Math. Phys. **47** (2001), no. 3, 431-462.

[101] W. Pusz and S.L. Woronowicz, *On quantum group of unitary operators. Quantum 'az+b' group.* Preprint University of Warsaw (2003).

[102] W. Pusz and S.L. Woronowicz, *A new quantum deformation of 'ax+b' group.* Preprint University of Warsaw (2004); new version Preprint (2005), to appear in Comm. Math. Phys.

[103] J. Quaegebeur and J. Verding, *A construction for weights on C^*-algebras. Dual weights for C^*-crossed products.* Internat. J. Math. **10** (1999), no. 1, 129-157.

[104] I. Raeburn and D.P. Williams, *Morita equivalence and continuous-trace C^*-algebras.* Math. Surveys and Monographs, 60, Amer. Math. Soc., Providence, RI, 1998.

[105] M. Reed and B. Simon, *Methods of modern mathematical physics. I. Functional analysis.* Second edition. Academic Press, New York, 1980.

[106] M.A. Rieffel, *Some solvable quantum groups.* Operator algebras and topology (Craiova, 1989), 146-159, Pitman Res. Notes Math. Ser., 270, Longman Sci. Tech., Harlow, 1992.

[107] M.A. Rieffel, *Compact quantum groups associated with toral subgroups. Representation theory of groups and algebras* 465-491, Contemp. Math., 145, Amer. Math. Soc., Providence, RI, 1993.

[108] M.A. Rieffel, *Non-compact quantum groups associated with abelian subgroups.* Comm. Math. Phys. **171** (1995), no. 1, 181-201.

[109] Małgorzata Rowicka *Unitary representations of the quantum 'ax+b' group.* Preprint University of Warsaw (2001). #QA/0102151

[110] Małgorzata Rowicka, *Quantum 'az+b' group at roots of unity: unitary representations.* Preprint University of Warsaw (2004). #QA/0108020

[111] Z.-J. Ruan, *Amenability of Hopf von Neumann algebras and Kac algebras.* J. Funct. Anal. **139** (1996), no. 2, 466-499.

[112] P. Sołtan, *A survey of new quantum 'az+b' groups.* Unpublished notes (2003).

[113] P. Sołtan, *New quantum 'az+b' groups.* Rev. Math. Phys. **17** (2005), no. 3, 313-364.

[114] P. Sołtan and S.L. Woronowicz, *A remark on manageable multiplicative unitaries.* Lett. Math. Phys. **57** (2001), no. 3, 239-252.

[115] S. Strătilă, *Modular Theory in Operator Algebras*, Abacus Press, Turnbridge Wells, 1981.

[116] S. Strătilă, L Zsidó, *Lectures on von Neumann algebras.* Abacus Press, Tunbridge Wells, 1979.

[117] W.F. Stinespring, *Integration theorems for gages and duality for unimodular groups.* Trans. Amer. Math. Soc. **90** (1959), 15-56.

[118] I. Szymczak and S. Zakrzewski, *Quantum deformations of the Heisenberg group obtained by geometric quantization.* J. Geom. Phys. **7** (1990), no. 4, 553-569.

[119] M. Takesaki, *Theory of operator algebras. I.* Springer-Verlag, New York-Heidelberg, 1979.

[120] M. Takesaki, *Theory of operator algebras. II.* Springer-Verlag, Berlin, 2003.

[121] M. Takesaki, *Duality and von Neumann algebras.* pp. 665-785, Lectures notes in Math., Vol. 247, Springer, Berlin 1972.

[122] D.C. Taylor, *The strict topology for double centralizer algebras.* Trans. Amer. Math. Soc. **150** (1970), 633-643.

[123] R. Tomatsu, *Amenable discrete quantum groups.* Preprint University of Tokyo (2002). #OA/0302222

[124] S. Vaes *Hopf C^*-algebras.* Master thesis KU Leuven (1998).

[125] S. Vaes, *Locally compact quantum groups.* Ph.D.-thesis, KU Leuven (2001). `http://www.math.jussieu.fr/~vaes/PhD.html`

[126] S. Vaes, *A Radon-Nikodym theorem for von Neumann algebras.* J. Operator Theory **46** (2001), no. 3, 477-489.

[127] S. Vaes, *The unitary implementation of a locally compact quantum group action.* J. Funct. Anal. **180** (2001), no. 2, 426-480.

[128] S. Vaes, *Examples of locally compact quantum groups through the bicrossed product construction.* XIIIth International Congress on Mathematical Physics (London, 2000), 341-348, Int. Press, Boston, MA, 2001.

[129] S. Vaes, *Strictly outer actions of groups and quantum groups.* J. Reine Angew. Math. **578** (2005), 147-184.

[130] S. Vaes, *A new approach to induction and imprimitivity results.* J. Funct. Anal., to appear.

[131] S. Vaes and L. Vaĭnerman, *Extensions of locally compact quantum groups and the bicrossed product construct.* Adv. Math. **175** (2003) no. 1, 1-101.

[132] S. Vaes and L. Vaĭnerman, *On low-dimensional locally compact quantum groups.* Locally Compact Quantum Groups and Groupoids (Strasbourg, 2002), 127-187, IRMA Lect. Math. Theor. Phys., 2, de Gruyter, Berlin, 2003.

[133] S. Vaes and A. Van Daele, *Hopf C^*-algebras.* Proc. London Math. Soc. (3) **82** (2001), no. 2, 337-384.

[134] S. Vaes and A. Van Daele, *The Heisenberg commutation relations, commuting squares and the Haar measure on locally compact quantum group.* Operator algebras and mathematical physics (Constanţa, 2001), 379-400, Theta, Bucharest, 2003.

[135] L. Vaĭnerman, 2*-cocycles and twisting of Kac algebras.* Comm. Math. Phys. **191** (1998), no. 3, 697-721.

[136] L. Vaĭnerman and G.I. Kac, *Nonunimodular ring groups and Hopf-von Neumann algebras.* Dokl. Akad. Nauk SSSR **211** (1973), 1031-1034.

[137] L. Vaĭnerman and G.I. Kac, *Nonunimodular ring groups and Hopf-von Neumann algebras.* Mat. Sb. (N.S.) **94(136)** (1974), 194-225.

[138] L.L. Vaksman and Y.S. Soĭbel'man, *An algebra of functions on the quantum group* $SU(2)$. Funct. Anal. Appl. **22** (1988), no. 3, 170-181.

[139] L. Vanheeswijck, *Duality in the theory of crossed products*. Math. Scand. **44** (1979), no. 2, 313-329.

[140] J.-M. Vallin, *C^*-algèbres de Kac et algèbres de Kac*. Proc. London Math. Soc. (3) **66** (1993), no. 3, 619-650.

[141] A. Van Daele, *Quantum deformation of the Heisenberg group*. Current topics in operator algebras (Nara, 1990), 314-325, World Sci. Publishing, River Edge, NJ, 1991; see also Preprint KU Leuven (1989) (unpublished version, with Haar weights).

[142] A. Van Daele, *The operator* $a \otimes b + b \otimes a^{-1}$ *when* $ab = \lambda ba$. Preprint KU Leuven (1990).

[143] A. Van Daele, *Dual pairs of Hopf *-algebras*. Bull. London Math. Soc. **25** (1993), no. 3, 209-230 (second version); see also Preprint KU Leuven (1990) (first version).

[144] A. Van Daele, *Multiplier Hopf algebras*. Trans. Amer. Math. Soc. **342** (1994), no. 2, 917-932.

[145] A. Van Daele, *The Haar measure on a compact quantum group*. Proc. Amer. Math. Soc. **123** (1995), no. 10, 3125-3128.

[146] A. Van Daele, *Discrete quantum groups*. J. Algebra **180** (1996), no. 2, 431-444.

[147] A. Van Daele, *An algebraic framework for group duality*. Adv. in Math. **140** (1998), no. 2, 323-366.

[148] A. Van Daele, *The Haar measure on some locally compact quantum groups*. Preprint KU Leuven (2001). #OA/0205284

[149] A. Van Daele and S.L. Woronowicz, *Duality for the quantum* $E(2)$ *group*. Pacific J. Math. **173** (1996), no. 2, 375-385.

[150] J. Verding, *Weights on C^*-algebras*. Ph.D.-thesis KU Leuven (1995).

[151] S. Wang, *Classification of quantum groups* $SU(n)$ J. London Math. Soc. (2) **59** (1999), no. 2, 669-680.

[152] N.E. Wegge-Olsen, *K-Theory and C^*-algebras. A friendly approach*. Oxford University Press, New York, 1993.

[153] S.L. Woronowicz, *Twisted* $SU(2)$ *group. An example of a noncommutative differential calculus.*. Publ. Res. Inst. Math. Sci. **23** (1987), no. 1, 117-181.

[154] S.L. Woronowicz, *Compact matrix pseudogroups*. Comm. Math. Phys. **111** (1987), no. 4, 613-665.

[155] S.L. Woronowicz, *A remark on compact matrix quantum groups*. Lett. Math. Phys. **21** (1991), no. 1, 35-39.

[156] S.L. Woronowicz, *Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(n)$ groups.*. Invent. Math. **93** (1988), no. 1, 35-76.

[157] S.L. Woronowicz, *Unbounded elements affiliated with C^*-algebras and non-compact quantum groups*. Comm. Math. Phys. **136** (1991), no. 2, 399-432.

[158] S.L. Woronowicz, *Operator equalities related to quantum $E(2)$ group*. Comm. Math. Phys. **144** (1992), no. 2, 417-428.

[159] S.L. Woronowicz, *Quantum $E(2)$ group and its Pontryagin dual*. Lett. Math. Phys. **23** (1991), no. 4, 251-263.

[160] S.L. Woronowicz, *Quantum $SU(2)$ and $E(2)$ groups. Contraction procedure*. Comm. Math. Phys. **149** (1992), no. 3, 637-652.

[161] S.L. Woronowicz, *C^*-algebras generated by unbounded elements*. Rev. Math. Phys. **7** (1995), no. 3, 481-521.

[162] S.L. Woronowicz, *From multiplicative unitaries to quantum groups*. Int. J. Math. **7** (1996), no. 1, 127-149.

[163] S.L. Woronowicz, *Compact quantum groups*. Symétries quantiques (Les Houches, 1995), 845-884, North-Holland, Amsterdam, 1998.

[164] S.L. Woronowicz, *Quantum exponential function*. Rev. Math. Phys. **12** (2000), no. 6, 873-920.

[165] S.L. Woronowicz, *Quantum 'az+b' group on complex plane*. Internat. J. Math. **12** (4) (2001), no. 4, 461-503.

[166] S.L. Woronowicz, *Haar weight on some quantum groups*. Proceedings of the 24th International Colloquium on Group Theoretical Methods in Physics, Paris, July 2002, Institute of Physics, Conference Series Number 173, 763-772.

[167] S.L. Woronowicz and K. Napiórkowski, *Operator theory in the C^*-algebra framework*. Rep. Math. Phys. **31** (1992), no. 3, 353-371.

[168] S.L. Woronowicz and S. Zakrzewski, *Quantum Lorentz group having Gauss decomposition property*. Publ. Res. Inst. Math. Sci. **28** (1992), no. 5, 809-824.

[169] S.L. Woronowicz and S. Zakrzewski, *Quantum 'ax+b' group*. Rev. Math. Phys. **14** (2002), no. 7-8, 797-828.

Index of notations

General notations

$B(H)$	Bounded operators on a Hilbert space H.
$B_0(H)$	Compact operators on a Hilbert space H.
$\omega_{\xi,\eta}$	Vector functional; $\omega_{\xi,\eta}(x) = \langle x\xi, \eta\rangle$.
ω_ξ	Positive vector functional; $\omega_\xi = \omega_{\xi,\xi}$.
$\mathrm{C}(X)$	$\mathrm{C}(X) = \{f : X \to \mathbb{C} \mid f \text{ is continuous}\}$.
$\mathrm{C}_b(X)$	$\mathrm{C}_b(X) = \{f \in \mathrm{C}(X) \mid f \text{ is bounded}\}$.
$\mathrm{C}_0(X)$	$\mathrm{C}_0(X) = \{f \in \mathrm{C}(X) \mid f \text{ vanishes at infinity}\}$.
$\mathrm{K}(X, A)$	$\mathrm{K}(X, A) = \{f : X \to A \mid f \text{ is continuous and has compact support}\}$.
$\mathrm{K}(X)$	$\mathrm{K}(X) = \{f \in \mathrm{C}(X) \mid f \text{ has compact support}\}$.
$\mathrm{L}^2(X,\nu), \mathrm{L}^2(G), \ell^2(G)$	Hilbert space of square integrable functions; see page 8.
$\mathrm{C}^*_r(G)$	Reduced group C*-algebra; see Example 1.2.4.
χ_Y	Characteristic function of a set Y.
$f\vert_{X_0}$	Restriction of $f \in \mathrm{C}(X)$ to a subset $X_0 \subseteq X$.
$\operatorname{span} X$	Linear span of a subset X in a Banach space.
$[X]$	Norm closed linear span of a subset X in a Banach space.
ι, ι_G	Identity map.
AB	$\operatorname{span}\{xy \mid x \in A, y \in B\}$.
$A \odot B$	Algebraic tensor product of two linear spaces A and B.
A, B, C	Denote C*-algebras (except in Section 2.1).
A^+	Cone of positive elements in a C*-algebra A.
A^*	Continuous linear functionals on a C*-algebra A.
A^*_+	Subset of the positive linear functionals in A^*.
$M(A)$	Multiplier algebra of a C*-algebra A; see Definition 1.5.2.
A^η	Set of elements affiliated to a C*-algebra A; see Notation 1.5.28.
A', A''	Commutant and bicommutant of a C*-algebra A.

$\mathrm{Aut}(A)$	Set of *-automorphisms on a C*-algebra A.
$A \cong B$	A is *-isomorphic with B.
M	Denotes a von Neumann algebra.
M_*	Normal functionals on a von Neumann algebra M.
M_*^+	Subset of positive functionals in M_*.
$\mathrm{Aut}(M)$	Set of normal *-automorphisms on a von Neumann algebra M.
$\bar{\omega}$	$\bar{\omega}(x) = \overline{\omega(x^*)}$.
$a\omega, \omega a$	$a\omega(x) = \omega(xa)$ and $\omega a(x) = \omega(ax)$.
$\omega\rho$	$\omega\rho = (\omega \otimes \rho)\Phi$; see Notation 1.1.3.
$D(T)$	Domain of a linear map T.
$T\|_D$	Restriction of a linear map T to D.
$\sigma(T)$	Spectrum of a linear operator T.
T^*	Adjoint of a linear operator T.
$R \dotplus S$	$R + S$ is densely defined and closable with closure $R \dotplus S$.
RS	The product of R and S is densely defined and closable with closure RS.
$R \otimes S$	Tensor product of two linear operators R and S.
z_T	z-transform of a closed operator T; see Definition 1.5.25.
$T \,\eta\, A$	T is affiliated to a C*-algebra A; see Definition 1.5.27.
$f(T)$	Functional calculus of a normal operator T; See Proposition B.3.4.
T^z	Complex power of a strictly positive operator; see Notation B.3.10.
$\sigma(S,T)$	Joint spectrum of S and T; See page 9.
$h(P,Q)$	Joint functional calculus of S and T; See Proposition B.3.20.
H, K	Denote Hilbert spaces.
$\overline{H}$	Complex conjugate Hilbert space of a Hilbert space H.
$\overline{x} \in \overline{H}$	Element corresponding to $x \in H$.
$m^\top$	Transpose of a linear operator m; see Notation 1.3.6.
$\omega^\top$	Transpose of a continuous linear functional ω; see Notation 1.3.6.
$H \otimes K$	Hilbert space tensor product of two Hilbert spaces H and K.
Σ	Flip map on a tensor product of Hilbert spaces.
$A \otimes B$	Minimal tensor product of two C*-algebras A and B (except in Section 2.1).
$M \bar{\otimes} N$	von Neumann algebraic tensor product of two von Neumann algebras M and N.
$\dot{\sigma}$	Flip map on a tensor product of C*-algebras; $\dot{\sigma}(x) = \Sigma x \Sigma$.
$\omega \otimes \iota, \iota \otimes \omega$	Slice maps on a C*-algebra tensor product.
$\omega \bar{\otimes} \iota, \iota \bar{\otimes} \omega$	Slice maps on a von Neumann algebra tensor product.
$\omega \otimes \rho$	Tensor product functional on a C*-algebra tensor product.
$\omega \bar{\otimes} \rho$	Tensor product functional on a von Neumann algebra tensor product.

φ, ψ	Denote weights.
$(H_\varphi, \pi_\varphi, \Lambda_\varphi)$	GNS-construction of a weight φ; see Proposition B.1.4.
$\mathfrak{M}_\varphi^+$	$\mathfrak{M}_\varphi^+ = \{a \in A^+ \mid \varphi(a) < +\infty\}$; see Notation B.1.2.
$\mathfrak{M}_\varphi$	$\mathfrak{M}_\varphi = \operatorname{span} \mathfrak{M}_\varphi^+$; see Notation B.1.2.
$\mathfrak{N}_\varphi$	$\mathfrak{N}_\varphi = \{a \in A^+ \mid \varphi(a^*a) < +\infty\}$; see Notation B.1.2.
$\mathcal{F}_\varphi$	$\mathcal{F}_\varphi = \big\{ \omega \in A_*^+ \mid \omega(x) \leq \varphi(x) \text{ for } x \in A^+ \big\}$; see Notation B.1.10.
$\mathcal{G}_\varphi$	$\mathcal{G}_\varphi = \big\{ \alpha\omega \mid \omega \in \mathcal{F}_\varphi, \alpha \in]0,1[\big\}$; see Notation B.1.10.
$\varphi_\delta, \psi_\delta$	See Propositions B.1.23 and B.1.32.

Tr	Canonical trace on a Hilbert space H; see Example B.1.33.
Tr_h	See Example B.1.33.

α_z, β_z	Analytic continuations of one-parameter groups; see Definitions B.2.5 and B.2.25.
$\bar{\alpha}_z$	Extension of α_z to the multiplier algebra; see Definitions B.2.13.
$\mathcal{S}(z)$	$\mathcal{S}(z) = \big\{ y \in \mathbb{C} \mid \operatorname{Im} y \in [0, \operatorname{Im} z] \big\}$.
$\mathcal{S}(z)^\circ$	$\mathcal{S}(z)^\circ = \big\{ y \in \mathbb{C} \mid \operatorname{Im} y \in]0, \operatorname{Im} z[\big\}$.

Notations used in the whole thesis

(A, Φ)	Denotes a bi-C*-algebra; see Definition 1.1.1.
$(A, \Phi)^{\mathrm{op}}$	Opposite quantum group of (A, Φ); see Corollary 1.1.12.
$(\mathrm{C}_0(G), \Phi)$	Commutative quantum group; see Notation 1.2.5.
$(\mathrm{C}_r^*(G), \hat{\Phi})$	Cocommutative quantum group; see Notation 1.2.5.

W	Denotes a multiplicative unitary (except in Section 2.1); see Definition 1.3.1.
$\hat{W}$	Dual multiplicative unitary of W; see Proposition 1.3.2.
$\mathcal{C}(W)$	$\mathcal{C}(W) = [\,(\omega \otimes \iota)(\Sigma W) \mid \omega \in B(H)_*]$.

μ	Deformation parameter; $0 < \mu < 1$.

S^1	Unit circle in $\mathbb{C}$; $S^1 = \{z \in \mathbb{C} \mid \lvert z \rvert = 1\}$.
$\mathbb{C}^\mu, \bar{\mathbb{C}}^\mu$	$\mathbb{C}^\mu = \{\mu^k z \mid k \in \mathbb{Z}, z \in S^1\}$ and $\bar{\mathbb{C}}^\mu = \mathbb{C}^\mu \cup \{0\}$.
$\mathbb{R}^\mu, \bar{\mathbb{R}}^\mu$	$\mathbb{R}^\mu = \{\mu^k \mid k \in \mathbb{Z}\}$ and $\bar{\mathbb{R}}^\mu = \mathbb{R}^\mu \cup \{0\}$.
$\mathbb{C}(\mu^{\frac{1}{2}})$	$\mathbb{C}(\mu^{\frac{1}{2}}) = \{\mu^{\frac{1}{2}k} z \mid k \in \mathbb{Z}, z \in S^1\}$.
$\bar{\mathbb{C}}(\mu^{\frac{1}{2}})$	$\bar{\mathbb{C}}(\mu^{\frac{1}{2}}) = \mathbb{C}(\mu^{\frac{1}{2}}) \cup \{0\}$.
$\mathbb{R}(\mu^{\frac{1}{2}})$	$\mathbb{R}(\mu^{\frac{1}{2}}) = \{\mu^{\frac{1}{2}k} \mid k \in \mathbb{Z}\}$.
$\bar{\mathbb{R}}(\mu^{\frac{1}{2}})$	$\bar{\mathbb{R}}(\mu^{\frac{1}{2}}) = \mathbb{R}(\mu^{\frac{1}{2}}) \cup \{0\}$.
E	$E = \big\{(p,q) \in \mathbb{R}(\mu^{\frac{1}{2}}) \times \bar{\mathbb{R}}(\mu^{\frac{1}{2}}) \mid pq \in \bar{\mathbb{R}}^\mu\big\}$; see Notation 2.5.19.

F_μ	$F_\mu : \bar{\mathbb{C}}^\mu \to S^1$; see Definition 2.2.1.
$B(k,n)$	See Definition 2.2.4.
χ	$\chi : \mathbb{R}(\mu^{\frac{1}{2}}) \times S^1 \to S^1$; see Definition 2.2.1.
$\boldsymbol{F}_\mu(\gamma)$	$F_\mu(\gamma ab \otimes cd)$.
$\boldsymbol{\chi}$	$\boldsymbol{\chi} = \chi(a \otimes 1, 1 \otimes c)$.
$f_{k,n}$	$f_{k,n} : \bar{\mathbb{C}}^\mu \to \mathbb{C}$; see Equation (2.15).
$g_{k,n}$	$g_{k,n} : \mathbb{R}(\mu^{\frac{1}{2}}) \to \mathbb{C}$; see Equation (2.18).
$\hat{F}_p$	$\hat{F}_p : \mathbb{C}_0 \to \mathbb{C}$; see Definition A.5.
D_μ	See Definition 2.2.8.

Notations in Sections 1.1 and 3.1

(A, Φ)	Locally compact quantum group; see Notation 1.1.14.
φ	Left Haar weight of (A, Φ); see Notation 1.1.14.
ψ	Right Haar weight of (A, Φ); see Notation 1.1.14.
$(H_\varphi, \pi_\varphi, \Lambda_\varphi)$	GNS-construction of φ; see Notation 1.1.14.
σ	Modular automorphism group of φ; see Notation 1.1.15.
σ'	Modular automorphism group of ψ; see Notation 1.1.15.
W	Left regular representation of (A, Φ); see Proposition 1.1.16.
S	Antipode of (A, Φ); see Proposition 1.1.19.
R	Unitary antipode of (A, Φ); see Proposition 1.1.19.
(τ_t)	Scaling group of (A, Φ); see Proposition 1.1.19.
$\bar{S}, (\bar{\tau}_t)$	Extensions of S and (τ_t) to $M(A)$; see Remark 1.1.20.
ν	Scaling constant of (A, Φ); see Proposition 1.1.23.
δ	Modular element of (A, Φ); see Proposition 1.1.25.
$(\hat{A}, \hat{\Phi})$	Dual quantum group of (A, Φ); see Proposition 1.1.28.
$(\widehat{A, \Phi})$	$(\widehat{A, \Phi}) = (\hat{A}, \hat{\Phi})$; see Notation 1.1.30.
$\hat{\varphi}, \hat{\psi}$	Left and Right Haar weight of $(\hat{A}, \hat{\Phi})$; see Proposition 1.1.28.
$\hat{W}$	Left regular representation of $(\hat{A}, \hat{\Phi})$; see Proposition 1.1.29.
$(\hat{\hat{A}}, \hat{\hat{\Phi}})$	Bidual quantum group of (A, Φ); see Proposition 1.1.33.
$\hat{\hat{\varphi}}, \hat{\hat{\psi}}$	Left and Right Haar weight of $(\hat{\hat{A}}, \hat{\hat{\Phi}})$; see Proposition 1.1.33.
(M, Φ)	von Neumann algebraic quantum group; see Definition 1.1.40 and Notation 3.1.1.
$(\hat{M}, \hat{\Phi})$	von Neumann algebraic quantum group associated to $(\hat{A}, \hat{\Phi})$; see Notation 3.1.1.
$\widetilde{W}$	Left regular representation of (A, Φ); see Notation 3.1.1.

Notations in Sections 1.3 and 1.4

W	Modular multiplicative unitary; see Notation 1.3.15.
(A, Φ)	Bi-C*-algebra associated to W; see Notation 1.3.15.
S	Antipode of (A, Φ); see Notation 1.3.15.
R	Unitary antipode of (A, Φ); see Notation 1.3.15.
(τ_t)	Scaling group of (A, Φ); see Notation 1.3.15.
$\bar{S}, (\bar{\tau}_t)$	Extensions of S and (τ_t) to $M(A)$; see Remark 1.3.13.
$Q, \hat{Q}$	See Definition 1.3.7 and Notation 1.3.15.
$\breve{W}$	See Definition 1.3.7 and Notation 1.3.15.
$I, \hat{I}$	See Definition 1.3.57 and Notation 1.3.61.
$\hat{W}$	Dual multiplicative unitary of W; see Notation 1.3.15.
$(\hat{A}, \hat{\Phi})$	Bi-C*-algebra associated to $\hat{W}$; see Notation 1.3.15.
$\hat{S}$	Antipode of $(\hat{A}, \hat{\Phi})$; see Notation 1.3.15.
$\hat{R}$	Unitary antipode of $(\hat{A}, \hat{\Phi})$; see Notation 1.3.15.
$(\hat{\tau}_t)$	Scaling group of $(\hat{A}, \hat{\Phi})$; see see Notation 1.3.15.
(M, Φ)	See Remark 1.3.14.
$(\hat{M}, \hat{\Phi})$	See Remark 1.3.14.
$\hat{N}$	See Notation 1.4.2.
ψ_1	$\psi_1 = \mathrm{Tr}_{\hat{Q}^2}$; see Definition 1.4.4 and Proposition 1.3.29.
$(H_{\psi_1}, \pi_{\psi_1}, \Lambda_{\psi_1})$	GNS-construction of ψ_1; see Definition 1.4.4.
∇_{ψ_1}	Modular operator of ψ_1; see Definition 1.4.4.
J_{ψ_1}	Modular conjugation of ψ_1; see Definition 1.4.4.
(σ_t)	Modular automorphism group of ψ_1; see Lemma 1.4.8.
$\pi'_{\varphi_1}(x)$	See Definition 1.4.10.
ψ_q	See Definitions 1.4.5 and Lemma 1.4.6.
ψ_∞	See Definition 1.3.47.
φ_q	See Definition 1.4.19.

Notations in Section 2.1

$A_0 \otimes B_0$	Algebraic tensor product of two algebras A_0 and B_0.
$[p, w]!$	See Proposition 2.1.9.
$G_{\mu^2}, G_{\mu^{-2}}$	See Lemma 2.1.10.

$(A, \Delta, \varepsilon, S)$	Hopf *-algebra underlying the quantum $E(2)$ group; see Proposition 2.1.1.
$(B, \hat{\Delta}, \hat{\varepsilon}, \hat{S})$	Hopf *-algebra underlying the quantum $\hat{E}(2)$ group; see Proposition 2.1.1.
$B \bar{\otimes} A$	The completion of $B \otimes A$ with respect to the weak topology induced by $A \otimes B$.
W	Element in $B \bar{\otimes} A$ carrying the duality; see Definition 2.1.6.
c, d	Generators of A; see Proposition 2.1.1.
a, b	Generators of B; see Proposition 2.1.1.
$\mathbf{x}, \mathbf{y}$	$\mathbf{x} = cd$ and $\mathbf{y} = ab$.
$\langle \cdot, \cdot \rangle$	Pairing between (A, Δ) and $(B, \hat{\Delta})$; see Proposition 2.1.4.
$S = R\tau_{-\frac{i}{2}}$	Polar decomposition of S; see Proposition 2.1.15.
$\hat{S} = \hat{R}\hat{\tau}_{-\frac{i}{2}}$	Polar decomposition of $\hat{S}$; see Proposition 2.1.16.

Notations in Chapter 2 (except Section 2.1), Section 3.2 and Chapter 4

H	$H = \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$; see Notation 2.3.1.
$(e_k)_{k \in \mathbb{Z}}$	An orthonormal basis in $\ell^2(\mathbb{Z})$; see Notation 2.3.1.
H_0	See Notation 2.3.1.
$\omega_{k,l,m,n}$	$\omega_{k,l,m,n} = \omega_{e_k \otimes e_l, e_m \otimes e_n}$; see Notation 2.3.2.
$\omega_{k,l}$	$\omega_{k,l} = \omega_{k,l,k,l}$; see Notation 2.3.2.
s, m	$se_k = e_{k+1}$ and $me_k = \mu^k e_k$ for $k \in \mathbb{Z}$; see Definition 2.3.3.
c, d	$c = s \otimes s$ and $d = s \otimes m^{-1}$; see Definition 2.3.3.
a, b	$a = m^{-\frac{1}{2}} \otimes m$ and $b = m^{\frac{1}{2}} \otimes s$; see Definition 2.3.3.
u, v, w	See Proposition 2.3.6 and Notation 2.5.35.
W	$W = F_\mu(ab \otimes cd)\chi(a \otimes 1, 1 \otimes c)$; see Definition 2.3.9.
$V(\gamma)$	$V(\gamma) = F_\mu(\gamma ab \otimes cd)\chi(a \otimes 1, 1 \otimes c)$; see Definition 2.3.20.
N	$N = m^2 \otimes m^{-2}$; see Definition 2.3.25.
$I, \hat{I}$	See Definition 2.3.25.
$\hat{N}, \hat{N}_0$	$\hat{N} = m^{-2} \otimes m^2$ and $\hat{N}_0 = 1 \otimes m^2$; see Definition 2.6.1.
(A, Φ)	The quantum $E(2)$ group; see Definition 2.4.1.
$(\hat{A}, \hat{\Phi})$	The quantum $\hat{E}(2)$ group; see Definition 2.4.2.
$M, \hat{M}$	$M = A''$ and $\hat{M} = \hat{A}''$; see Notation 2.7.1.
R	Unitary antipode of (A, Φ); see Definition 2.6.8.
(τ_t)	Scaling group of (A, Φ); see Definition 2.6.9.
S	Antipode of (A, Φ); see Definition 2.6.10.
$\bar{S}, (\bar{\tau}_t)$	Extensions of S and (τ_t) to $M(A)$; see Lemma 2.6.12.
$\hat{R}$	Unitary antipode of $(\hat{A}, \hat{\Phi})$; see Definition 2.6.33.
$(\hat{\tau}_t)$	Scaling group of $(\hat{A}, \hat{\Phi})$; see Definition 2.6.35.
$\hat{S}$	Antipode of $(\hat{A}, \hat{\Phi})$; see Definition 2.6.38.

S_k	$S_k = \{z \in \bar{\mathbb{C}}^\mu \mid \lvert z\rvert = \mu^k\}$; see Notation 2.6.23.
$h_{k,n}$	$h_{k,n} : S_k \to \mathbb{C}$ and $h_{k,n}(z) = z^n$; see Notation 2.6.23.
$D_{n,1}, D_{n,2}, D_{n,3}, D_n$	See Notation 2.6.23.
$D, \mathbf{D}$	See Notation 2.6.26.
$\hat{S}_k$	$\hat{S}_k = \{z \in \bar{\mathbb{C}}(\mu^{\frac{1}{2}}) \mid \lvert z\rvert = \mu^{\frac{1}{2}k}\}$; see Notation 2.6.52.
$\hat{h}_{k,n}$	$\hat{h}_{k,n} : \hat{S}_k \to \mathbb{C}$ and $\hat{h}_{k,n}(z) = z^n$; see Notation 2.6.52.
$\hat{D}_{n,1}, \hat{D}_{n,2}, \hat{D}_{n,3}, \hat{D}_n$	See Notation 2.6.52.
Tr	Canonical trace on the Hilbert space H; see Notation 2.7.2.
ψ_1	$\psi_1 = \mathrm{Tr}_N$; see Definition 2.7.5.
q	$q \in \hat{M} \cap \hat{M}'$; see Definition 2.7.6.
ψ	Haar weight of (A, Φ) with $\psi(x) = \psi_1(qxq)$; see Definition 2.7.8.
(σ_t)	Modular automorphism group of ψ; see Definition 2.7.15.
$(H_\psi, \pi_\psi, \Lambda_\psi)$	GNS-construction of ψ; see Proposition 2.7.17.
∇_ψ	Modular operator of ψ; see Proposition 2.7.17.
J_ψ	Modular conjugation of ψ; see Proposition 2.7.17.
$\hat{\varphi}$	Left Haar weight of (A, Φ) with $\hat{\varphi} = \mathrm{Tr}_N$; see Definition 2.7.21.
$\hat{\psi}$	Right Haar weight of $(\hat{A}, \hat{\Phi})$ with $\hat{\psi} = \mathrm{Tr}_{\hat{N}_0}$; see Definition 2.7.21.
$(\hat{\sigma}_t)$	Modular automorphism group of $\hat{\varphi}$; see Definition 2.7.29.
$(H_{\hat{\varphi}}, \pi_{\hat{\varphi}}, \Lambda_{\hat{\varphi}})$	GNS-construction of $\hat{\varphi}$; see Proposition 2.7.32.
$\nabla_{\hat{\varphi}}$	Modular operator of $\hat{\varphi}$; see Proposition 2.7.32.
$J_{\hat{\varphi}}$	Modular conjugation of $\hat{\varphi}$; see Proposition 2.7.32.
$(\hat{\sigma}'_t)$	Modular automorphism group of $\hat{\psi}$; see Notation 2.7.29.
$(H_{\hat{\psi}}, \pi_{\hat{\psi}}, \Lambda_{\hat{\psi}})$	GNS-construction of $\hat{\psi}$; see Proposition 2.7.33.
$\nabla_{\hat{\psi}}$	Modular operator of $\hat{\psi}$; see Proposition 2.7.33.
$J_{\hat{\psi}}$	Modular conjugation of $\hat{\psi}$; see Proposition 2.7.33.
δ	Modular element of (A, Φ); see Proposition 2.8.3.
ν	Scaling constant of (A, Φ); see Proposition 2.8.3.
$\hat{\delta}$	Modular element of $(\hat{A}, \hat{\Phi})$; see Proposition 2.8.4.
$\hat{\nu}$	Scaling constant of $(\hat{A}, \hat{\Phi})$; see Proposition 2.8.4.
$\widetilde{W}, \widetilde{V}$	Left and right regular representation of (A, Φ); see Definition 2.8.7.
$X, U_0, \mathcal{U}$	See Definition 2.8.12.
$\boldsymbol{\alpha}, \boldsymbol{\beta}$	See Definition 2.8.13.
$(B, \Phi_B), (\hat{B}, \hat{\Phi}_{\hat{B}})$	See Definition 2.8.15.
$(A_0, \Phi_0), (\hat{A}_0, \hat{\Phi}_0)$	See Definition 2.8.30.

T_0, T	See Definition 2.8.37.
h_t	See Definition 2.8.39.
$\mathcal{N}$	See Definition 2.8.41.
F'_k, F_k	See Notation 2.8.42.
C	See Definition 2.8.45.
γ	$\gamma : \mathcal{N} \to M \bar{\otimes} \mathcal{N}$; see Definition 2.8.49.
(M, Φ)	See Definition 2.9.1.
$(\hat{M}, \hat{\Phi})$	See Definition 2.9.1.
ψ	Haar weight of (M, Φ); See Definition 2.9.2.
$\hat{\varphi}, \hat{\psi}$	Left and right Haar weight of $(\hat{M}, \hat{\Phi})$; See Definition 2.9.5.
$\xi_n, \xi_{n,p,q,r,s}$	See Definitions 3.2.2 and 3.2.46.
$\xi'_n, \xi'_{n,p,q}$	See Definitions 3.2.4 and 3.2.22.
θ, θ'	See Definition 3.2.6.
ε	Bounded counit on (A, Φ); see Definition 3.2.9.
$m, m_{p,q}$	Invariant means on (A, Φ); see Definitions 3.2.16 and 3.2.24.
$\hat{\varepsilon}$	Bounded counit on $(\hat{A}, \hat{\Phi})$; see Definition 3.2.31.
$\hat{m}, \hat{m}_{p,q}$	Invariant means on $(\hat{A}, \hat{\Phi})$; see Definitions 3.2.39 and 3.2.48.
α	$\alpha : \mathbb{Z} \to \mathrm{Aut}(\mathrm{C}_0(\bar{\mathbb{C}}^\mu))$; See Definition 4.1.1.
α''	$\alpha'' : \mathbb{Z} \to \mathrm{Aut}(L^\infty(\bar{\mathbb{C}}^\mu))$; See Definition 4.1.9.
β	$\beta : \mathbb{Z} \to \mathrm{Aut}(\mathrm{C}_0(E))$; See Definition 4.1.12.
β''	$\beta'' : \mathbb{Z} \to \mathrm{Aut}(L^\infty(E))$; See Definition 4.1.20.
$\hat{A}_{\mathrm{ex}}, \hat{A}_{\mathrm{cp}}$	See Definition 2.5.28.
$\tilde{\pi}, \breve{\pi}$	See Definitions 4.1.4 and 4.1.15.
ν_1	See Definition 4.1.7.
$L^\infty(\bar{\mathbb{C}}^\mu)$	$L^\infty(\bar{\mathbb{C}}^\mu) = L^\infty(\bar{\mathbb{C}}^\mu, \nu_1)$; see Notation 4.1.8.
$\hat{\nu}_1$	See Definition 4.1.18.
$L^\infty(E)$	$L^\infty(E) = L^\infty(E, \hat{\nu}_1)$; see Notation 4.1.8.
ω_z	$\omega_z(c) = z$ and $\omega_z(d) = 0$; see Corollary 4.2.4.
π_z	$\pi_z(c) = s^*$ and $\pi_z(d) = zm$; see Proposition 4.3.1.
$\hat{\omega}_k$	$\hat{\omega}_k(c) = \mu^{\frac{1}{2}k}$ and $\hat{\omega}_k(b) = 0$; see Corollary 4.2.11.
$\hat{\pi}_k$	$\hat{\pi}_k(c) = m$ and $\hat{\pi}_k(b) = \mu^k s$; see Proposition 4.3.2.
$\hat{\pi}'_k$	$\hat{\pi}'_k(c) = \mu^{\frac{1}{2}} m$ and $\hat{\pi}'_k(b) = \mu^k s$; see Proposition 4.3.2.

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Nederlandse samenvatting

Inleiding

In deze thesis bestuderen we de *kwantum $E(2)$ groep* van Woronowicz [158, 159] als een lokaal compacte kwantumgroep zoals gedefinieerd door J. Kustermans en S. Vaes [65, 66]. We beschouwen ook de duale kwantum $\hat{E}(2)$ groep. We geven een korte studie van beide niet-compacte kwantumgroepen.

De hedendaagse theorie van lokaal compacte kwantumgroepen (bestudeerd in een operatoralgebraïsch kader) is opgebouwd vanuit de *definitie van Kustermans en Vaes* (zie Definitie N.1.3). In zijn huidige vorm is deze studie vrij uitgewerkt. Dankzij het werk van vele onderzoekers zijn nu heel wat aspecten van de algemene theorie goed begrepen en zijn er toepassingen in verschillende gebieden. We formuleren in Sectie N.1 de basisdefinitie en we geven dan ook een kort overzicht van de belangrijkste resultaten van de algemene theorie.

Het is uiteraard heel belangrijk om voldoende (niet-triviale) voorbeelden van kwantumgroepen te hebben die de algemene theorie illustreren. Met deze thesis leveren we een bijdrage hiertoe doordat we een gedetailleerde studie geven van één dergelijk voorbeeld. We richten onze aandacht op de kwantum $E(2)$ groep. Het construeren van voorbeelden is een volledig aparte tak in de theorie van kwantumgroepen. Bij de constructie van elk niet-triviaal voorbeeld is het nodig om typische technieken te gebruiken. Dit is dan ook de reden dat een hele thesis nodig is voor de volledige uitwerking van één enkel voorbeeld.

Het doel van deze doctoraatsthesis is om al de kenmerken van de kwantum $E(2)$ groep op een accurate manier te beschrijven. In de Sectie N.2 beschrijven we (heel kort) de complete constructie van deze niet-compacte kwantumgroep en de bijbehorende objecten. We richten hierbij onze aandacht ook naar specifieke eigenschappen van de kwantum $E(2)$ groep en zijn duale.

Vooraleer we ons concentreren op de kwantum $E(2)$ groep, betreden we de wereld van de (modulaire) multiplicatieve unitairen. Het al of niet bestaan van *Haargewichten* binnen dit kader is hierbij het belangrijkste aandachtspunt.

Deze studie van multiplicatieve unitairen leidt dan uiteindelijk naar een nieuwe techniek voor het construeren van invariante gewichten.

Enkele beschouwingen vooraf

De twee kwantumgroepen $E(2)$ en $\hat{E}(2)$ zijn het eerst ingevoerd en bestudeerd door S.L. Woronowicz in [158, 159]. Hij construeerde twee paren (A, Φ) en $(\hat{A}, \hat{\Phi})$ bestaande uit een C*-algebra en een covermenigvuldiging. Het beginpunt van zijn constructie was een kwantumdeformatie van de groep van transformaties van het (Euclidische) vlak. De twee beschouwde bi-C*-algebra's hadden genoeg eigenschappen om ze *'lokaal compacte kwantumgroepen'* te noemen.

S. Baaj bestudeerde de kwantum $E(2)$ group verder in [1, 2]. Hij was de eerste die een formule vond voor de Haargewichten. Door geavanceerde technieken te gebruiken, is hij erin geslaagd om links en rechts invariante gewichten te construeren op beide bi-C*-algebra's (A, Φ) en $(\hat{A}, \hat{\Phi})$. Hij bewees bovendien ook een uniciteitseigenschap. Deze resultaten vervolledigden (in zekere) zin de kwantumgroep-structuur van de kwantum $E(2)$ groep.

De resultaten van Woronowicz en Baaj stammen uit 1991 en 1992. In die tijd was er nog geen algemene theorie van kwantumgroepen. Deze kwam er pas in 1999 toen J. Kustermans en S. Vaes de artikels [65, 66, 67, 68] publiceerden. De kwantum $E(2)$ groep is dus een voorbeeld dat er al was voor de theorie.

De laatste jaren is de theorie van kwantumgroepen sterk veranderd. Er zijn tal van nieuwe ontwikkelingen gekomen in zowel de algemene theorie als dat deel van de theorie dat is toegespitst op het construeren van voorbeelden.

Door alle nieuwe resultaten in verband met kwantumgroepen denken we dat het zeker interessant is om het voorbeeld van de kwantum $E(2)$ groep opnieuw te bekijken en aan te passen aan de huidige theorie. We geven in de thesis een kort bewijs van het feit dat de kwantum $E(2)$ groep een lokaal compacte kwantumgroep is volgens de definitie van Kustermans en Vaes.

Het schrijven van deze thesis heeft 3 basisredenen. De eerste reden is dus om een hedendaagse behandeling te geven van de kwantum $E(2)$ groep. Verder bekijken we in deze thesis de Van Daele-techniek [148] om invariantie van Haargewichten aan te tonen. In Sectie N.1 voeren we een veralgemening van deze techniek in die in concrete voorbeelden van kwantumgroepen kan gebruikt worden om Haargewichten te construeren. Tenslotte maken we van de gelegenheid gebruik om de gekende eigenschappen van kwantumgroepen in detail te bekijken in het specifieke geval van de kwantum $E(2)$ groep.

In deze Nederlandse samenvatting spelen (omwille van beknoptheid) vooral de eerste twee van de bovenvermelde redenen een rol. De specifieke eigenschappen van de kwantum $E(2)$ groep krijgen hier heel weinig aandacht.

We merken op dat de eigen inbreng betreffende de kwantum $E(2)$ groep niet is dat we revolutionaire nieuwe dingen bewijzen, maar dat we een directe en complete studie geven van dit belangrijke voorbeeld. We bewijzen enkele nieuwe eigenschappen en vereenvoudigen bewijzen van gekende resultaten.

Het belang van de thesis ligt verder ook in het opbouwen van de techniek om Haargewichten te construeren zoals we dat doen in Sectie N.1.

We beschrijven in deze Nederlandse samenvatting de kwantum $E(2)$ groep op een korte en duidelijke manier en bekijken hierbij ook enkele aspecten van deze niet-compacte kwantumgroep. De details staan enkel in de volledig uitgewerkte studie in het Engelstalige deel van de thesis.

Het is hier dus niet de bedoeling om volledig te zijn en we laten daarom de bewijzen achterwege. We krijgen zo een mooie, beknopte tekst die de meest interessante resultaten van de thesis samenbundelt.

Structuur van de Nederlandse samenvatting

We geven een kort overzicht van de *structuur* en de *inhoud* van de Nederlandse samenvatting. We merken nog eens op dat deze samenvatting zeer beknopt is. Meer informatie kan gevonden worden in de Engelstalige tekst.

De Nederlandse samenvatting bestaat uit 3 secties.

Sectie N.1 - Lokaal compacte kwantumgroepen. We beginnen, in Sectie N.1, met een korte inleiding tot de algemene theorie van *lokaal compacte kwantumgroepen*. We formuleren de definitie van Kustermans en Vaes en we vermelden de standaard-eigenschappen van kwantumgroepen. We beschrijven dan ook de basisvoorbeelden van kwantumgroepen en bespreken heel kort enkele verschillende methoden voor het construeren van andere voorbeelden.

We geven verder een studie van *multiplicatieve unitairen*. Hierin is het vooral de *manageability theorie* van Woronowicz [162] die aan bod komt. We formuleren dan zowel de basisresultaten hieruit als enkele extra eigenschappen.

De volgende stap is dan om deze manageability resultaten te combineren met de Van Daele-techniek [148]. We creëren op die manier een vrij algemeen kader waarin invariantie van gewichten kan bestudeerd worden. We geven een concrete beschrijving van een familie gewichten ψ_q die sterk rechts invariant zijn. Elk van deze gewichten is een goede kandidaat om een Haargewicht te zijn.

Sectie N.2 - De kwantum $E(2)$ groep en zijn duale. Deze Sectie N.2 is geheel gewijd aan de constructie van de kwantum $E(2)$ groep. We behandelen dit op een korte en duidelijke manier. Het belangrijkste resultaat is dat de kwantum $E(2)$ group een lokaal compacte kwantumgroep is in de betekenis van Kustermans en Vaes; zie Stelling N.2.17. We construeren ook de duale kwantum $\hat{E}(2)$ groep. We geven niet enkel de constructie van deze twee voorbeelden, maar we bekijken ook hun typische eigenschappen.

Een heel belangrijk instrument is de *multiplicatieve unitaire* W. We hebben dat W *manageable* is in de betekenis van Woronowicz [162]. Dit geeft ons de mogelijkheid om de constructie van de kwantum $E(2)$ groep (A, Φ) te vereenvoudigen. Uit de manageability theorie vinden we direct de basiseigenschappen van de C*-algebra A en de covermenigvuldiging Φ. We krijgen automatisch ook dezelfde resultaten voor de kwantum $\hat{E}(2)$ groep $(\hat{A}, \hat{\Phi})$.

De volgende stap is de constructie van de *Haargewichten.* Dit is een eenvoudige toepassing van de algemene resultaten uit Sectie N.1. De kwantum $E(2)$ groep is unimodulair; de duale kwantum $\hat{E}(2)$ groep is niet unimodulair.

Als we al deze resultaten combineren, dan vinden we dat zowel de kwantum $E(2)$ groep als zijn duale lokaal compacte kwantumgroepen zijn volgens Definitie N.1.3. Eens we dit belangrijke resultaat hebben, richten we verder onze aandacht op de basiskenmerken van deze twee kwantumgroepen.

Sectie N.3 - Conclusies en open problemen. In Sectie N.3 trekken we de besluiten. We vermelden ook enkele open problemen. Deze zijn eventueel van belang voor toekomstig onderzoek over de kwantum $E(2)$ groep.

N.1 Lokaal compacte kwantumgroepen

In deze Sectie N.1 geven we een korte introductie tot the algemene theorie van lokaal compacte kwantumgroepen. We formuleren de belangrijkste definities en resultaten uit de theorie van Kustermans en Vaes.

We richten onze aandacht ook op de manageability theorie. We bewijzen nieuwe eigenschappen van modulaire multiplicatieve unitairen en we introduceren een nieuwe techniek om Haargewichten te construeren. We gebruiken deze resultaten in Sectie N.2 in onze studie van de kwantum $E(2)$ groep.

Deze inleiding tot de algemene theorie is zeer beknopt. We verwijzen naar de Engelstalige tekst en de artikels [65, 66, 67, 68, 125, 63, 69] voor meer uitgebreide behandelingen van de theorie van lokaal compacte kwantumgroepen.

De definitie van een lokaal compacte kwantumgroep

We bekijken dus vooreerst de algemene theorie van lokaal compacte kwantumgroepen in zijn huidige vorm. Zoals reeds gezegd is deze theorie opgebouwd vanuit de *definitie van Kustermans en Vaes.*

J. Kustermans en S. Vaes formuleerden in 1999 de onderstaande Definitie N.1.3 van een (gereduceerde) C*-algebraïsche kwantumgroep; zie [65, 66]. Ze is nu wijd verspreid en aanvaard als een 'goede' definitie. We verwijzen naar de Engelstalige tekst voor de voorgeschiedenis van deze belangrijke definitie.

De definitie van Kustermans en Vaes is zowel eenvoudig als effectief. Ze vormt een stevige basis voor het onderzoek in de theorie van kwantumgroepen.

We formuleren eerst enkele inleidende definities.

Definitie N.1.1 *Stel A een C^*-algebra en stel $\Phi : A \to M(A \otimes A)$ een niet-gedegenereerd *-homomorfisme. We noemen dan (A, Φ) een* bi-C*-algebra *als Φ* coassociatief *is. Dit wil precies zeggen dat $(\Phi \otimes \iota)\Phi = (\iota \otimes \Phi)\Phi$.*

In dit geval noemen we Φ een covermenigvuldiging *op de C^*-algebra A.*

Definitie N.1.2 *Beschouw een bi-C^*-algebra (A, Φ) en een gewicht φ op A^+.*

- *We noemen φ* links invariant *als $\varphi\big((\omega \otimes \iota)\Phi(a)\big) = \omega(1)\varphi(a)$ voor alle $a \in \mathfrak{M}_\varphi^+$ en $\omega \in A_+^*$. Als deze gelijkheid bovendien geldt voor alle $a \in A^+$ en $\omega \in A_+^*$, dan noemen we φ* sterk *links invariant;*
- *We noemen φ* rechts invariant *als $\varphi\big((\iota \otimes \omega)\Phi(a)\big) = \omega(1)\varphi(a)$ voor alle $a \in \mathfrak{M}_\varphi^+$ en $\omega \in A_+^*$. Als deze gelijkheid bovendien geldt voor alle $a \in A^+$ en $\omega \in A_+^*$, dan noemen we φ* sterk *rechts invariant.*

We formuleren de *definitie van Kustermans en Vaes.* Dit is het beginpunt van de hedendaagse theorie van lokaal compacte kwantumgroepen.

Definitie N.1.3 (Kustermans en Vaes) *Beschouw een C^*-algebra A en een niet-gedegenereerd *-homomorfisme $\Phi : A \to M(A \otimes A)$ zodanig dat*

- $(\Phi \otimes \iota)\Phi = (\iota \otimes \Phi)\Phi$*;*
- $A = [\,(\omega \otimes \iota)\Phi(a) \mid \omega \in A^*, a \in A] = [\,(\iota \otimes \omega)\Phi(a) \mid \omega \in A^*, a \in A]$.

Veronderstel daarenboven het bestaan van

- *een trouw links invariant bijna-KMS gewicht φ op (A, Φ);*
- *een rechts invariant bijna-KMS gewicht ψ op (A, Φ).*

Dan noemen we (A, Φ) een lokaal compacte kwantumgroep.

De Stelling N.1.6 maakt de volgende terminologie mogelijk.

Terminologie N.1.4 *Stel (A, Φ) een lokaal compacte kwantumgroep. Stel dat φ en ψ de gewichten zijn die voorkomen in Definitie N.1.3.*

We noemen φ het linkse Haargewicht *en ψ het* rechtse Haargewicht.

We vermelden de basisresultaten van de algemene theorie. Deze resultaten zijn bewezen door Kustermans en Vaes in [65, 66, 68].

De Stellingen N.1.5 and N.1.6 bevatten zeer belangrijke eigenschappen.

Stelling N.1.5 *Stel (A, Φ) een lokaal compacte kwantumgroep. Stel φ en ψ het linkse en rechtse Haargewicht. We hebben de volgende eigenschappen:*

- *De deelruimten $\Phi(A)(A \otimes 1)$ en $\Phi(A)(1 \otimes A)$ zijn dicht in $A \otimes A$;*
- *Het gewicht φ is een trouw, sterk links invariant KMS-gewicht op (A, Φ);*
- *Het gewicht ψ is een trouw, sterk rechts invariant KMS-gewicht op (A, Φ).*

Stelling N.1.6 *Stel* (A, Φ) *een lokaal compacte kwantumgroep. Stel* φ *en* ψ *het linkse en rechtse Haargewicht. Zij* η *een niet-nul, dicht gedefinieerd, beneden semi-continu gewicht op* A^+. *De volgende eigenschappen gelden:*

- *Als* η *links invariant is, dan bestaat een getal* $r > 0$ *zodat* $\eta = r\varphi$*;*
- *Als* η *rechts invariant is, dan bestaat een getal* $r > 0$ *zodat* $\eta = r\psi$*.*

We beschrijven de multiplicative unitaire W die standaard geassocieerd is aan een kwantumgroep (A, Φ). Dit levert meer inzicht in de algemene theorie.

Eerst leggen we de Notatie N.1.7 vast.

Notatie N.1.7 *Stel* (A, Φ) *een lokaal compacte kwantumgroep. Stel* φ *het linkse Haargewicht en* ψ *het rechtse Haargewicht. De GNS-constructie van* φ *noteren we met* $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$. *Omdat* π_φ *trouw is, kunnen we de* C^**-algebra's* A *en* $\pi_\varphi(A)$ *identificeren. Dit zullen we dan ook doen.*

Het eerste object dat Kustermans en Vaes opbouwen vanuit Definitie N.1.3 is de (links) *reguliere representatie* W. Dit is zonder twijfel één van de meest cruciale objecten die aan (A, Φ) zijn geassocieerd.

Propositie N.1.8 *Er bestaat een unieke unitaire operator* $W \in B(H_\varphi \otimes H_\varphi)$ *zo dat*

$$W^*\big(\Lambda_\varphi(x) \otimes \Lambda_\varphi(y)\big) = (\Lambda_\varphi \otimes \Lambda_\varphi)\big(\Phi(y)(x \otimes 1)\big)$$

voor alle $x, y \in \mathfrak{N}_\varphi$.

De unitaire operator W *is multiplicatief. Dit wil zeggen dat* W *voldoet aan de* pentagonvergelijking

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

Bovendien hebben we dat W *manageable is.*

We noemen W *de* links reguliere representatie *van* (A, Φ).

De volgende Propositie N.1.1 toont het grote belang van W. We zien dat de complete structuur van de bi-C*-algebra (A, Φ) kan beschreven worden met behulp van W. Dit is uiteraard een belangrijke eigenschap.

Het is duidelijk dat de formules in Propositie N.1.1 zeer nuttig zijn.

Proposition N.1.1 *We hebben dat*

$$A = [\,(\iota \otimes \omega)W \mid \omega \in B(H_\varphi)_*].$$

Voor alle $x \in A$ *geldt*

$$\Phi(x) = W^*(1 \otimes x)W.$$

Voorbeelden van kwantumgroepen

We richten onze aandacht dan nu op *voorbeelden* van kwantumgroepen. Het is natuurlijk van een heel groot belang om voldoende (niet-triviale) voorbeelden te hebben van lokaal compacte kwantumgroepen.

De twee basisvoorbeelden van kwantumgroepen zijn de lokaal compacte groepen en hun dualen. Deze voorbeelden zijn zeer goed gekend en uitgewerkt.

Het is vooral S.L. Woronowicz die een heel belangrijke rol heeft gespeeld in de zoektocht naar nieuwe voorbeelden. Hij construeerde in 1987 zijn beroemde en revolutionaire voorbeeld van de kwantum $SU(2)$ groep [153].

S.L. Woronowicz construeerde in 1991 de kwantum $E(2)$ groep. Dit is een zeer verfijnd voorbeeld van een niet-compacte kwantumgroep. We geven in Sectie N.2 een uitgebreide studie van de kwantum $E(2)$ groep en zijn duale.

We bekijken hier enkele algemeenheden in verband met voorbeelden van kwantumgroepen. Deze tak van de kwantumgroepen-theorie bevat enkele specifieke technieken. Het is interessant om daar eens naar te kijken.

Het voorbeeld N.1.9 beschrijft de commutatieve kwantumgroepen.

Voorbeeld N.1.9 *Stel G een lokaal compacte groep. We gebruiken dx om de linkse Haarmaat op G aan te duiden.*

Bekijk de commutatieve C^*-algebra $A = \mathrm{C}_0(G)$. Dan geldt $A \otimes A = \mathrm{C}_0(G \times G)$. We hebben verder dat $M(A) = \mathrm{C}_b(G)$ en $M(A \otimes A) = \mathrm{C}_b(G \times G)$.

Een covermenigvuldiging Φ op $A \otimes A$ kan gedefinieerd worden door

$$\Phi(f)(s,t) = f(st)$$

met hierin $f \in \mathrm{C}_0(G)$ en $s, t \in G$.

Het volgt uit de schrappingswet op G dat de deelruimten

$$\Phi\big(\mathrm{C}_0(G)\big)\big(\mathrm{C}_0(G) \otimes 1\big) \quad \text{en} \quad \Phi\big(\mathrm{C}_0(G)\big)\big(1 \otimes \mathrm{C}_0(G)\big)$$

dicht zijn in $\mathrm{C}_0(G \times G)$.

We kunnen een trouw KMS-gewicht φ op A definiëren door te integreren met betrekking tot de linkse Haarmaat. Dus, voor alle $f \in \mathrm{C}_0(G)$, hebben we

$$\varphi(f) = \int_G f(x)\, dx.$$

We hebben dan dat φ een links Haargewicht is op (A, Φ).

Het is dan verder meteen duidelijk dat we een rechts Haargewicht ψ krijgen door te integreren met betrekking tot de rechtse Haarmaat.

We besluiten dat (A, Φ) een lokaal compacte kwantumgroep is.

We richten nu onze aandacht op niet-commutatieve voorbeelden. We geven een aantal algemeenheden in verband met het construeren van nieuwe voorbeelden. Enkele typische technieken komen hierbij aan bod.

We beperken ons in deze Nederlandse samenvatting tot het bespreken van de zogeheten 'atomische' voorbeelden. Dit zijn voorbeelden die opgebouwd zijn via de 'drie-stappen aanpak' van Woronowicz.

De drie stappen die voorkomen in zo'n constructie zijn:

1. Het Hopf *-algebra-niveau;
2. Het Hilbertruimte-niveau;
3. Het C*-algebra-niveau.

Het Hopf *-algebra-niveau dient voornamelijk als een intuïtie. Het geeft een algebraïsche motivatie voor latere resultaten. Het Hilbertruimte-niveau is er om af te rekenen met de technische details. Tenslotte hebben we dan het C*-algebra-niveau. Dit bevat de meest belangrijke en nuttige resultaten. We bestuderen hierin het voorbeeld als een lokaal compacte kwantumgroep.

Er moeten meestal een heel aantal moeilijkheden overwonnen worden vooraleer we de constructie van een voorbeeld kunnen voltooien. Deze problemen zijn bijna altijd zeer typisch voor het bekeken voorbeeld. Het is daarom nodig om voor elk voorbeeld apart naar mogelijke oplossingen te zoeken.

We hebben gelukkig enkele handige technieken tot onze beschikking. Deze zijn vaak een grote hulp bij het construeren van een voorbeeld. Later in deze Sectie N.1 bestuderen we kort de meest nuttige van deze technieken.

Bij het construeren van een atomische kwantumgroep (A, Φ) is er bijna altijd een cruciale rol weggelegd voor een multiplicative unitaire W die alle informatie bevat over A en Φ. Deze W vormt dan de basis van de constructie.

De opbouw van (A, Φ) gebeurt meestal als volgt:

1. Probeer een 'goede' multiplicatieve unitaire $W \in B(H \otimes H)$ te vinden.
2. Definieer een bi-C*-algebra (A, Φ) door de onderstaande formules:
   - $A = [\,(\omega \otimes \iota)W \mid \omega \in B(H)_*]$,
   - $\Phi(x) = W(x \otimes 1)W^*$ voor $x \in A$.
3. Gebruik de Stelling N.1.20 om de Haargewichten te construeren.

We geven nog een lijst met de meest belangrijke atomische voorbeelden. Bij elk voorbeeld geven we referenties naar uitgebreide behandelingen:

– kwantum $SU(2)$ groep [153, 154],

– kwantum $SU(n)$ groep [156, 83, 76],

- kwantum $E(2)$ groep [158, 159, 149, 1]; zie Sectie N.2,
- kwantum $az+b$-groep [165, 101, 113, 148],
- kwantum $ax+b$-groep [169, 102, 148, 166],
- kwantum $\widetilde{SU}_q(1,1)$-groep [54, 55].

Multiplicatieve unitairen

We geven een systematische studie van *multiplicatieve unitairen*. Het is al eerder opgemerkt in deze Nederlandse samenvatting dat deze operatoren een zeer groot belang hebben. En dus verdienen ze voldoende aandacht.

De basis van de hedendaagse theorie van multiplicatieve unitairen werd gelegd door Baaj en Skandalis in hun artikel [4]. S.L. Woronowicz gaf in 1996 een heel belangrijke bijdrage met zijn *manageability theorie* [162, 114].

We maken in de thesis uitvoerig gebruik van de resultaten omtrent manageable multiplicatieve unitairen. De manageability theorie zal dan ook als een rode draad lopen doorheen het verdere verloop van Sectie N.1.

Eerst bekijken we enkele gekende basiseigenschappen. We vermelden daarna ook een eigen bijdrage. We introduceren hierin een nieuwe techniek die kan gebruikt worden om Haargewichten te construeren; zie Stelling N.1.19.

We starten met de Definitie N.1.10 van een multiplicatieve unitaire.

Definitie N.1.10 *Stel $W \in B(H \otimes H)$ een unitaire operator. We noemen W een* multiplicatieve unitaire *als W voldoet aan de* pentagonvergelijking

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

S.L. Woronowicz definieerde in [162] de notie van *manageabiliteit* voor multiplicatieve unitairen. Later werd dit afgezwakt tot *modulariteit* [114]. We maken in deze Sectie N.1 geen onderscheid tussen beide begrippen.

We formuleren eerst de Definitie N.1.11 van manageabiliteit. We geven niet de versie van Woronowicz, maar een licht aangepaste versie. Het precieze verband met de originele definitie is uitgelegd in Sectie 1.3.

Deze Definitie N.1.11 maakt een iets eenvoudiger formulering mogelijk.

Definitie N.1.11 *Stel $W \in B(H \otimes H)$ een multiplicatieve unitaire. We zeggen dat W* manageable *is als er twee strikt positieve operatoren Q en $\hat{Q}$ op H en twee anti-unitaire operatoren I and $\hat{I}$ op H bestaan zo dat*

1. *We hebben de volgende gelijkheden:*
   (a) $W(\hat{Q} \otimes Q)W^* = \hat{Q} \otimes Q$,
   (b) $(\hat{I} \otimes I)W^*(\hat{I}^* \otimes I^*) = W$.

2. *Voor alle $x, z \in H$ en $u \in D(Q)$, $y \in D(Q^{-1})$ geldt dat*
$$\big\langle W(z \otimes y), x \otimes u \big\rangle = \big\langle W^*(z \otimes IQu), x \otimes IQ^{-1}y \big\rangle.$$

De volgende Stelling N.1.12 bevat de centrale resultaten uit de manageability theorie. Ze werd bewezen door P. Soltan en S.L. Woronowicz [114].

We verwijzen hierbij naar Sectie 1.3 voor een meer uitgebreide versie.

Stelling N.1.12 ([162, 114]) *Stel $W \in B(H \otimes H)$ een manageable multiplicatieve unitaire. We definiëren dan*
$$A = [\,(\omega \otimes \iota)W \mid \omega \in B(H)_*] \quad \textit{en} \quad \hat{A} = [\,(\iota \otimes \omega)W \mid \omega \in B(H)_*].$$

De onderstaande eigenschappen gelden:

1. *We hebben dat A en $\hat{A}$ niet-gedegenereerde C^*-algebra's in $B(H)$ zijn,*
2. *Er geldt $W \in M(B_0(H) \otimes A)$ en $W \in M(\hat{A} \otimes B_0(H))$,*
3. *Er geldt $W \in M(\hat{A} \otimes A)$,*
4. *Er bestaat een niet-gedegenereerd *-homomorfisme $\Phi : A \to M(A \otimes A)$ dat uniek gedefinieerd is door*
$$(\iota \otimes \Phi)W = W_{12}W_{13}.$$
We hebben bovendien de volgende resultaten:
   (a) *Er geldt $\Phi(x) = W(x \otimes 1)W^*$ voor alle $x \in A$,*
   (b) *We hebben dat Φ* coassociatief *is; dus $(\Phi \otimes \iota)\Phi = (\iota \otimes \Phi)\Phi$,*
   (c) *De deelruimten $\Phi(A)(1 \otimes A)$ en $\Phi(A)(A \otimes 1)$ zijn dicht in $A \otimes A$.*

Een formule voor de Haargewichten

Het is meteen duidelijk dat Stelling N.1.12 heel erg belangrijk is. We vinden dat elke manageable multiplicatieve unitaire ons al een eind op weg brengt naar de constructie van een lokaal compacte kwantumgroep.

De Haargewichten zijn de enige objecten die nog ontbreken om een kwantumgroep te krijgen. We hebben een techniek ontwikkelt die helpt bij het zoeken naar deze gewichten. We lichten kort de belangrijkste resultaten toe.

Eerst leggen we de Notatie N.1.13 vast.

Notatie N.1.13 *Stel voor de rest van deze Sectie N.1 dat W een manageable multiplicatieve unitaire is. We gebruiken dan de objecten die geassocieerd zijn aan W zoals ingevoerd in Stelling N.1.12.*

De notaties uit Stelling N.1.12 worden hierbij identiek overgenomen.

We gebruiken ook de volgende Notatie N.1.14

Notatie N.1.14 *Stel H een Hilbertruimte met (e_i) een orthonormale basis.*
We noteren met Tr *het canonieke spoor op $B(H)^+$. Dus voor $x \in B(H)^+$ geldt*

$$\mathrm{Tr}(x) = \sum_i \langle xe_i, e_i \rangle.$$

Stel nu dat h een strikt positieve operator is.
We bedoelen dan met Tr_h *het gewicht op $B(H)^+$ dat informeel gegeven wordt door de formule*

$$\mathrm{Tr}_h(x) = \mathrm{Tr}(h^{\frac{1}{2}} x h^{\frac{1}{2}}).$$

Met de Definities N.1.15 en N.1.16 definiëren we twee families van gewichten. Deze spelen de hoofdrol in de zoektocht naar de Haargewichten.

We herhalen dat we gebruik maken van Notatie N.1.13.

Definitie N.1.15 *Stel $q \in \hat{A}'$. We definiëren dan een beneden semi-continu gewicht ψ_q op A^+ door de formule*

$$\psi_q(x) = \mathrm{Tr}_{\hat{Q}^2}(q^* x q).$$

Definitie N.1.16 *Stel $q \in \hat{A}'$. We definiëren dan een beneden semi-continu gewicht φ_q op A^+ door de formule*

$$\varphi_q(x) = \psi_q(I^* x^* I).$$

We voeren nog de Notatie N.1.17 in.

Notatie N.1.17 *We noteren met ψ_∞ het gewicht op A^+ gedefinieerd door*

$$\psi_\infty(x) = \begin{cases} 0 & \textit{als } x = 0, \\ +\infty & \textit{als } x \neq 0. \end{cases}$$

We formuleren dan de belangrijkste resultaten in verband met de gewichten ψ_q. Deze eigenschappen zijn bewezen in Sectie 1.4.

Het blijkt dat we de gewichten ψ_q kunnen classificeren in drie categorieën.

Propositie N.1.18 *Stel $q \in \hat{A}'$. Enkel de volgende mogelijkheden komen voor:*

1. *We hebben dat $\psi_q = 0$,*
2. *We hebben dat $\psi_q = \psi_\infty$,*
3. *We hebben dat ψ_q een trouw KMS-gewicht is.*

De volgende Stelling N.1.19 geeft de belangrijkste eigenschap van ψ_q.

Stelling N.1.19 *Stel $q \in \hat{A}'$ met $q \neq 0$. We hebben dan dat ψ_q een trouw, beneden semi-continu gewicht is dat sterk rechts invariant is. We hebben dus voor alle $x \in A^+$ en $\omega \in A^*_+$ dat*

$$\psi_q\big((\iota \otimes \omega)\Phi(x)\big) = \omega(1)\psi_q(x).$$

We beschrijven dan hoe de bovenstaande gewichten ψ_q en φ_q kunnen helpen om van (A, Φ) een kwantumgroep te maken.

Stelling N.1.20 *Stel dat elementen $q \in \hat{A}'$ en $x \in A^+$ bestaan zo dat $\psi_q(x) = 1$. Dan is (A, Φ) een lokaal compacte kwantumgroep.*

We hebben dat ψ_q het rechtse Haargewicht is en φ_q het linkse Haargewicht.

N.2 De kwantum $E(2)$ groep en zijn duale

Deze Sectie N.2 is het centrale deel van de Nederlandse samenvatting. Het bevat de details betreffende de constructie van de kwantum $E(2)$ group. We zullen zien in Stelling N.2.17 dat de kwantum $E(2)$ groep inderdaad een lokaal compacte kwantumgroep is zoals gedefinieerd door Kustermans en Vaes.

We bouwen de kwantum $E(2)$ groep op volgens de 'atomische' constructiemethode (cf. Sectie N.1). We volgen hierbij het schema op pagina 290.

We starten dus met een multiplicatieve unitaire W en dan definiëren we een bi-C*-algebra (A, Φ) zoals in Stelling N.1.12. De resultaten uit Sectie N.1 geven dan vrij direct dat (A, Φ) een lokaal compacte kwantumgroep is.

De duale kwantum $\hat{E}(2)$ groep $(\hat{A}, \hat{\Phi})$ wordt analoog geconstrueerd.

Eens de kwantum $E(2)$ groep (A, Φ) en zijn duale zijn opgebouwd, kijken we naar enkele specifieke eigenschappen. We geven bijvoorbeeld de belangrijkste formules in verband met de covermenigvuldigingen en de Haargewichten.

We beperken ons evenwel sterk in het bespreken van de eigenschappen van de kwantum $E(2)$ groep en zijn duale. Er zijn heel wat interessante resultaten die we niet behandelen. We verwijzen hiervoor naar de Engelstalige tekst.

We fixeren een getal $\mu \in \mathbb{R}$ met $0 < \mu < 1$.

Het Hopf *-algebra niveau

We starten de constructie van de kwantum $E(2)$ groep op het *Hopf *-algebra niveau*. Het is al vermeld in Sectie N.1 dat dit niveau puur algebraïsch is en voornamelijk dient als een intuïtie voor latere resultaten.

We beschrijven in Propositie N.2.1 twee Hopf *-algebra's (A, Δ) en $(B, \hat{\Delta})$. Men kan aantonen dat deze duaal zijn ten opzichte van elkaar.

De Hopf *-algebra (A, Δ) vormt de algebraïsche intuïtie die we gebruiken om de kwantum $E(2)$ groep te construeren.

Propositie N.2.1 *Bekijk de unitale *-algebra A die gegenereerd is door een unitair element c en een normaal element d zo dat $cd = \mu dc$.*

De *-algebra A wordt een Hopf *-algebra als we een covermenigvuldiging Δ definiëren door de formules

$$\Delta(c) = c \otimes c \qquad \text{and} \qquad \Delta(d) = c \otimes d + d \otimes c^*.$$

Bekijk ook de unitale *-algebra B die gegenereerd is door een zelf-toegevoegd, inverteerbaar element a en een normaal element b zo dat $ab = \mu ba$.

De *-algebra B wordt een Hopf *-algebra als we een covermenigvuldiging $\hat{\Delta}$ definiëren door de formules

$$\hat{\Delta}(a) = a \otimes a \qquad \text{and} \qquad \hat{\Delta}(b) = a \otimes b + b \otimes a^{-1}.$$

De multiplicatieve unitaire

Het is nu de bedoeling om een C*-algebra-versie op te bouwen van de twee Hopf *-algebra's (A, Δ) en $(B, \hat{\Delta})$. We hebben al opgemerkt in Sectie N.1 dat de theorie van multiplicatieve unitaire heel nuttig is bij een dergelijke constructie.

Het is dan ook helemaal geen verrassing dat het construeren van een multiplicatieve unitaire W het eerste is wat we nu gaan doen. De constructie van W is de belangrijkste stap in de opbouw van de kwantum $E(2)$ groep.

Met behulp van W kunnen we dan een standaard procedure gebruiken om zowel de kwantum $E(2)$ groep (A, Φ) als zijn duale te construeren.

We beschrijven de manier om W te definiëren. Omdat we dit op een correcte manier doen, is het nodig dat we eerst enkele objecten invoeren.

De twee functies F_μ en χ die we definiëren in de Definitie N.2.3 zijn van een groot belang in de constructie van W.

Definitie N.2.2 *We bekijken de verzamelingen*

$$\begin{aligned} \mathbb{C}^\mu &= \{\mu^k z \mid k \in \mathbb{Z}, z \in S^1\}, & \mathbb{C}(\mu^{\frac{1}{2}}) &= \{\mu^{\frac{1}{2}k} z \mid k \in \mathbb{Z}, z \in S^1\}, \\ \mathbb{R}^\mu &= \{\mu^k \mid k \in \mathbb{Z}\}, & \mathbb{R}(\mu^{\frac{1}{2}}) &= \{\mu^{\frac{1}{2}k} \mid k \in \mathbb{Z}\}. \end{aligned}$$

Notaties zoals $\bar{\mathbb{C}}^\mu$ betekenen dat we de sluiting nemen; dus $\bar{\mathbb{C}}^\mu = \mathbb{C}^\mu \cup \{0\}$.

Definitie N.2.3 ([158]) *We definiëren een functie $F_\mu \in \mathrm{C}_b(\bar{\mathbb{C}}^\mu)$ op de volgende manier. Voor alle $z \in \bar{\mathbb{C}}^\mu \setminus \{-\mu^{-2k} \mid k \in \mathbb{N}\}$ stellen we dat*

$$F_\mu(z) = \prod_{k=0}^{\infty} \frac{1 + \mu^{2k}\overline{z}}{1 + \mu^{2k} z}$$

Verder stellen we $F_\mu(z) = -1$ als $z \in \{-\mu^{-2k} \mid k \in \mathbb{N}\}$.

We definiëren een functie $\chi \in \mathrm{C}_b(\mathbb{R}(\mu^{\frac{1}{2}}) \times S^1)$ door te stellen dat

$$\chi(\mu^{\frac{1}{2}k}, z) = z^k$$

als $k \in \mathbb{Z}$ en $z \in S^1$.

De Definitie N.2.5 representeert dan de generatoren a, b, c, d uit Propositie N.2.1 als operatoren op een Hilbertruimte. We merken hierbij op dat we dezelfde notaties gebruiken als in Propositie N.2.1. Omdat we vanaf nu enkel nog op het C*-algebra-niveau werken, zorgt dit niet voor verwarring.

De operatoren a, b, c, d in Definitie N.2.5 komen later terug als de generatoren van de kwantum $E(2)$ groep en zijn duale; zie de resultaten beneden.

Notatie N.2.4 *We bekijken de Hilbertruimtes $\ell^2(\mathbb{Z})$ en $H = \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$.*

Stel dan $\{e_k \mid k \in \mathbb{Z}\}$ een orthonormale basis van $\ell^2(\mathbb{Z})$.

Definitie N.2.5 *We definiëren een begrensde operator s en een gesloten operator m op $\ell^2(\mathbb{Z})$ zoals volgt. Voor alle $k \in \mathbb{Z}$ stellen we*

$$se_k = e_{k+1} \qquad \text{en} \qquad me_k = \mu^k e_k.$$

We hebben dat s een unitaire operator is en dat m een strikt positieve operator is. Het is eenvoudig om te controleren dat

$$ms = \mu sm.$$

We definiëren dan operatoren a, b, c, d op H door

$$\begin{aligned} a &= m^{-\frac{1}{2}} \otimes m, & b &= m^{\frac{1}{2}} \otimes s, \\ c &= s \otimes s, & d &= s \otimes m^{-1}. \end{aligned}$$

Het volgende Lemma N.2.6 geeft een aantal basiseigenschappen.

Lemma N.2.6 *De operator c is unitair en de operator a is strikt positief. De operatoren b en d zijn niet-singulier en normaal.*

De spectra van de operatoren a, b, c, d worden gegeven door

$$\sigma(a) = \bar{\mathbb{R}}(\mu^{\frac{1}{2}}), \qquad \sigma(b) = \bar{\mathbb{C}}(\mu^{\frac{1}{2}}), \qquad \sigma(c) = S^1, \qquad \sigma(d) = \bar{\mathbb{C}}^\mu.$$

We hebben dat

$$cd = \mu dc \qquad en \qquad ab = \mu ba.$$

De laatste commutatieregel betekent hierbij dat $a^{it} b a^{-it} = \mu^{it} b$ voor alle $t \in \mathbb{R}$.

We hebben ook dat a en $|b|$ sterk commuteren.

Er geldt dat $ab \otimes cd$ niet-singulier en normaal is. We hebben

$$\sigma(ab \otimes cd) \subseteq \bar{\mathbb{C}}^{\mu}.$$

We kunnen nu de Definitie N.2.1 van W geven.

Definition N.2.1 ([159]) *We definiëren een unitaire $W \in B(H \otimes H)$ door*

$$W = F_{\mu}(ab \otimes cd)\chi(a \otimes 1, 1 \otimes c).$$

We formuleren dan meteen ook de belangrijke Stelling N.2.7. Dit is één van de centrale eigenschappen in deze Sectie N.2.

Het vergt vrij veel werk om deze Stelling N.2.7 te bewijzen.

Stelling N.2.7 *De unitaire operator W is* multiplicatief. *Dit wil precies zeggen dat W voldoet aan de* pentagonale vergelijking

$$W_{12} W_{13} W_{23} = W_{23} W_{12}.$$

Het belang van W kan eigenlijk niet overschat worden. Het is het belangrijkste instrument in onze constructie van de kwantum $E(2)$ groep.

We bekijken dan ook nog de manageability eigenschappen van W. Deze worden beschreven in Propositie N.2.9 en Stelling N.2.10.

De Definitie N.2.8 voert eerst enkele operatoren in.

Definitie N.2.8 *Noteer $N = m^2 \otimes m^{-2}$. Dan is N een strikt positieve operator op H. We definiëren verder ook twee anti-unitaire operatoren I en $\hat{I}$ op H door te stellen dat*

- $I(e_k \otimes e_l) = (-1)^{k+l}(e_{-k+2l} \otimes e_l)$,
- $\hat{I}(e_k \otimes e_l) = (-1)^{l}(e_k \otimes e_{k-l})$ *en* $\hat{I}^*(e_k \otimes e_l) = (-1)^{k+l}(e_k \otimes e_{k-l})$.

voor alle $k, l \in \mathbb{Z}$.

De onderstaande Propositie N.2.9 geeft aan op welke manier dat N, I en $\hat{I}$ verbonden zijn met W. Het is verder duidelijk dat de Stelling N.2.10 gewoon de meest relevante informatie uit Propositie N.2.9 selecteert.

Propositie N.2.9 *Er geldt dat het vijftal $(W, N^{\frac{1}{2}}, N^{\frac{1}{2}}, I, \hat{I})$ voldoet aan de voorwaarden in de Definitie N.1.11.*

Stelling N.2.10 *De multiplicatieve unitaire W is manageable.*

De definitie van de kwantum $E(2)$ groep

We gebruiken nu de theorie in Sectie N.1 om via W twee bi-C*-algebra's (A, Φ) en $(\hat{A}, \hat{\Phi})$ op te bouwen. Dit leidt tot de Definities N.2.11 en N.2.11.

We geven de definitie van zowel de *kwantum* $E(2)$ *groep* (A, Φ) als zijn duale.

Definitie N.2.11 *We definiëren een bi-C*-algebra* (A, Φ) *door*

- $A = [\,(\omega \otimes \iota)W \mid \omega \in B(H)_*]$,
- $\Phi : A \to M(A \otimes A) : x \mapsto W(x \otimes 1)W^*$.

We noemen (A, Φ) *de* kwantum $E(2)$ groep.

Definitie N.2.12 *We definiëren een bi-C*-algebra* $(\hat{A}, \hat{\Phi})$ *door*

- $\hat{A} = [\,(\iota \otimes \omega)W \mid \omega \in B(H)_*]$,
- $\hat{\Phi} : \hat{A} \to M(\hat{A} \otimes \hat{A}) : y \mapsto W^*(1 \otimes y)W$.

We noemen $(\hat{A}, \hat{\Phi})$ *de* kwantum $\hat{E}(2)$ groep.

Het is dan vrij eenvoudig om te bewijzen dat (A, Φ) en $(\hat{A}, \hat{\Phi})$ lokaal compacte kwantumgroepen zijn volgens de Definitie N.1.3 van Kustermans en Vaes. Dit zal blijken uit de hiernavolgende behandelingen.

We merken hierbij op dat de onderstaande resultaten ook illustreren dat de combinatie van de Stellingen N.1.12 en N.1.20 een krachtig techniek geeft om voorbeelden van kwantumgroepen te construeren.

We bekijken de kwantum $E(2)$ groep (A, Φ) en zijn duale elk apart.

De kwantum $E(2)$ groep

Het is meteen duidelijk uit Stelling N.1.12 dat we enkel nog Haargewichten nodig hebben om te bewijzen dat (A, Φ) een lokaal compacte kwantumgroep is volgens de Definitie N.1.3 van Kustermans en Vaes.

We gebruiken dan de techniek uit Sectie N.1 om een Haargewicht ψ op (A, Φ) te vinden. Het zal blijken dat dit gaat op de vooropgestelde manier.

We beginnen daarom meteen met het construeren van een gewicht ψ dat is van de vorm ψ_q zoals beschreven in de Definitie N.1.15.

Eerst definiëren we een element $q \in \hat{A}'$.

Definitie N.2.13 *Stel* p *de projectie op de deelruimte* $\mathbb{C}e_0$. *We definiëren dan een projectie* $q \in B(H)$ *door de formule* $q = p \otimes 1$.

Lemma N.2.14 *Er geldt dat* $q \in \hat{A}'$.

De volgende Definitie N.2.15 voert het gewicht ψ in.

Definitie N.2.15 *We definiëren een gewicht ψ op A^+ door de formule*

$$\psi(x) = \mathrm{Tr}_N(qxq).$$

Voor alle $x \in A^+$ hebben we dan

$$\psi(x) = \sum_{k\in\mathbb{Z}} \mu^{-2k}\langle x(e_0 \otimes e_k), e_0 \otimes e_k\rangle.$$

Met behulp van Propositie N.1.18 en Stelling N.1.19 is het dan mogelijk om, via een eenvoudige berekening, de volgende Stelling N.2.16 aan te tonen.

Stelling N.2.16 *We hebben dat ψ een trouw KMS-gewicht is op A^+. Er geldt bovendien dat ψ sterk links en sterk rechts invariant is. We hebben dus dat*

$$\psi\big((\omega \otimes \iota)\Phi(x)\big) = \psi\big((\iota \otimes \omega)\Phi(x)\big) = \psi(x).$$

voor alle $x \in A^+$ en $\omega \in A^*_+$.

De Stelling N.2.17 geeft dan de centrale eigenschap. Het volgt uit de combinatie van de twee Stellingen N.1.12 en N.2.16.

Deze Stelling N.2.17 besluit de opbouw van de kwantum $E(2)$ groep.

Stelling N.2.17 *Het paar (A, Φ) is een lokaal compacte kwantumgroep.*

We geven dan enkele belangrijke eigenschappen van (A, Φ). Het is hierbij ook zeker belangrijk om de gelijkenis met Propositie N.2.1 op te merken.

Eerst beschrijven we de C*-algebra A in termen van de generatoren c, d.

We hebben de volgende twee Stellingen N.2.18 en N.2.19.

Stelling N.2.18 *We hebben dat*

$$A = \Big[\sum_{k\in\mathbb{Z}} c^k f_k(d) \;\big|\; f_k \in \mathrm{C}_0(\bar{\mathbb{C}}^\mu) \text{ voor alle } k \text{ en } f_k \neq 0 \text{ voor eindig veel } k\Big].$$

De C-algebra A heeft dus geen eenheid*

Stelling N.2.19 *Er geldt dat $c \in M(A)$ en $d\,\eta\,A$. We hebben bovendien dat de C*-algebra A is gegenereerd door de geaffilieerde elementen $c, d\,\eta\,A$.*

De covermenigvuldiging Φ heeft nu enkele mooie eigenschappen.

Stelling N.2.20 *Er geldt dat Φ het enige niet-gedegenereerde *-homomorfisme van A naar $M(A \otimes A)$ is dat voldoet aan de formules*

$$\Phi(c) = c \otimes c \qquad \text{en} \qquad \Phi(d) = c \otimes d \dotplus d \otimes c^*.$$

We hebben verder dat

$$\Phi(c^*) = c^* \otimes c^*, \qquad \Phi(d^*) = c^* \otimes d^* \dotplus d^* \otimes c,$$
$$\Phi(cd) = c^2 \otimes cd \dotplus cd \otimes 1.$$

Propositie N.2.21 *Stel* $\sum c^k f_k(d) \in A$. *We hebben dan*

$$\Phi\big(\sum c^k f_k(d)\big) = \sum (c^k \otimes c^k) f_k(c \otimes d \dotplus d \otimes c^*).$$

De Propositie N.2.22 geeft dat (A, Φ) voldoet aan een soort *'semigroep-gedrag'*. Dit is een heel opmerkelijke en intrigerende eigenschap.

Propositie N.2.22 ([160]) *Er geldt dat*

$$\Phi(A) \subseteq A \otimes A.$$

Het Haargewicht ψ voldoet aan de formules in Propositie N.2.23.

Propositie N.2.23 *Stel* $x = \sum c^k f_k(d) \in A$. *We hebben dan*

$$\psi(x^* x) = \frac{1}{2\pi} \sum_{k,m} \mu^{-2m} \int_0^{2\pi} |f_k(\mu^{-m} e^{it})|^2 \, dt.$$

De kwantum $\hat{E}(2)$ groep

We bekijken dan de bi-C*-algebra $(\hat{A}, \hat{\Phi})$. We gaan analoog te werk als bij (A, Φ) om aan te tonen dat $(\hat{A}, \hat{\Phi})$ een lokaal compacte kwantumgroep is.

Na de opbouw van de Haargewichten bekijken we enkele specifieke eigenschappen van $(\hat{A}, \hat{\Phi})$. Dit is ook (bijna) analoog als in het geval van (A, Φ), al zijn de formules soms iets minder gestroomlijnd.

We beginnen hier met de Definitie N.2.24 van twee gewichten.

Definitie N.2.24 *We definiëren twee gewichten* $\hat{\varphi}, \hat{\psi}$ *op* $\hat{A}^+$ *door de formules*

$$\hat{\varphi}(x) = \mathrm{Tr}_N(x) \qquad en \qquad \hat{\psi}(x) = \mathrm{Tr}_{\hat{N}_0}(x).$$

Voor alle $x \in \hat{A}^+$ *hebben we dan*

$$\hat{\varphi}(x) = \sum_{k,l \in \mathbb{Z}} \mu^{2(k-l)} \langle x(e_k \otimes e_l), e_k \otimes e_l \rangle,$$
$$\hat{\psi}(x) = \sum_{k,l \in \mathbb{Z}} \mu^{2l} \langle x(e_k \otimes e_l), e_k \otimes e_l \rangle.$$

De volgende Stelling N.2.25 kan bewezen worden door Propositie N.1.18 en Stelling N.1.19 te combineren met Stelling N.1.12.

Stelling N.2.25 *We hebben dat $\hat{\varphi}$ en $\hat{\psi}$ trouwe KMS-gewichten zijn op A^+. Er geldt bovendien dat $\hat{\varphi}$ sterk links invariant is en dat $\hat{\psi}$ sterk rechts invariant is. We hebben dus dat*

$$\hat{\varphi}\big((\omega \otimes \iota)\hat{\Phi}(x)\big) = \omega(1)\hat{\varphi}(x) \quad \textit{en} \quad \hat{\psi}\big((\iota \otimes \omega)\hat{\Phi}(x)\big) = \omega(1)\hat{\psi}(x).$$

voor alle $x \in A^+$ en $\omega \in A^*_+$.

We hebben dan de volgende Stelling N.2.26. Het bewijs wordt gegeven door de resultaten uit de twee Stellingen N.1.12 en N.2.25 te combineren.

Deze Stelling N.2.26 besluit de opbouw van de kwantum $\hat{E}(2)$ groep.

Stelling N.2.26 *Het paar $(\hat{A}, \hat{\Phi})$ is een lokaal compacte kwantumgroep.*

We beschrijven dan de C*-algebra $\hat{A}$ in termen van de generatoren a, b.

Notatie N.2.27 *Noteer $E = \big\{(p,q) \in \mathbb{R}(\mu^{\frac{1}{2}}) \times \bar{\mathbb{R}}(\mu^{\frac{1}{2}}) \mid pq \in \bar{\mathbb{R}}^{\mu}\big\}$.*

Notatie N.2.28 *Stel dat $b = u|b|$ de polaire decompositie van b is.*

Stelling N.2.29 *We hebben dat*

$$\hat{A} = \Big[\sum_{k\in\mathbb{Z}} u^k g_k(a,|b|) \;\Big|\; g_k \in \mathrm{C}_0(E) \textit{ voor elke } k,\; g_k \neq 0 \textit{ voor eindig veel } k$$
$$\textit{en } g_k(s,0) = 0 \textit{ voor alle } s \in \mathbb{R}(\mu^{\frac{1}{2}}) \textit{ als } k \neq 0\Big].$$

De C-algebra $\hat{A}$ heeft dus geen eenheid*

Stelling N.2.30 *Er geldt dat $a, a^{-1}, b \,\eta\, \hat{A}$. We hebben dat de C*-algebra $\hat{A}$ is gegenereerd door de geaffilieerde elementen $a, a^{-1}, b \,\eta\, \hat{A}$.*

De covermenigvuldiging $\hat{\Phi}$ voldoet nu aan de volgende eigenschap.

Stelling N.2.31 *Er geldt dat $\hat{\Phi}$ het enige niet-gedegenereerde *-homomorfisme van $\hat{A}$ naar $M(\hat{A} \otimes \hat{A})$ is dat voldoet aan de formules*

$$\hat{\Phi}(a) = a \otimes a \qquad \textit{en} \qquad \hat{\Phi}(b) = a \otimes b \dotplus b \otimes a^{-1}.$$

We hebben verder dat

$$\hat{\Phi}(a^{-1}) = a^{-1} \otimes a^{-1}, \qquad \hat{\Phi}(b^*) = a \otimes b^* \dotplus b^* \otimes a^{-1},$$
$$\hat{\Phi}(ab) = a^2 \otimes ab \dotplus ab \otimes 1.$$

De Haargewichten $\hat{\varphi}$ an $\hat{\psi}$ voldoen aan de formules in Propositie N.2.32.

Propositie N.2.32 *Stel $x = \sum u^k g_k(a, |b|) \in \hat{A}$. We hebben dan*

$$\hat{\varphi}(x^* x) = \sum_{k,m,n} \mu^{2(m-n)} \left| g_k(\mu^{-\frac{1}{2}m+n}, \mu^{\frac{1}{2}m}) \right|^2,$$
$$\hat{\psi}(x^* x) = \sum_{k,m,n} \mu^{2n} \left| g_k(\mu^{-\frac{1}{2}m+n}, \mu^{\frac{1}{2}m}) \right|^2.$$

N.3 Conclusies en open problemen

In deze laatste Sectie N.3 denken we na over de betekenis van ons werk. Als we naar de thesis kijken, dan lijkt het alsof we hebben gevonden wat we zochten. Een volledige uitwerking van de kwantum $E(2)$ groep.

We merken hierbij op dat de Nederlandse samenvatting slechts een heel beknopte weergave is van de resultaten in de thesis. In de Engelstalige tekst komen veel meer onderwerpen aan bod en geven we bovendien alle details.

We gebruiken deze korte Sectie N.3 ook om de lezer te wijzen op een aantal vragen die interessant lijken om verder over na te denken. We vermelden hierbij enkel die problemen die (op het eerste zicht) oplosbaar lijken.

De thesis omvat twee belangrijke delen. Elk van deze delen bevat een bijdrage tot één specifieke tak van de kwantumgroepen-theorie. De Sectie N.1 behelst een studie van de manageability theorie. We geven verder een complete studie van de kwantum $E(2)$ groep. Dit wordt gedaan in Sectie N.2.

De beide delen vormen elk een afgerond geheel. We hebben geprobeerd om alles zo goed mogelijk te brengen. Maar toch zijn er dus nog enkele open problemen. We vermelden beneden enkele vragen die we zijn tegengekomen. Dit kan een stimulans zijn voor verder onderzoek. De uitkomst van deze vragen is meestal onzeker en zouden eventueel kunnen leiden tot nieuwe inzichten.

We hopen uiteraard dat er in de nabije toekomst antwoorden zullen komen op de vragen die we formuleren in de benedenstaande lijst. We zouden natuurlijk zeer verheugd zijn als enkele van onze resultaten een (kleine) poort zouden openen naar nieuw onderzoek over de kwantum $E(2)$ groep.

Enkele gedachten over de Sectie N.1

We hebben in Sectie N.1 een beknopt overzicht gegeven van de theorie van manageable multiplicatieve unitairen. Zowel gekende als nieuwe resultaten zijn hierbij aan bod gekomen. We hebben in het bijzonder enkele nuttige resultaten geformuleerd in verband met het bestaan van Haargewichten.

We denken dat deze resultaten een vrij goed kader vormen dat nuttig kan zijn bij de constructie van voorbeelden van kwantumgroepen.

De volgende vragen zijn intrigerende open problemen:

- Stel dat W een manageable multiplicatieve unitaire is. Kan W dan steeds gebruikt worden om een lokaal compacte kwantumgroep te construeren? Dus, als (A, Φ) de bi-C*-algebra geassocieerd aan W is, bestaat er dan altijd een (links) Haargewicht ψ op (A, Φ)?
- Een sterkere vraag is of er altijd een Haargewicht ψ op (A, Φ) bestaat dat is van de vorm ψ_q zoals geformuleerd in Definitie N.1.15.

Enkele gedachten over de Sectie N.2

De Sectie N.2 geeft een studie van de kwantum $E(2)$ groep. We gebruiken de techniek die we geïntroduceerd hebben in Sectie N.1 om de kwantum $E(2)$ groep op te bouwen als een lokaal compacte kwantumgroep.

We hebben verder de basiskenmerken van dit voorbeeld bestudeerd met speciale aandacht voor de Haargewichten.

De volgende onderwerpen kwamen op in ons onderzoek:

- We vragen ons af of ook de kwantum $\hat{E}(2)$ groep een soort *semigroep-gedrag* heeft. Dus of er een duale versie is van Propositie N.2.22.
- We maken in de centrale Sectie N.2 gebruik van een deformatieparameter μ met $0 < \mu < 1$. Het is misschien mogelijk om de constructieprocedure van de kwantum $E(2)$ groep (A, Φ) aan te passen op zo een manier dat ook andere waarden voor μ zijn toegelaten.

  Deze vraag kan gezien worden in de lijn van recent onderzoek op bijvoorbeeld de kwantum $az+b$ groep; zie [165, 113].
- Als laatste en onvermijdelijke gedachte is er de open vraag of het mogelijk is om een 'kwantum $E(3)$ groep' te construeren.

  De zoektocht naar een 'kwantum $E(n)$ groep' is nog meer ambitieus.