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A COMPARISON BETWEEN A PRIMAL AND A DUAL
CUTTING PLANE ALGORITHM FOR POSYNOMIAL
GEOMETRIC PROGRAMMING PROBLEMS.

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1. INTRODUCTION

In this paper we present cutting plane algorithms for the solution of posynomial geometric programming problems.

As we will show the deepest dual cut is in essence the Kelley cut [see Ref. 10]. However, some important modifications are required because of two basic problems:

- (i) the nondifferentiability of the objective function at a point where some of the variables are equal to zero, and
- (ii) the fact that each "block of dual variables" must have all components positive or all components equal to zero at optimality.

Both these problems will be more fully discussed in section 2.

Section 3 shall be devoted to the derivation of the dual cutting plane algorithm.

In section 4 we describe the primal cutting plane method, while in section 5 a comparison is made between both methods.

2. POSYNOMIAL GEOMETRIC PROGRAMS

A posynomial geometric program can be defined as :

$$(P) \quad \text{Min } g_0(\underline{t})$$

$$\text{s.t.} \quad g_k(\underline{t}) \leq 1 \quad k = 1, \dots, p$$

$$\underline{t} > 0$$

$$g_k(\underline{t}) = \sum_{i \in J_k} u_{ki} \quad k = 0, 1, \dots, p \quad J_k = \{1, 2, \dots, T_k\}$$

Furthermore it is a property of any finite optimal solution that for a given k , either $x_{ki} = 0 \quad \forall i \in J_k$ or $x_{ki} > 0 \quad \forall i \in J_k$. (see Ref. 7)

The components $G_i(X)$ of the gradient of the objective function can be computed as :

$$G_i(X) = \ln(c_{ki} \lambda_k / x_{ki}) \quad i \in J_k \quad k = 0, 1, \dots, p$$

$$\text{if } x_{ki} > 0 \quad i \in J_k \quad k = 1, \dots, p$$

3. THE DUAL CUTTING PLANE

3.1. The subdifferential

Since cutting plane methods usually make use of the gradient of the objective function, the nondifferentiability of the objective function at points where $x_{ki} = 0$ for some i and k , causes problems.

The way we propose to circumvent this difficulty is to make use of the subdifferential for those gradient components that are not defined.

A vector X^* is said to be a subgradient of a concave function f at a point x if [see Ref. 11] :

$$f(Z) \leq f(x) + X^{*T} (Z-x) \quad \forall Z \in \text{dom } f$$

The set of all subgradients of f at x is called the subdifferential $\partial f(x)$ of f at x .

Suppose $x_{ki} = 0 \quad \forall i \in J_k$, some $k \in \{1, 2, \dots, p\}$

Let us derive the subdifferential of "block" k of $V(x)$ at this point.

x^* is a subgradient if :

$$\sum_{i \in J_k} Z_{ki} \ln(c_{ki} \gamma_k / Z_{ki}) \leq \sum_{i \in J_k} x_i^* Z_{ki} \quad \forall Z_{ki} \geq 0 \quad \gamma_k = \sum_{i \in J_k} Z_{ki} \quad k \in \{1, 2, \dots, p\}$$

$$\text{or} \quad \sum_{i \in J_k} Z_{ki} \ln c_{ki} + \sum_{i \in J_k} Z_{ki} \ln(\gamma_k / Z_{ki}) \leq \sum_{i \in J_k} x_i^* Z_{ki} \quad \forall Z_{ki} \geq 0 \quad (1)$$

$\sum_{i \in J_k} Z_{ki} \ln(\gamma_k / Z_{ki})$ is a concave function in Z_{ki} [see Ref. 7] .

Since a function is concave over some convex region R iff it holds that :

$$f(x) \leq f(x_1) + \nabla f(x_1) (x - x_1) \quad \text{for} \quad \forall x, x_1 \in R,$$

we can write the following :

$$\begin{aligned} \sum_{i \in J_k} Z_{ki} \ln(\gamma_k / Z_{ki}) &\leq \sum_{i \in J_k} \alpha_i \ln\left(\sum_{r \in J_k} \alpha_r / \alpha_i\right) \\ &+ \sum_{i \in J_k} [\ln\left(\sum_{r \in J_k} \alpha_r / \alpha_i\right)] [Z_{ki} - \alpha_i] \quad \text{for all } \alpha > 0 \end{aligned} \quad (2)$$

Define $y_i = \alpha_i / \sum_{r \in J_k} \alpha_r$, then (2) reduces to :

$$\sum_{i \in J_k} Z_{ki} \ln(\gamma_k / Z_{ki}) \leq \sum_{i \in J_k} Z_{ki} \ln\left(\frac{1}{y_i}\right) \quad \forall y_i > 0 \quad \sum_{i \in J_k} y_i = 1$$

The lefthandside of (1) becomes :

$$\sum_{i \in J_k} Z_{ki} \ln c_{ki} + \sum_{i \in J_k} Z_{ki} \ln \left(\frac{y_k}{Z_{ki}} \right) \leq \sum_{i \in J_k} Z_{ki} \ln c_{ki} + \sum_{i \in J_k} Z_{ki} \ln \left(\frac{1}{y_i} \right)$$

$$\text{for all } y_i > 0 \quad \sum_{i \in J_k} y_i = 1.$$

Therefore,

$$x_i^* = \ln \frac{c_{ki}}{y_i} \quad \text{for any } y_i > 0 \quad \sum_{i \in J_k} y_i = 1$$

satisfies equation (1).

$$\Rightarrow \partial f(0) = \{ x^* \mid x_i^* = \ln \frac{c_{ki}}{y_i}, \quad i \in J_k, \quad y_i > 0, \quad \sum_{i \in J_k} y_i = 1 \}$$

3.2. The principle of the convex cutting plane algorithm applied to the dual program

The convex cutting plane algorithm is applicable to any problem of the form

$$(I) \quad \text{Max } f(x)$$

$$\text{s.t. } g_i(x) \leq 0 \quad i = 1, \dots, m$$

with $f(\cdot)$ concave, $g_i(\cdot)$, $i = 1, \dots, m$ convex.

First the problem is transformed into :

$$(II) \quad \text{Max. } Z$$

$$\text{s.t. } Z - f(x) \leq 0$$

$$g_i(x) \leq 0 \quad i = 1, \dots, m$$

Since the constraints of (D) are all linear in the case of geometric programming, (II) becomes :

$$\begin{aligned}
 \text{(III)} \quad & \text{Max } Z \\
 \text{s.t.} \quad & Z - V(x) \leq 0 \\
 & A^T x = 0 \\
 & \sum_{i \in J_0} x_{oi} = 1 \\
 & x \geq 0
 \end{aligned} \tag{3}$$

The algorithm proceeds as follows :

$$\begin{aligned}
 \text{Step}_1 : \quad & \text{Set } k = 1 \\
 & \text{Solve (IV)} \quad \text{Max } Z \\
 & \text{s.t.} \quad A^T x = 0 \\
 & \quad \quad \quad \sum_{i \in J_0} x_{oi} = 1 \\
 & \quad \quad \quad x \geq 0 \\
 & \quad \quad \quad -M \leq Z \leq M
 \end{aligned}$$

Let (Z^1, x^1) be the optimal solution.

Step_2 : If (Z^k, x^k) satisfies constraint (3) within a predefined tolerance, Stop.
 (Z^k, x^k) is optimal. Otherwise go to step 3.

Step_3 : Add a linear constraint to (IV), called a cut, in such a way that (Z^k, x^k) is no longer in the feasible set.

Set $k = k + 1$.

Solve the augmented problem IV.

Let (z^k, x^k) be the solution. Go to step 2.

According to which method the cut in step 2 is selected, a number of different cutting plane algorithms result.

The best known cut for convex nonlinear programming is the one by Kelley [see Ref. 10].

In the next paragraph we shall derive a so-called "deepest" cut. This means a hyperplane is selected that cuts off "as much infeasible points as possible".

3.3. The deepest cutting Plane

Consider the constraint

$$g(\delta) = z - V^1(x) \leq 0 \quad \delta = (z, x) \quad (3)$$

$V^1(x)$ is derived from $V(x)$ by dropping blocks k for which
 $x_{ki} = 0 \quad i \in J_k$

The points satisfying this constraint obviously form a convex set. Therefore this set can be represented as the intersection of all its supporting half-spaces, i.e.

$$(\bar{y} - \delta)^T \nabla g(\delta) \leq 0 \quad \text{for all } \delta$$

such that $g(\delta) = 0$

Consider a point $\bar{y} = (\bar{z}, \bar{x})$ not satisfying (3)

Then we select as cutting plane, the halfspace which is most violated in the sense that

$$(\bar{y} - \delta^*)^T \nabla g(\delta^*) = \begin{cases} \text{Max}_{\delta} (\bar{y} - \delta)^T \nabla g(\delta) \\ \text{S.t. } g(\delta) = 0 \end{cases}$$

In Appendix I we show that the resulting deepest cut is identical to the Kelley cutting plane :

$$Z - \sum_{k=0}^p \sum_{i \in J_k} x_{ki} \ln c_{ki} \frac{\bar{\lambda}_k}{x_{ki}} \leq 0 \quad (4)$$

3.4. Adapting the cutting plane for geometric programming

As we pointed out in section 2, it is a property of the optimal solution that for a given k , either

$$\text{all } x_{ki} = 0 \quad \forall i \in J_k$$

$$\text{or } \text{all } x_{ki} > 0 \quad \forall i \in J_k$$

In order to account for this property, we will add some constraints which will make sure that at every iteration in the algorithm all variables belonging to the same block are either all zero or all positive.

These constraints are :

$$x_{ki} \geq \epsilon \sum_{r \in J_k} x_{kr} \quad i \in J_k \quad k = 1, 2, \dots, p$$

with ϵ a very small positive number.

The second problem we mentioned earlier is the nondifferentiability of the objective function at points where some blocks are equal to zero.

It can be seen that the coefficients of the cutting plane (4) are exactly the gradient components of the objective function evaluated at the point $\bar{y} = (\bar{Z}, \bar{x})$.

We therefore replace the gradient components by the subgradient for those blocks k which are zero.

The cutting plane then becomes :

$$Z - \sum_{k \in P} \sum_{i \in J_k} x_{ki} \ln c_{ki} \frac{\bar{\lambda}_k}{x_{ki}} - \sum_{k \in Z} \sum_{i \in J_k} x_{ki} \ln \frac{c_{ki}}{y_{ki}} \leq 0 \quad (5)$$

for any $y_{ki} > 0$ such that $\sum_{i \in J_k} y_{ki} = 1$

where P is the index set of positive blocks,

Z is the index set of zero blocks.

In order to obtain the deepest cut we have to choose the y_{ki} such that $y_{ki}, i \in J_k, k \in Z$ is the solution to :

$$\text{Max} \sum_{i \in J_k} \ln y_{ki}$$

$$\text{subject to} \quad \sum_{i \in J_k} y_{ki} = 1$$

The optimal solution for this problem is : $y_{ki} = \frac{1}{T_k} \quad \forall i \in J_k$

So the cutting plane eventually becomes :

$$Z - \sum_{k \in P} \sum_{i \in J_k} x_{ki} \ln c_{ki} \frac{\bar{\lambda}_k}{x_{ki}} - \sum_{k \in Z} \sum_{i \in J_k} x_{ki} \ln (c_{ki} T_k) \leq 0$$

It is this cut that is used in the algorithm described in section 3.2.

Since this cutting plane consists of a combination of gradient and subgradient, convergence of the algorithm is assured [see Ref. 1] .

4. THE PRIMAL CUTTING PLANE

4.1. Selecting the deepest cut

Consider again primal program P as defined in section 2 and its equivalent transform P1 :

Program P1

Min to

$$\text{s.t. } g_0^1(t) = t_0^{-1} g_0(t) \leq 1 \quad (6)$$

$$g_k(t) \leq 1 \quad k = 1, \dots, p$$

$$t > 0$$

The primal cutting plane we shall apply is the so called approximating hypersurface, described in Cole et al. [Ref. 6] .

In Appendix II we show that after a logarithmic transformation this hypersurface is equivalent with the cutting plane as constructed in Gochet en Smeers (1974) [Ref. 9] , where it is moreover proven to be parallel but deeper than the usual Kelley cut.

Let us briefly reconsider this approximating hypersurface.

Suppose $t^x = (t_j^x, j = 1, \dots, m)$ does not satisfy all of the constraints (6). Then k is selected as the index of the most violated constraint.

Compute :

$$\bar{u}_k = (\bar{u}_{kj}(t), j \in J_k) = \left[\frac{u_{kj}^x(t)}{\sum_{i \in J_k} u_{ki}^x(t)}, j \in J_k \right]$$

The hypersurface that is considered is of the form :

$$\prod_{i \in J_k} u_{ki}^{\alpha_i} = \beta_k$$

such that :

- (i) the point $\bar{u}_k$ is on the hypersurface ;
- (ii) the hypersurface is tangent to the hyperplane

$$\sum_{r \in J_k} u_{kr} = 1 \text{ at the point } \bar{u}_k$$

It is shown in Cole et al. that, under the normalization $\sum_{i \in J_k} \alpha_i = 1$,

the exponents α_i , $i \in J_k$, and the constant β_k in the hypersurface satisfying conditions (i) and (ii) above, are defined by

$$\alpha_i = \bar{u}_{ki} \quad i \in J_k$$

$$\beta_k = \prod_{i \in J_k} \alpha_i$$

This hypersurface can be seen to be linear in $\ln t$. In figure 1 the approximating surface is illustrated in case of a constraint with two terms.

4.2. The primal algorithm

Step 1

Set $i = 1$

Replace all of the constraints (6) by the following :

$$\prod_{i \in J_k} u_{ki}^{\alpha_i} \leq \beta_k \quad k = 0, 1, \dots, p$$

where $\prod_{i \in J_k} u_{ki}^{\alpha_i} = \beta_k \quad k = 0, 1, \dots, p$ is

a hypersurface at the barycenter of the simplex

$$\{ u_{ki} : i \in J_k \mid \sum_{i \in J_k} u_{ki} = 1, \quad u_{ki} \geq 0 \}$$

Taking logarithms this results in a linear program in $\ln t$, ALP.

Let t^i be the optimal solution.

Go to step 2.

Step 2

If t^i satisfies constraints (6) within a predefined tolerance, stop.
 t^i is optimal.

Otherwise go to step 3.

Step 3

Select k^* , the index of the most violated constraint.

Construct a hypersurface as explained in section 4.1 and add the

$$\text{constraint} \quad \ln \prod_{i \in J_{k^*}} u_{ki}^{\alpha_i} \leq \ln \beta_{k^*}$$

to program ALP. Solve the augmented program.

Set $i = i + 1$.

Let t^i be the solution.

Go to step 2.

5. COMPARISON OF PRIMAL AND DUAL ALGORITHM

Both algorithms were applied to a number of test problems, which are cited in Appendix III together with the solutions obtained.

Table 1 gives a number of characteristics of these problems, while Table 2 constitutes the core of the comparison.

Problem #	# Prim. Var.	# Dual Var.	# Prim. Constr.	# Dual ⁽¹⁾ Constr.
1	3	7	2	4
2	5	9	2	6
3	4	21	4	5
4a b	3	7	2	4
5	7	18	4	8
6a b c	7	18	3	8
7	7	13	1	8
8	7	48	7	8
9	4	8	3	5
10	8	17	7	9

Table 1 : Some characteristics of problems.

(1) Constraints which assure all variables of a block to be positive or zero (see 3.4), not included.

Problem #	# Primal iterations		# Dual iter.	Primal solution		Dual	Resulting	Primal	CPU-time in sec.		Dual
	feas. = 0.01	feas. = 0.0005	feas. = 0.01	feas. = 0.01	feas. = 0.0005	solution	solution	infeas.	feas. = 0.01	-0.0005	
1	8	13	19	90993.971	91246.018	91.620	91.620	feasible	0.32	0.36	0.68
2	26	37	2	36.892	37.216	37.224	-	-	1.06	1.89	0.12
3	16	24	149 ^c	265.237	267.729	274.570	-	-	0.56	0.79	110.59
4a	6	9	24	278.518	280.623	281.062	-	0.005	0.26	0.29	0.93
b	9	15	15	245.073	246.109	246.634	246.634	feasible	0.35	0.45	0.49
5	24	24 ^d 30 ^e	58	1776.889	1800.735	1810.487	1810.421	6.42	1.04	1.71	24.22
6a	24	33	130	154.261	155.728	156.447	180.224	feasible	1.37	2.15	79.92
b	21	32	86	157.837	159.259	159.703	164.419	feasible	1.20	2.07	34.52
c	21	31	{ 88 88 95	190.670	191.837	{ 192.761 192.761 192.339	{ 199.458 202.870 13824.725	{ feasible feasible feasible	1.28	2.25	{ 39.64 39.31 48.06
7	39	65	44	133.962	136.396	137.442	137.441	feasible	3.45	11.36	6.67
8a	39	62	g	177.440	178.425				5.07	15.71	
b	36	53	g	183.224	185.825				4.53	16.78	
9	6	8	34	11.903	11.946	11.956	11.956	0.008	0.28	0.30	1.91
10	32	53	20	548.050	550.642	551.891	551.890	0.003	2.24	7.66	2.47

Table 2 : Main results of conducted comparison.

c. failed to reach feasibility degree

d. feasible within 0.008

e. feasible within 0.005

f. different primal solutions due to different bases

g. failed due to LP-code

h. CPU-time on IBM 3033

A few comments might be useful for understanding Table 2.

Columns 2 and 3 give the number of iterations required by the primal algorithm to reach a degree of feasibility of respectively 0.01 and 0.0005. By degree of feasibility we mean the maximal difference between the value of the left-handside of the most violated constraint and one.

Columns 5 and 6 denote the objective function value of the primal algorithm for both degrees of feasibility.

Columns 8 and 9 give the objective function value and the maximal infeasibility of the primal program for the dual solution after conversion from dual to primal.

As main result we can state that the performance of both algorithms is about equivalent for small problems with one or two constraints.

For larger problems however the primal method clearly outperforms the dual. Overall the dual was only better in three out of the 13 problems solved.

Table 3 shows the primal algorithm even bears comparison with first order (gradient) methods specially adapted for solving the dual by Beck and Ecker [Ref. 4] and by Ecker, Gochat and Smeers [Ref. 8].

More important perhaps is the robustness of the codes.

While the primal method only failed in one occasion near the optimum due to parallel cuts (problem 5), the dual gave rise to a number of problems.

Problem #	Number of iter. feas. = 0.0005	Beck & Ecker feas. = 0.001	Gochat & Smeers feas. = 0.001
2	37	13	11
3	24	149	147
5	30 ^a	37	165
6c	31	169	14
7	65	118	139
8	62	459	338 ^b
10	53	26	12

Table 3 : Comparison with two dual first order methods.

a. feasible within 0.005

b. feasible within 0.025

A possible cause of difficulty, e.g. in problem 2, arises when less than m (being the number of primal variables) dual variables are positive in the optimal solution. When this occurs no estimate can be made of the primal variables, and one must reside in solving subsidiary problems (see Ref. 2).

Another source of problems might precisely be the inaccurate estimates of the primal variables. This occurred in problems 5 and 6.

For problem 6c we tried a number of different bases to derive the primal variables, each of them resulting in feasible solutions but with quite unreliable estimates of the primal objective function value. Problem 5 on the other hand resulted in a completely infeasible primal solution.

The rate of convergence near the optimum can be very slow. Problem 3 e.g. failed to reach the required feasibility degree after about 2 minutes computing time.

6. CONCLUSION

In this paper cutting plane algorithms were presented for the primal and the dual geometric posynomial programming problem.

The dual method circumvents the problem of nondifferentiability by using a subgradient.

Although the resulting problem seems easier to solve because of the linear constraints, the primal cutting plane method clearly outperforms the dual in execution time and number of iterations.

Furthermore the primal method is robuster in the sense that it is not subject to a number of problems resulting from the conversion from dual to primal solutions.

APPENDIX I : Derivation of the deepest dual cut.

$$\begin{aligned} \text{Consider : } \quad & \text{Max}_{\delta} \quad (\bar{y} - \delta)^T \nabla g(\delta) \\ & \text{s.t.} \quad g(\delta) = 0 \quad (1) \text{ with } g(\delta) = Z - V^1(x) \\ & \delta = (Z, x) \\ & \bar{y} = (\bar{Z}, \bar{x}) \end{aligned}$$

The Lagrangian function is :

$$\begin{aligned} \mathcal{L}(\delta, u) &= (\bar{y} - \delta)^T \nabla g(\delta) + u \cdot g(\delta) \\ &= \bar{Z} - Z - \sum_{k=0}^p \sum_{i=1}^{T_k} (\bar{x}_{ki} - x_{ki}) \ln \left(\frac{c_{ki} \lambda_k}{x_{ki}} \right) \\ &\quad + u \left(Z - \sum_k \sum_i x_{ki} \ln \frac{c_{ki} \lambda_k}{x_{ki}} \right) \end{aligned}$$

Taking first derivatives and setting them equal to zero :

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial Z} &= -1 + u = 0 \quad \text{or} \quad u = 1 \\ \frac{\partial \mathcal{L}}{\partial x_{ki}} &= -\frac{\bar{\lambda}_k}{\lambda_k} + \frac{\bar{x}_{ki}}{x_{ki}} = 0 \quad \text{or} \quad \frac{\bar{x}_{ki}}{x_{ki}} = \frac{\bar{\lambda}_k}{\lambda_k} \end{aligned} \quad (2)$$

The objective function value for any point satisfying (1) and (2) is :

$$\bar{Z} - \sum_k \sum_i \bar{x}_{ki} \ln (c_{ki} \bar{\lambda}_k / \bar{x}_{ki}) > 0$$

We can therefore choose :

$$\begin{aligned} x_{ki} &= \bar{x}_{ki} \\ Z &= \sum_k \sum_i \bar{x}_{ki} \ln (c_{ki} \bar{\lambda}_k / \bar{x}_{ki}) \end{aligned}$$

So the cutting plane becomes :

$$Z - \sum_k \sum_i \bar{x}_{ki} \ln (c_{ki} \bar{\lambda}_k / \bar{x}_{ki}) - \sum_k \sum_i (x_{ki} - \bar{x}_{ki}) \ln (c_{ki} \bar{\lambda}_k / \bar{x}_{ki}) \leq 0$$

$$\text{or } Z - \sum_k \sum_i x_{ki} \ln (c_{ki} \bar{\lambda}_k / \bar{x}_{ki}) \leq 0 \quad (3)$$

Now consider the usual Kelley-cut

$$g(\bar{\delta}) + (\delta - \bar{\delta})^T \nabla g(\bar{\delta}) \leq 0$$

$$\bar{Z} - \sum_k \sum_i \bar{x}_{ki} \ln c_{ki} \frac{\bar{\lambda}_k}{\bar{x}_{ki}} + (Z - \bar{Z}) - \sum_k \sum_i (x_{ki} - \bar{x}_{ki}) \ln c_{ki} \frac{\bar{\lambda}_k}{\bar{x}_{ki}} \leq 0$$

$$\text{or : } Z - \sum_k \sum_i x_{ki} \ln c_{ki} \frac{\bar{\lambda}_k}{\bar{x}_{ki}} \leq 0$$

which can be seen to be the same cut.

APPENDIX II : The primal cutting plane

We will show here that the primal cut we described in section 4 is the same cut as derived in Gochet & Smeers (1974) Ref.[9].

The cut we use is :

$$\sum_{i \in J_k} \alpha_{ki} \ln u_{ki} \leq \ln \beta_k \quad k \text{ being the most violated constraint}$$

Replacing α_{ki} , β_k and u_{ki} by their respective expressions :

$$\sum_{i \in J_k} \frac{\bar{u}_{ki}}{\sum_{r \in J_k} \bar{u}_{kr}} \left[\ln c_{ki} + \sum_{j=1}^m a_{kij} \ln t_j \right] \leq \sum_{i \in J_k} \frac{\bar{u}_{ki}}{\sum_{r \in J_k} \bar{u}_{kr}} \ln \frac{\bar{u}_{ki}}{\sum_{r \in J_k} \bar{u}_{kr}}$$

Multiplying by $\sum_{r \in J_k} \bar{u}_{kr}$ and simplifying the right-hand side :

$$\sum_{i \in J_k} \bar{u}_{ki} \left[\ln c_{ki} + \sum_{j=1}^m a_{kij} \ln t_j \right] \leq \sum_{i \in J_k} \bar{u}_{ki} \ln \bar{u}_{ki} - \left(\sum_{r \in J_k} \bar{u}_{kr} \right) \ln \left(\sum_{r \in J_k} \bar{u}_{kr} \right) \quad (1)$$

If we substitute : $X_i = \ln c_{ki} + \sum_{j=1}^m a_{kij} \ln t_j$

$$\ln \bar{u}_{ki} = \ln c_{ki} + \sum_{j=1}^m a_{kij} \ln t_j = \bar{X}_i \quad (1) \text{ becomes}$$

$$\sum_{i \in J_k} X_i \exp(\bar{X}_i) \leq \sum_{i \in J_k} \exp(\bar{X}_i) \bar{X}_i - \left[\sum_{i \in J_k} \exp(\bar{X}_i) \right] \left[\ln \left(\sum_{i \in J_k} \exp(\bar{X}_i) \right) \right]$$

which is exactly the same cut as in Ref. 9, where this cut is shown to be parallel but deeper than the Kelley cut.

APPENDIX III : Test problems

1. Source : Beightler and Phillips (1976), p. 274

$$\text{Min } 314.16 X_1 X_2 + 408.41 X_1^2 + 100 X_3$$

$$\text{s.t. } 1592.36 X_1^{-2} X_2^{-1} + .125 X_1^{-2} X_2^{-1} X_3 \leq 1$$

$$X_2^2 X_3^{-2} + 4 X_1^2 X_3^{-2} \leq 1$$

$$X_i > 0$$

$$\text{Solution : } F = 91246.018$$

$$X_1 = 8.648$$

$$X_2 = 21.339$$

$$X_3 = 27.468$$

2. Source : Beck & Ecker [Ref. 3]

$$\text{Min } 10 X_1^{-0.25} X_2 X_4^{-1} X_5^{-0.5} + 20 X_2^{-1} X_3 X_4^2 X_5^{-0.5} + 15 X_1^{0.5} X_2^{-1} X_3^{-1} X_5^{1.5}$$

$$\text{s.t. } 0.01 X_1 X_2 + 0.02 X_2^{-2} X_4 X_5 \leq 1$$

$$0.04 X_1 X_3^{-3} + 0.02 X_2^{-2} X_3 + 0.02 X_4 X_5^{-3} + 0.06 X_1^{-1} X_2 X_4^{-0.5} X_5^{0.5} \leq 1$$

$$X_i > 0$$

$$\text{Solution } F = 37.216$$

$$X_1 = 11.425$$

$$X_3 = 0.787$$

$$X_5 = 1.179$$

$$X_2 = 8.747$$

$$X_4 = 2.401$$

3. Source : Beck & Ecker [Ref. 3]

$$\text{Min } 20 X_1 + 10 X_2 + 30 X_3 + 15 X_4 + 1500 X_1^{-1} X_2^{-1} X_3^{-1} X_4^{-1}$$

$$\text{s.t. } 0.16667 X_1 + 0.08333 X_2 + 0.08333 X_3 + 0.016667 X_4 \leq 1$$

$$0.05405 X_1 + 0.16216 X_2 + 0.21622 X_3 + 0.10811 X_4 \leq 1$$

$$0.08888 X_1 + 0.06667 X_2 + 0.15556 X_3 + 0.31111 X_4 \leq 1$$

$$0.06667 X_1 + 0.06667 X_2 + 0.06667 X_3 + 0.06667 X_4 \leq 1$$

$$X_i > 0$$

$$\text{Solution } F = 267.729$$

$$X_1 = 2.624 \quad X_3 = 1.326$$

$$X_2 = 2.712 \quad X_4 = 1.222$$

4. a) Source : Beightler & Phillips (1976), p. 408

$$\text{Min } 400 X_1^{0.4} + 4.183 X_1^{-1.2} X_3^{1.2} + 10 X_1^{-0.5} X_2 X_3^{0.5}$$

$$\text{s.t. } X_3^{-1} + X_2 \leq 1$$

(1)

$$0.1975 X_1^{-0.5} X_2^{-0.5} + 0.1975 X_1^{-0.5} X_2^{-0.5} X_3^{-1} \leq 1$$

$$X_i > 0$$

$$\text{Solution : } F = 280.623$$

$$X_1 = 0.238$$

$$X_3 = 1.704$$

$$X_2 = 0.413$$

b) Constraint (1) is replaced by :

$$x_2^{-1} + x_1 \leq 1$$

Solution : $F = 246.109$

$$x_1 = 0.159 \quad x_3 = 0.604$$

$$x_2 = 1.726$$

5. Source : Beck & Ecker [Ref. 3]

$$\text{Min } 10 x_1 x_2^{-1} x_4^2 x_6^{-3} x_7^{-0.25} + 15 x_1^{-1} x_2^{-2} x_3 x_4 x_5^{-1} x_7^{-0.5}$$

$$+ 20 x_1^{-2} x_2 x_4^{-1} x_6 + 25 x_1^2 x_2^2 x_3^{-1} x_5^{0.5} x_6^{-2} x_7$$

$$\text{s.t. } 0.5 x_1^{0.5} x_3^{-1} x_6^{-2} x_7 + 0.7 x_1^3 x_2 x_3^{-2} x_6 x_7^{0.5} + 0.2 x_2^{-1} x_3 x_4^{-0.5}$$

$$+ 0.2 x_2^{-1} x_3 x_4^{-0.5} x_6^{2/3} x_7^{0.25} \leq 1$$

$$1.3 x_1^{-0.5} x_2 x_3^{-1} x_5^{-1} x_6 + 0.8 x_3 x_4^{-1} x_5^{-1} x_6^2$$

$$+ 3.1 x_1^{-1} x_2^{0.5} x_4^{-2} x_5^{-1} x_6^{1/3} \leq 1$$

$$2 x_1 x_3^{-1.5} x_5 x_6^{-1} x_7^{1/3} + 0.1 x_2 x_3^{-0.5} x_5 x_6^{-1} x_7^{-0.5}$$

$$+ x_1^{-1} x_2 x_3^{0.5} x_5 + 0.65 x_2^{-2} x_3 x_5 x_6^{-1} x_7 \leq 1$$

$$0.2 x_1^{-2} x_2 x_4^{-1} x_5^{0.5} x_7^{1/3} + 0.3 x_1^{0.5} x_2^2 x_3 x_4^{1/3} x_5^{-2/3} x_7^{0.25}$$

$$+ 0.4 x_1^{-3} x_2^{-2} x_3 x_5 x_7^{0.75} + 0.5 x_3^{-2} x_4 x_7^{0.5} \leq 1$$

$$x_i > 0$$

Solution $F = 1800.735$

feasible within 0.005

$$X_1 = 2.837$$

$$X_5 = 0.997$$

$$X_2 = 0.646$$

$$X_6 = 1.415$$

$$X_3 = 2.126$$

$$X_7 = 0.035$$

$$X_4 = 5.368$$

6. c) Source Beck & Ecker [Ref. 3]

$$\begin{aligned} \text{Min } 10 X_1 X_3 X_5^2 X_6^{0.5} X_7^{-2} + 20 X_1^{-1} X_4^{-1} X_5^{-1} X_6^{0.5} X_7 \\ + 30 X_2 X_4 X_5^{-1} X_7^2 + 40 X_2^{-1} X_3^{-1} X_6^{-1} X_7^{-1} \end{aligned}$$

$$\text{s.t. } 2 t_1 + 2 t_2 \leq 1 \quad (1)$$

$$2 t_1 + 4 t_2 + t_3 + 2 t_4 + t_5 + 3 t_6 + 4 t_7 \leq 20$$

$$\begin{aligned} 0.2 t_1 t_2 t_3 + 0.3 t_2 t_3 t_4 + 0.4 t_3 t_4 t_5 + 0.3 t_3 t_5 t_6 \\ + 0.2 t_5 t_6 t_7 \leq 1 \end{aligned} \quad (2)$$

$$X_i > 0$$

a) Constraint (1) is replaced by :

$$2 t_1 + t_2 \leq 1$$

b) Constraints (1) and (2) become :

$$2 t_1 + t_2 \leq 1$$

$$\begin{aligned} 0.2 t_1 t_2 t_3 + 0.3 t_2 t_3 t_4 + 0.4 t_3 t_4 t_5 + 0.3 t_4 t_5 t_6 \\ + 0.2 t_5 t_6 t_7 \leq 1 \end{aligned}$$

Solutions : a) $F = 155.728$

$$x_1 = 0.265 \quad x_5 = 0.374$$

$$x_2 = 0.471 \quad x_6 = 3.533$$

$$x_3 = 0.860 \quad x_7 = 0.349$$

$$x_4 = 2.184$$

b) $F = 159.259$

$$x_1 = 0.304 \quad x_5 = 0.510$$

$$x_2 = 0.393 \quad x_6 = 1.553$$

$$x_3 = 2.167 \quad x_7 = 0.447$$

$$x_4 = 0.939$$

c) $F = 191.837$

$$x_1 = 0.294 \quad x_5 = 0.324$$

$$x_2 = 0.206 \quad x_6 = 2.226$$

$$x_3 = 3.078 \quad x_7 = 0.345$$

$$x_4 = 1.133$$

7. Source : Beck & Ecker [Ref. 3]

$$\text{Min } 10 x_2^{-1} x_5^{-1} x_6^{-0.5} x_7^{-2} + 15 x_1^{-2} x_2 x_4^{-1} x_5^{-0.5} x_6^{0.5} x_7^{-1}$$

$$+ 25 x_1 x_2^{-1} x_5^{-1} x_6^{-1} x_7^{-1} + 5 x_3^{-1} x_5^{0.5} x_6^{0.25} x_7^{-0.5}$$

$$+ 5 x_1^{0.25} x_2^{0.5} x_3 x_4 x_5^{0.25} x_6^2 x_7^2$$

$$+ 20 x_1^{-1} x_2^{-0.5} x_3^{0.5} x_4^{-0.5} x_5^{-0.25} x_6^{-0.5} x_7^{0.25}$$

$$\text{s.t. } 5 x_1 + 10 x_2 + 6 x_3 + 3 x_4 + 4 x_5 + 2 x_6 + 15 x_7 \leq 30$$

$$x_i > 0$$

Solution : $F = 136.396$

$$X_1 = 0.628 \quad X_5 = 1.309$$

$$X_2 = 0.397 \quad X_6 = 1.438$$

$$X_3 = 0.258 \quad X_7 = 0.693$$

$$X_4 = 0.949$$

8. a) Source : Beck & Ecker [Ref. 3]

$$\begin{aligned} \text{Min } & 10 X_1^{-1} X_2^{-1} X_3^{-1} X_4 + 20 X_1^{-1} X_4^{-1} X_5 X_6 + 30 X_2 X_3 X_4 \\ & + 100 X_1^{-1} X_2^{-1} X_3^{-1} X_4^{-1} X_5^{-1} X_6^{-1} X_7^{-1} + 5 X_1^2 X_2^2 X_3 X_5 X_6^{1.5} X_7^2 \\ & + 50 X_3^{-0.5} X_4^{-0.5} X_5^{-0.5} + 25 X_3^2 X_4^2 X_5^{-1} X_6^{-1} X_7^{-1} + 10 X_3^{0.5} X_4^{0.5} X_5 X_6 X_7 \end{aligned}$$

$$\text{s.t. } 0.1 X_1^2 X_2^2 X_3 + 0.05 X_4 X_5^{0.5} + 0.15 X_6^{0.5} X_7^{0.5} \leq 1$$

$$\begin{aligned} & 0.1 X_1 X_4 X_7 + 0.05 X_1 X_2^{-1} X_3^{-1} X_5 X_6 X_7^{0.5} + 0.05 X_2^2 X_3^2 X_4^{-1} \\ & + 0.15 X_1^{-0.5} X_2^{-0.3} X_3 X_5^{0.5} + 0.1 X_5 X_6 + 0.1 X_4^2 + 0.2 X_1 X_2 X_3 \leq 1 \end{aligned}$$

$$\sum_{j=1}^7 X_j \leq 10$$

$$\sum_{j=1}^7 X_1 X_j \leq 50$$

$$\sum_{j=1}^6 X_j X_{j+1}^{-1} \leq 100$$

$$\sum_{j=1}^5 x_j x_{j+2}^{-2} \leq 10$$

$$\sum_{j=1}^5 x_{j+2} x_j^{-0.5} \leq 50$$

$$x_i > 0$$

b) The constraints become :

$$0.1 x_1^2 x_2^2 x_3 + 0.05 x_4 x_5^{0.5} + 0.15 x_6^{0.5} x_7^{0.5} \leq 1$$

$$0.1 x_1 x_4 x_7 + 0.05 x_1 x_2^{-1} x_3^{-1} x_5 x_6 x_7^{0.5} + 0.05 x_2^2 x_3^2 x_4^{-1} \\ + 0.15 x_1^{-0.5} x_2^{-0.3} x_3 x_5^{0.5} + 0.1 x_5 x_6 + 0.1 x_4^2 \leq 1$$

$$0.1 (x_1 + x_2 + x_3 + x_4 + x_5 + x_6) + 0.2 x_1 x_2 x_3 \leq 1$$

$$x_1^2 + 0.02 \left(\sum_{j=2}^6 x_1 x_j \right) + 0.1 x_7 \leq 1$$

$$0.01 \left(\sum_{j=1}^5 x_j x_{j+1}^{-1} \right) + 0.02 x_1 x_7 \leq 1$$

$$0.1 \left(\sum_{j=1}^4 x_j x_{j+2}^{-2} \right) + 0.01 x_6 x_7^{-1} \leq 1$$

$$0.02 \left(\sum_{j=1}^4 x_{j+2} x_j^{-0.5} \right) + 0.1 x_5 x_7^{-2} \leq 1$$

$$x_i > 0$$

Solution : a) $F = 178.425$

$$\begin{array}{ll} X_1 = 1.353 & X_5 = 3.144 \\ X_2 = 0.986 & X_6 = 0.406 \\ X_3 = 0.873 & X_7 = 1.514 \\ X_4 = 0.927 & \end{array}$$

b) $F = 185.825$

$$\begin{array}{ll} X_1 = 0.836 & X_5 = 3.488 \\ X_2 = 1.188 & X_6 = 0.4196 \\ X_3 = 0.695 & X_7 = 1.823 \\ X_4 = 1.322 & \end{array}$$

9. Source : Gochet & Smeers (1974) [Ref. 9]

$$\text{Min } X_1 X_3^{-1} X_4 + 2 X_1 + X_1 X_4 + X_2^{-1}$$

$$\text{s.t. } X_1^{-1} X_2 + X_1^{-1} \leq 1$$

$$X_1 X_4^{-1} \leq 1$$

$$X_1 X_3 \leq 1$$

$$X_1 > 0$$

Solution : $F = 11.946$

$$\begin{array}{ll} X_1 = 1.267 & X_3 = 0.789 \\ X_2 = 0.267 & X_4 = 1.267 \end{array}$$

10. Source : Beck & Ecker [Ref. 3]

$$\begin{aligned} \text{Min } & 19.4 X_1^{-1.47} + 16.8 X_2^{-1.66} + 91.5 X_3^{-0.3} + 19.4 X_4^{-1.47} \\ & + 86 X_5^{-0.38} + 152 X_6^{-0.27} + 16.8 X_7^{-1.66} + 27.4 X_8^{-0.63} \end{aligned}$$

$$\text{s.t. } 0.537 X_1 X_2 X_3 + 4.47 X_4 X_5 X_6 + 0.386 X_7 X_8 \leq 1$$

$$0.65 X_1^{-1} \leq 1$$

$$0.15 X_1^{-1} X_2^{-1} X_3^{-1} \leq 1$$

$$0.65 X_4^{-1} \leq 1$$

$$0.15 X_4^{-1} X_5^{-1} \leq 1$$

$$0.60 X_7^{-1} \leq 1$$

$$0.20 X_7^{-1} X_8^{-1} \leq 1$$

$$X_1 > 0$$

Solution $F = 550.642$

$$X_1 = 0.937$$

$$X_5 = 0.331$$

$$X_2 = 0.901$$

$$X_6 = 0.467$$

$$X_3 = 0.676$$

$$X_7 = 1.096$$

$$X_4 = 0.671$$

$$X_8 = 0.554$$

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