A Lanczos-like reduction of symmetric structured matrices to semiseparable form

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Abstract

An algorithm that transforms symmetric matrices to similar semiseparable ones was recently proposed [19]. As with the Householder reduction, the latter algorithm works without taking into account the structure of the original matrix. In this paper we propose a Lanczos-like algorithm to transform a symmetric matrix to a similar semiseparable one by making use of a product of the original matrix times a vector at each step. An efficient algorithm can be achieved if the original matrix is sparse or structured.

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A Lanczos-like reduction of symmetric structured matrices to semiseparable form*

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Abstract. An algorithm that transforms symmetric matrices to similar semi-separable ones was recently proposed [19]. As with the Householder reduction, the latter algorithm works without taking into account the structure of the original matrix. In this paper we propose a Lanczos-like algorithm to transform a symmetric matrix to a similar semiseparable one by making use

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of a product of the original matrix times a vector at each step. An efficient algorithm can be achieved if the original matrix is sparse or structured.

1 Introduction

An algorithm that transforms symmetric matrices to similar semiseparable form was recently proposed [19]. The algorithm combines the properties of the Lanczos tridiagonalization [5, pp. 470–499] with those of the subspace iteration method [5, pp. 332–334], with the size of the subspace increasing by one dimension at each step of the algorithm. Hence, in many cases, after a few steps of the algorithm, the extreme eigenvalues, and the corresponding eigenvectors, are already well approximated. Unfortunately, as with the Householder reduction, the latter algorithm destroys the structure of the original matrix.

In this paper we propose a Lanczos-like algorithm to transform a symmetric matrix to a similar semiseparable one that, as with Lanczos tridiagonalization, uses a product of the original matrix times a vector at each step. Therefore it can be used efficiently if the original matrix is sparse or structured.

The matrices dealt with in this paper are symmetric. However, there is no loss of generality, because, as shown below, similar techniques can be applied to general real matrices as well. Moreover, an extension of this algorithm to reduce rectangular matrices to upper triangular semiseparable ones in order to compute the singular value decomposition [21] is straightforward.

As an application, the proposed algorithm has been considered for computing the kernel step of a state-space method called HLSVD [1, 8, 18], in which the subspace associated to the largest singular values of a Hankel matrix is computed.

The paper is organized as follows. In § 2 the basic steps of the classical Lanczos algorithm are described. This is followed by our Lanczos algorithm for reducing symmetric matrices into semiseparable ones in § 3. Numerical experiments are given in § 4 followed by the conclusion and a discussion of future work.

2 Lanczos-like algorithm for semiseparable matrices

The Lanczos algorithm is a simple and effective method for finding extreme eigenvalues and corresponding eigenvectors of symmetric matrices. Because it only accesses the matrix through matrix-vector multiplications, it is commonly used when matrices are sparse or structured. The algorithm can be summarized in the following steps.

Lanczos algorithm

$A \in \mathbb{R}^{n \times n}$ symmetric and $r_0 \in \mathbb{R}^n$ the initial guess.

$\beta_0 = \|r_0\|_2$ and $q_0 = 0$.

For $i = 1, 2, \dots$

(a) $q_i = r_{i-1} / \|r_{i-1}\|_2$

(b) $p = Aq_i$

(c) $\alpha_i = q_i^T p$

(d) $r_i = p - \alpha_i q_i - \beta_{i-1} q_{i-1}$

(e) $\beta_i = \|r_i\|_2$.

Let $Q_k \equiv [q_1, q_2, \dots, q_k]$ and

$$T_k = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & \ddots & \ddots & \\ & & \ddots & \alpha_{n-1} & \beta_{k-1} \\ & & & \beta_{k-1} & \alpha_k \end{bmatrix},$$

and let $T_k = U_k \Theta_k U_k^T$ be its spectral decomposition, where $\Theta_k = \text{diag}(\theta_1, \theta_2, \dots, \theta_k)$ and $U_k \in \mathbb{R}^{k \times k}$ is orthogonal. Then

$$A(Q_k U_k) = (Q_k U_k) \Theta_k + \beta_k q_{k+1} e_k^T U_k,$$

where e_k is the last vector of the canonical basis of $\mathbb{R}^k$. It turns out (see, e.g., [5, pp. 475]) that

$$\min_{\mu \in \lambda(A)} |\theta_i - \mu| \leq |\beta_k| |u_{k,i}|, \quad i = 1, \dots, k.$$

Hence β_k and the last components of the eigenvectors of T_k associated to θ_i give an indication of the convergence of the latter eigenvalues to the eigenvalues of A .

To prevent the loss of orthogonality in the computation of the Krylov basis Q_k , many techniques have been developed (see, e.g., [5] and its references and [12–15]).

The Lanczos-like algorithm for semiseparable matrices is based on the classical Lanczos algorithm for reducing symmetric matrices into similar symmetric tridiagonal ones [19].

Before considering it, we introduce the definition of generator representable semiseparable matrices.

Definition 1 A matrix S is called a generator representable semiseparable matrix if

$$S = \begin{pmatrix} v_1 u_1 & v_1 u_2 & v_1 u_3 & \dots & v_1 u_n \\ v_1 u_2 & v_2 u_2 & v_2 u_3 & \dots & v_2 u_n \\ v_1 u_3 & v_2 u_3 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ v_1 u_n & v_2 u_n & \dots & \dots & v_n u_n \end{pmatrix},$$

i.e., the lower triangular part is equal to the lower triangular part of the rank-one matrix uv^T and, symmetrically, the upper triangular part is equal to the upper triangular part of vu^T . The vectors $u = [u_1, \dots, u_n]^T$ and $v = [v_1, \dots, v_n]^T$ are the generators of S .

Definition 2 A semiseparable matrix is a block diagonal matrix, whose diagonal blocks are generator representable semiseparable matrices.

Remark 1 A comprehensive treatment of *semiseparable-like* matrices can be found in [22]. A more stable and efficient representation of semiseparable matrices in the context of eigenvalue problems can be found in the latter reference as well.

For $A \in \mathbb{R}^{n \times n}$, the i th step, $i = 1, \dots, n - 1$, of the algorithm described in [19] to reduce a symmetric matrix into a symmetric semiseparable one can be summarized as follows.

Reduction of a symmetric matrix to symmetric semiseparable form

Let $A_0 \equiv A$.

For $i = 1, \dots, n - 1$

- Compute the Householder matrix H_i in order to annihilate the entries of A_{i-1} in the i th column below the $(i + 1)$ th row
- Compute $H_i A_{i-1} H_i^T$
- Compute the orthogonal matrix Z_i such that the leading principal submatrix of order $i + 1$ of $Z_i H_i A_{i-1} H_i^T Z_i^T$ is symmetric semiseparable
- $A_i = Z_i H_i A_{i-1} H_i^T Z_i^T$.

Remark 2 The leading principal submatrices of order i of $H_i A_{i-1} H_i^T$ already have symmetric semiseparable structure. Therefore step (c) in this algorithm can be interpreted as an updating step, i.e., increasing by one the order of the leading principal semiseparable submatrix.

2.1 Updating of semiseparable matrices

Let $S_k \in \mathbb{R}^{k \times k}$ be a symmetric semiseparable matrix. We now describe how the augmented matrix

$$\hat{S}_{k+1} = \left[\begin{array}{c|c} S_k & \begin{matrix} 0 \\ \vdots \\ 0 \\ \beta_k \end{matrix} \\ \hline 0 \cdots 0 & \beta_k \end{array} \middle| \begin{matrix} 0 \\ \vdots \\ 0 \\ \beta_k \end{matrix} \right], \quad (1)$$

with $\beta_k \neq 0$, can be updated, i.e., reduced in symmetric semiseparable form by orthogonal similarity transformations working on the first k rows (and columns).

For simplicity, suppose that $k = 4$,

$$\hat{S}_5 = \left[\begin{array}{cccc|c} u_1 v_1 & u_1 v_2 & u_1 v_3 & u_1 v_4 & 0 \\ u_1 v_2 & u_2 v_2 & u_2 v_3 & u_2 v_4 & 0 \\ u_1 v_3 & u_2 v_3 & u_3 v_3 & u_3 v_4 & 0 \\ u_1 v_4 & u_2 v_4 & u_3 v_4 & u_4 v_4 & \beta_4 \\ \hline 0 & 0 & 0 & \beta_4 & \alpha_5 \end{array} \right].$$

Define $\delta_4 \equiv \beta_4$. Let $\hat{G}_3$ be the Givens rotation

$$G_3 = \begin{bmatrix} c_3 & s_3 \\ -s_3 & c_3 \end{bmatrix} \quad \text{such that} \quad G_3 \begin{bmatrix} v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} \hat{v}_3 \\ 0 \end{bmatrix}$$

and let

$$\hat{G}_3 = \begin{bmatrix} I_2 & & \\ & G_3 & \\ & & I_1 \end{bmatrix},$$

where I_k is the identity matrix of order k . Then,

$$\hat{G}_3 \hat{S}_5 = \left[\begin{array}{cccc|c} u_1 v_1 & u_1 v_2 & u_1 v_3 & u_1 v_4 & 0 \\ u_1 v_2 & u_2 v_2 & u_2 v_3 & u_2 v_4 & 0 \\ u_1 \hat{v}_3 & u_2 \hat{v}_3 & u_3 \hat{v}_3 & \rho_3 & s_3 \delta_4 \\ 0 & 0 & 0 & \delta_3 & c_3 \delta_4 \\ \hline 0 & 0 & 0 & \delta_4 & \alpha_5 \end{array} \right],$$

where

$$\begin{bmatrix} \rho_3 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} (c_3 u_3 + s_3 u_4) v_4 \\ (-s_3 u_3 + c_3 u_4) v_4 \end{bmatrix}.$$

Therefore,

$$S_5^{(1)} \equiv \hat{G}_3 \hat{S}_5 \hat{G}_3^T = \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 \hat{v}_3 & 0 & 0 \\ u_1 v_2 & u_2 v_2 & u_2 \hat{v}_3 & 0 & 0 \\ u_1 \hat{v}_3 & u_2 \hat{v}_3 & \eta_3 & s_3 \delta_3 & s_3 \delta_4 \\ 0 & 0 & s_3 \delta_3 & c_3 \delta_3 & c_3 \delta_4 \\ 0 & 0 & s_3 \delta_4 & c_3 \delta_4 & \alpha_5 \end{bmatrix},$$

where $\eta_3 = u_3 \hat{v}_3 c_3 + \rho_3 s_3$. The sub-block matrices $S_5^{(1)}(1 : 3, 1 : 3)$ and $S_5^{(1)}(3 : 5, 3 : 5)$ turn out to be symmetric semiseparable.

Let

$$G_2 = \begin{bmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{bmatrix} \quad \text{such that} \quad G_2 \begin{bmatrix} v_2 \\ \hat{v}_3 \end{bmatrix} = \begin{bmatrix} \hat{v}_2 \\ 0 \end{bmatrix},$$

and

$$\hat{G}_2 = \begin{bmatrix} I_1 & \\ & G_2 \\ & & I_2 \end{bmatrix}.$$

Multiplying $\hat{G}_3 \hat{S}_5 \hat{G}_3^T$ on the left by $\hat{G}_2$ and on the right by $\hat{G}_2^T$, we obtain

$$S_5^{(2)} = \hat{G}_2 \hat{G}_3 \hat{S}_5 \hat{G}_3^T \hat{G}_2^T = \begin{bmatrix} u_1 v_1 & u_1 \hat{v}_2 & 0 & 0 & 0 \\ u_1 \hat{v}_2 & \eta_2 & s_2 \delta_2 & s_2 s_3 \delta_3 & s_2 s_3 \delta_4 \\ 0 & s_2 \delta_2 & c_2 \delta_2 & c_2 s_3 \delta_3 & c_2 s_3 \delta_4 \\ 0 & s_2 s_3 \delta_3 & c_2 s_3 \delta_3 & c_3 \delta_3 & c_3 \delta_4 \\ 0 & s_2 s_3 \delta_4 & c_2 s_3 \delta_4 & c_3 \delta_4 & \alpha_5 \end{bmatrix},$$

where

$$\eta_2 = u_2 \hat{v}_2 c_2 + \rho_2 s_2 \quad \text{and} \quad \begin{bmatrix} \rho_2 \\ \delta_2 \end{bmatrix} \equiv \begin{bmatrix} c_2 u_2 \hat{v}_3 + s_2 \eta_3 \\ -s_2 u_2 \hat{v}_3 + c_2 \eta_3 \end{bmatrix}.$$

Therefore the sub-block matrices $S_5^{(2)}(1 : 2, 1 : 2)$ and $S_5^{(2)}(2 : 5, 2 : 5)$ are symmetric semiseparable. To end the updating, consider the Givens rotation

$$\hat{G}_1 = \begin{bmatrix} G_1 & \\ & I_3 \end{bmatrix}$$

with

$$G_1 = \begin{bmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{bmatrix} \quad \text{such that} \quad G_1 \begin{bmatrix} v_1 \\ \hat{v}_2 \end{bmatrix} = \begin{bmatrix} \hat{v}_1 \\ 0 \end{bmatrix}.$$

Then

$$S_5^{(3)} \equiv \hat{G}_1 \hat{G}_2 \hat{G}_3 \hat{S}_5 \hat{G}_3^T \hat{G}_2^T \hat{G}_1^T = \begin{bmatrix} \eta_1 & s_1 \delta_1 & s_1 s_2 \delta_2 & s_1 s_2 s_3 \delta_3 & s_1 s_2 s_3 \delta_4 \\ s_1 \delta_1 & c_1 \delta_1 & c_1 s_2 \delta_2 & c_1 s_2 s_3 \delta_3 & c_1 s_2 s_3 \delta_4 \\ s_1 s_2 \delta_2 & c_1 s_2 \delta_2 & c_2 \delta_2 & c_2 s_3 \delta_3 & c_2 s_3 \delta_4 \\ s_1 s_2 s_3 \delta_3 & c_1 s_2 s_3 \delta_3 & c_2 s_3 \delta_3 & c_3 \delta_3 & c_3 \delta_4 \\ s_1 s_2 s_3 \delta_4 & c_1 s_2 s_3 \delta_4 & c_2 s_3 \delta_4 & c_3 \delta_4 & \alpha_5 \end{bmatrix}$$

is symmetric semiseparable with

$$\eta_1 = u_1 \hat{v}_1 c_1 + \rho_1 s_1 \quad \text{and} \quad \begin{bmatrix} \rho_1 \\ \delta_1 \end{bmatrix} = \begin{bmatrix} c_1 u_1 \hat{v}_2 + s_1 \eta_1 \\ -s_1 u_1 \hat{v}_2 + c_1 \eta_1 \end{bmatrix}.$$

Hence the generators of $S_5^{(3)}$ are

$$u \equiv \left[\frac{\eta_1}{s_1 s_2 s_3}, \frac{\delta_1}{s_2 s_3}, \frac{\delta_2}{s_3}, \delta_3, \delta_4 \right]^T, \\ v \equiv \left[s_1 s_2 s_3, c_1 s_2 s_3, c_2 s_3, c_3, \frac{\alpha_5}{\delta_4} \right]^T.$$

The updating of a symmetric semiseparable matrix of order k has $O(k)$ computational complexity and needs $O(k)$ storage.

Having shown how the semiseparable structure in the matrix (1) can be updated, we will see that is straightforward to modify the Lanczos algorithm so as to compute semiseparable matrices.

Lanczos reduction to semiseparable matrices

Let r_0 be the initial guess, $\beta_0 = \|r_0\|_2$, $q_0 = 0$ and S_0 the empty matrix.

For $i = 1, 2, \dots$

(a) $q_i = r_{i-1} / \|r_{i-1}\|_2$

(b) $p = Aq_i$

(c) $\alpha_i = q_i^T p$

(d) if

$$i = 1,$$

$$S_i = [\alpha_1]$$

else

Compute S_i ,

i.e., reduce into semiseparable form the augmented matrix

$$\left[\begin{array}{c|c} S_{i-1} & \beta_{i-1} e_{i-1} \\ \hline \beta_{i-1} e_{i-1}^T & \alpha_i \end{array} \right] \quad \text{with} \quad e_i = \underbrace{[0, \dots, 0, 1]}_{i-1}^T$$

(e) $r_i = p - \alpha_i q_i - \beta_{i-1} q_{i-1}$

(f) $\beta_i = \|r_i\|_i.$

With the initial guess r_0 fixed, the Krylov subspace generated by the Lanczos reduction to tridiagonal form is the same as the one generated by the Lanczos reduction to semiseparable form because the Givens rotations for performing the updating step (d) do not change this subspace. The Lanczos reduction to tridiagonal matrices halts before complete tridiagonalization if the initial guess r_0 is contained in a proper invariant subspace. In this case, one of the β_j , $j \in \{1, \dots, n-1\}$, is equal to zero. However,

the Lanczos reduction to tridiagonal form can be continued by choosing a vector r_j orthogonal to the already computed Krylov basis which yields a block diagonal matrix with tridiagonal blocks [5]. The Lanczos reduction to semiseparable form with the same initial guess r_0 has a similar behavior, breaking down at the same step j . Also in this case the algorithm can be run to completion by choosing a vector r_j orthogonal to the already computed Krylov basis. The final reduced matrix is, in this case, a block diagonal matrix with generator representable semiseparable blocks.

Step (d) in the Lanczos reduction to semiseparable matrices corresponds to applying one iteration of the QR -method without shift to the matrix S_{i-1} [19]. This is accomplished by applying $i - 2$ Givens rotations. Step (d) could be replaced by applying one step of the implicitly shifted QR -method, with the shift chosen so as to improve the convergence of the sequence of the generated semiseparable matrices towards a similar block diagonal one [23].

We observe that it is not necessary to compute the product of the Givens rotations at each step. The Givens coefficients are stored in a matrix and the product is computed only when the convergence of the sequence of semiseparable matrices to a block diagonal form has occurred. As a consequence, the Krylov basis is then updated by multiplying Q_k by the latter orthogonal matrix.

The Lanczos reduction, as proposed here, generates all possible orthogonally similar semiseparable matrices to a given symmetric matrix by varying the initial vector (or initial vectors when there is a breakdown in the Lanczos reduction, i.e., when one of the β_k is equal to zero). We give the proof only for nonsingular matrices.

Theorem 1 *Let A be a nonsingular symmetric matrix. Then each semiseparable matrix $\tilde{S}$ that is orthogonally similar to A can be found by applying the Lanczos reduction as described in this section for a certain initial vector (or initial vectors when the Lanczos algorithm has found an invariant subspace).*

Proof The semiseparable matrix can always be decomposed as a direct sum of (generator representable) semiseparable matrices $\tilde{S}_i$, $i = 1, 2, \dots, l$, each having its bottom-left element different from zero. When the original matrix A is nonsingular, these semiseparable matrices are also nonsingular. On each of these semiseparable matrices $\tilde{S}_i$ we execute the updating step in reverse direction, i.e., we make the last row of $\tilde{S}_i$ zero except possibly for the last two elements; similarly the last column is made zero by similarity transformation except possibly for the last two elements. Note that because the first element in the last row is different from zero, all the Givens rotations are nontrivial, i.e., not equivalent to the identity transformation. Because the matrix $\tilde{S}_i$ is nonsingular, these Givens rotation are just those uniquely

determined for performing an RQ factorization of the leading principal semiseparable submatrix $\tilde{S}'_i$ having order one less than $\tilde{S}_i$:

$$\tilde{S}'_i = R_i Q_i.$$

Here the diagonal elements of R_i are different from zero. Hence, performing the similarity transformation gives

$$\tilde{S}''_i = Q_i \tilde{S}'_i Q_i^T$$

with the first element on the last row of $\tilde{S}''_i$ different from zero (nontrivial Givens). Hence, we can repeat the procedure, i.e., reversing the updating step and so on. Finally, we find, for each matrix $\tilde{S}_i$, the corresponding unreduced tridiagonal matrix T_i . Thus, e.g., when $l = 2$, we arrive at the following situation:

$$A \begin{bmatrix} Q' & Q'' \end{bmatrix} = \begin{bmatrix} Q' & Q'' \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}.$$

Hence, T_1 can be found by the Lanczos algorithm using as initial vector the first vector of Q' . Therefore, taking the Implicit Q Theorem for symmetric tridiagonal matrices [5, pp. 416–417] into account, we see that T_1 is entirely determined by the first column of Q' . Similarly, at the point where the Lanczos algorithm breaks down, the first vector of Q'' is taken as initial vector. $\square$

In the case of a singular matrix A , to generate all possible orthogonally similar matrices $\tilde{S}$, we use another degree of freedom in the algorithm. In this case, when performing the updating procedure, it can happen that the Givens transformation is not uniquely determined. By taking this degree of freedom into account, all semiseparable matrices orthogonally similar to a symmetric singular matrix can be generated.

The reduction of a symmetric matrix to a similar semiseparable one proposed in this paper has the same properties as the algorithm proposed in [19]. Therefore, if gaps are present in the spectrum of the original matrix, they are “revealed” after some steps of the algorithm [11] making the matrices S_i “numerically” block diagonal with generator representable semiseparable blocks. This property makes the proposed algorithm suitable for computing the largest eigenvalues and the corresponding eigenvectors of sparse or structured matrices, if the large eigenvalues are “quite well” separated from the rest of the spectrum. Indeed, the most computationally expensive part, at each step of the proposed algorithm, is the matrix-by-vector product, which can be efficiently performed if the matrix is sparse or structured.

3 Numerical experiments

In the previous section, the Lanczos reduction of a symmetric matrix to semiseparable form was described. An extension for reducing a general matrix to an upper triangular semiseparable one, very useful in the applications in order to compute the singular value decomposition, is straightforward (for the sake of space, we omit the details). The extended algorithm is used in the following examples and we refer to it as `lansvdSS`.

The proposed algorithm has been implemented in `matlab` in a straightforward way. To prevent the loss of orthogonality of the Lanczos method [5, pp. 479–484], the proposed method is based on the `PROPACK` implementation of the Lanczos algorithm with partial reorthogonalization [6, 7, 14] and compared with `lansvd`, an implementation of the Lanczos algorithm with partial reorthogonalization in `PROPACK` [7].

As previously noted, the Givens rotations used to update the semiseparable matrix in step (d) of the algorithm are stored and their product is computed only when the semiseparable matrix becomes numerically block diagonal with each block in semiseparable form and the desired eigenvalues are the eigenvalues of the leading principal blocks of the semiseparable matrix. The updating of the semiseparable matrix at each step of the algorithm is done with the number of floating point operations proportional to the size of the involved semiseparable matrix.

Since, on the blocks below the main diagonal of the semiseparable matrix, we have the relation

$$\|S(i : n, 1 : i - 1)\|_2 = \|u(i : n)\|_2 \|v(1 : i - 1)\|_2, \quad i = 2, \dots, n,$$

the convergence of the computed semiseparable matrix to block diagonal semiseparable form is checked by a simple recurrence relation in linear time (see [11, 20] for details). Moreover, to prevent loss of accuracy the algorithm has been implemented using the Givens–vector representation of a semiseparable matrix instead of using the generator representation.

Example 1 Here we consider a simulated Magnetic Resonance Spectroscopy (MRS) signal derived from an in vivo spectrum measured in the healthy peripheral zone tissue of the prostate [9]. The discrete simulated signal, consisting of $N = 256$ data points, was modelled as the sum of $K = 8$ exponentially damped sinusoids (Fig.1),

$$y_n = \sum_{k=1}^K a_k e^{j\phi_k} e^{(-d_k + j2\pi f_k)t_n}, \quad n = 0, 1, \dots, N - 1, \quad (2)$$

with $j = \sqrt{-1}$, a_k the amplitude, ϕ_k the phase, d_k the damping factor, f_k the frequency of the k th sinusoid $k = 1, 2, \dots, K$, K the number of sinusoids,

Fig. 1. Original signal (real part)

$t_n = n\Delta t + t_0$ with Δt the sampling interval, and t_0 the time between the effective time origin and the first data point. A realistic realization of a noisy signal is obtained by adding to the latter signal a white Gaussian noise with zero mean and standard deviation equal to $2.5e + 6$ (see Fig.2). An approximation of the original signal is computed by using a state-space method called HLSVD [1,8,18], that yields an approximation of the parameters characterizing the signal, i.e., the amplitudes, the phases, the damping factors and the frequencies of the damped sinusoids in (2). The computationally most intensive part of this method is the computation of the K largest singular values and the corresponding left singular vectors of a Hankel matrix of size $(\lfloor N/2 \rfloor + 1) \times (N - \lfloor N/2 \rfloor)$, whose entries in the first column and last row are the equidistant and consecutive samples of the noisy signal. The approximation of the original signal is then constructed from the K left singular vectors corresponding to the largest K singular values of the Hankel matrix. In Fig.3 the original and the reconstructed signals are shown together. Both algorithms, `lansvd` and `lansvdSS`, compute the K largest singular values and the corresponding left singular vectors of the constructed Hankel matrix to full accuracy and require 17 and 13 steps of the algorithm, respectively.

As described in [11, 19, 21], the proposed algorithm is very efficient if gaps are present in the distribution of the singular values of the matrix.

Fig. 2. Noisy signal obtained by adding a white Gaussian noise with zero mean and standard deviation equal to $2.5e + 6$ (real part)

To show this feature, `lansvdSS` was used to compute the first $K = 4$ singular values of 7 complex Hankel matrices, whose first rows and last columns are constructed by considering the original signal perturbed by a white Gaussian noise with zero mean and standard deviation $\text{st. dev.} = 1e + k, k = 0, 1, \dots, 6$. The results are given in Table 1. In the first, second and third columns, the level of the standard deviation used to construct the Gaussian noise, the number of steps needed by the proposed algorithm `lansvdSS` to compute the first 4 singular values to full accuracy, and the ratio between the 4th and 5th singular values, i.e., the gap between the “signal” singular values and the “noise” singular values, respectively, are reported.

We observe that the larger the gap, the faster the convergence is to the largest eigenvalues.

Example 2 In this example, the matrix `illc1033.mat` in [16], of dimension 1033×320 , is considered. This matrix has a pronounced gap between the 13th and 14th singular values (see Fig.4). The number of steps of the algorithm `lansvd` in PROPACK [7] to compute the largest 13 singular values of the matrix to full accuracy is 53. The proposed algorithm `lansvdSS` requires only 32 steps.

Fig. 3. Original signal (continuous line) vs signal reconstructed by HLSVD (dashed line) (real part)

Table 1. Results obtained computing the first 4 singular values of 7 complex Hankel matrices, constructed with different levels of noise

st. dev.	# steps	σ_8/σ_9
1	8	$2.0232e+06$
$1e+1$	8	$2.0221e+05$
$1e+2$	8	$1.8447e+04$
$1e+3$	8	$1.9438e+03$
$1e+4$	9	$1.5069e+02$
$1e+5$	10	$2.1517e+01$
$1e+6$	13	$1.7023e+01$

4 Conclusion and future work

In this paper, an algorithm that reduces a symmetric matrix to a symmetric semiseparable one is described. The algorithm is based on the Lanczos method. Therefore each iteration relies on the product of the original matrix times a vector. All the techniques developed to prevent the loss of orthogonality in the Krylov basis (full reorthogonalization, partial reorthogonalization, ...) can be used in this case as well. An extension for reducing a matrix to an upper semiseparable form to compute the singular value decomposition is straightforward. Two numerical examples are given to show the effectiveness of the proposed method.

Fig. 4. Distribution of the singular values of the matrix `illc1033.mat` in logarithmic scale

Further research will consist of extending the proposed algorithm for computing the rank revealing factorization of sparse and structured matrices.

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