Infinitesimal Automorphisms of VB-Groupoids and Algebroids

Chiara Esposito^{*}

Dipartimento di Matematica
Università degli Studi di Salerno
via Giovanni Paolo II, 123
84084 Fisciano (SA)
Italy

Alfonso Giuseppe Tortorella[†]

Department of Mathematics
Katholieke Universiteit (KU) Leuven
Celestijnenlaan 200B box 2400
3001 Leuven
Belgium

Luca Vitagliano[‡]

Dipartimento di Matematica
Università degli Studi di Salerno
via Giovanni Paolo II, 123
84084 Fisciano (SA)
Italy

Abstract

VB-groupoids and algebroids are *vector bundle objects* in the categories of Lie groupoids and Lie algebroids respectively, and they are related via the *Lie functor*. VB-groupoids and algebroids play a prominent role in Poisson and related geometries. Additionally, they can be seen as models for vector bundles over singular spaces. In this paper we study their infinitesimal automorphisms, i.e. vector fields on them generating a flow by diffeomorphisms preserving both the linear and the groupoid/algebroid structures. For a special class of VB-groupoids/algebroids coming from representations of Lie groupoids/algebroids, we prove that infinitesimal automorphisms are the same as multiplicative sections of a certain derivation VB-groupoid/algebroid.

^{*}chesposito@unisa.it

[†]alfonsogiuseppe.tortorella@kuleuven.be

[‡]lvitagliano@unisa.it

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Introduction

This is the first in a series of papers aiming at studying (multi)differential operators (DO) and jets on Lie groupoids. Here we study a class of first order DOs, namely *derivations of vector bundles* (also known as *covariant differential operators*). Derivations can be conveniently regarded as infinitesimal automorphisms of the vector bundle structure. Thus, in the realm of Lie groupoids, it is natural to look at infinitesimal automorphisms of VB-groupoids and their infinitesimal counterparts, VB-algebroids.

VB-groupoids are *groupoid objects in the category of vector bundles* or, equivalently, *vector bundle objects in the category of Lie groupoids* [7, 28, 46]. They play an important role in Poisson geometry and, more generally, in Lie groupoid theory. The tangent and the cotangent bundle of a Lie groupoid provide canonical examples. More generally, VB-groupoids can be seen as intrinsic models of two-term representations up to homotopy of Lie groupoids [28]. For instance, the whole information about the *adjoint representation (up to homotopy)* of a Lie groupoid [2] is encoded by its tangent VB-groupoid. Accordingly, deformations of a Lie groupoid G are controlled by the *VB-groupoid complex* of TG [14].

The infinitesimal counterparts of VB-groupoids are VB-algebroids, i.e. *Lie algebroid objects in the category of vector bundles* or, equivalently, *vector bundle objects in the category of Lie algebroids* [7, 23, 27, 46]. The tangent and the cotangent bundle of a Lie algebroid provide canonical examples. Similarly to VB-groupoids, VB-algebroids can be seen as intrinsic models of two-term representations up to homotopy of Lie algebroids [27]. The *adjoint representation (up to homotopy)* of a Lie algebroid is encoded by its tangent VB-algebroid and deformations of a Lie algebroid A are controlled by the *VB-algebroid complex* of TA [15]. Additionally, the deformation complex of a Lie groupoid G and the deformation complex of its Lie algebroid A are intertwined by a Van Est map [12, 64] (see also [5, 9] for *homogeneous versions* of the Van Est theorem), which is yet another manifestation of the action of the *Lie functor* [14].

Because of their importance in Lie groupoid/algebroid theory, it is natural to look at infinitesimal automorphisms of VB-groupoids and VB-algebroids and at how the Lie functor intertwines them. Notice that infinitesimal automorphisms of a Lie groupoid are the same as *multiplicative vector fields* on it [47]. Similarly, infinitesimal automorphisms of a VB-groupoid are (equivalent to) its *multiplicative derivations*. Multiplicative vector fields on Lie groupoids are important objects. In particular, their theory includes the classical liftings of a vector field on the tangent and the cotangent bundle, which are of general interest in differential geometry [47]. We expect multiplicative derivations to have similar importance. In this respect, this paper is also part of the research line aiming at studying *multiplicative structures* on Lie groupoids and their infinitesimal counterparts (see [38] for a recent review), i.e. those geometric data which are compatible with the groupoid, resp. algebroid, structure.

The study of multiplicative structures was originally motivated by Poisson-Lie groups [21] and symplectic groupoids [11, 63, 67, 68] and was pursued by several authors. Recent research has concentrated on multiplicative structures of tensorial type: multiplicative differential forms [6] and multivectors [29], multiplicative (1,1) tensors [56], multiplicative generalized complex structures [36] (see also [3, 13]), and, encompassing all the previous ones, generic multiplicative tensors [8], with few exceptions: multiplicative foliations [33, 35], multiplicative Dirac structures [34, 52, 53], multiplicative vector-bundle valued differential forms [16, 22]. Non-tensorial multiplicative structures (other than the few examples listed above) have not been considered yet. In particular, a systematic investigation (including a definition) of multiplicative (multi)DOs, multiplicative jets, and their infinitesimal counterparts, is still unavailable.

There are two additional motivations to study multiplicative (multi)DOs and jets. One comes from Jacobi geometry and is represented, on one hand, by Jacobi-Lie groups [31], a wide generalization of Poisson-Lie groups and, on another hand, contact groupoids (i.e. Lie groupoids equipped with a multiplicative contact structure [17, 37, 43]). A Jacobi structure on a manifold M is a line bundle $L \rightarrow$

M together with a Lie bracket on its sections which is also a first order biDO (see, e.g., [48]). Jacobi structures encompass Poisson, contact, and locally conformal symplectic structures as particular cases. The case when the line bundle $L \rightarrow M$ is equipped with a global trivialization has often been an object of study. In particular Iglesias and Marrero adopt this approach to their definition of a Jacobi groupoid, i.e. a Lie groupoid with a compatible Jacobi structure [30, 32]. However, probably, this is not the best choice from a conceptual point of view (see, e.g., [4]). Not to mention that there are important Jacobi structures whose underlying line bundle is not at all trivial. In [4] the authors study Jacobi manifolds intrinsically, viewing them as homogeneous Poisson manifolds, in particular they define Jacobi groupoids as *homogeneous Poisson groupoids*. There is an alternative, more algebraic approach, requiring the technology of derivations of vector bundles (see, e.g. [41, 58]). In particular, in order to apply the latter approach to a Jacobi groupoid, one needs to understand multiplicative (multi)derivations of VB-groupoids first. Here, we begin this program by studying multiplicative derivations of a VB-groupoid. Multiplicative multiderivations will be addressed in a subsequent paper [24] (see also [61, Section 2.7]).

Our second motivation comes from the geometric theory of PDEs. Geometrically, a PDE is a submanifold in a jet space. Symmetries provide important geometric invariants of a PDE. Symmetries of a PDE form a Lie pseudogroup and the latter can be seen as a Lie subgroupoid of a jet groupoid. There are several natural (multi)DOs and jets attached to a jet space, in particular a jet groupoid [10, 16, 54, 66]. One should expect that natural DOs/jets on jet groupoids are also multiplicative, so falling into the class our program aims to define and study. Additionally, zero loci of multiplicative DOs will provide an important class of PDEs on sections of vector bundles over Lie groupoids.

Finally, it is worth to mention that it has been recently shown that VB-groupoids can be seen as linear atlases on vector bundles over differentiable stacks [18], so paving the way for a theory of vector bundles over singular spaces such as orbifolds, leaf spaces of foliations, orbit spaces of group actions, etc. This paper can then be seen as a study of *derivations of vector bundles over stacks*.

The paper is divided into four main sections. Section 1 is a short review of VB-groupoids and VB-algebroids. Besides the book by Mackenzie [46], our main references for this material are [7, 27, 28]. In Section 2 we define and begin our study of *multiplicative derivations*. Let $\mathcal{W} \rightrightarrows E$ be a VB-groupoid over a Lie groupoid $G \rightrightarrows M$, and let $W \rightrightarrows E$ be a VB-algebroid over a Lie algebroid $A \rightrightarrows M$. We also write $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ and $(W \rightrightarrows E; A \rightrightarrows M)$ for these data. We say that a derivation of $\mathcal{W} \rightarrow G$ is *multiplicative* if it generates a flow by VB-groupoid automorphisms. Similarly, a derivation of $W \rightarrow A$ is *infinitesimally multiplicative (IM)* if it generates a flow by VB-algebroid automorphisms. It immediately follows that if the Lie functor maps $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ to $(W \rightrightarrows E; A \rightrightarrows M)$, then it also maps multiplicative derivations of $\mathcal{W} \rightarrow G$ to IM derivations of $W \rightarrow A$. Additionally, if $G \rightrightarrows M$ is source simply connected (hence $\mathcal{W} \rightrightarrows E$ is source simply connected [7]) then the Lie functor establishes a one-to-one correspondence between multiplicative derivations of $\mathcal{W} \rightarrow G$ and IM derivations of $W \rightarrow A$. Our main result in this section is an algebraic description of IM derivations of a VB-algebroid $(W \rightrightarrows E; A \rightrightarrows M)$ (Theorem 2.24) which parallels a result by Mackenzie and Xu in [47] stating that there is a one-to-one correspondence between IM vector fields on a Lie algebroid and Lie algebroid derivations. We also prove that multiplicative derivations (resp. IM derivations) are 1-cocycles in a certain *linear subcomplex* of the *deformation complex* of the Lie groupoid $\mathcal{W}$ (resp. Lie algebroid W) (Proposition 2.28, resp. Corollary 2.32).

Derivations of a vector bundle $E \rightarrow M$ are sections of the *gauge algebroid* $DE \rightarrow M$ (see Section 2.1). Unfortunately, unlike the tangent bundle of a groupoid (resp. algebroid), the gauge algebroid of a VB-groupoid (resp. algebroid) is not a groupoid itself in general. Hence, unlike multiplicative vector fields [47], multiplicative (resp. IM) derivations of a VB-groupoid $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ (resp. algebroid $(W \rightrightarrows E; A \rightrightarrows M)$) can not be characterized, in general, as groupoid (resp. algebroid) morphisms $G \rightarrow D\mathcal{W}$ (resp. $A \rightarrow DW$). In Section 3, we isolate a particular class of VB-groupoids (resp. algebroids) for which the associated gauge algebroid is a VB-groupoid (resp. algebroid) itself,

namely either *trivial-core* or *full-core VB-groupoids* (resp. *algebroids*) and prove that in this case multiplicative (resp. IM) derivations can be characterized as morphic sections of the *gauge VB-groupoid* (resp. *algebroid*) (Theorems 3.17, 3.19). This parallels similar results for multiplicative (resp. IM) vector fields and the tangent VB-groupoid (resp. algebroid) [47]. Finally, Section 4 contains an application of the results of Section 3 to representations up to homotopy.

1 Preliminaries

1.1 VB-groupoids

In this section, we recall basic aspects of the theory of VB-groupoids and VB-algebroids. We refer to [7, 27, 28, 46] for more details. Given a Lie groupoid $G \rightrightarrows M$, we usually denote by $s, t : G \rightarrow M$ the source and target respectively, by $u : M \rightarrow G$ the unit, by $G^{(2)} = G \times_s G \subset G \times G$ the submanifold of composable arrows, by $m : G^{(2)} \rightarrow G$ the multiplication, and by $i : G \rightarrow G$ the inversion. Alternatively, we denote the unit by $x \mapsto 1_x$, the multiplication by $(g, h) \mapsto gh$, and the inversion by $g \mapsto g^{-1}$. For $g \in G$, we write $g : x_1 \rightarrow x_2$ to mean that $s(g) = x_1$ and $t(g) = x_2$. Finally, we identify M with its image under u . Consider a diagram

$$\begin{array}{ccc} \mathcal{W} & \rightrightarrows & E \\ q_{\mathcal{W}} \downarrow & & \downarrow q \\ G & \rightrightarrows & M \end{array} \quad (1.1)$$

where the rows are Lie groupoids and the columns are vector bundles. We denote by $\tilde{s}, \tilde{t}, \tilde{u}, \tilde{m}, \tilde{i}$ the structure maps of the top row (source, target, unit, multiplication, inversion respectively) and by s, t, u, m, i the structure maps of the bottom row. Additionally, we denote by $+_{\mathcal{W}} : \mathcal{W} \times_G \mathcal{W} \rightarrow \mathcal{W}$ and $+_E : E \times_M E \rightarrow E$ the additions, by $\lambda \cdot_{\mathcal{W}} : \mathcal{W} \rightarrow \mathcal{W}$ and $\lambda \cdot_E : E \rightarrow E$ the scalar multiplications by $\lambda \in \mathbb{R}$, by $0_{\mathcal{W}} : G \rightarrow \mathcal{W}$ and $0_E : M \rightarrow E$ the zero sections. Notice that throughout this paper, for any vector bundle $q : V \rightarrow N$, we will denote by s_x the value of a section $s \in \Gamma(V)$ at a point $x \in N$.

Definition 1.1 (VB-groupoid) A *VB-groupoid* is a vector bundle object in the category of Lie groupoids, i.e. a diagram like (1.1) such that $(q_{\mathcal{W}}, q)$ and $(+_{\mathcal{W}}, +_E)$ are Lie groupoid morphisms. Equivalently, a VB-groupoid is a *Lie groupoid object in the category of vector bundles*, i.e. a diagram (1.1) such that $(\tilde{s}, s)$, $(\tilde{t}, t)$ and $(\tilde{m}, m)$ are vector bundle morphisms.

It follows from the definition, that $(\lambda \cdot_{\mathcal{W}}, \lambda \cdot_E)$ and $(0_{\mathcal{W}}, 0_E)$ are also Lie groupoid morphisms, for all $\lambda \in \mathbb{R}$ and that $(\tilde{u}, u)$ and $(\tilde{i}, i)$ are also vector bundle morphisms. When there is no risk of ambiguity, we simply denote by $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ (or even just $\mathcal{W}$) a VB-groupoid, thus understanding all structure maps.

Definition 1.2 (VB-groupoid morphism) Let $(\mathcal{W}_1 \rightrightarrows E_1; G_1 \rightrightarrows M_1)$ and $(\mathcal{W}_2 \rightrightarrows E_2; G_2 \rightrightarrows M_2)$ be VB-groupoids. A *morphism of VB-groupoids*

$$(F_{\mathcal{W}}, \tilde{F}; F_G, F) : (\mathcal{W}_1 \rightrightarrows E_1; G_1 \rightrightarrows M_1) \rightarrow (\mathcal{W}_2 \rightrightarrows E_2; G_2 \rightrightarrows M_2) \quad (1.2)$$

is a (commutative) diagram

$$\begin{array}{ccccc}
 & & \mathcal{W}_1 & \xrightarrow{F_{\mathcal{W}}} & \mathcal{W}_2 \\
 & \swarrow & \downarrow & & \swarrow \\
 E_1 & \xrightarrow{\tilde{F}} & E_2 & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & \swarrow & G_1 & \xrightarrow{F_G} & G_2 \\
 & \downarrow & \downarrow & & \downarrow \\
 M_1 & \xrightarrow{F} & M_2 & &
 \end{array} \tag{1.3}$$

such that the following conditions are satisfied:

- i.) $(F_{\mathcal{W}}, \tilde{F})$ and (F_G, F) are morphisms of groupoids,
- ii.) $(F_{\mathcal{W}}, F_G)$ and $(\tilde{F}, F)$ are morphisms of vector bundles.

Notice that the map $F_{\mathcal{W}}$ in (1.2) determines the morphism completely. Accordingly, we also say that $F_{\mathcal{W}} : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ is a morphism of VB-groupoids.

The (*right*) *core* of a VB-groupoid $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ is the vector bundle $C \rightarrow M$ defined by setting

$$C := \{\omega \in \mathcal{W} : \tilde{s}(\omega) = 0, \text{ and } q_{\mathcal{W}}(\omega) = 1\}. \tag{1.4}$$

In other words, $C = u^*(\ker \tilde{s}) = \ker \tilde{s}|_M$ is the pull-back of the vector subbundle $\ker \tilde{s} \subset \mathcal{W}$ along the unit. There is a similar notion of left core where the target is in place of the source. It turns out that the right and the left cores are canonically isomorphic as vector bundles. Hence they are manifestations of a unique vector bundle which can be safely referred to as the *core*.

The dual vector bundle $\mathcal{W}^*$ is a Lie groupoid over C^* , and, actually, $(\mathcal{W}^* \rightrightarrows C^*; G \rightrightarrows M)$ is a VB-groupoid, the *dual VB-groupoid* of $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$. Additionally the VB-groupoids $\mathcal{W}$ and $\mathcal{W}^{**}$ are canonically isomorphic. We refer to [46] for more details on this construction. Here we only notice that given a VB-groupoid isomorphism $F_{\mathcal{W}} : \mathcal{W}_1 \rightarrow \mathcal{W}_2$, its transpose $F_{\mathcal{W}}^* : \mathcal{W}_2^* \rightarrow \mathcal{W}_1^*$ is a VB-groupoid isomorphism as well. In particular, VB-groupoid automorphisms $\mathcal{W} \rightarrow \mathcal{W}$ are equivalent to VB-groupoid automorphisms $\mathcal{W}^* \rightarrow \mathcal{W}^*$.

Example 1.3 (Tangent VB-groupoid) Let $G \rightrightarrows M$ be a Lie groupoid with Lie algebroid A . Then $(TG, TM; G, M)$ is a VB-groupoid, called the *tangent VB-groupoid to G* , in a natural way: the structure maps are the tangent maps to the structure maps of G . The core of $(TG \rightrightarrows TM; G \rightrightarrows M)$ is A .

1.2 VB-algebroids

We now turn to the infinitesimal counterpart of VB-groupoids, the so-called VB-algebroids. First of all, given a Lie algebroid $A \Rightarrow M$, we usually denote by $\rho : A \rightarrow TM$ the anchor and by $[-, -] : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$ the Lie bracket. Consider a diagram

$$\begin{array}{ccc}
 W & \xrightarrow{\tilde{p}} & E \\
 q_W \downarrow & & \downarrow q \\
 A & \xrightarrow{p} & M
 \end{array} \tag{1.5}$$

where the rows and the columns are vector bundles. We denote by $+_{W,A} : W \times_A W \rightarrow W$, $+_{W,E} : W \times_E W \rightarrow W$ the additions, by $\lambda_{\cdot, W, A} : W \rightarrow W$ and $\lambda_{\cdot, W, E} : W \rightarrow W$ the scalar multiplications by

$\lambda \in \mathbb{R}$ (in the fibers over A and E respectively), and by $0_{W,A} : A \rightarrow W$ and $0_{W,E} : E \rightarrow W$ the zero sections. If the maps (q_W, q) and $(+_{W,A}, +_E)$ are vector bundle morphisms, or, equivalently $(\tilde{p}, p)$ and $(+_{W,E}, +_A)$ are vector bundle morphisms, then (1.5) is a *double vector bundle* [46, Chapter 9], denoted $(W \rightarrow E; A \rightarrow M)$, and it follows that $(\lambda \cdot_{W,A}, \lambda \cdot_E)$, $(0_{W,A}, 0_E)$, $(\lambda \cdot_{W,E}, \lambda \cdot_A)$ and $(0_{W,E}, 0_A)$ are also vector bundle morphisms.

Remark 1.4 (Core and linear sections of a double vector bundle) Recall that the core of a double vector bundle (1.5) is the vector bundle

$$C := \ker \tilde{p} \cap \ker q_W \rightarrow M. \quad (1.6)$$

Vector bundle $W \rightarrow E$ has two distinguished classes of sections:

i.) *core sections*: for every section χ of the core $C \rightarrow M$, there is a section $\hat{\chi}$ of $W \rightarrow E$ defined by:

$$\hat{\chi}_e := (0_{W,E})_e +_{W,A} \chi_{q(e)} \quad (1.7)$$

for all $e \in E$.

ii.) *linear sections*: they are sections $\tilde{a}$ of $W \rightarrow E$ such that $(\tilde{a}, a)$ is a vector bundle morphism for a (necessarily unique) section a of A .

We denote by $\Gamma(W, E)$ the space of all sections of $W \rightarrow E$, by $\Gamma_c(W, E)$ the subspace of core sections and by $\Gamma_\ell(W, E)$ the subspace of linear sections. Linear and core sections generate $\Gamma(W, E)$ as a $C^\infty(E)$ -module. Actually, linear sections alone generate $\Gamma(W, E)$ away from the (image of) the zero section of E , i.e., for every point $e \in E \setminus 0_E(M)$, there exists an open neighborhood $\mathcal{U}$ of e such that the $C^\infty(\mathcal{U})$ -module of local sections of $W \rightarrow E$ defined on $\mathcal{U}$ is spanned by linear sections. This is easily seen in local coordinates adapted to the double vector bundle structure.

From now on, we also assume that the rows of (1.5) possess Lie algebroid structures $W \rightrightarrows E$, and $A \rightrightarrows M$. We denote by $\tilde{\rho} : W \rightarrow TE$ the anchor.

Definition 1.5 (VB-algebroid) A *VB-algebroid* is a *vector bundle object in the category of Lie algebroids*, i.e. a diagram like (1.5) such that (q_W, q) and $(+_{W,A}, +_E)$ are Lie algebroid morphisms. Equivalently, a VB-algebroid is a *Lie algebroid object in the category of vector bundles*, i.e. a diagram (1.5) such that $(\tilde{p}, p)$, $(+_{W,E}, +_A)$ and $(\tilde{\rho}, \rho)$ are morphisms of vector bundles, and the Lie bracket in $\Gamma(W, E)$ satisfies the following *linearity conditions*:

- i.) $[\Gamma_\ell(W, E), \Gamma_\ell(W, E)] \subset \Gamma_\ell(W, E)$,
- ii.) $[\Gamma_\ell(W, E), \Gamma_c(W, E)] \subset \Gamma_c(W, E)$,
- iii.) $[\Gamma_c(W, E), \Gamma_c(W, E)] = 0$.

We denote simply by $(W \rightrightarrows E; A \rightrightarrows M)$ a VB-algebroid and when there is no risk of ambiguity we only use W .

Definition 1.6 (VB-algebroid morphism) Let $(W_1 \rightrightarrows E_1; A_1 \rightrightarrows M_1)$ and $(W_2 \rightrightarrows E_2; A_2 \rightrightarrows M_2)$ be VB-algebroids. A *morphism of VB-algebroids*

$$(F_W, \tilde{F}; F_A, F) : (W_1 \rightrightarrows E_1; A_1 \rightrightarrows M_1) \rightarrow (W_2 \rightrightarrows E_2; A_2 \rightrightarrows M_2) \quad (1.8)$$

is a (commutative) diagram

$$\begin{array}{ccccc}
 & W_1 & \xrightarrow{F_W} & W_2 & \\
 & \swarrow & & \swarrow & \\
 E_1 & \xrightarrow{\tilde{F}} & E_2 & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & A_1 & \xrightarrow{F_A} & A_2 & \\
 \downarrow & & \downarrow & & \downarrow \\
 M_1 & \xrightarrow{F} & M_2 & &
 \end{array} \tag{1.9}$$

such that the following conditions are satisfied:

- i.) $(F_W, \tilde{F})$ and (F_A, F) are morphisms of Lie algebroids
- ii.) (F_W, F_A) and $(\tilde{F}, F)$ are morphisms of vector bundles.

The map F_W in (1.9) determines the morphism completely. Accordingly, we sometimes say that $F_W : W_1 \rightarrow W_2$ is a morphism of VB-algebroids.

Similarly as VB-groupoids, VB-algebroids can be dualized. Let $(W \Rightarrow E; A \Rightarrow M)$ be a VB-algebroid with core $C \rightarrow M$. Then the dual W^* of the vector bundle $W \rightarrow A$ is a VB-algebroid and $(W^* \Rightarrow C^*; A \Rightarrow M)$ is a VB-algebroid, called the *dual* VB-algebroid of $(W \Rightarrow E; A \Rightarrow M)$. Additionally the VB-algebroids W and W^{**} are canonically isomorphic. We refer to [46] again for more details. Here we only notice that similarly as for VB-groupoids, given a VB-algebroid isomorphism $F_W : W_1 \rightarrow W_2$, its transpose $F_W^* : W_2^* \rightarrow W_1^*$ is a VB-algebroid isomorphism as well. Hence VB-algebroid automorphisms $W \rightarrow W$ are equivalent to VB-algebroid automorphisms $W^* \rightarrow W^*$.

Example 1.7 Let $E \rightarrow M$ be a vector bundle. Then $(TE \Rightarrow E; TM \Rightarrow M)$ is a VB-algebroid in the obvious way.

Example 1.8 (Tangent VB-algebroid) Let $A \Rightarrow M$ be a Lie algebroid. Then $(TA \Rightarrow TM; A \Rightarrow M)$ is a VB-algebroid, called the *tangent VB-algebroid to A* , in a natural way. The projection $TA \rightarrow TM$ and the anchor $\tilde{\rho} : TA \rightarrow TTM$ are the tangent map to the projection $A \rightarrow M$, and the composition of the tangent map to the anchor $\rho : A \rightarrow TM$, followed by the canonical *involution* $TTM \rightarrow TTM$ (see, e.g., [46, Section 9.6]). The Lie bracket

$$[-, -] : \Gamma(TA, TM) \times \Gamma(TA, TM) \rightarrow \Gamma(TA, TM) \tag{1.10}$$

is uniquely defined by its action on certain distinguished sections, as follows. First notice that $(TA \rightarrow TM; A \rightarrow M)$ is a double vector bundle with core canonically isomorphic to A itself. The isomorphism identifies the tangent space to a fiber of $A \rightarrow M$ at the origin, with the fiber itself. Now, consider the following distinguished classes of sections of $TA \rightarrow TM$:

- i.) core sections,
- ii.) linear sections of the form $da : TM \rightarrow TA$ for some $a \in \Gamma(A)$.

We denote by $\hat{a}_T : TM \rightarrow TA$ the core section corresponding to $a \in \Gamma(A)$, seen as a section of the core. We have

$$(\hat{a}_T)_v = (d0_A)(v) + a_{(0_A)_x}^\uparrow \tag{1.11}$$

for all $v \in T_x M$, $x \in M$, where $a^\uparrow \in \mathfrak{X}(A)$ is the *vertical lift of a* (see also Remark 2.1). Then bracket (1.10) is completely determined by

$$\begin{aligned} [da, db] &= d[a, b], \\ [da, \widehat{b}_T] &= [\widehat{a}, \widehat{b}]_T, \\ [\widehat{a}_T, \widehat{b}_T] &= 0, \end{aligned} \tag{1.12}$$

for all $a, b \in \Gamma(A)$. Finally, notice that the VB-algebroid structures on $(TA \rightrightarrows A; TM \rightrightarrows M)$ and $(TA \rightrightarrows TM; A \rightrightarrows M)$ are compatible in the sense that they provide an instance of a *double Lie algebroid* [44, 45].

Remark 1.9 (VB-groupoids/algebroids and homogeneity structures) There is an alternative definition of a VB-groupoid/algebroid that is often useful in practice. Namely, a vector bundle is the same as a manifold E equipped with a *regular homogeneity structure*, i.e. a *regular action* $h : \mathbb{R} \times E \rightarrow E$, written $(\lambda, e) \mapsto h_\lambda(e)$, of the monoid $(\mathbb{R}, \cdot)$. The base manifold is then $M = h_0(E)$ and the projection $E \rightarrow M$ is h_0 . The infinitesimal generator of $h_{\exp \lambda}$ is the *Euler vector field* on E . We will not explain here the word *regular*: we refer to [26] for more details (see also [7, Definition 2.1.1]). Similarly, a VB-groupoid (resp. algebroid) is equivalent to a groupoid $\mathcal{W} \rightrightarrows E$ (resp. algebroid $W \rightarrow E$) equipped with a regular action $h : \mathbb{R} \times \mathcal{W} \rightarrow \mathcal{W}$ (resp. $h : \mathbb{R} \times W \rightarrow W$) of the monoid $(\mathbb{R}, \cdot)$ with the additional property that $h_\lambda : \mathcal{W} \rightarrow \mathcal{W}$ (resp. $h_\lambda : W \rightarrow W$) is a Lie groupoid (resp. algebroid) morphism for all t . In particular, a diagram like (1.1) is a VB-groupoid if and only if $\lambda \cdot_{\mathcal{W}} : \mathcal{W} \rightarrow \mathcal{W}$ (multiplication by λ in the fibers of $\mathcal{W} \rightarrow G$) is a groupoid morphism for all λ , and similarly for VB-algebroids. Notice that using this point of view a diagram like (1.2) (resp. (1.8)) is a morphism if and only if $F_{\mathcal{W}}$ (resp. F_W) commutes with the homogeneity structures.

1.3 Differentiation and integration

We conclude this section discussing briefly *differentiation* and *integration* issues. We refer to [7] for details. Denote by Lie the Lie functor mapping Lie groupoids to Lie algebroids (and Lie groupoid morphisms to Lie algebroid morphisms). Let $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ be a VB-groupoid. We denote by $W = \text{Lie}(\mathcal{W})$ the Lie algebroid of $\mathcal{W}$, and by $A = \text{Lie}(G)$ the Lie algebroid of G . Then $(W \rightrightarrows E; A \rightrightarrows M)$ is a VB-algebroid in a natural (and obvious) way, and it is called the *VB-algebroid of $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$* . We also denote the latter by $\text{Lie}(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$. Additionally,

- i.) $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ and $(W \rightrightarrows E; A \rightrightarrows M)$ have canonically isomorphic cores (we denote both by C), and
- ii.) $\text{Lie}(\mathcal{W}^* \rightrightarrows C^*; G \rightrightarrows M) \cong (W^* \rightrightarrows C^*; A \rightrightarrows M)$ canonically.

Similarly, the Lie functor maps morphisms of VB-groupoids to morphisms of VB-algebroids. Conversely, let $(W \rightrightarrows E; A \rightrightarrows M)$ be a VB-algebroid. If W is integrable and $\mathcal{W} \rightrightarrows E$ is its source simply connected integration, then 1) A is integrable as well, and, denoted by $G \rightrightarrows M$ its source simply connected integration, 2) $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ is a VB-groupoid in a natural way. Finally, if $(\mathcal{W}' \rightrightarrows E'; G' \rightrightarrows M')$ is another VB-groupoid, $(W' \rightrightarrows E'; A' \rightrightarrows M')$ is its VB-algebroid and $F_W : W \rightarrow W'$ is a morphism of VB-algebroids, then F_W integrates to a unique morphism $F_{\mathcal{W}} : \mathcal{W} \rightarrow \mathcal{W}'$ of VB-groupoids.

Remark 1.10 Let $G \rightrightarrows M$ be a Lie groupoid and let $A = \text{Lie}(G)$ be its Lie algebroid. Then the VB-algebroid of the tangent VB-groupoid to G , is (canonically isomorphic to) the tangent VB-algebroid to A : $\text{Lie}(TG \rightrightarrows TM; G \rightrightarrows M) \cong (TA \rightrightarrows TM; A \rightrightarrows M)$.

2 Infinitesimal automorphisms of VB-groupoids and algebroids

2.1 Derivations

Let $E \rightarrow M$ be a vector bundle, and let $x \in M$. A *derivation* at x is an $\mathbb{R}$ -linear map $\Delta : \Gamma(E) \rightarrow E_x$ such that

$$\Delta(fe) = v(f)e_x + f(x)\Delta(e) \quad (2.1)$$

for all $f \in C^\infty(M)$, $e \in \Gamma(E)$ and some (necessarily unique) tangent vector $v \in T_x M$. We also call v the symbol of Δ , and denote it by $\sigma(\Delta)$.

It is useful to mention that there is an alternative (more geometric) description of derivations at x . Let $\varepsilon \mapsto \gamma(\varepsilon)$ be a smooth curve in M such that $\gamma(0) = x$ and let $\varepsilon \mapsto \gamma_E(\varepsilon) : E_x \rightarrow E_{\gamma(\varepsilon)}$ be a smooth curve of linear isomorphisms (covering γ) such that $\gamma_E(0) = \text{id}_{E_x}$. The *velocity* of γ_E is the following derivation at x

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \gamma_E(\varepsilon) : \Gamma(E) \rightarrow E_x, \quad e \mapsto \left(\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \gamma_E(\varepsilon) \right) e := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \gamma_E(\varepsilon)^{-1} e_{\gamma(\varepsilon)}. \quad (2.2)$$

Additionally, the symbol of $\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \gamma_E(\varepsilon)$ is $\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \gamma(\varepsilon)$. Every derivation at x arises in this way.

Derivations of E at x form a vector space denoted $D_x E$. If we let x vary, we get a vector bundle $DE \rightarrow M$. Sections of DE identify with *derivations* of E , i.e. $\mathbb{R}$ -linear operators $\Delta : \Gamma(E) \rightarrow \Gamma(E)$ such that

$$\Delta(fe) = X(f)e + f\Delta(e) \quad (2.3)$$

for all $f \in C^\infty(M)$, $e \in \Gamma(E)$, and some (necessarily unique) vector field $X \in \mathfrak{X}(M)$. We call X the *symbol* of Δ and denote it by $\sigma(\Delta)$ or σ_Δ . In the literature, derivations of a vector bundle are referred to with different names, including *der-operators* [20] and *covariant differential operators* [46]. From now on we identify sections of $DE \rightarrow M$ and derivations of E in the obvious way. In particular, we denote by $\Gamma(DE)$ the space of derivations of E .

The vector bundle DE carries a structure of Lie algebroid: the *Atiyah* or *gauge algebroid* of E . The Lie bracket of two derivations is the commutator and the anchor is the symbol $\sigma : DE \rightarrow TM$, $\Delta \mapsto \sigma(\Delta)$. Notice that a *representation* of a Lie algebroid $A \rightarrow M$ can be regarded as a vector bundle $E \rightarrow M$ equipped with a Lie algebroid morphism $A \rightarrow DE$. The kernel of the symbol $\sigma : DE \rightarrow TM$ is the endomorphism bundle of E . Accordingly, there is a short exact sequence of vector bundles:

$$0 \longrightarrow \text{End } E \longrightarrow DE \xrightarrow{\sigma} TM \longrightarrow 0. \quad (2.4)$$

The correspondence $E \mapsto DE$ is functorial. To see this, first recall that a vector bundle morphism is called *regular* if it is an isomorphism on fibers. A regular vector bundle morphism $\phi_E : E_N \rightarrow E$ covering a smooth map $\phi : N \rightarrow M$ allows us to *pull-back* sections of E to sections of E_N . Specifically, let $e \in \Gamma(E)$. Its pull-back along ϕ_E is the section $\phi_E^* e \in \Gamma(E_N)$ defined by

$$(\phi_E^* e)_x = \phi_E|_{(E_N)_x}^{-1}(e_{\phi(x)}), \quad (2.5)$$

for all $x \in N$. Notice that we can use a regular vector bundle morphism $\phi_E : E_N \rightarrow E$ covering $\phi : N \rightarrow M$ to identify E_N with the pull-back bundle $\phi^* E = N \times_M E$ in a canonical way. In the following, we always make use of this identification. In particular, we denote $\phi^* E = E_N$, and denote by $\phi_E : \phi^* E \rightarrow E$ the canonical map. Sometimes, if there is no risk of confusion, we also denote simply by $\phi^* e$ the pull-back section $\phi_E^* e$. Now, let $E \rightarrow M$ and $E_N \rightarrow N$ be vector bundles, and let $\phi_E : E_N \rightarrow E$ be a *regular* vector bundle morphism, covering the smooth map $\phi : N \rightarrow M$. Morphism ϕ_E determines a (generically non regular) vector bundle morphism

$$D\phi_E : DE_N \rightarrow DE, \quad (2.6)$$

covering $\phi : N \rightarrow M$ by

$$D\phi_E(\Delta)(e) = (\phi_E \circ \Delta)(\phi^*e) \quad (2.7)$$

for all $\Delta \in DE_N$ and $e \in \Gamma(E)$. If

$$\Delta = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \gamma_E(\varepsilon) \quad (2.8)$$

for some curve $\varepsilon \mapsto \gamma_E(\varepsilon)$ of fiber isomorphisms, then

$$D\phi_E(\Delta) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \phi_E \circ \gamma_E(\varepsilon) \circ \phi_E|_{(E_N)_x}^{-1}, \quad (2.9)$$

where x is the base point of Δ , i.e. $\Delta \in D_x E_N$.

Clearly, diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{End } E_N & \longrightarrow & DE_N & \xrightarrow{\sigma} & TN \longrightarrow 0 \\ & & \downarrow \text{End } \phi & & \downarrow D\phi_E & & \downarrow d\phi \\ 0 & \longrightarrow & \text{End } E & \longrightarrow & DE & \xrightarrow{\sigma} & TM \longrightarrow 0 \end{array} \quad (2.10)$$

commutes. Here $\text{End } \phi = \phi_{\text{End } E} : \text{End } E \rightarrow \text{End } E$ is the obvious regular vector bundle morphism (covering ϕ). Even more, $D\phi_E : DE_N \rightarrow DE$ is actually a morphism of Lie algebroids (covering ϕ). We will often denote $D\phi_E$ simply by $D\phi$, if there is no risk of confusion.

It is easy to see that the rightmost square in the diagram (2.10) is a pull-back diagram. In particular, DE_N is canonically isomorphic to a pull-back bundle:

$$DE_N \cong TN {}_{d\phi} \sigma DE. \quad (2.11)$$

More precisely, DE_N is the pull-back Lie algebroid of DE along the smooth map $\phi : N \rightarrow M$ (which makes sense, because, as the anchor $\sigma : DE \rightarrow TM$ is surjective, ϕ is automatically transverse to it).

The case when $\phi : N \rightarrow M$ is the inclusion of a submanifold is of a special interest. In this case $E_N = E|_N$ and $D\phi : DE_N \rightarrow DE$ is an embedding whose image consists of derivations of E with symbol tangent to N .

There is another way to understand derivations of a vector bundle $p : E \rightarrow M$. They can be regarded as *infinitesimal automorphisms* of E . A vector field $\tilde{X}$ on the total space E of the vector bundle $E \rightarrow M$ is an infinitesimal automorphism if it generates a flow by vector bundle automorphisms. In particular, if $\tilde{X}$ is an infinitesimal automorphism of E , then it projects onto a vector field $p_*\tilde{X}$ on M .

Remark 2.1 There are several characterizations of infinitesimal automorphisms of the vector bundle $E \rightarrow M$ that we now discuss. First of all, let $h : \mathbb{R} \times E \rightarrow E$ be the (regular) homogeneity structure encoding the vector bundle structure (see Remark 1.9), and let $\mathcal{E}_E$ be the infinitesimal generator of $h_{\exp \lambda}$, i.e. the Euler vector field on E . Take $\tilde{X}$ a vector field on E . The following conditions are equivalent:

- i.) $\tilde{X}$ is an infinitesimal automorphism;
- ii.) for every fiber-wise linear function φ on E , the Lie derivative $\tilde{X}(\varphi)$ is fiber-wise linear;
- iii.) for every fiber-wise constant vector field Y on E , the commutator $[\tilde{X}, Y]$ is fiber-wise constant;
- iv.) $(h_\lambda)_* X = X$ for all $\lambda \neq 0$;
- v.) $[\mathcal{E}_E, X] = 0$;

vi.) when regarded as a section of $TE \rightarrow E$, $\tilde{X}$ is a *linear section* of (the top row) of the double vector bundle $(TE \rightarrow E; TM \rightarrow M)$, i.e. $\tilde{X} \in \Gamma_\ell(TE, E)$.

Point iii.) needs some explanations. First of all, there is a canonical one-to-one correspondence between fiber-wise linear functions on E and sections of the dual vector bundle $E^* \rightarrow M$. Now, let e be a section of E , and let $e^\uparrow \in \mathfrak{X}(E)$ be its vertical lift, i.e. $e^\uparrow$ is the vector field defined by

$$e^\uparrow_{e_0} := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} (e_0 + \varepsilon e_{p(e_0)}), \quad (2.12)$$

for all $e_0 \in E$. Notice that $e^\uparrow$ is uniquely determined by the identity

$$e^\uparrow(\varphi) = \langle \varphi, e \rangle \quad (2.13)$$

where $\varphi \in \Gamma(E^*)$ is also understood as a fiber-wise linear function on E . A *fiber-wise constant* vector field on E is a vector field of the form $e^\uparrow$.

If $\tilde{X}$ satisfies one of the above equivalent conditions we also say that $\tilde{X}$ is *linear*.

We conclude this remark noticing that a function $f \in C^\infty(E)$ is fiber-wise constant if and only if $h_\lambda^* f = f$, and it is fiber-wise linear if and only if $h_\lambda^* f = \lambda \cdot f$ for all λ . Equivalently, f is fiber-wise constant if and only if $\mathcal{E}_E(f) = 0$, and it is fiber-wise linear if and only if $\mathcal{E}_E(f) = f$. Similarly, a vector field $\tilde{X} \in \mathfrak{X}(E)$ is fiber-wise constant if and only if $(h_\lambda)_* \tilde{X} = \lambda \cdot \tilde{X}$, and (as already remarked) it is linear if and only if $(h_\lambda)_* \tilde{X} = \tilde{X}$, for all $\lambda \neq 0$. Equivalently, $\tilde{X}$ is fiber-wise constant if and only if $[\mathcal{E}_E, \tilde{X}] = -\tilde{X}$, and it is linear if and only if $[\mathcal{E}_E, \tilde{X}] = 0$.

The following lemma is familiar to experts.

Lemma 2.2 *Infinitesimal automorphisms of E correspond bijectively to derivations of E .*

PROOF: We report a proof for completeness. Recall that the gauge algebroids DE and DE^* are canonically isomorphic. The isomorphism identifies a derivation $\Delta : \Gamma(E) \rightarrow \Gamma(E)$ with derivation Δ^* of E^* defined by

$$\langle \Delta^* \varphi, e \rangle = \sigma(\Delta) \langle \varphi, e \rangle - \langle \varphi, \Delta e \rangle, \quad (2.14)$$

for all $\varphi \in \Gamma(E^*)$ and $e \in \Gamma(E)$, where $\langle -, - \rangle : E^* \otimes E \rightarrow \mathbb{R}$ is the duality pairing. Here $\mathbb{R}_M := M \times \mathbb{R} \rightarrow M$ denotes the trivial line bundle. It is easy to see that $\Delta^{**} = \Delta$ for all $\Delta \in \Gamma(DE)$. Now, it follows from point ii.) in Remark 2.1 that there is a bijection between infinitesimal automorphisms and derivations of E , mapping $\tilde{X}$ to $\Delta_{\tilde{X}} := \tilde{X}|_{\Gamma(E^*)}^*$ (where we identify fiber-wise linear functions on E and sections of E^*). $\square$

Let $\tilde{X}$ and $\Delta_{\tilde{X}}$ be as in the proof of Lemma 2.2. Clearly, $\sigma(\Delta_{\tilde{X}}) = p_* \tilde{X}$. Additionally, bijection $\tilde{X} \mapsto \Delta_{\tilde{X}}$ is $C^\infty(M)$ -linear and commutator-preserving. It follows that the flow $\{\tilde{\phi}_\varepsilon\}_\varepsilon$ of $\tilde{X}$ is uniquely determined by the following ODE

$$\Delta e = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \tilde{\phi}_\varepsilon^* e \quad (2.15)$$

In this sense, Δ generates the flow $\{\tilde{\phi}_\varepsilon\}_\varepsilon$ of automorphisms of E . We denote by $\Delta \mapsto X_\Delta$ the inverse correspondence.

Remark 2.3 There is another way to describe the bijection $\tilde{X} \mapsto \Delta_{\tilde{X}}$, which is often useful. Namely, let $\tilde{X}$ be an infinitesimal automorphism of E . Then, it follows from point iii.) in Remark 2.1 that the derivation $\Delta_{\tilde{X}}$ is uniquely determined by

$$(\Delta_{\tilde{X}} e)^\uparrow = [\tilde{X}, e^\uparrow] \quad (2.16)$$

for all $e \in \Gamma(E)$.

In view of the Lemma 2.2, we can make sense of “compatibility of two derivations with respect to a, non-necessarily regular, vector bundle morphism”. Namely, let $E_1 \rightarrow M_1$ and $E_2 \rightarrow M_2$ be vector bundles, and let $\tilde{F} : E_1 \rightarrow E_2$ that be a vector bundle morphism covering a smooth map $F : M_1 \rightarrow M_2$. If Δ_1 and Δ_2 are derivations of E_1 and E_2 , respectively, we say that Δ_1 and Δ_2 are $\tilde{F}$ -related if X_{Δ_1} and X_{Δ_2} are $\tilde{F}$ -related. The latter condition can be actually restated in terms of the sole Δ_1, Δ_2 . Indeed, the pull-back along $\tilde{F}$ maps fiber-wise linear functions on E_2 to fiber-wise linear functions on E_1 . Accordingly, there is a well-defined map $\tilde{F}^* : \Gamma(E_2^*) \rightarrow \Gamma(E_1^*)$, and Δ_1 and Δ_2 are $\tilde{F}$ -related if and only if

$$\Delta_1^* \circ \tilde{F}^* = \tilde{F}^* \circ \Delta_2^*, \quad (2.17)$$

$$\sigma(\Delta_1) \circ F^* = F^* \circ \sigma(\Delta_2). \quad (2.18)$$

If $\tilde{F}$ is a surjective submersion, then (2.18) follows directly from (2.17), Δ_1 completely determines Δ_2 via (2.17) and we write $\Delta_2 = \tilde{F}_* \Delta_1$. In this case $\sigma(\Delta_2) = F_* \sigma(\Delta_1)$. On another hand, if $\tilde{F} : E_1 \rightarrow E_2$ is the inclusion of a subbundle (over a possibly smaller base), $\sigma(\Delta_2)$ is tangent to M_1 , Δ_2 completely determines Δ_1 and we write $\Delta_1 = \Delta_2|_{E_1}$. In this case $\sigma(\Delta_1) = \sigma(\Delta_2)|_{M_1}$.

2.2 Infinitesimal automorphisms and multiplicative derivations

Recall that a vector field $X \in \mathfrak{X}(G)$ on a Lie groupoid $G \rightrightarrows M$ is *multiplicative* if it generates a flow by groupoid automorphisms. We begin this section adapting this notion to the case of VB-groupoids. Let $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ be a VB-groupoid as in (1.1).

Definition 2.4 (Infinitesimal automorphisms of VB-groupoids) A vector field $X_{\mathcal{W}}$ on $\mathcal{W}$ is said to be an *infinitesimal automorphism* of $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ if it generates a flow by VB-groupoid automorphisms.

Example 2.5 Let $G \rightrightarrows M$ be a Lie groupoid and let X be a multiplicative vector field on G . Recall that the *tangent lift* of X is the vector field $\tilde{X}$ on TG whose flow is the differential $\{d\phi_t\}$ of the flow $\{\phi_t\}$ of X (see, e.g., [65]). Then, $\tilde{X}$ is an infinitesimal automorphism of the tangent VB-groupoid $(TG \rightrightarrows TM; G \rightrightarrows M)$.

Remark 2.6 Let $X_{\mathcal{W}} \in \mathfrak{X}(\mathcal{W})$ be an infinitesimal automorphism of $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$. It immediately follows from the definition that $X_{\mathcal{W}}$ projects onto vector fields

- $X_G = (q_{\mathcal{W}})_* X_{\mathcal{W}}$ on G ,
- $X_E = \tilde{s}_* X_{\mathcal{W}} = \tilde{t}_* X_{\mathcal{W}}$ on E ,
- $X = s_* X_G = t_* X_G = q_* X_E$ on M .

Again from the definition, $X_{\mathcal{W}}$ is tangent to the core C , and we denote by $X_C = X_{\mathcal{W}}|_C$ the restriction. We also have that X_C projects onto X . Additionally, X_G is a multiplicative vector field on G and X_E (resp. X_C) is an infinitesimal automorphism of the vector bundle $E \rightarrow M$ (resp. $C \rightarrow M$). From another point of view, $X_{\mathcal{W}}$ corresponds to a derivation $\Delta_{\mathcal{W}}$ of the vector bundle $\mathcal{W} \rightarrow G$, X_E corresponds to a derivation Δ_E of the vector bundle $E \rightarrow M$, and X_C corresponds to a derivation Δ_C of the vector bundle $C \rightarrow M$. Finally $\Delta_{\mathcal{W}}$ and Δ_E are both $\tilde{s}$ and $\tilde{t}$ -related: $\Delta_E = \tilde{s}_* \Delta_{\mathcal{W}} = \tilde{t}_* \Delta_{\mathcal{W}}$. Similarly, $\Delta_C = \Delta_{\mathcal{W}}|_C$.

Definition 2.7 (Multiplicative derivations) A derivation $\Delta_{\mathcal{W}}$ of $\mathcal{W} \rightarrow G$ is *multiplicative* if the corresponding linear vector field $X_{\Delta_{\mathcal{W}}} \in \mathfrak{X}(\mathcal{W})$ is an infinitesimal automorphism of the VB-groupoid $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$.

The map $\Delta_{\mathcal{W}} \mapsto X_{\Delta_{\mathcal{W}}}$ establishes a one-to-one correspondence between multiplicative derivations and infinitesimal automorphisms of $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$. Consider a multiplicative derivation $\Delta_{\mathcal{W}}$ of $\mathcal{W} \rightarrow G$. It follows from Remark 2.6 that

- there is a derivation Δ_E of $E \rightarrow M$ such that $\Delta_E = \tilde{s}_* \Delta_{\mathcal{W}} = \tilde{t}_* \Delta_{\mathcal{W}}$,
- there is a derivation Δ_C of $C \rightarrow M$ such that $\Delta_C = \Delta_{\mathcal{W}}|_C$,
- the symbol of $\Delta_{\mathcal{W}}$ is a multiplicative vector field on G projecting on the symbols of both Δ_E and Δ_C (via both s and t).

Let $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ be a VB-groupoid with VB-algebroid $(W \rightrightarrows E; A \rightrightarrows M)$. Take a linear section $\tilde{a} : E \rightarrow W$ covering a section a of $A \rightarrow M$, and let $\vec{\tilde{a}}$ and $\overleftarrow{\tilde{a}}$ be the associated right and left invariant vector fields on $\mathcal{W}$.

Proposition 2.8 *Both $\vec{\tilde{a}}$, $\overleftarrow{\tilde{a}}$ are linear vector fields on $\mathcal{W}$.*

PROOF: Consider $\vec{\tilde{a}}$ first. According to Remark 2.1, point *iv.*), it is enough to show that $(h_{\lambda})_* \vec{\tilde{a}} = \vec{\tilde{a}}$, for all $\lambda \neq 0$. So let $\omega \in \mathcal{W}$, with $\tilde{t}(\omega) = e$, and compute

$$(h_{\lambda*} \vec{\tilde{a}})_{\omega} = dh_{\lambda}(dR_{\lambda^{-1}\omega}(\tilde{a}_{\lambda^{-1}e})) = dR_{\omega}(dh_{\lambda}(\tilde{a}_{\lambda^{-1}e})) = dR_{\omega}(\tilde{a}_e) = \vec{\tilde{a}}_{\omega}, \quad (2.19)$$

where R denotes right translations in $\mathcal{W}$, and we used the linearity of $\tilde{m}$ and $\tilde{a}$ in the form of the identities $h_{\lambda} \circ R_{\lambda^{-1}\omega} = R_{\omega} \circ h_{\lambda}$ and $dh_{\lambda}(\tilde{a}_{\lambda^{-1}e}) = \tilde{a}_e$ respectively. A similar argument (with left translations in place of right translations) shows that $h_{\lambda*} \overleftarrow{\tilde{a}} = \overleftarrow{\tilde{a}}$, for all $\lambda \neq 0$, i.e. $\overleftarrow{\tilde{a}}$ is a linear vector field as well. $\square$

Remark 2.9 It follows from Proposition 2.8 that $\vec{\tilde{a}}$ and $\overleftarrow{\tilde{a}}$ correspond to derivations of $\mathcal{W} \rightarrow G$, denoted $\vec{\Delta}_{\tilde{a}}$ and $\overleftarrow{\Delta}_{\tilde{a}}$, respectively. Moreover, $\vec{\tilde{a}} + \overleftarrow{\tilde{a}}$ is a linear vector field which is additionally multiplicative. Hence it corresponds to a multiplicative derivation of $\mathcal{W}$, namely $\vec{\Delta}_{\tilde{a}} + \overleftarrow{\Delta}_{\tilde{a}}$, also denoted $\delta\tilde{a}$, and called an *internal derivation* of $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$. By duality, $\vec{\tilde{a}} + \overleftarrow{\tilde{a}}$ does also correspond to a multiplicative derivation of $\mathcal{W}^*$. Similarly, a linear section of $W^* \rightarrow C^*$ determines a multiplicative derivation of both $\mathcal{W}$ and $\mathcal{W}^*$.

Remark 2.10 There is an important characterization of multiplicative vector fields on a Lie groupoid $G \rightrightarrows M$. Namely, a vector field $X \in \mathfrak{X}(G)$ is multiplicative if and only if, when regarded as a section $X : G \rightarrow TG$ of the tangent VB-groupoid, it is a morphism of Lie groupoids [47]. This suggests the following definition. Given a VB-groupoid $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$, a section w of $\mathcal{W} \rightarrow G$ is *multiplicative* if it is a Lie groupoid morphism. So, a vector field $X_{\mathcal{W}} \in \mathfrak{X}(\mathcal{W})$ is an infinitesimal automorphism of $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ if and only if the following conditions are satisfied:

- i.) it is a linear vector field with respect to the vector bundle structure $\mathcal{W} \rightarrow G$,
- ii.) when regarded as a section $X_{\mathcal{W}}$ of $T\mathcal{W} \rightarrow \mathcal{W}$, it is a multiplicative section of the (tangent) VB-groupoid $(T\mathcal{W} \rightrightarrows TE; \mathcal{W} \rightrightarrows E)$.

Yet in other words, infinitesimal automorphisms of $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ are the same as morphisms between the VB-groupoids $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ and $(T\mathcal{W} \rightrightarrows TE; TG \rightrightarrows TM)$ inverting the projection $T\mathcal{W} \rightarrow \mathcal{W}$ on the right.

Remark 2.11 A similar characterization as for multiplicative vector fields (and infinitesimal automorphism of VB-groupoids, see Remark 2.10) cannot take place, in general, for multiplicative derivations. The reason is that the gauge algebroid $D\mathcal{W}$ is *not* a VB-groupoid itself, if $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ is

a *generic* VB-groupoid. In Section 3 we discuss two simple but interesting classes of VB-groupoids, those with either trivial core or trivial side vector bundle $0_M \rightarrow M$ (we call the latter *full-core* VB-groupoids). Trivial-core VB-groupoids $(E_G \rightrightarrows E; G \rightrightarrows M)$ come from representations E of Lie groupoids G and possess the nice property that their gauge algebroid DE_G fits in a VB-groupoid $(DE_G \rightrightarrows DE; G \rightrightarrows M)$ itself. It turns out that multiplicative derivations of a trivial-core VB-groupoid $(E_G \rightrightarrows E; G \rightrightarrows M)$ are the same as multiplicative sections of $DE_G \rightarrow G$ (see Theorem 3.17). As duality establishes a one-to-one correspondence between trivial-core and full-core VB-groupoids, similar considerations hold true for multiplicative derivation of full-core VB-groupoids.

Remark 2.12 We already have a simple and conceptually satisfactory description of infinitesimal automorphisms of a VB-groupoid $(W \rightrightarrows E; G \rightrightarrows M)$ as morphisms between the VB-groupoids $(W \rightrightarrows E; G \rightrightarrows M)$ and $(TW \rightrightarrows TE; TG \rightrightarrows TM)$ (Remark 2.10), so the reader may wonder: *why do we care so much about multiplicative derivations* (Remark 2.11)? There are at least two reasons for that. First of all, the target space TW of an infinitesimal automorphism is an example of a triple structure: it possesses a groupoid structure $TW \rightrightarrows TE$, and two compatible vector bundle structures $TW \rightarrow W$ and $TW \rightarrow TG$. All these structures play a role when dealing with infinitesimal automorphisms of $(W \rightrightarrows E; G \rightrightarrows M)$. This could potentially lead to complications, especially in view of the fact that a detailed theory of triple structures like TW is still unavailable (and developing it would take us too far from our original scopes). On the other hand, derivations provide a simpler, more algebraic, and somehow minimal description of the same objects. Second, and, probably, more important, the reader should remember that this paper represents just the first step in a program aiming at studying (higher order) DOs on groupoids and algebroids. In this respect, derivations are an instance of first order DOs. Now, derivations can be seen as linear vector fields, but there is no similar description for higher order DOs. So working with derivations directly is a better training to learn how to work with generic DOs.

We now pass to VB-algebroids. First recall that a vector field $X \in \mathfrak{X}(A)$ on a Lie algebroid $A \Rightarrow M$ is *infinitesimally multiplicative* (IM) if it generates a flow by Lie algebroid automorphisms. Let $(W \rightrightarrows E; A \rightrightarrows M)$ be a VB-algebroid. As in (1.5), we denote by $\tilde{p} : W \rightarrow E$, and $p : A \rightarrow M$ the projections.

Definition 2.13 (Infinitesimal automorphisms of VB-algebroids) A vector field X_W on W is an *infinitesimal automorphism* of $(W \rightrightarrows E; A \rightrightarrows M)$ if it generates a flow by VB-algebroid automorphisms.

Remark 2.14 Let $X_W \in \mathfrak{X}(W)$ be an infinitesimal automorphism of $(W \rightrightarrows E; A \rightrightarrows M)$. It immediately follows from the definition that X_W projects onto vector fields

- $X_A = (q_W)_* X_W$ on A ,
- $X_E = \tilde{p}_* X_W$ on E ,
- $X = p_* X_A = q_* X_E$ on M .

Again from the definition, X_W is tangent to the core C and we denote by $X_C = X_W|_C$ the restriction. We also have that X_C projects onto X . Additionally, X_A is an IM vector field on A , and X_E (resp. X_C) is an infinitesimal automorphism of the vector bundle $E \rightarrow M$ (resp. $C \rightarrow M$). From another point of view, X_W corresponds both to a derivation $\Delta_{W,A}$ of the vector bundle $W \rightarrow A$, and a derivation $\Delta_{W,E}$ of $W \rightarrow E$. In the same way, X_E corresponds to a derivation Δ_E of the vector bundle $E \rightarrow M$, X_C corresponds to a derivation Δ_C of the vector bundle $C \rightarrow M$, and X_A corresponds to a derivation Δ_A of A (sharing the same symbol as Δ_E and Δ_C). Finally $\Delta_{W,A}$ and Δ_E are $\tilde{p}$ -related, Δ_C agrees with the restrictions of both $\Delta_{W,A}$ and $\Delta_{W,E}$ to $C \rightarrow M$, and $\Delta_{W,E}$ and Δ_A are q_W -related: $\Delta_E = \tilde{p}_* \Delta_{W,A}$,

$\Delta_C = \Delta_{W,A}|_C = \Delta_{W,E}|_C$ and $\Delta_A = (q_W)_* \Delta_{W,E}$. We call $\Delta_{W,A}$ the *vertical derivation* and $\Delta_{W,E}$ the *horizontal derivation* corresponding to X_W . In the following we provide a *more algebraic* description of both Δ_E and Δ_C in terms of $\Delta_{W,E}$ (see Theorem 2.24).

Definition 2.15 (IM derivations) A derivation $\Delta_{W,A}$ of $W \rightarrow A$ is an *IM derivation* if the corresponding linear vector field $X_W \in \mathfrak{X}(W)$ is an infinitesimal automorphism of the VB-algebroid $(W \Rightarrow E; A \Rightarrow M)$.

Recall from [47] that if $A \Rightarrow M$ is a Lie algebroid, a *Lie algebroid derivation* of $A \Rightarrow M$ is a derivation Δ of $A \rightarrow M$ such that

$$\Delta[a, b] = [\Delta a, b] + [a, \Delta b], \quad (2.20)$$

for all $a, b \in \Gamma(A)$. It follows from (2.20) that Δ is also compatible with the anchor, i.e.

$$\rho(\Delta a) = [\sigma(\Delta), \rho(a)], \quad (2.21)$$

for all $a \in \Gamma(A)$. The commutator of two Lie algebroid derivations is clearly a Lie algebroid derivation as well. Now, the usual map $X \mapsto \Delta_X$ establishes a one-to-one, commutator-preserving, correspondence between IM vector fields on A and Lie algebroid derivations of A (see [47]). So, Lie algebroid derivations provide an algebraic description of IM vector fields. Notice that Lie algebroid derivations of A are the same as 1-differentials of A (cf. [29]).

Let $(W \Rightarrow E; A \Rightarrow M)$ be a VB-algebroid. Take a linear section $\tilde{a}$ of $W \rightarrow E$ and consider the adjoint operator $[\tilde{a}, -]$. It is a derivation of $W \rightarrow E$. Additionally, it is a Lie algebroid derivation. Thus, from the above discussion it follows that $X_{[\tilde{a}, -]}$ is an infinitesimal automorphism of $W \Rightarrow E$.

Proposition 2.16 *A vector field $X_{[\tilde{a}, -]}$ is an infinitesimal automorphism of the VB-algebroid $(W \Rightarrow E; A \Rightarrow M)$.*

Before proving Proposition 2.16 we need some general remarks which will be useful in the next Section 2.3 as well. Let $\mathcal{E}_{W,A}$ and $\mathcal{E}_{W,E}$ be the Euler vector fields of vector bundles $W \rightarrow A$ and $W \rightarrow E$, respectively. As W is a double vector bundle, $\mathcal{E}_{W,A}$ and $\mathcal{E}_{W,E}$ commute. In particular, it follows from Remark 2.1, point *v.*), that $\mathcal{E}_{W,A}$ is an infinitesimal automorphism of the vector bundle $W \rightarrow E$. Hence, it corresponds to a derivation $\Delta_{\mathcal{E}}$ of $W \rightarrow E$. The symbol of $\Delta_{\mathcal{E}}$ is $\tilde{p}_* \mathcal{E}_{W,A}$ and coincides with $\mathcal{E}_E$, the Euler vector field of E . The following lemma parallels the characterization of fiber-wise constant (resp. linear) functions and vector fields on a vector bundle in terms of the Euler vector field (as discussed in Remark 2.1) and can be easily proved in local coordinates adapted to the double vector bundle structure of W (see e.g. [26, 27]).

Lemma 2.17 *A section w of $W \rightarrow E$ is a core section if and only if $\Delta_{\mathcal{E}} w = -w$ and it is linear if and only if $\Delta_{\mathcal{E}} w = 0$. Additionally, there are no non-trivial sections w of $W \rightarrow E$ such that $\Delta_{\mathcal{E}} w = -iw$ for some integer $i > 1$.*

PROOF (OF PROPOSITION 2.16): As the vector field $X_{[\tilde{a}, -]}$ is already known to be an infinitesimal automorphism of the Lie algebroid $W \Rightarrow E$, it remains to check that it is linear with respect to the vector bundle structure $W \rightarrow A$. Now a vector field on W is linear with respect to $W \rightarrow A$ if and only if it commutes with $\mathcal{E}_{W,A}$ (cf. Remark 2.1, point *v.*). In order to see that $X_{[\tilde{a}, -]}$ commutes with $\mathcal{E}_{W,A}$ it is enough to check that the operator $[\tilde{a}, -]$ commutes with $\Delta_{\mathcal{E}}$ (cf. Lemma 2.2). In other words we have to show that the commutator $\square$ of $[\tilde{a}, -]$ and $\Delta_{\mathcal{E}}$ vanishes. To do this it is enough to check that

- i.)* the symbol of $\square$ vanishes,
- ii.)* $\square$ vanishes on generators (over $C^\infty(E)$).

Now, for any linear section $\tilde{a}$ of $W \rightarrow E$, the symbol of $[\tilde{a}, -]$ is $\tilde{\rho}(\tilde{a})$ and it is a linear vector field on E . On the other hand the symbol of Δ is the Euler vector field $\mathcal{E}_E$ of E . So $\sigma(\square) = [\tilde{\rho}(\tilde{a}), \mathcal{E}_E] = 0$. Finally, from Definition 1.5 and Lemma 2.17, $\square$ vanishes on both core and linear sections. $\square$

Remark 2.18 It follows from Proposition 2.16 that the vector field $X_{[\tilde{a}, -]}$ corresponds to an IM derivation of $W \rightarrow A$, denoted $\delta\tilde{a}$ and called an *internal derivation*, and by duality to an IM derivation of $W^* \rightarrow A$. Similarly, a linear section of $W^* \rightarrow C^*$ determines an IM derivation of both W and W^* .

Note that if $(W \rightrightarrows E; A \rightrightarrows M)$ is the VB-algebroid of a VB-groupoid $(\mathcal{W} \rightrightarrows W; G \rightrightarrows M)$, then for any $\tilde{a} \in \Gamma_\ell(W, E)$ the vector field $X_{[\tilde{a}, -]} \in \mathfrak{X}(W)$ is the image of $\overrightarrow{\tilde{a}} + \overleftarrow{\tilde{a}}$ under the Lie functor. We use the same symbol $\delta\tilde{a}$ to denote the derivations corresponding to both. It will be always clear from the context if we are referring to one or the other.

Remark 2.19 Let $A \rightarrow M$ be a Lie algebroid. As proved in [47], a vector field $X \in \mathfrak{X}(A)$ is IM if and only if, when regarded as a section $X : A \rightarrow TA$ of the tangent VB-algebroid, it is a morphism of Lie algebroids. This suggests the following definition. Given a VB-algebroid $(W \rightrightarrows E; A \rightrightarrows M)$ a section $s : A \rightarrow W$ of $W \rightarrow A$ is an *IM section* if it is a Lie algebroid morphism. Thus, similarly as for infinitesimal automorphisms of VB-groupoids (Remark 2.10), a vector field $X_W \in \mathfrak{X}(W)$ is an infinitesimal automorphism of $(W \rightrightarrows E; A \rightrightarrows M)$ if and only if it is a linear vector field with respect to the vector bundle structure $W \rightarrow A$, and when regarded as a section of $TW \rightarrow W$, it is an IM section of the tangent VB-algebroid $(TW \rightrightarrows TE; W \rightrightarrows E)$. In other words, infinitesimal automorphisms of $(W \rightrightarrows E; A \rightrightarrows M)$ are the same as morphisms between the VB-algebroids $(W \rightrightarrows E; A \rightrightarrows M)$ and $(TW \rightrightarrows TE; TA \rightrightarrows TM)$ inverting the projection $TW \rightarrow W$ on the right.

Remark 2.20 A similar characterization as for IM vector fields cannot take place, in general, for IM derivations because the gauge algebroid DW of $W \rightarrow A$ is *not* a VB-algebroid itself, when $(W \rightrightarrows E; A \rightrightarrows M)$ is a *generic* VB-algebroid. However, it turns out that the gauge algebroid DE_A of a *trivial-core* VB-algebroid $(E_A \rightrightarrows E; A \rightrightarrows M)$ (see Section 3) fits in a VB-algebroid $(DE_A, \rightrightarrows DE; A \rightrightarrows M)$ itself, and IM derivations of $(E_A \rightrightarrows E; A \rightrightarrows M)$ are the same as IM sections of $DE_A \rightarrow A$ (Theorem 3.19). Similar considerations hold true for IM derivations of trivial-core VB-algebroids.

The following proposition proves that IM derivations of VB-algebroids are the infinitesimal counterpart of multiplicative derivations of VB-groupoids. It follows immediately from Definitions 2.4, 2.7, 2.13, 2.15 and the general discussion in Subsection 1.3 (see [7] for more details).

Proposition 2.21 *Let $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ be a VB-groupoid with VB-algebroid $(W \rightrightarrows E; A \rightrightarrows M)$. The Lie functor maps infinitesimal automorphisms of $\mathcal{W}$ to infinitesimal automorphisms of W . If, additionally, $\mathcal{W}$ is source simply connected, then any infinitesimal automorphism of W integrates to a unique infinitesimal automorphism of $\mathcal{W}$. In particular, there is a one-to-one correspondence between multiplicative derivations of $\mathcal{W}$ and IM derivations of W .*

We conclude this subsection presenting an alternative description of IM derivations of a VB-algebroid in the same spirit as the algebraic description of IM vector fields on a Lie algebroid provided by Lie algebroid derivations.

Remark 2.22 Let $G \rightrightarrows M$ be a Lie groupoid with Lie algebroid $A \rightrightarrows M$. The Lie functor maps a multiplicative vector field X_G on G to an IM vector field X_A on A . In its turn X_A corresponds to a Lie algebroid derivation Δ_A of A . It has been shown in [47, Theorem 3.9] that for every right invariant vector field Y on G , $[X_G, Y]$ is a right invariant vector field and, more importantly, X_G and Δ_A are related by the following formula:

$$\overrightarrow{\Delta_A a} = [X_G, \overrightarrow{a}], \quad (2.22)$$

for all $a \in \Gamma(A)$.

Now, let $(W \Rightarrow E; A \Rightarrow M)$ be a VB-algebroid, and let X_W be an infinitesimal automorphism of it. The horizontal derivation $\Delta_{W,E}$ corresponding to X_W is a Lie algebroid derivation of $W \Rightarrow E$. We want to provide an alternative description of those Lie algebroid derivations of $W \Rightarrow E$ that arise in this way. In order to do this, we need to recall some extra features of VB-algebroids.

The space $\Gamma_\ell(W, E)$ of linear sections of $W \rightarrow E$ (see Remark 1.4) is a $C^\infty(M)$ -module in an obvious way. Actually, it is the module of sections of a vector bundle $\hat{A} \rightarrow M$. Specifically, a point of $\hat{A}$ over a point $x \in M$ is a pair (a, h) consisting of a point a of A_x and a linear map $h : E_x \rightarrow W_a$ which inverts the projection $W_a \rightarrow E_x$ on the right. As linear sections project onto sections of $A \rightarrow M$, there is a vector bundle epimorphism $\pi : \hat{A} \rightarrow A$. In view of the canonical splitting $W|_M = C \oplus E$, linear sections of $W \rightarrow E$ taking their values in the kernel of $\pi : \hat{A} \rightarrow A$ identify with vector bundle morphisms $E \rightarrow C$. So there is a short exact sequence

$$0 \longrightarrow \text{Hom}(E, C) \longrightarrow \hat{A} \xrightarrow{\pi} A \longrightarrow 0, \quad (2.23)$$

and in what follows we understand the inclusion $\text{Hom}(E, C) \hookrightarrow \hat{A}$. In this regard notice that for any $\chi \in \Gamma(C)$ and $\varphi \in \Gamma(E^*)$ the latter inclusion identifies $\varphi \otimes \chi \in \Gamma(E^* \otimes C)$ with the linear section $\varphi\hat{\chi} \in \Gamma_\ell(W, E)$ given by the product of the fiber-wise linear function φ and the core section $\hat{\chi}$.

The Lie bracket on $\Gamma(W, E)$ restricts to linear sections. As a consequence, there is a Lie algebroid structure $\hat{A} \Rightarrow M$ with anchor given by $\rho \circ \pi$. This Lie algebroid is the so-called *fat algebroid* of $(W \Rightarrow E; A \Rightarrow M)$ (see [27]). Vector bundle $E \rightarrow M$ carries a canonical representation of the fat algebroid, called the *side representation* and denoted by ψ^s . Let $\tilde{a}$ be a section of $\hat{A}$ or, equivalently, a linear section of $W \rightarrow E$ projecting onto a section a of $A \rightarrow M$. The side representation is implicitly given by

$$\langle \varphi, \psi_a^s e \rangle = \rho(a) \langle \varphi, e \rangle - \langle \tilde{\rho}(\tilde{a})(\varphi), e \rangle, \quad (2.24)$$

for all $\varphi \in \Gamma(E^*)$, and $e \in \Gamma(E)$. Here, as usual, φ is also regarded as a fiber-wise linear function on E . Recall that given a linear section $\tilde{a}$ of $W \rightarrow E$, vector field $\tilde{\rho}(\tilde{a})$ is an infinitesimal automorphism of the vector bundle E , hence its Lie derivative preserves vertical lifts of sections of E (see Remark 2.3) and

$$(\psi_a^s e)^\uparrow = [\tilde{\rho}(\tilde{a}), e^\uparrow], \quad (2.25)$$

for all $e \in \Gamma(E)$. Yet in other words, ψ_a^s is nothing but the derivation of E corresponding to the infinitesimal automorphism $\tilde{\rho}(\tilde{a})$ of E (Lemma 2.2). The core $C \rightarrow M$ does also carry a representation of the fat algebroid, called the *core representation* and denoted by ψ^c . The core representation is implicitly given by

$$\widehat{\psi_a^c \chi} = [\tilde{a}, \hat{\chi}] \quad (2.26)$$

for all $\tilde{a} \in \Gamma(\hat{A})$ and $\chi \in \Gamma(C)$, where $\hat{\chi}$ denotes the core section of $W \rightarrow E$ corresponding to χ . Finally, a distinguished vector bundle morphism $\partial : C \rightarrow E$, called the *core-anchor*, is defined by

$$\langle \partial \chi, \varphi \rangle = \tilde{\rho}(\hat{\chi})(\varphi), \quad (2.27)$$

for all $\chi \in \Gamma(C)$ and $\varphi \in \Gamma(E^*)$. Notice that our convention on the core-anchor differs by a sign from that of [27]. There is an equivalent way to describe the core-anchor. Actually, in view of Definition 1.5 and Lemma 2.17, for any $\chi \in \Gamma(C)$, vector field $\tilde{\rho}(\hat{\chi}) \in \mathfrak{X}(E)$ is the vertical lift of a (necessarily unique) section of E (see Remark 2.3) and

$$(\partial \chi)^\uparrow = \tilde{\rho}(\hat{\chi}). \quad (2.28)$$

Example 2.23 Let $E \rightarrow M$ be a vector bundle, and recall that its *tangent double vector bundle* $(TE \rightarrow E; TM \rightarrow M)$ is actually a VB-algebroid $(TE \Rightarrow E; TM \Rightarrow M)$. It follows from Remark 2.1, point *vi.*, and Lemma 2.2, that its fat algebroid is precisely the gauge algebroid DE of E . Additionally, both the side vector bundle and the core of $(TE \Rightarrow E; TM \Rightarrow M)$ are (canonically isomorphic to) E (see also Example 1.8) and the core/side representations both agree with the tautological action of DE on E . Finally the core-anchor is just the identity $\text{id} : E \rightarrow E$.

We now show that infinitesimal automorphisms of $(W \Rightarrow E; A \Rightarrow M)$ or equivalently IM derivations of $W \rightarrow A$ can be characterized in terms of certain algebraic data, in a similar way as IM vector fields on a Lie algebroid can be characterized in terms of Lie algebroid derivations. So, let X_W be an infinitesimal automorphism of $(W \Rightarrow E; A \Rightarrow M)$ and let $\Delta_{W,A}$ and $\Delta_{W,E}$ be its vertical and horizontal derivations respectively. Since X_W is, in particular, an IM vector field on W , then $\Delta_{W,E}$ is a Lie algebroid derivation of the Lie algebroid $W \Rightarrow E$. Additionally, X_W is also an infinitesimal automorphism of the vector bundle $W \rightarrow A$, i.e. it commutes with $\mathcal{E}_{W,A}$ (cf. Remark 2.1, point *v.*). It follows that $\Delta_{W,E}$ commutes with Δ_E (see Lemma 2.2). Hence, in view of Lemma 2.17, $\Delta_{W,E}$ preserves linear sections of $W \rightarrow E$ and it restricts to a derivation $\Delta_{\hat{A}} : \Gamma(\hat{A}) \rightarrow \Gamma(\hat{A})$ of the fat algebroid. Obviously, $\Delta_{\hat{A}}$ is a Lie algebroid derivation. For the same reason, $\Delta_{W,E}$ preserves core sections of $W \rightarrow E$ and determines a derivation $\Delta_C : \Gamma(C) \rightarrow \Gamma(C)$ of the core. Consider also the derivations Δ_A, Δ_E of A, E , respectively, as in Remark 2.14. We are now ready to state the main result in this section.

Theorem 2.24 *Let $(W \Rightarrow E; A \Rightarrow M)$ be a VB-algebroid. The assignments $X_W \mapsto \Delta_{W,A}$ and $X_W \mapsto (\Delta_{\hat{A}}, \Delta_E, \Delta_C)$ establish one-to-one correspondences between*

- *infinitesimal automorphisms of $(W \Rightarrow E; A \Rightarrow M)$,*
- *IM derivations of $W \rightarrow A$,*
- *triples $(\Delta_{\hat{A}}, \Delta_E, \Delta_C)$ consisting of*
 - i.) a Lie algebroid derivation $\Delta_{\hat{A}}$ of the fat algebroid $\hat{A}$,*
 - ii.) a derivation Δ_E of E ,*
 - iii.) a derivation Δ_C of C ,*

such that

$$\sigma(\Delta_{\hat{A}}) = \sigma(\Delta_E) = \sigma(\Delta_C), \quad (2.29)$$

$$\Delta_{\hat{A}}\Phi = \Delta_C \circ \Phi - \Phi \circ \Delta_E, \quad (2.30)$$

for all vector bundle morphisms $\Phi : E \rightarrow C$ (also regarded as sections of $\hat{A}$ via the inclusion $\text{Hom}(E, C) \hookrightarrow \hat{A}$), and

$$\partial \circ \Delta_C = \Delta_E \circ \partial, \quad (2.31)$$

$$[\Delta_E, \psi_a^s] = \psi_{\Delta_{\hat{A}}\tilde{a}}^s, \quad (2.32)$$

$$[\Delta_C, \psi_a^c] = \psi_{\Delta_{\hat{A}}\tilde{a}}^c, \quad (2.33)$$

for all $\tilde{a} \in \Gamma(\hat{A})$.

PROOF: The correspondence $X_W \mapsto \Delta_{W,A}$ between infinitesimal automorphisms and IM derivations is one-to-one by definition. Now, begin with X_W and consider the associated data $(\Delta_{\hat{A}}, \Delta_E, \Delta_C)$. The discussion immediately before the statement shows that $\Delta_{\hat{A}}$ is a Lie algebroid derivation. Property (2.29) follows from the fact that all terms agree with the projection X of X_W on M . For property (2.30), it is enough to consider Φ of the form $\Phi = \varphi \otimes \chi$ (or equivalently $\Phi = \varphi \hat{\chi}$), where $\varphi \in \Gamma(E^*)$ and $\chi \in \Gamma(C)$. From the Leibniz rule for $\Delta_{W,E}$ it follows immediately that

$$\Delta_{\hat{A}}\Phi = (\Delta_E^*\varphi) \otimes \chi + \varphi \otimes \Delta_C\chi \in \Gamma(\text{Hom}(E, C)). \quad (2.34)$$

Hence $\Delta_{\hat{A}}$ restricts to a derivation of $\text{Hom}(E, C)$. Additionally, from (2.29) we get also

$$\langle \Delta_{\hat{A}}\Phi, e \rangle = \langle \Delta_E^*\varphi, e \rangle \hat{\chi} + \langle \varphi, e \rangle \widehat{\Delta_C\chi} = -\langle \varphi, \Delta_E e \rangle \hat{\chi} + \Delta_C(\langle \varphi, e \rangle \hat{\chi}) = \langle \Delta_C \circ \Phi - \Phi \circ \Delta_E, e \rangle, \quad (2.35)$$

for all $e \in \Gamma(E)$, i.e. (2.30) holds. For (2.31), fix $\chi \in \Gamma(C)$ and compute

$$(\partial \Delta_C \chi)^\dagger = \tilde{\rho}(\Delta_{W,E} \hat{\chi}) = [\sigma(\Delta_{W,E}), \tilde{\rho}(\hat{\chi})] = [X_{\Delta_E}, (\partial \chi)^\dagger] = (\Delta_E \partial \chi)^\dagger, \quad (2.36)$$

where we used the fact that $\Delta_{W,E}$ is a Lie algebroid derivation of W whose symbol is the linear vector field on E corresponding to Δ_E . For the same reason, for all $a \in \Gamma(\hat{A})$ and $e \in \Gamma(E)$

$$(\psi_{\Delta_{\hat{A}} \tilde{a}}^s e)^\dagger = [\tilde{\rho}(\Delta_{W,E} \tilde{a}), e^\dagger] = [[\sigma(\Delta_{W,E}), \tilde{\rho}(\tilde{a})], e^\dagger] = ([\Delta_E, \psi_a^s] e)^\dagger, \quad (2.37)$$

which proves (2.32) and for all $\chi \in \Gamma(C)$,

$$\widehat{\psi_{\Delta_{\hat{A}} \tilde{a}}^c \chi} = [\Delta_{W,E} \tilde{a}, \hat{\chi}] = \Delta_{W,E}[\tilde{a}, \hat{\chi}] - [\tilde{a}, \Delta_{W,E} \hat{\chi}] = [\Delta_C, \psi_a^c] \chi, \quad (2.38)$$

which proves (2.33). Conversely, let $(\Delta_{\hat{A}}, \Delta_E, \Delta_C)$ be as in the statement. From (2.29), and (2.30), $(\Delta_{\hat{A}}, \Delta_E, \Delta_C)$ extends uniquely to an infinitesimal automorphism X_W of the double vector bundle $(W \rightrightarrows E; A \rightrightarrows M)$. From (2.31), (2.32), (2.33) and the fact that $\Delta_{\hat{A}}$ is a Lie algebroid derivation, X_W is an infinitesimal automorphism of the VB-algebroid structure as well. The details can be easily checked in local coordinates. $\square$

Let $(\Delta_{\hat{A}}, \Delta_E, \Delta_C)$ be a triple as in the above theorem. Then, derivations Δ_E, Δ_C agree with those in Remark 2.14. Notice that it follows from (2.30) that both pairs $(\Delta_{\hat{A}}, \Delta_E)$ and $(\Delta_{\hat{A}}, \Delta_C)$ determine the whole triple. Additionally $\Delta_{\hat{A}}$ and derivation Δ_A from Remark 2.14, are π -related.

Remark 2.25 Equations (2.29) and (2.30) express compatibility with the double vector bundle structure, while Equations (2.31), (2.32), (2.33) express (the residual) compatibility with the algebroid structure.

From Proposition 3.16, Remark 2.22 and Theorem 2.24 the theorem stated below easily follows.

Theorem 2.26 *Let $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ be a VB-groupoid with VB-algebroid $(W \rightrightarrows E; A \rightrightarrows M)$. Every infinitesimal automorphism X_W of $\mathcal{W}$ determines a triple $(\Delta_{\hat{A}}, \Delta_E, \Delta_C)$ as in Theorem 2.24, via the following formulas*

$$\overrightarrow{\Delta_{\hat{A}} \tilde{a}} = [X_W, \overrightarrow{\tilde{a}}], \quad X_{\Delta_E} = \tilde{s}_* X_W = \tilde{t}_* X_W \quad \text{and} \quad X_{\Delta_C} = X_W|_C, \quad (2.39)$$

or, equivalently,

$$\overrightarrow{\Delta_{\hat{A}} \tilde{a}} = [\Delta_W, \overrightarrow{\tilde{a}}], \quad \Delta_E = \tilde{s}_* \Delta_W = \tilde{t}_* \Delta_W, \quad \text{and} \quad \Delta_C = \Delta_W|_C, \quad (2.40)$$

for all $\tilde{a} \in \Gamma(\hat{A})$. If $\mathcal{W} \rightrightarrows E$ is source simply connected, this correspondence is one-to-one.

Consider X_W to be an infinitesimal automorphism of the VB-groupoid $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ and X_W the infinitesimal automorphism of $(W \rightrightarrows E; A \rightrightarrows M) = \text{Lie}(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ associated to X_W by the Lie functor. Finally, let $(\Delta_{\hat{A}}, \Delta_E, \Delta_C)$ be the algebraic data corresponding to X_W via Theorem 2.24. The above theorem then states that derivation Δ_E (resp. Δ_C) agrees with that of Remark 2.6.

2.3 The linear deformation complex of a VB-groupoid/algebroid

Multiplicative vector fields on a Lie group G can be seen as 1-cocycles in the Lie group cohomology complex of G with coefficients in the adjoint representation. More generally, multiplicative vector fields on a Lie groupoid $G \rightrightarrows M$ are 1-cocycles in the *deformation complex* of G [14]. At the infinitesimal level, IM vector fields on a Lie algebroid $A \rightrightarrows M$ are 1-cocycles in the *deformation complex* of A [15]. Additionally, the deformation complex of a Lie groupoid and the deformation

complex of the associated Lie algebroid are intertwined by a Van Est map [14], which is yet another manifestation of the Lie functor. In this section we show that there is a natural cochain complex $C_{\text{def},\text{lin}}^\bullet(\mathcal{W})$ (resp. $C_{\text{def},\text{lin}}^\bullet(W)$) attached to every VB-groupoid $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ (resp. VB-algebroid $(W \rightrightarrows E; A \rightrightarrows M)$) in such a way that multiplicative (resp. IM) derivations are the same as 1-cocycles, while internal derivations are equivalent to 1-coboundaries. Specifically, $C_{\text{def},\text{lin}}^\bullet(\mathcal{W})$ (resp. $C_{\text{def},\text{lin}}^\bullet(W)$) is a subcomplex of the deformation complex of the Lie groupoid $\mathcal{W} \rightrightarrows E$ (resp. Lie algebroid $W \rightarrow E$) consisting of cochains that are *linear* in a suitable sense (see below). Additionally we show that the Van Est map restricts to a cochain map intertwining $C_{\text{def},\text{lin}}^\bullet(\mathcal{W})$ and $C_{\text{def},\text{lin}}^\bullet(W)$. Notice that this is similar to what Cabrera and Drummond do in [9]. There is one main difference though. While they work with the Lie groupoid/algebroid cohomology of the total space of a VB-groupoid/algebroid, we work with the deformation complex. In this respect we recall that the deformation complex $C_{\text{def}}^\bullet(\mathcal{W})$ of a Lie groupoid $\mathcal{W} \rightrightarrows E$ with Lie algebroid $W \rightrightarrows E$, can be seen as a subcomplex in the Lie groupoid complex $C^\bullet(T^*\mathcal{W})$ of the cotangent groupoid $T^*\mathcal{W} \rightrightarrows W_E^*$ [14] (where $W_E^* \rightarrow E$ is the dual of the vector bundle $W \rightarrow E$). Specifically, $C_{\text{def}}^\bullet(\mathcal{W})$ is the subcomplex consisting of cochains that are *linear* with respect to the vector bundle structure $T^*\mathcal{W} \rightarrow \mathcal{W}$, plus an additional projectability condition. If $\mathcal{W}$ is the total space of a VB-groupoid, then $T^*\mathcal{W}$ is actually a *triple structure*: a double vector bundle in the category of groupoids. This means that the results in this sections are analogous to those in [9] but for groupoids equipped with two (not just one) compatible vector bundle structures.

Let us recall how is the deformation complex defined. Let $G \rightrightarrows M$ be a Lie groupoid with Lie algebroid A . Following [14], we denote by $\bar{m}$ the division map in G , i.e.

$$\bar{m}(g, h) = gh^{-1} \quad (2.41)$$

for all $g, h \in G$ such that $s(g) = s(h)$. For any $k \geq 0$, we also denote by $G^{(k)}$ the space of k -tuples of composable arrows. In particular, $G^{(0)} = M$ and $G^{(1)} = G$. The *deformation complex* of G [14] is the pair $(C_{\text{def}}^\bullet(G), \delta)$, where, for $k > 0$, $C_{\text{def}}^k(G)$ is the space of smooth maps

$$c : G^{(k)} \longrightarrow TG, \quad (g_1, \dots, g_k) \longmapsto c(g_1, \dots, g_k) \in T_{g_1}G, \quad (2.42)$$

which are *source-projectable*, i.e. there exists a (necessarily unique) smooth map $\sigma_c : G^{(k-1)} \rightarrow TM$, $(g_1, \dots, g_{k-1}) \mapsto \sigma_c(g_1, \dots, g_{k-1}) \in T_{t(g_1)}M$, called the *source projection* of c , such that $(ds \circ c)(g_1, g_2, \dots, g_k) = \sigma_c(g_2, \dots, g_k)$. So 1-cochains are exactly source-projectable vector fields on G . The differential δc of $c \in C_{\text{def}}^k(G)$ is defined by

$$(\delta c)(g_1, \dots, g_{k+1}) = -d\bar{m}(c(g_1g_2, g_3, \dots, g_{k+1}), c(g_2, g_3, \dots, g_{k+1})) \quad (2.43)$$

$$+ \sum_{i=2}^k (-)^i c(g_1, \dots, g_i g_{i+1}, \dots, g_{k+1}) + (-)^{k+1} c(g_1, \dots, g_k). \quad (2.44)$$

For $k = 0$, $C_{\text{def}}^0(G) := \Gamma(A)$ and for $a \in \Gamma(A)$, $\delta a := \overrightarrow{a} + \overleftarrow{a}$. The terminology is motivated by the fact that $(C_{\text{def}}^\bullet(G), \delta)$ controls deformations of $G \rightrightarrows M$ [14]. It is easy to see that 1-cocycles in $(C_{\text{def}}^\bullet(G), \delta)$ are the same as multiplicative vector fields on G (see [14, Proposition 4.3]). We also consider the *normalized deformation complex* of G as the subcomplex $\hat{C}_{\text{def}}^\bullet(G) \subset C_{\text{def}}^\bullet(G)$ consisting of *normalized cochains*. For $k \geq 2$, a normalized k -cochain is a k -cochain c such that for all $x \in M$

$$c(1_x, g_2, \dots, g_k) = \sigma_c(g_2, \dots, g_k) \quad \text{and} \quad c(g_1, g_2, \dots, 1_x, \dots, g_k) = 0. \quad (2.45)$$

In degree 1, the only condition is that $c(1_x) = (\sigma_c)_x$, and in degree 0 there are no conditions.

Now, let $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ be a VB-groupoid with VB-algebroid $(W \rightrightarrows E; A \rightrightarrows M)$. Consider the subspace $C_{\text{def},\text{lin}}^\bullet(\mathcal{W})$ of $C_{\text{def}}^\bullet(\mathcal{W})$ defined as follows. For $k > 0$, $C_{\text{def},\text{lin}}^k(\mathcal{W})$ consists of those cochains $\tilde{c} \in C_{\text{def}}^k(\mathcal{W})$ which are *linear*, i.e. $\tilde{c} : \mathcal{W}^{(k)} \rightarrow T\mathcal{W}$ is a vector bundle morphism covering a

(necessarily unique) smooth map $c : G^{(k)} \rightarrow TG$:

$$\begin{array}{ccc} \mathcal{W}^{(k)} & \xrightarrow{\tilde{c}} & T\mathcal{W} \\ \downarrow & & \downarrow \\ G^{(k)} & \xrightarrow{c} & TG \end{array} . \quad (2.46)$$

In particular $C_{\text{def,lin}}^1(\mathcal{W})$ consists of those source-projectable vector fields on the Lie groupoid $\mathcal{W}$ which are also infinitesimal automorphisms of the vector bundle $\mathcal{W} \rightarrow G$. For $k = 0$, $C_{\text{def,lin}}^0(\mathcal{W}) := \Gamma_\ell(W, E) = \Gamma(\hat{A})$, where $\hat{A} \Rightarrow M$ is the fat algebroid of $(W \Rightarrow E; A \Rightarrow M)$.

Lemma 2.27

- i.) $C_{\text{def,lin}}^\bullet(\mathcal{W})$ is a subcomplex of $(C_{\text{def}}^\bullet(\mathcal{W}), \delta)$.
- ii.) If $\tilde{c} : \mathcal{W}^{(k)} \rightarrow T\mathcal{W}$ belongs to $C_{\text{def,lin}}^k(\mathcal{W})$, then its projection $c : G^{(k)} \rightarrow TG$ belongs to $C_{\text{def}}^k(G)$ and $\delta\tilde{c}$ projects to δc .

PROOF: Let $\tilde{a} \in C_{\text{def,lin}}^0(\mathcal{W})$ be a linear section of $\mathcal{W} \rightarrow E$. Then $\delta\tilde{a} = \overrightarrow{\tilde{a}} + \overleftarrow{\tilde{a}}$ is an infinitesimal automorphism of $(\mathcal{W} \Rightarrow E; G \Rightarrow M)$ (Remark 2.9). In particular, it belongs to $C_{\text{def,lin}}^1(\mathcal{W})$. Now, for $k > 0$, let $\tilde{c} : \mathcal{W}^{(k)} \rightarrow T\mathcal{W}$ belong to $C_{\text{def,lin}}^k(\mathcal{W})$, and show that $\delta\tilde{c}$ belongs to $C_{\text{def,lin}}^{k+1}(\mathcal{W})$. This follows, after a straightforward computation, from the linearity of $\tilde{c}$ and the obvious fact that like the (other) structure maps, the division $\tilde{m} = \mathcal{W}_{\tilde{s} \times \tilde{s}} \mathcal{W} \rightarrow \mathcal{W}$ in a VB-groupoid $(\mathcal{W} \Rightarrow E; G \Rightarrow M)$ is a vector bundle morphism (covering the division $\bar{m} : G_{s \times s} G \rightarrow G$ in G). The second part of the statement immediately follows from the axioms of VB-groupoids and the, easy to check, commutativity of the following diagram:

$$\begin{array}{ccccc} & \mathcal{W}^{(k)} & \xrightarrow{d\tilde{s} \circ \tilde{c}} & TE & \\ & \swarrow & \searrow \sigma_{\tilde{c}} & \downarrow & \\ \mathcal{W}^{(k-1)} & & & & \\ \downarrow & & & & \\ & G^{(k)} & \xrightarrow{ds \circ c} & TM & \\ & \swarrow & \searrow \sigma_c & \downarrow & \\ G^{(k-1)} & & & & \end{array} . \quad (2.47)$$

Here the projection $\mathcal{W}^{(k)} \rightarrow \mathcal{W}^{(k-1)}$ consists in dropping the first entry (and similarly for the projection $G^{(k)} \rightarrow G^{(k-1)}$) and $\sigma_c = dq \circ \sigma_{\tilde{c}} \circ 0_{\mathcal{W}^{(k-1)}}$, where $0_{\mathcal{W}^{(k-1)}} : G^{(k-1)} \rightarrow \mathcal{W}^{(k-1)}$ is the zero section. $\square$

Proposition 2.28

- i.) Infinitesimal automorphisms of $(\mathcal{W} \Rightarrow E; G \Rightarrow M)$ are 1-cocycles in $(C_{\text{def,lin}}^\bullet(\mathcal{W}), \delta)$.
- ii.) Internal derivations of $(\mathcal{W} \Rightarrow E; G \Rightarrow M)$ are (equivalent to) 1-coboundaries in $(C_{\text{def,lin}}^\bullet(\mathcal{W}), \delta)$.

PROOF: It immediately follows from Remark 2.9 and [14, Proposition 4.3]. $\square$

We now pass to the infinitesimal picture. Recall from [15] that the *deformation complex* of a Lie algebroid $A \Rightarrow M$ is the pair $(C_{\text{def}}^\bullet(A), \delta)$, where, for $k > 0$, $C_{\text{def}}^k(A)$ consists of (skew-symmetric) multiderivations

$$c : \Gamma(A) \times \cdots \times \Gamma(A) \longrightarrow \Gamma(A) \quad (2.48)$$

with k -entries, and multilinear symbol, i.e. c is a skew-symmetric $\mathbb{R}$ -multilinear map, and there exists a (necessarily unique) vector bundle morphism $\sigma_c : \wedge^{k-1} A \rightarrow TM$, called the *symbol* of c , such that

$$c(a_1, \dots, a_{k-1}, fa_k) = fc(a_1, \dots, a_{k-1}, a_k) + \sigma_c(a_1, \dots, a_{k-1})(f)a_k, \quad (2.49)$$

for all $a_1, \dots, a_k \in \Gamma(A)$ and $f \in C^\infty(M)$. In particular 1-cochains are just derivations of the vector bundle $A \rightarrow M$. The differential δc of $c \in C_{\text{def}}^k(A)$ is defined by

$$\begin{aligned} (\delta c)(a_1, \dots, a_{k+1}) &= \sum_{i=1}^{k+1} (-)^{i+1} [a_i, c(a_1, \dots, \widehat{a}_i, \dots, a_{k+1})] \\ &\quad + \sum_{i < j} (-)^{i+j} c([a_i, a_j], a_1, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, a_{k+1}), \end{aligned} \quad (2.50)$$

for all $a_1, \dots, a_{k+1} \in \Gamma(A)$. A straightforward computation then shows that

$$\begin{aligned} \sigma_{\delta c}(a_1, \dots, a_k) &= \sum_{i=1}^k (-)^{i+1} [\rho(a_i), \sigma_c(a_1, \dots, \widehat{a}_i, \dots, a_k)] \\ &\quad + \sum_{i < j} (-)^{i+j} \sigma_c([a_i, a_j], a_1, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, a_k) - (-)^k \rho(c(a_1, \dots, a_k)). \end{aligned} \quad (2.51)$$

for all $a_1, \dots, a_k \in \Gamma(A)$. For $k = 0$, $C_{\text{def}}^0(A) := \Gamma(A)$ and for $a \in \Gamma(A)$, $\delta a := [a, -]$. The terminology is motivated by the fact that $(C_{\text{def}}^\bullet(A), \delta)$ controls deformations of A [15] (see also [39, Section 1.1]). We remark, however, that the complex $(C_{\text{def}}^\bullet(A), \delta)$ first appeared in [25] under a different name, and for different purposes (see Theorem 6 loc. cit.).

It is immediate that 1-cocycles in $(C_{\text{def}}^\bullet(A), \delta)$ are the same as Lie algebroid derivations of A (cf. [15, Section 3.1]). Hence they are in one-to-one correspondence with IM vector fields on A .

Remark 2.29 The graded space $C_{\text{def}}^\bullet(A)$ is not just a cochain complex. Actually, up to a shift, it becomes a dg-Lie algebra when additionally equipped with the *Schouten-Jacobi bracket* $\llbracket -, - \rrbracket$ (see [41, 42]), originally named the *Gerstenhaber bracket* in [15]. Differential δ is then $\llbracket b, - \rrbracket$, where $b = [-, -]$ is the Lie bracket on $\Gamma(A)$.

When A is the Lie algebroid of a Lie groupoid $G \rightrightarrows M$, then the deformation complex of A and the normalized deformation complex of G are intertwined by a *Van Est map* [14], i.e. the cochain map

$$\mathcal{V} : (\widehat{C}_{\text{def}}^\bullet(G), \delta) \longrightarrow (C_{\text{def}}^\bullet(A), \delta) \quad (2.52)$$

defined by

$$(\mathcal{V}c)(a_1, \dots, a_k) := \sum_{\tau \in S_k} (-)^{\tau} (R_{a_{\tau(1)}} \circ \dots \circ R_{a_{\tau(k)}})(c), \quad (2.53)$$

for all $c \in \widehat{C}_{\text{def}}^k(G)$ and $a_1, \dots, a_k \in \Gamma(A)$. Here, for any $a \in \Gamma(A)$,

$$R_a : \widehat{C}_{\text{def}}^{\bullet+1}(G) \rightarrow \widehat{C}_{\text{def}}^\bullet(G) \quad (2.54)$$

is the map defined as follows. For $c \in \widehat{C}_{\text{def}}^1(G)$, $R_a c := [c, \vec{a}]|_M$ and for $c \in \widehat{C}_{\text{def}}^{k+1}(G)$, $k > 0$,

$$(R_a c)(g_1, \dots, g_k) := (-)^k \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} c(g_1, \dots, g_k, \phi_\varepsilon^a(s(g_k))^{-1}), \quad (2.55)$$

where $\{\phi_\varepsilon^a\}$ is the flow of $\vec{a}$. Now, let $(W \rightrightarrows E; A \rightrightarrows M)$ be a VB-algebroid. Consider the subspace $C_{\text{def}, \text{lin}}^\bullet(W)$ of $C_{\text{def}}^\bullet(W)$ defined as follows. For $k > 0$, $C_{\text{def}, \text{lin}}^k(W)$ consists of *linear cochains*, i.e. those multiderivations $\tilde{c} : \Gamma(W, E) \times \dots \times \Gamma(W, E) \rightarrow \Gamma(W, E)$ such that

$$\tilde{c}(\tilde{a}_1, \dots, \tilde{a}_k) \text{ is a linear section,} \quad (2.56)$$

$$\tilde{c}(\tilde{a}_1, \dots, \tilde{a}_{k-1}, \hat{\chi}_1) \text{ is a core section,} \quad (2.57)$$

$$\tilde{c}(\tilde{a}_1, \dots, \tilde{a}_{k-i}, \hat{\chi}_1, \dots, \hat{\chi}_i) = 0, \quad (2.58)$$

and, additionally

$$\sigma_{\tilde{c}}(\tilde{a}_1, \dots, \tilde{a}_{k-1}) \text{ is a linear vector field,} \quad (2.59)$$

$$\sigma_{\tilde{c}}(\tilde{a}_1, \dots, \tilde{a}_{k-2}, \hat{\chi}_1) \text{ is the vertical lift of a section of } E, \quad (2.60)$$

$$\sigma_{\tilde{c}}(\tilde{a}_1, \dots, \tilde{a}_{k-i-1}, \hat{\chi}_1, \dots, \hat{\chi}_i) = 0, \quad (2.61)$$

for all linear sections $\tilde{a}_1, \dots, \tilde{a}_k$, all core sections $\hat{\chi}_1, \dots, \hat{\chi}_i$ of $W \rightarrow E$, and all $i \geq 2$. For $k = 0$, we define $C_{\text{def,lin}}^0(W) := \Gamma_\ell(W, E) = \Gamma(\hat{A})$.

Lemma 2.30 $C_{\text{def,lin}}^\bullet(W)$ is a subcomplex, and a Lie subalgebra, in $C_{\text{def}}^\bullet(W)$.

PROOF: It immediately follows from the explicit formula for δ , Definition 1.5, and the explicit formula for the Schouten-Jacobi bracket (see [15, Proposition 1]). $\square$

Linear 0-cochains in $C_{\text{def}}^\bullet(W)$ are already characterized by Lemma 2.17. For higher degree cochains the situation is illustrated by the following

Lemma 2.31 For any $k \geq 0$, a cochain $\tilde{c} \in C_{\text{def}}^k(W)$ is linear if and only if $[\Delta_\varepsilon, \tilde{c}] = 0$, where $[\![\cdot, \cdot]\!]$ is the Schouten-Jacobi bracket (see Remark 2.29).

PROOF: First recall that a multiderivation is completely determined by its own action and its symbol action on generators. From the explicit formula for the Schouten-Jacobi bracket [15, Proposition 1] it follows that

$$[\![\Delta_\varepsilon, \tilde{c}]\!](w_1, \dots, w_k) = \Delta_\varepsilon(\tilde{c}(w_1, \dots, w_k)) - \sum_{i=1}^k \tilde{c}(w_1, \dots, \Delta_\varepsilon w_i, \dots, w_k), \quad (2.62)$$

and

$$\sigma_{[\![\Delta_\varepsilon, \tilde{c}]\!] }(w_1, \dots, w_{k-1}) = [\sigma(\Delta_\varepsilon), \sigma_{\tilde{c}}(w_1, \dots, w_{k-1})] - \sum_{i=1}^{k-1} \sigma_{\tilde{c}}(w_1, \dots, \Delta_\varepsilon w_i, \dots, w_{k-1}), \quad (2.63)$$

for all $w_1, \dots, w_k \in \Gamma(W, E)$. Using (2.62) and Lemma 2.17 we immediately see that if $\tilde{c}$ is linear, then $[\![\Delta_\varepsilon, \tilde{c}]\!]$ and its symbol vanish on linear and core sections. Hence $[\![\Delta_\varepsilon, \tilde{c}]\!] = 0$. Conversely, suppose that $[\![\Delta_\varepsilon, \tilde{c}]\!] = 0$. Then (2.56), (2.57), and (2.58) follow from (2.62) and Lemma 2.17. Similarly, (2.59), (2.60), and (2.61) follow from (2.63), the fact that $\sigma(\Delta_\varepsilon)$ is the Euler vector field on E (cf. Remark 2.1) and that there are no non-trivial vector fields X on E such that $[\sigma(\Delta_\varepsilon), X] = -iX$ for some integer $i > 1$ (cf. Lemma 2.17). $\square$

Corollary 2.32

- i.) Infinitesimal automorphisms of $(W \Rightarrow E; A \Rightarrow M)$ are equivalent to 1-cocycles in $(C_{\text{def,lin}}^\bullet(W), \delta)$.
- ii.) Internal derivations of $(W \Rightarrow E; A \Rightarrow M)$ are equivalent to 1-coboundaries in $(C_{\text{def,lin}}^\bullet(W), \delta)$.

PROOF: It immediately follows from Remark 2.18, Lemma 2.31 and [15, Section 3.1]. $\square$

Now we discuss the analogue of Theorem 2.24 for cochains in $C_{\text{def,lin}}^k(W)$. Namely, let $\tilde{c} \in C_{\text{def,lin}}^k$, $k > 1$. By definition, $\tilde{c}$ preserves linear sections, hence it determines by restriction a k -cochain $c_{\hat{A}}$ in the fat algebroid $\hat{A}$. Additionally, $\sigma_{\tilde{c}}$ maps linear sections to linear vector fields and we set

$$c_E(\tilde{a}_1, \dots, \tilde{a}_{k-1}) := \Delta_{\sigma_{\tilde{c}}(\tilde{a}_1, \dots, \tilde{a}_{k-1})}, \quad (2.64)$$

for all $\tilde{a}_1, \dots, \tilde{a}_{k-1} \in \Gamma(\widehat{A})$ (recall that Δ_X denotes the derivation corresponding to a linear vector field X). Moreover, $\tilde{c}$ maps $k-1$ linear sections and a core sections to core sections and we set

$$c_C(\widehat{\tilde{a}_1, \dots, \tilde{a}_{k-1}})(\chi) := \tilde{c}(\tilde{a}_1, \dots, \tilde{a}_{k-1}, \widehat{\chi}), \quad (2.65)$$

for all $\tilde{a}_1, \dots, \tilde{a}_{k-1} \in \Gamma(\widehat{A})$ and $\chi \in \Gamma(C)$ (recall that $\widehat{\chi}$ is the core section of $W \rightarrow E$ corresponding to χ). Finally, we define $d_{\widehat{A}}$ by

$$d_{\widehat{A}}(\tilde{a}_1, \dots, \tilde{a}_{k-2})(\chi)^\uparrow := \sigma_{\tilde{c}}(\tilde{a}_1, \dots, \tilde{a}_{k-2}, \widehat{\chi}), \quad (2.66)$$

(recall that $e^\uparrow$ denotes the vertical lift of a section $e \in \Gamma(E)$). For $k=1$, a cochain $\tilde{c} \in C_{\text{def, lin}}^1(W)$ is a derivation of $W \rightarrow E$ with the additional property of preserving linear sections. It is then obvious that the corresponding linear vector field $X_{\tilde{c}}$ on W is an infinitesimal automorphism of the double vector bundle $(W \rightarrow E; A \rightarrow M)$. In particular, we can associate to $X_{\tilde{c}}$ a triple $(\Delta_{\widehat{A}}, \Delta_E, \Delta_C)$ exactly as we did in Theorem 2.24, except for the fact that as $X_{\tilde{c}}$ is not an infinitesimal automorphism of the VB-algebroid in general, $(\Delta_{\widehat{A}}, \Delta_E, \Delta_C)$ do only satisfy Equations (2.29) and (2.30), and do not satisfy (2.31), (2.32) and (2.33) in general (see also Remark 2.25). In this section we will denote by $(c_{\widehat{A}}, c_E, c_C)$ the triple $(\Delta_{\widehat{A}}, \Delta_E, \Delta_C)$ to stress that it comes from a (non-necessarily closed) 1-cochain in $C_{\text{def, lin}}^\bullet(W)$.

Theorem 2.33

i.) For $k > 1$, the assignment $\tilde{c} \mapsto (c_{\widehat{A}}, c_E, c_C, d_{\widehat{A}})$ establishes a one-to-one correspondence between cochains in $C_{\text{def, lin}}^k(W)$ and 4-tuples consisting of

- a k -cochain $c_{\widehat{A}}$ in the deformation complex of the fat algebroid $\widehat{A}$,
- a vector bundle morphism $c_E : \wedge^{k-1} \widehat{A} \rightarrow DE$,
- a vector bundle morphism $c_C : \wedge^{k-1} \widehat{A} \rightarrow DC$ and
- a vector bundle morphism $d_{\widehat{A}} : \wedge^{k-2} \widehat{A} \rightarrow \text{Hom}(C, E)$, such that

$$\sigma \circ c_E = \sigma \circ c_C = \sigma_{c_{\widehat{A}}} \quad (2.67)$$

where $\sigma : DE \rightarrow TM$ (resp. $\sigma : DC \rightarrow TM$) is the symbol map and additionally

$$c_{\widehat{A}}(\tilde{a}_1, \dots, \tilde{a}_{k-1}, \Phi) = c_C(\tilde{a}_1, \dots, \tilde{a}_{k-1}) \circ \Phi - \Phi \circ c_E(\tilde{a}_1, \dots, \tilde{a}_{k-1}), \quad (2.68)$$

$$c_E(\tilde{a}_1, \dots, \tilde{a}_{k-2}, \Phi) = -d_{\widehat{A}}(\tilde{a}_1, \dots, \tilde{a}_{k-2}) \circ \Phi, \quad (2.69)$$

$$c_C(\tilde{a}_1, \dots, \tilde{a}_{k-2}, \Phi) = -\Phi \circ d_{\widehat{A}}(\tilde{a}_1, \dots, \tilde{a}_{k-2}), \quad (2.70)$$

$$d_{\widehat{A}}(\tilde{a}_1, \dots, \tilde{a}_{k-3}, \Phi) = 0, \quad (2.71)$$

for all $\tilde{a}_1, \dots, \tilde{a}_{k-1} \in \Gamma(\widehat{A})$ and all $\Phi \in \Gamma(\text{Hom}(E, C))$.

If $\tilde{c}$ corresponds to $(c_{\widehat{A}}, c_E, c_C, d_{\widehat{A}})$, then $\delta \tilde{c}$ corresponds to $(\delta c_{\widehat{A}}, c'_E, c'_C, d'_{\widehat{A}})$ where

$$\begin{aligned} c'_E(\tilde{a}_1, \dots, \tilde{a}_k) &= \sum_{i=1}^k (-)^{i+1} [\psi_{\tilde{a}_i}^s, c_E(\tilde{a}_1, \dots, \widehat{\tilde{a}_i}, \dots, \tilde{a}_k)] \\ &\quad + \sum_{i < j} (-)^{i+j} c_E([\tilde{a}_i, \tilde{a}_j], \tilde{a}_1, \dots, \widehat{\tilde{a}_i}, \dots, \widehat{\tilde{a}_j}, \dots, \tilde{a}_k) - (-)^k \psi_{c_{\widehat{A}}(\tilde{a}_1, \dots, \tilde{a}_k)}^s, \end{aligned} \quad (2.72)$$

and like-wise for c'_C and

$$\begin{aligned}
& d'_A(\tilde{a}_1, \dots, \tilde{a}_{k-1}) \\
&= \sum_{i=1}^{k-1} (-)^{i+1} \left(\psi_{\tilde{a}_i}^s \circ d_{\hat{A}}(\tilde{a}_1, \dots, \hat{\tilde{a}}_i, \dots, \tilde{a}_{k-1}) - d_{\hat{A}}(\tilde{a}_1, \dots, \hat{\tilde{a}}_i, \dots, \tilde{a}_{k-1}) \circ \psi_{\tilde{a}_i}^c \right) \\
&+ \sum_{i < j} (-)^{i+j} d_{\hat{A}}([\tilde{a}_i, \tilde{a}_j], \tilde{a}_1, \dots, \hat{\tilde{a}}_i, \dots, \hat{\tilde{a}}_j, \dots, \tilde{a}_{k-1}) \\
&+ (-)^k (c_E(\tilde{a}_1, \dots, \tilde{a}_{k-1}) \circ \partial - \partial \circ c_C(\tilde{a}_1, \dots, \tilde{a}_{k-1})),
\end{aligned} \tag{2.73}$$

for all $\tilde{a}_1, \dots, \tilde{a}_k \in \Gamma(\hat{A})$.

ii.) For $k = 1$, the assignment $\tilde{c} \mapsto (c_{\hat{A}}, c_E, c_C)$ establishes a one-to-one correspondence between 1-cochains in $C_{\text{def}, \text{lin}}^\bullet(W)$ and triples $(c_{\hat{A}}, c_E, c_C)$, where $c_{\hat{A}}, c_E, c_C$ are derivations of the vector bundles $\hat{A}, E, C$ respectively, such that

$$\sigma(c_E) = \sigma(c_C) = \sigma(c_{\hat{A}}) \tag{2.74}$$

and

$$c_{\hat{A}}(\Phi) = c_C \circ \Phi - \Phi \circ c_E \tag{2.75}$$

for all $\Phi \in \Gamma(\text{Hom}(E, C))$. If $\tilde{c}$ corresponds to $(c_{\hat{A}}, c_E, c_C)$, then $\delta\tilde{c} \in C_{\text{def}, \text{lin}}^2(W)$ corresponds to the 4-tuple $(\delta c_{\hat{A}}, c'_E, c'_C, d'_{\hat{A}})$, where

$$d'_{\hat{A}} = \partial \circ c_C - c_E \circ \partial \tag{2.76}$$

and for all $\tilde{a} \in \Gamma(\hat{A})$,

$$c'_E(\tilde{a}) = [\psi_{\tilde{a}}^s, c_E] + \psi_{c_{\hat{A}}(\tilde{a})}^s, \quad c'_C(\tilde{a}) = [\psi_{\tilde{a}}^c, c_C] + \psi_{c_{\hat{A}}(\tilde{a})}^c. \tag{2.77}$$

iii.) For $k = 0$, the 0-cochains $\tilde{c}$ in $C_{\text{def}, \text{lin}}^\bullet(W)$ are the same as sections of the fat algebroid $\hat{A} \rightarrow M$. If $c \in C_{\text{def}, \text{lin}}^0(W)$ is given by $\tilde{a} \in \Gamma(\hat{A})$, then δc corresponds to the triple $(\delta\tilde{a}, \psi_{\tilde{a}}^s, \psi_{\tilde{a}}^c)$.

PROOF: The second and the third claims are straightforward. Let us prove the first one. Let $\tilde{c} \in C_{\text{def}, \text{lin}}^k(W)$. It is clear from the definition that $c_{\hat{A}}$ is a multiderivation, both c_E and $d_{\hat{A}}$ are multi-linear and (2.67) holds. In order to prove the multi-linearity of c_C (in its first $k-1$ arguments), let $f \in C^\infty(M)$ and compute

$$\tilde{c}(\tilde{a}_1, \dots, \tilde{a}_{k-2}, f\tilde{a}_{k-1}, \hat{\chi}) = f\tilde{c}(\tilde{a}_1, \dots, \tilde{a}_{k-2}, \tilde{a}_{k-1}, \hat{\chi}) - \sigma_{\tilde{c}}(\tilde{a}_1, \dots, \tilde{a}_{k-2}, \hat{\chi})(f)\tilde{a}_{k-1}. \tag{2.78}$$

Since $\sigma_{\tilde{c}}(\tilde{a}_1, \dots, \tilde{a}_{k-2}, \hat{\chi})$ is a vertical vector field on E , the last summand vanishes and we get $C^\infty(M)$ -linearity in $\tilde{a}_{k-1}$. Now, we check (2.68)–(2.71). It is enough to consider Φ of the form $\Phi = \varphi \otimes \chi$, where $\varphi \in \Gamma(E^*)$ and $\chi \in \Gamma(C)$. Hence Φ corresponds to the linear section $\varphi\hat{\chi}$, where, as usual, we interpret φ as a fiber-wise linear function on E , and (2.68)–(2.70) easily follow from either the Leibniz rule for $\tilde{c}$ or the multi-linearity of $\sigma_{\tilde{c}}$, while (2.71) follows from (2.61). As already recalled, a multiderivation c of a vector bundle is completely determined by its action and the action of σ_c on generators of the module of sections of the vector bundle. As linear and core sections generate $\Gamma(W, E)$ as a $C^\infty(E)$ -module, then the data $(c_{\hat{A}}, c_E, c_C, d_{\hat{A}})$ determine $\tilde{c} \in C_{\text{def}, \text{lin}}^\bullet(W)$ completely, i.e. the correspondence $\tilde{c} \mapsto (c_{\hat{A}}, c_E, c_C, d_{\hat{A}})$ is injective. To see that every 4-tuple $(c_{\hat{A}}, c_E, c_C, d_{\hat{A}})$ as in the statement arises in this way, use local coordinates to show that $\tilde{c} \mapsto (c_{\hat{A}}, c_E, c_C, d_{\hat{A}})$ is a(n injective) $C^\infty(M)$ -linear map between modules of sections of two vector bundles over M of the same rank. Finally, (2.72) and (2.73) follow from (2.50) and (2.51) by a straightforward computation. $\square$

Let $(c_{\hat{A}}, c_E, c_C, d_{\hat{A}})$ be a 4-tuple as in the statement of the above theorem. Then condition (2.67) says that c_E and c_C are *derivation valued skew-symmetric k -forms on $\hat{A}$* in the sense of [59, Section 2.4]. Additionally, it follows from (2.68) that $c_{\hat{A}}$ descends to a k -cochain in the deformation complex of A , i.e.

$$c_A(\pi(-), \dots, \pi(-)) = \pi \circ c_{\hat{A}}, \quad (2.79)$$

for some $c_A \in C_{\text{def}}^k(A)$, where $\pi : \hat{A} \rightarrow A$ is the projection. Finally, from (2.71), $d_{\hat{A}}$ descends to a vector bundle morphism $d_A : \wedge^{k-1} A \rightarrow \text{Hom}(C, E)$, i.e.

$$d_{\hat{A}} = d_A(\pi(-), \dots, \pi(-)). \quad (2.80)$$

We conclude this subsection showing that the deformation complex of a VB-groupoid and the deformation complex of its VB-algebroid are intertwined by a *linear* Van Est map.

Theorem 2.34 (Linear Van Est map) *Let $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ be a VB-groupoid with (associated) VB-algebroid $(W \rightrightarrows E; A \rightrightarrows M)$. The Van Est map (2.52) restricts to a cochain map*

$$\mathcal{V} : (\hat{C}_{\text{def,lin}}^{\bullet}(\mathcal{W}), \delta) \rightarrow (C_{\text{def,lin}}^{\bullet}(W), \delta), \quad (2.81)$$

where $\hat{C}_{\text{def,lin}}^{\bullet}(\mathcal{W})$ denotes linear, normalized cochains, i.e. $\hat{C}_{\text{def,lin}}^{\bullet}(\mathcal{W}) = \hat{C}_{\text{def}}^{\bullet}(\mathcal{W}) \cap C_{\text{def,lin}}^{\bullet}(W)$.

PROOF: When the side bundle E has positive rank the proof can be significantly simplified by noticing that in this case a cochain $\tilde{c} \in C_{\text{def}}^{\bullet}(\mathcal{W})$ is linear if and only if it preserves linear sections. This means that if $\text{rank } E > 0$, only condition (2.56) is needed in the definition of linear cochains, while conditions (2.57)-(2.61) are redundant. This can be seen, for instance, in local coordinates. In the rest of this proof we assume $\text{rank } E > 0$. A complete proof will be presented in [40].

So, it is enough to show that if $\tilde{a}$ is a linear section of $\mathcal{W} \rightarrow E$, then $R_{\tilde{a}}$ maps $\hat{C}_{\text{def,lin}}^{\bullet+1}(\mathcal{W})$ to $\hat{C}_{\text{def,lin}}^{\bullet}(\mathcal{W})$. But this immediately follows from (2.55) and the fact that from linearity the right invariant vector field $\vec{\tilde{a}} \in \mathfrak{X}(\mathcal{W})$ is a linear vector field on $\mathcal{W}$ (Proposition 2.8), hence it generates a flow $\{\phi_{\varepsilon}^{\tilde{a}}\}_{\varepsilon}$ by vector bundle automorphisms. $\square$

3 Multiplicative derivations of trivial-core VB-groupoids and algebroids

3.1 VB-groupoids/algebroids and representations

In this section we discuss the properties of some special class of VB-groupoids/algebroids. We have mentioned in the introduction that VB-groupoids/algebroids are intrinsic models of 2-term representations up to homotopy (see [27, 28], see also Section 4 for more details). For instance, the isomorphism class of the tangent VB-groupoid/algebroid corresponds to the isomorphism class of the adjoint representation (up to homotopy). In this picture, 2-term representations up to homotopy concentrated in one single degree (-1 or 0) are genuine representations and correspond to the class of VB-groupoids/algebroids that we aim to discuss here.

Definition 3.1 (Trivial-core VB-groupoid) A VB-groupoid is *trivial-core* if its core is trivial.

Trivial-core VB-groupoids are also known as *vacant*. Let $(E_G \rightrightarrows E; G \rightrightarrows M)$ be a trivial-core VB-groupoid. In this case, the source $\tilde{s} : E_G \rightarrow E$ is a regular vector bundle morphism (covering $s : G \rightarrow M$). Hence E_G is canonically isomorphic to s^*E as a vector bundle over G . In the following we will understand the latter isomorphism and denote $E_G = s^*E$.

Example 3.2 Let $G \rightrightarrows M$ be a Lie groupoid and let $E \rightarrow M$ be a vector bundle equipped with a left representation of G . Then there is a trivial core VB-groupoid structure $(E_G = s^*E \rightrightarrows E; G \rightrightarrows M)$. The structure maps of the top groupoid $E_G \rightrightarrows E$ are the following:

- i.) the source and target maps are given by $\tilde{s}(g, e) = e$ and $\tilde{t}(g, e) = g.e$, respectively,
- ii.) the multiplication $\tilde{m}$ is given by $\tilde{m}((g, e), (g', e')) = (gg', e')$,
- iii.) the unit and the inversion are given by $\tilde{u}(e) = (1_{q(e)}, e)$ and $\tilde{i}(g, e) = (g^{-1}, g.e)$.

In other words, $E_G \rightrightarrows E$ is the *action groupoid* corresponding to the action of G on the fibered manifold $E \rightarrow M$. It is easy to see that every trivial-core VB-groupoid arises in this way. In [28] trivial-core VB-groupoids $(E_G \rightrightarrows E; G \rightrightarrows M)$ correspond to representations of G on E , seen as representations up to homotopy concentrated in degree 0.

Definition 3.3 (Full-core VB-groupoid) A VB-groupoid $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ is *full-core* if its core is $C = \mathcal{W}|_M$.

Let $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ be a full-core VB-groupoid with core C . In this case, $E = 0_M$, the rank 0 vector bundle over M , and $\mathcal{W}$ is canonically isomorphic to t^*C .

Example 3.4 Let $G \rightrightarrows M$ be a Lie groupoid and let $q : C \rightarrow M$ be a vector bundle carrying a left representation of G . Denote by $0_M \rightarrow M$ the rank zero vector bundle. Then there is a full-core VB-groupoid structure $(t^*C \rightrightarrows 0_M; G \rightrightarrows M)$. The structure maps of the top groupoid $t^*C \rightrightarrows 0_M$ are the following:

- i.) the source and the target are given by $\tilde{s}(g, \chi) = 0_{s(g)}$ and $\tilde{t}(g, \chi) = 0_{t(g)} = 0_{q(\chi)}$,
- ii.) the multiplication is given by $\tilde{m}((g, \chi), (g', \chi')) = (gg', \chi + g.\chi')$,
- iii.) the unit and the inversion are given by $\tilde{u}(0_x) = (1_x, 0_x)$, for all $x \in M$, and $\tilde{i}(g, \chi) = (g^{-1}, -g^{-1}.\chi)$.

In other words, $t^*C \rightrightarrows 0_M = M$ is the *representation groupoid* corresponding to the representation C of G . Actually, every full-core VB-groupoid arises in this way. In [28] full-core VB-groupoids correspond to representations of G seen as representations up to homotopy concentrated in degree -1 .

Clearly, full-core VB-groupoids are dual to trivial-core VB-groupoids and vice-versa. If $(E_G \rightrightarrows E; G \rightrightarrows M)$ is the trivial-core VB-groupoid corresponding to a left representation of G on E , then the dual full-core VB-groupoid corresponds to the transpose representation of G on $C = E^*$. Similar definitions exist in the realm of Lie algebroids. Specifically, we give the following

Definition 3.5 (Trivial-core and full-core VB-algebroids) A VB-algebroid $(W \rightrightarrows E; A \rightrightarrows M)$ is *trivial-core* if its core is trivial, and it is *full-core* if its core is $C = W|_M$.

Example 3.6 Let $A \rightrightarrows M$ be a Lie algebroid, denote by $p : A \rightarrow M$ the projection, and let $q : E \rightarrow M$ be a vector bundle carrying a representation of A . Denote by ∇ the flat A -connection in E defining the representation. For every section $a \in \Gamma(A)$ the operator ∇_a is a derivation of E . Hence, it corresponds to a linear vector field on E , that we denote by X_a . The vector field X_a projects onto $\rho(a)$. It is easy to see that there is a trivial-core VB-algebroid structure $(E_A := p^*E \rightrightarrows E; A \rightrightarrows M)$. The structure maps of the top Lie algebroid $E_A \rightrightarrows E$ are the following:

- i.) the projection $\tilde{p} = p_A : E_A \rightarrow E$ is the one induced by p ,
- ii.) the anchor $\tilde{\rho} : E_A \rightarrow TE$ is given by $\tilde{\rho}(a_x, e) = (X_a)_e$ for all $a \in \Gamma(A)$, $x \in M$, and $e \in E_x$,
- iii.) the Lie bracket $[-, -] : \Gamma(E_A, E) \times \Gamma(E_A, E) \rightarrow \Gamma(E_A, E)$ is uniquely defined by the anchor and

$$[q^*a, q^*b] = q^*[a, b], \quad (3.1)$$

for all pull-back sections $q^*a, q^*b \in \Gamma(E_A, E)$, with $a, b \in \Gamma(A)$.

In other words, $E_A \Rightarrow E$ is the *action algebroid* corresponding to the action of A on the fibered manifold $E \rightarrow M$. Every trivial-core VB-algebroid arises in this way. In [27] trivial-core VB-algebroids $(E_A \Rightarrow E; A \Rightarrow M)$ correspond to representations of A on E seen as representations up to homotopy concentrated in degree 0.

Example 3.7 Let A be as in Example 3.6 and let $C \rightarrow M$ be a vector bundle with a flat A -connection ∇ . Then there is a full-core VB-algebroid structure $(A \times_M C \Rightarrow 0_M; A \Rightarrow M)$. The structure maps of the top Lie algebroid $A \times_M C \Rightarrow 0_M = M$ are the following:

- i.) the projection $\tilde{p} : A \times_M C \rightarrow 0_M$ is the obvious one,
- ii.) the anchor $\tilde{\rho} : A \times_M C \rightarrow TM$ is given by $\tilde{\rho}(a, e) = \rho(a)$, and
- iii.) the Lie bracket

$$[-, -] : \Gamma(A \times_M C, M) \times \Gamma(A \times_M C, M) \rightarrow \Gamma(A \times_M C, M) \quad (3.2)$$

is given by

$$[(a, \chi), (a', \chi')] = ([a, a'], \nabla_a \chi' - \nabla_{a'} \chi) \quad (3.3)$$

for all $a, a' \in \Gamma(A)$, and $\chi, \chi' \in \Gamma(C)$.

In other words, $A \times_M C \rightarrow M$ is the *representation algebroid* corresponding to the representation C of A , i.e. the *semidirect product of A and C* . All full-core VB-algebroids arise in this way. In [27] they correspond to representations of A seen as representations up to homotopy concentrated in degree -1 .

Full-core VB-algebroids are dual to trivial-core VB-algebroids and vice-versa. If $(E_A \Rightarrow E; A \Rightarrow M)$ is the trivial-core VB-algebroid corresponding to a representation of A on E , the dual full-core VB-algebroid corresponds to the transpose representation of A on $C = E^*$.

The proof of the following proposition is straightforward.

Proposition 3.8

- i.) Let G be a Lie groupoid with Lie algebroid A , and let E, C be representations of G (hence of A). The VB-algebroid of the trivial-core (resp. full-core) VB-groupoid determined by G and E (resp. C) is the trivial-core (resp. full-core) VB-algebroid corresponding to A and E (resp. C).
- ii.) Let A be an integrable Lie algebroid with source simply connected integration G , and let E, C be representations of A , hence of G . The trivial-core (resp. full-core) VB-algebroid determined by A and E (resp. C) is integrable and its source simply connected integration is the trivial-core (resp. full-core) VB-groupoid determined by G and E (resp. C).

Example 3.9 A VB-groupoid $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ (resp. a VB-algebroid $(W \Rightarrow E; A \Rightarrow M)$) such that $\mathcal{W}$ (resp. W) is a line bundle over G (resp. A) is necessarily either trivial-core or full-core.

3.2 The gauge algebroid of trivial-core/full-core VB-groupoids/algebroids

Let $G \rightrightarrows M$ be a Lie groupoid and let $q : E \rightarrow M$ be a vector bundle carrying a representation of G . As discussed in the previous section, starting with this data we can construct two VB-groupoids, a trivial-core one $(E_G \rightrightarrows E; G \rightrightarrows M)$ and a full-core one $(t^*E \rightrightarrows 0_M; G \rightrightarrows M)$. Actually, $E_G = s^*E \rightarrow G$ and $t^*E \rightarrow G$ are canonically isomorphic as vector bundles over G (see, e.g., [60]). The vector bundle isomorphism $\tau : E_G \rightarrow t^*E$ is given by

$$\tau(g, e) = (g, g.e), \quad (3.4)$$

for all $(g, e) \in G_s \times_q E = E_G$. Hence the gauge algebroids DE_G and Dt^*E are canonically isomorphic as Lie algebroids over G .

In this section, we show that the gauge algebroid DE_G fits itself in a VB-groupoid $(DE_G \rightrightarrows DE; G \rightrightarrows M)$ (Proposition 3.10). Using the isomorphism τ we immediately get that Dt^*E fits in a VB-groupoid as well. In other words, the situation is basically the same for trivial-core and full-core VB-groupoids, exactly as one would expect from the fact that they are both equivalent to the same object: a representation of a Lie groupoid.

Proposition 3.10 *There is a natural VB-groupoid structure $(DE_G \rightrightarrows DE; G \rightrightarrows M)$ and a short exact sequence of VB-groupoids:*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{End}E_G & \longrightarrow & DE_G & \xrightarrow{\sigma} & TG \longrightarrow 0 \\
& & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow \\
0 & \longrightarrow & \text{End}E & \longrightarrow & DE & \xrightarrow{\sigma} & TM \longrightarrow 0 \\
& & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow \\
& & G & \longrightarrow & G & \longrightarrow & G & \longrightarrow 0 \\
& & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow \\
& & M & \longrightarrow & M & \longrightarrow & M & \longrightarrow 0
\end{array} \tag{3.5}$$

Here $(\text{End}E_G \rightrightarrows \text{End}E; G \rightrightarrows M)$ is the trivial-core VB-groupoid corresponding to the induced action of G on $\text{End}E$.

PROOF: Denote by

$$\mathbf{a} : G \times_q E = E_G \rightarrow E, \quad (g, e) \mapsto \mathbf{a}_g e = g.e = \tilde{t}(g, e), \tag{3.6}$$

the action of G on E . We begin with a remark (for which we thank the referee): *the action $\mathbf{a}$ induces an action $\mathcal{A}$ of the tangent groupoid $TG \rightrightarrows TM$ on $\sigma : DE \rightarrow TM$.* To see this, let $g \in G$, let $V \in T_g G$, and let $\Delta \in D_{s(g)}E$ be such that:

$$ds(V) = \sigma(\Delta). \tag{3.7}$$

Choose a curve $\varepsilon \mapsto \Gamma(\varepsilon)$ in G such that $\Gamma(0) = g$ and

$$V = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \Gamma(\varepsilon), \tag{3.8}$$

and denote $\gamma = s \circ \Gamma$. Choose also a curve $\varepsilon \mapsto \gamma_E(\varepsilon) : E_{s(g)} \rightarrow E_{\gamma(\varepsilon)}$ of isomorphisms such that

$$\Delta = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \gamma_E(\varepsilon). \tag{3.9}$$

As the symbol of Δ is exactly $ds(V)$ this is possible. Now, put

$$\mathcal{A}(V, \Delta) := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathbf{a}_{\Gamma(\varepsilon)} \circ \gamma_E(\varepsilon) \circ \mathbf{a}_{g^{-1}}. \tag{3.10}$$

It immediately follows that since $\mathbf{a}$ is an action, $\mathcal{A}$ is an action as well, as claimed.

Now, recall from Section 2.1 that the gauge algebroid DE_N of the pull-back vector bundle $E_N = \phi^*E \rightarrow N$ along a smooth map $\phi : N \rightarrow M$, is (canonically isomorphic to) the pull-back $DE_N \cong TN \times_{d\phi \times \sigma} DE$. In particular,

$$DE_G \cong TG \times_{ds \times \sigma} DE. \tag{3.11}$$

The action of $TG \rightrightarrows DE$ then gives DE_G the structure of an action groupoid $DE_G \rightrightarrows DE$. It is now easy to check that with all its structure maps, $(DE_G \rightrightarrows DE; G \rightrightarrows M)$ is indeed a VB-groupoid fitting in the exact sequence (3.5). We leave details to the reader (see also Remark 3.11). $\square$

Remark 3.11 The structure maps of the groupoid $DE_G \rightrightarrows DE$ are those of the action groupoid. For instance, the source $s_D : DE_G \rightarrow DE$ is just the projection of $DE_G = TG_{ds \times \sigma} DE$ onto the second factor. In other words $s_D = D\tilde{s}$. The target $t_D : DE_G \rightarrow DE$ is just the action map $A : DE_G = TG_{ds \times \sigma} DE \rightarrow DE$, and a direct check exploiting (2.9) reveals that $t_D = D\tilde{t}$. It is easy to check that the other structure maps (multiplication m_D , unit u_D , inversion i_D) are all induced by those of the VB-groupoid $(E_G \rightrightarrows E; G \rightrightarrows M)$ via application of the functor D . The multiplication may need a little explanation. Consider the space $E_G^{(2)} = E_{G\tilde{s} \times \tilde{t}} E_G$ of composable arrows of $E_G \rightrightarrows E$. It is a vector bundle over $G^{(2)}$ in the obvious way, and the projections $\tilde{pr}_i : E_G^{(2)} \rightarrow E_G$ over the factors are regular vector bundle morphisms covering the projections $pr_i : G^{(2)} \rightarrow G$, $i = 1, 2$. Similarly, the multiplication $\tilde{m} : E_G^{(2)} \rightarrow E_G$ is a regular vector bundle morphism covering $m : G^{(2)} \rightarrow G$. The space of composable arrows of $DE_G \rightrightarrows DE$ is

$$(DE_G)^{(2)} = DE_{G \text{ D}\tilde{s} \times \text{D}\tilde{t}} DE_G, \quad (3.12)$$

and it is a vector bundle over $G^{(2)}$ as well. Finally, the map

$$D(E_G^{(2)}) \rightarrow (DE_G)^{(2)}, \quad \square \mapsto (D\tilde{pr}_1(\square), D\tilde{pr}_2(\square)) \quad (3.13)$$

is an isomorphism of vector bundles over $G^{(2)}$. Under this identification, $m_D = D\tilde{m}$. Similarly, $u_D = D\tilde{u}$ and $i_D = D\tilde{i}$. This shows a close analogy between the groupoid $DE_G \rightrightarrows DE$ and the tangent groupoid $TG \rightrightarrows TM$.

Remark 3.12 The core C of the VB-groupoid $(DE_G \rightrightarrows DE; G \rightrightarrows M)$ is canonically isomorphic to A . The isomorphism $A \rightarrow C$ maps a to $\mathbb{D}_a$, where $\mathbb{D}$ is the obvious flat $(\ker ds)$ -connection in E_G .

A similar situation occurs for trivial-core/full-core VB algebroids: let $A \rightrightarrows M$ be a Lie algebroid with projection $p : A \rightarrow M$, and let $E \rightarrow M$ be a vector bundle carrying a representation of A . We denote by ∇ the flat A -connection in E . Beware that from now on DE_A will always indicate the gauge algebroid of the vector bundle $E_A \rightarrow A$ (not the vector bundle $E_A \rightarrow E$). We will now show that DE_A fits in a VB-algebroid $(DE_A \rightrightarrows DE; A \rightrightarrows M)$ (Proposition 3.13). Finally, if $(E_A \rightrightarrows E; A \rightrightarrows M)$ is the VB-algebroid of the VB-groupoid $(E_G \rightrightarrows E; G \rightrightarrows M)$ then $(DE_A \rightrightarrows DE; A \rightrightarrows M)$ is the VB-algebroid of $(DE_G \rightrightarrows DE; G \rightrightarrows M)$ (Proposition 3.16).

Proposition 3.13 *There is a natural VB-algebroid structure $(DE_A \rightrightarrows DE; A \rightrightarrows M)$ and a short exact sequence of VB-algebroids:*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{End}E_A & \longrightarrow & DE_A & \xrightarrow{\sigma} & TA & \longrightarrow & 0 \\
& & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \\
0 & \longrightarrow & \text{End}E & \longrightarrow & DE & \xrightarrow{\sigma} & TM & \longrightarrow & 0 \\
& & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \\
& & A & \xrightarrow{\quad} & A & \xrightarrow{\quad} & A & & \\
& & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \\
& & M & \xrightarrow{\quad} & M & \xrightarrow{\quad} & M & &
\end{array} \quad (3.14)$$

Here $(\text{End}E_A \rightrightarrows \text{End}E; A \rightrightarrows M)$ is the trivial-core VB-algebroid corresponding to the induced action of A on $\text{End}E$.

We propose a proof in the same spirit of that of Proposition 3.10. We conjecture that there exists a proof exploiting the fact that DE_A is the fat algebroid of the VB-algebroid $(TE_A \rightrightarrows E_A; TA \rightrightarrows A)$. Actually, TE_A is a triple structure: it is a Lie algebroid over E_A , over TA , and over TE , and all these structures are compatible. We expect the conjectured proof to require a preliminary analysis of the fat algebroid of a triple structure like that but this goes beyond the scopes of the present paper.

PROOF: We begin remarking that similarly as in the groupoid case *the flat connection ∇ induces an (infinitesimal) action*

$$\mathfrak{A} : \Gamma(TA, TM) \rightarrow \mathfrak{X}(DE) \quad (3.15)$$

of the tangent algebroid $TA \Rightarrow TM$ on $\sigma : DE \rightarrow TM$. To see this, recall from Example 1.8, that sections of $TA \rightarrow TM$ are generated by core sections $\widehat{a}_T$, and linear sections of the form da , for some $a \in \Gamma(A)$. Now, every section a of A determines two vector fields on DE as follows. Correspondence $\Delta \mapsto [\nabla_a, \Delta]$ is a derivation of DE . Hence, it corresponds to a (linear) vector field Y_a on (the total space of) DE itself. Section a does also determine another vector field on DE : the vertical lift V_a of $\nabla_a \in \Gamma(DE)$. We put

$$\mathfrak{A}(da) = Y_a \quad \text{and} \quad \mathfrak{A}(\widehat{a}_T) = V_a, \quad (3.16)$$

for all $a \in \Gamma(A)$. It is easy to see that this uniquely defines an infinitesimal action as required.

Now, the gauge algebroid DE_A is the pull-back $DE_A \cong TA \times_{dp \times \sigma} DE$. The infinitesimal action $\mathfrak{A}$ then gives DE_A the structure of an action algebroid $DE_A \Rightarrow DE$. It is easy to check that $(DE_A \Rightarrow DE; A \Rightarrow M)$ with all its structure maps is indeed a VB-groupoid fitting in the exact sequence (3.14). We leave details to the reader (see also Remark 3.14). $\square$

Remark 3.14 The structure maps of the algebroid $DE_A \Rightarrow DE$ are those of the action algebroid. For instance, the projection $p_D : DE_A \rightarrow DE$ is just the projection of $DE_A = TA \times_{dp \times \sigma} DE$ onto the second factor. In other words $p_D = Dp$. In order to describe the anchor $\rho_D : DE_A \rightarrow TDE$ first notice that sections of $DE_A \rightarrow DE$ are generated by pull-backs of sections of $TA \rightarrow TM$. The pull-back of a core section $\widehat{a}_T$ is a core section that we denote by $\widehat{a}_D$, while the pull-back of a linear section of the form da is simply Da , $a \in \Gamma(A)$. Hence the anchor $\rho_D : DE_A \rightarrow TDE$ is uniquely determined by

$$\rho_D(Da) = Y_a \quad \text{and} \quad \rho_D(\widehat{a}_D) = V_a, \quad (3.17)$$

for all $a \in \Gamma(A)$. Finally,

$$\begin{aligned} [Da, Db] &= D[a, b], \\ [Da, \widehat{b}_D] &= [\widehat{a}, b]_D, \\ [\widehat{a}_D, \widehat{b}_D] &= 0, \end{aligned} \quad (3.18)$$

for all $a, b \in \Gamma(A)$.

Remark 3.15 The core C of the VB-algebroid $(DE_A \Rightarrow DE; A \Rightarrow M)$ is canonically isomorphic to A . The isomorphism $A \rightarrow C$ maps a to $\mathbb{D}_a$, where $\mathbb{D}$ is the flat $(\ker dp)$ -connection in E_A .

Proposition 3.16 *Let $(E_G \rightrightarrows E; G \rightrightarrows M)$ be a trivial-core VB-groupoid, and let $(E_A \rightrightarrows E; A \rightrightarrows M)$ be its trivial-core VB-algebroid (equivalently, let $(t^*E \rightrightarrows 0_M; G \rightrightarrows M)$ be a full-core VB-groupoid and let $(A \times_M E \rightrightarrows 0_M; A \rightrightarrows M)$ be its full-core VB-algebroid). Then $(DE_A \Rightarrow DE; A \Rightarrow M)$ is canonically isomorphic to the VB-algebroid of the VB-groupoid $(DE_G \rightrightarrows DE; G \rightrightarrows M)$.*

PROOF: It is enough to check that the Lie functor intertwines the action $\mathcal{A}$ in the proof of Proposition 3.10 and the infinitesimal action $\mathfrak{A}$ in the proof of Proposition 3.13. To do this, denote by

$$\dot{\mathcal{A}} : \Gamma(TA, TM) \rightarrow \mathfrak{X}(DE) \quad (3.19)$$

the infinitesimal action induced by $\mathcal{A}$. This means that for every section $w \in \Gamma(TA, TM)$, every $x \in M$ and every point $\Delta \in D_x E$

$$\dot{\mathcal{A}}(w)_\Delta := \left. \frac{d}{d\lambda} \right|_{\lambda=0} \mathcal{A}(\phi_\lambda^w(v), \Delta) \quad (3.20)$$

where $\{\phi_\lambda^w\}$ is the flow of $\vec{w}$, the right invariant vector field on TG corresponding to w , $v = \sigma(\Delta)$, and we identify v with $1_v \in TG$.

Now, we compute (3.20) in the cases $w = \hat{a}_T$ and $w = da$, for some $a \in \Gamma(A)$. We begin with the first one. For every $x \in M$, and every $v \in T_x M$, we have (see [46])

$$\phi_\lambda^{\hat{a}_T}(v) = v + \lambda a_x. \quad (3.21)$$

Hence

$$\begin{aligned} \dot{\mathcal{A}}(\hat{a}_T)_\Delta &= \left. \frac{d}{d\lambda} \right|_{\lambda=0} \mathcal{A}(v + \lambda a_x, \Delta) = \left. \frac{d}{d\lambda} \right|_{\lambda=0} \mathcal{A}(v + \lambda a_x, \Delta + 0) = \left. \frac{d}{d\lambda} \right|_{\lambda=0} \mathcal{A}(v, \Delta) + \lambda \mathcal{A}(a_x, 0) \\ &= \left. \frac{d}{d\lambda} \right|_{\lambda=0} \Delta + \lambda \mathcal{A}(a_x, 0), \end{aligned} \quad (3.22)$$

where we used the (easy to check) fact that $\mathcal{A} : TG_{ds \times \sigma} DE \rightarrow DE$ is a vector bundle morphism covering the target $t : G \rightarrow M$. Now, for every section $e \in \Gamma(E)$,

$$\nabla_{a_x} e = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathbf{a}_{\phi_\varepsilon^a(x)} e, \quad (3.23)$$

where $\{\phi_\varepsilon^a\}$ is the flow of $\vec{a}$, and a direct computation exploiting (3.23) and (3.10) then shows that $\mathcal{A}(a_x, 0) = \nabla_{a_x}$. We conclude that

$$\dot{\mathcal{A}}(\hat{a}_T)_\Delta = \left. \frac{d}{d\lambda} \right|_{\lambda=0} \Delta + \lambda \nabla_{a_x} = (V_a)_\Delta = \mathfrak{A}(\hat{a}_T)_\Delta. \quad (3.24)$$

Finally, let $w = da$. We have

$$\phi_\lambda^{da} = d\phi_\lambda^a, \quad (3.25)$$

where, as usual, $\{\phi_\lambda^a\}$ is the flow of $\vec{a}$. Hence

$$\dot{\mathcal{A}}(da)_\Delta := \left. \frac{d}{d\lambda} \right|_{\lambda=0} \mathcal{A}(d\phi_\lambda^a(v), \Delta). \quad (3.26)$$

Choose a curve $\varepsilon \mapsto \gamma(\varepsilon)$ in G such that $\gamma(0) = x$, and

$$v = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \gamma(\varepsilon), \quad (3.27)$$

and a curve $\varepsilon \mapsto \gamma_E(\varepsilon) : E_x \rightarrow E_{\gamma(\varepsilon)}$ of isomorphisms such that

$$\Delta = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \gamma_E(\varepsilon). \quad (3.28)$$

Then

$$d\phi_\lambda^a(v) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \phi_\lambda^a(\gamma(\varepsilon)), \quad (3.29)$$

and from (3.10),

$$\mathcal{A}(d\phi_\lambda^a(v), \Delta) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathbf{a}_{\phi_\lambda^a(\gamma(\varepsilon))} \circ \gamma_E(\varepsilon) \circ \mathbf{a}_{\phi_\lambda^a(x)}^{-1} = D\Phi_\lambda^a(\Delta), \quad (3.30)$$

where Φ_λ^a is the vector bundle isomorphism

$$\Phi_\lambda^a : E \rightarrow E, \quad e \mapsto \mathbf{a}_{\phi_\lambda^a(q(e))} e. \quad (3.31)$$

This shows that the vector field $\dot{\mathcal{A}}(da) \in \mathfrak{X}(DE)$ generates the flow $\{D\Phi_\lambda^a\}$. Notice that $\{\Phi_\lambda^a\}$ is the flow of ∇_a . It now easily follows that $\{D\Phi_\lambda^a\}$ is the flow of the adjoint operator $[\nabla_a, -]$. In other words

$$\dot{\mathcal{A}}(da) = Y_a = \mathfrak{A}(da). \quad (3.32)$$

This concludes the proof. $\square$

3.3 Multiplicative derivations as multiplicative sections

Recall that a vector field X on a Lie groupoid $G \rightrightarrows M$ is multiplicative if and only if, when regarded as a section $X : G \rightarrow TG$ of $TG \rightarrow G$, it is a morphism of Lie groupoids. A similar statement holds true for IM vector fields on Lie algebroids. In this section we prove similar results for multiplicative derivations of trivial-core and full-core VB-groupoids and IM derivations of trivial-core and full-core VB-algebroids.

Theorem 3.17 *Let $(E_G \rightrightarrows E; G \rightrightarrows M)$ be a trivial-core VB-groupoid. A derivation of $E_G \rightarrow G$ is multiplicative if and only if it is a multiplicative section of $DE_G \rightarrow G$.*

PROOF: Let $G \rightrightarrows M$ be a Lie groupoid, and let $E \rightarrow M$ be a vector bundle equipped with a representation of G . Then $(E_G \rightrightarrows E; G \rightrightarrows M)$ is a trivial-core VB-groupoid with structure maps $\tilde{s}, \tilde{t}, \tilde{m}, \tilde{u}, \tilde{i}$ as in Example 3.2. Similarly the dual representation of G on $E^* \rightarrow M$ determines the trivial-core VB-groupoid $(E_G^* \rightrightarrows E^*; G \rightrightarrows M)$ with structure maps denoted by $\tilde{s}^*, \tilde{t}^*, \tilde{m}^*, \tilde{u}^*, \tilde{i}^*$. Applying Proposition 3.10 to both $(E_G \rightrightarrows E; G \rightrightarrows M)$ and $(E_G^* \rightrightarrows E^*; G \rightrightarrows M)$, we get VB-groupoids $(DE_G \rightrightarrows DE; G \rightrightarrows M)$ and $(DE_G^* \rightrightarrows DE^*; G \rightrightarrows M)$ with structure maps denoted by s_D, t_D, m_D, u_D, i_D and $s_D^*, t_D^*, m_D^*, u_D^*, i_D^*$ respectively. It is straightforward to see that isomorphism $DE_G \rightarrow DE_G^*, \Delta \mapsto \Delta^*$ (see Lemma 2.2) is actually a VB-groupoid isomorphism $(DE_G \rightrightarrows DE; G \rightrightarrows M) \rightarrow (DE_G^* \rightrightarrows DE^*; G \rightrightarrows M)$ (covering isomorphism $DE \rightarrow DE^*$). As a consequence, a derivation $\Delta \in \Gamma(DE_G)$ is a multiplicative section of $DE_G \rightarrow DE$ if and only if its “dual” derivation $\Delta^* \in \Gamma(DE_G^*)$ is a multiplicative section of $DE_G^* \rightarrow DE^*$. So it remains to check that the following conditions are equivalent:

- i.) X_Δ is a multiplicative section of $TE_G \rightarrow E_G$,
- ii.) Δ^* is a multiplicative section of $DE_G^* \rightarrow G$.

First show that $i.) \Rightarrow ii.)$. Condition $i.)$ means that there is a (necessarily unique) vector field $\underline{X} \in \mathfrak{X}(E)$ such that

$$d\tilde{s}(X_\Delta)_{(g,e)} = \underline{X}_e, \quad d\tilde{t}(X_\Delta)_{(g,e)} = \underline{X}_{g.e}, \quad (3.33)$$

for all $(g, e) \in E_G$, and, additionally, for every pair of composable arrows $((g, e), (g', e')) \in (E_G)^{(2)}$, we have

$$(X_\Delta)_{(gg', e')} = d\tilde{m}((X_\Delta)_{(g,e)}, (X_\Delta)_{(g', e')}). \quad (3.34)$$

In this case, it follows from (3.33), that $\underline{X}$ is also a linear vector field, i.e. $\underline{X} = X_\Delta$ for some (necessarily unique) derivation $\underline{\Delta} \in \Gamma(DE)$. Hence we can rephrase (3.33) in the following equivalent way

$$s_D^* \Delta_g^* = \underline{\Delta}_{s(g)}, \quad t_D^* \Delta_g^* = \underline{\Delta}_{t(g)}, \quad (3.35)$$

for all $g \in G$. assume the first one of (3.33), then Equation (3.34) holds if and only if its sides have the same image through both $d\tilde{s}$ and the tangent maps to the projections $q_G : E_G \rightarrow G$. The first condition is guaranteed by the first one of (3.33). As X_Δ is q_G -related to $\sigma(\Delta) = \sigma(\Delta^*)$, in view of (3.5) the second condition is equivalent to

$$\sigma(\Delta_{gg'}^*) = \sigma(m_D^*(\Delta_g^*, \Delta_{g'}^*)), \quad (3.36)$$

for all $(g, g') \in G^{(2)}$. Finally, from the isomorphism $DE_G = TG \times_{ds} \sigma DE$, Equation (3.36), together with the first one of (3.35), gives

$$\Delta_{gg'}^* = m_D^*(\Delta_g^*, \Delta_{g'}^*). \quad (3.37)$$

for all $(g, g') \in G^{(2)}$. Equations (3.35) and (3.37) exactly mean that Condition $ii.)$ holds. Implication $ii.) \Rightarrow i.)$ can be proved by retracing the above steps in the reverse order. $\square$

Theorem 3.18 *Let $(t^*C \rightrightarrows 0_M; G \rightrightarrows M)$ be a full-core VB-groupoid. A derivation of $t^*C \rightarrow G$ is multiplicative if and only if it corresponds, via the vector bundle isomorphism $\tau : C_G \rightarrow t^*C$, to a multiplicative section of $DC_G \rightarrow G$.*

PROOF: Let Δ be a derivation of $t^*C \rightarrow G$. Then Δ is multiplicative if and only if it generates a flow by automorphisms of $(t^*C \rightrightarrows 0_M; G \rightrightarrows M)$ if and only if the dual derivation Δ^* generates a flow by automorphisms of the dual VB-groupoid $(C_G^* \rightrightarrows C^*; G \rightrightarrows M)$ if and only if Δ^* is a multiplicative derivation of $C_G^* \rightarrow G$. Now $(C_G^* \rightrightarrows C^*; G \rightrightarrows M)$ is a trivial-core VB-groupoid, hence Δ is multiplicative if and only if Δ^* is a multiplicative section of $(DC_G^* \rightrightarrows DC^*; G \rightrightarrows M)$. Finally, recall from the proof of Theorem 3.17, that the correspondence $\Delta \mapsto \Delta^*$ is an isomorphism of the VB-groupoids $(DC_G \rightrightarrows DC; G \rightrightarrows M)$ and $(DC_G^* \rightrightarrows DC^*; G \rightrightarrows M)$. This concludes the present proof. $\square$

Theorem 3.19 *Let $(E_A \rightrightarrows E; A \rightrightarrows M)$ be a trivial-core VB-algebroid. A derivation of $E_A \rightarrow A$ is IM if and only if it is an IM section of $DE_A \rightarrow A$.*

PROOF: Let Δ be a derivation of $E_A \rightarrow A$ and let $X_\Delta \in \mathfrak{X}(E_A)$ be the corresponding linear vector field. Regard Δ as a section of $DE_A \rightarrow A$, and X_Δ as a section of $E_A \rightarrow TE_A$. In view of Remark 2.19 it is enough to show that Δ is an IM section of $DE_A \rightarrow A$ if and only if X_Δ is an IM section of $E_A \rightarrow TE_A$. This can be checked with a long, but straightforward, computation in local coordinates, that we sketch below. Let (x^i) be local coordinates on M , let (e_A) be a local basis of $\Gamma(E)$, and let (v^A) be the associated, linear, fiber coordinates on E . Similarly, let (ε_α) be a local basis of $\Gamma(A)$, and let (u^α) be the associated, linear, fiber coordinates on A . So E_A is coordinatized by (x^i, u^α, v^A) . A derivation $\Delta : \Gamma(E) \rightarrow E_x$ of $E \rightarrow M$ at the point $x \in M$, is locally of the form

$$\Delta(f^A e_A) = \left(\dot{x}^i \frac{\partial}{\partial x^i} \Big|_x f^B + f^A(x) v_A^B \right) e_{B,x} \quad (3.38)$$

for some real numbers $(\dot{x}^i, v_A^B)$ that serve as linear, fiber coordinates on the total space of $DE \rightarrow M$. Note that $\sigma(\Delta) = \dot{x}^i \frac{\partial}{\partial x^i} \Big|_x$. Similarly, a derivation $\Delta : \Gamma(E_A) \rightarrow E_x$ of $E_A \rightarrow A$ at the point $a \in A$, is locally of the form

$$\Delta(f^A p^* e_A) = \left(\dot{x}^i \frac{\partial}{\partial x^i} \Big|_a f^B + \dot{u}^\alpha \frac{\partial}{\partial u^\alpha} \Big|_a f^B + f^A(x) v_A^B \right) e_{B,x} \quad (3.39)$$

for some real numbers $(\dot{x}^i, \dot{u}^\alpha, v_A^B)$ that serve as linear, fiber coordinates on the total space of $DE_A \rightarrow A$, and $\sigma(\Delta) = \dot{x}^i \frac{\partial}{\partial x^i} \Big|_a + \dot{u}^\alpha \frac{\partial}{\partial u^\alpha} \Big|_a$. The Lie algebroid structure in $A \rightarrow M$ is encoded by the de Rham differential d_A in $\Gamma(\wedge^\bullet A^*) = \wedge^\bullet \Gamma(A)^*$ which does locally look like:

$$d_A = \rho_\alpha^i u^\alpha \frac{\partial}{\partial x^i} - \frac{1}{2} c_{\beta\gamma}^\alpha u^\beta \wedge u^\gamma \frac{\partial}{\partial u^\alpha}, \quad (3.40)$$

for some functions (ρ_α^i) encoding the anchor $\rho : A \rightarrow TM$, and some (structure) functions $(c_{\beta\gamma}^\alpha)$ encoding the Lie bracket in $\Gamma(A)$. Finally, the flat A -connection ∇ in E is locally encoded by “Christoffel symbols” $(\Gamma_{\alpha A}^B)$ such that

$$\nabla_{\varepsilon_\alpha} e_A = \Gamma_{\alpha A}^B e_B. \quad (3.41)$$

It is now straightforward to check that the Lie algebroid structure in $E_A \rightarrow E$ is encoded by a de Rham differential d_{E_A} in $\wedge^\bullet \Gamma(E_A, E)^*$ locally given by:

$$d_{E_A} = \rho_\alpha^i u^\alpha \frac{\partial}{\partial x^i} - \Gamma_{\alpha A}^B v^B u^\alpha \frac{\partial}{\partial v^B} - \frac{1}{2} c_{\beta\gamma}^\alpha u^\beta \wedge u^\gamma \frac{\partial}{\partial u^\alpha}. \quad (3.42)$$

The (tangent) Lie algebroid structure in $TE_A \rightarrow TE$ is then encoded by a differential d_{TE_A} in $\wedge^\bullet \Gamma(TE_A, TE)^*$ locally given by the “tangent lift” of (3.42), i.e.

$$\begin{aligned} d_{TE_A} = & \rho_\alpha^i u^\alpha \frac{\partial}{\partial x^i} + \left(\frac{\partial \rho_\alpha^i}{\partial x^j} \dot{x}^j u^\alpha + \rho_\alpha^i \dot{u}^\alpha \right) \frac{\partial}{\partial \dot{x}^i} \\ & - \Gamma_{\alpha A}^B v^B u^\alpha \frac{\partial}{\partial v^B} - \left(\frac{\partial \Gamma_{\alpha B}^A}{\partial x^i} \dot{x}^i v^B u^\alpha + \Gamma_{\alpha B}^A \dot{v}^B u^\alpha + \Gamma_{\alpha B}^A v^B \dot{u}^\alpha \right) \frac{\partial}{\partial \dot{v}^A} \\ & - \frac{1}{2} c_{\beta\gamma}^\alpha u^\beta \wedge u^\gamma \frac{\partial}{\partial u^\alpha} - \left(c_{\beta\gamma}^\alpha u^\beta \wedge \dot{u}^\gamma + \frac{1}{2} \frac{\partial c_{\beta\gamma}^\alpha}{\partial x^i} \dot{x}^i u^\beta \wedge u^\gamma \right) \frac{\partial}{\partial \dot{u}^\alpha}. \end{aligned} \quad (3.43)$$

Finally the Lie algebroid structure in $DE_A \rightarrow DE$ is encoded by a differential d_{DE_A} in $\wedge^\bullet \Gamma(DE_A, DE)^*$ locally given by

$$\begin{aligned} d_{DE_A} = & \rho_\alpha^i u^\alpha \frac{\partial}{\partial x^i} + \left(\frac{\partial \rho_\alpha^i}{\partial x^j} \dot{x}^j u^\alpha + \rho_\alpha^i \dot{u}^\alpha \right) \frac{\partial}{\partial \dot{x}^i} \\ & + \left(\frac{\partial \Gamma_{\alpha B}^A}{\partial x^i} \dot{x}^i u^\alpha + \Gamma_{\alpha B}^C v_C^A u^\alpha - \Gamma_{\alpha C}^A v_B^C u^\alpha + \Gamma_{\alpha B}^A \dot{u}^\alpha \right) \frac{\partial}{\partial v_B^A} \\ & - \frac{1}{2} c_{\beta\gamma}^\alpha u^\beta \wedge u^\gamma \frac{\partial}{\partial u^\alpha} - \left(c_{\beta\gamma}^\alpha u^\beta \wedge \dot{u}^\gamma + \frac{1}{2} \frac{\partial c_{\beta\gamma}^\alpha}{\partial x^i} \dot{x}^i u^\beta \wedge u^\gamma \right) \frac{\partial}{\partial \dot{u}^\alpha}. \end{aligned} \quad (3.44)$$

Now, let Δ be a section of $DE_A \rightarrow A$ locally given by

$$\Delta : \begin{cases} \dot{x}^i &= X^i(x, u) \\ \dot{u}^\alpha &= U^\alpha(x, u) \\ v_B^A &= V_B^A(x, u) \end{cases}. \quad (3.45)$$

The associated linear vector field on E_A is the section X_Δ of $TE_A \rightarrow E_A$ locally given by

$$X_\Delta : \begin{cases} \dot{x}^i &= X^i(x, u) \\ \dot{u}^\alpha &= U^\alpha(x, u) \\ \dot{v}^A &= -V_B^A(x, u) v^B \end{cases}. \quad (3.46)$$

By definition, Δ is an IM section if and only if

(IM1) Δ is a morphism of the vector bundles $A \rightarrow M$ and $DE_A \rightarrow DE$, and

(IM2) $\Delta^* \circ d_{DE_A} = d_A \circ \Delta^*$,

where $\Delta^* : \wedge^\bullet \Gamma(DE_A, DE)^* \rightarrow \wedge^\bullet \Gamma(A)^*$ is the obvious graded algebra map induced by Δ in view of (IM1). Similarly, X_Δ is an IM section if and only if

(IM'1) X_Δ is a morphism of the vector bundles $E_A \rightarrow E$ and $TE_A \rightarrow TE$, and

(IM'2) $X_\Delta^* \circ d_{TE_A} = d_{E_A} \circ X_\Delta^*$,

where $X_\Delta^* : \wedge^\bullet \Gamma(TE_A, TE)^* \rightarrow \wedge^\bullet \Gamma(E_A, E)^*$ is the graded algebra map induced by X_Δ in view of (IM'1). A direct computation shows that (IM1) and (IM'1) are both equivalent to

$$\begin{aligned} X^i &= X^i(x), \\ V_B^A &= V_B^A(x), \\ U^\alpha &= U_\beta^\alpha(x) u^\beta. \end{aligned} \quad (3.47)$$

Finally, $(IM'1)$ and $(IM'2)$ are both equivalent to the following system of PDEs:

$$\begin{aligned}\rho_\alpha^j \frac{\partial X^i}{\partial x^j} &= \frac{\partial \rho_\alpha^i}{\partial x^i} X^i + \rho_\beta^i U_\alpha^\beta, \\ \rho_\alpha^i \frac{\partial V_B^A}{\partial x^i} &= \frac{\partial \Gamma_{\alpha B}^A}{\partial x^i} X^i + \Gamma_{\alpha B}^C V_C^A - \Gamma_{\alpha C}^A V_B^C + \Gamma_{\beta A}^B U_\alpha^\beta, \\ \rho_\beta^i \frac{\partial U_\gamma^\alpha}{\partial x^i} - \rho_\gamma^i \frac{\partial U_\beta^\alpha}{\partial x^i} &= c_{\delta\beta}^\alpha U_\gamma^\delta - c_{\delta\gamma}^\alpha U_\beta^\delta + c_{\beta\gamma}^\delta U_\delta^\alpha - \frac{\partial c_{\beta\gamma}^\alpha}{\partial x^i} X^i.\end{aligned}\tag{3.48}$$

This concludes the proof. $\square$

By an analogous computation we can prove the following

Theorem 3.20 *Let $(A \times_M C \rightrightarrows 0_M; A \rightrightarrows M)$ be a full-core VB-algebroid. A derivation of $A \times_M C \rightarrow A$ is IM if and only if it is an IM section of $DC_A \rightarrow A$.*

We showed in this section that the gauge algebroid of a trivial-core or a full-core VB-groupoid is a VB-groupoid itself and likewise for VB-algebroids. For a generic VB-groupoid $(W \rightrightarrows E; G \rightrightarrows M)$ the gauge algebroid is *not* a VB-groupoid. To see this notice that DW contains $\text{End } W = W^* \otimes W$ as a vector subbundle (and a Lie subalgebroid). But the tensor product of two VB-groupoids over $G \rightrightarrows M$ is not a VB-groupoid in general. In fact it is part of a *higher VB-groupoid* or, more precisely, a *simplicial vector bundle over the nerve of G* [19]. We conjecture that DW is a higher VB-groupoid in the same sense. In the case of a VB-algebroid $(W \rightrightarrows E; A \rightrightarrows M)$ we can say a little bit more. The Lie algebroid $A \rightrightarrows M$ is equivalent to the degree 1 $\mathbb{N}Q$ -manifold $(A[1], d_A)$. Similarly, as proved by T. Voronov [62], $(W \rightrightarrows E; A \rightrightarrows M)$ is equivalent to the vector bundle $(W[1]_E \rightarrow A[1], d_W)$ in the category of degree 1 $\mathbb{N}Q$ -manifolds. Here $W[1]_E$ is obtained from W shifting by one the degree in the fibers over E . Now, not only d_W is a homological vector field, but it is a linear vector field, hence it corresponds to a degree 1 derivation Δ_W of $W[1]_E \rightarrow A[1]$, such that $\Delta_W^2 = 0$. Denote by $\mathcal{D}W[1]_E$ the module of all graded derivations of $W[1]_E \rightarrow A[1]$, so that $(\mathcal{D}W[1]_E, [\Delta_W, -])$ is a DG Lie algebra. More precisely it is the DG Lie algebra of sections of a DG Lie algebroid $D_A W[1]_E \rightrightarrows A[1]$. The cochain complex $(\mathcal{D}W[1]_E, [\Delta_W, -])$ is canonically isomorphic to the linear deformation complex of $(W \rightrightarrows E; A \rightrightarrows M)$ [24, 39], so that IM derivations of $W \rightarrow A$ are the same as degree zero derivations of $W[1]_E \rightarrow A[1]$ commuting with Δ_W . In this way we encoded IM derivations in $D_A W[1]_E \rightrightarrows A[1]$. After a suitable shift, the latter is itself a vector bundle in the category of $\mathbb{N}Q$ -manifolds, however it is not of degree 1, but of degree 2. So it is a degree 2 Lie algebroid module in the sense of Vaňtrob [57] and corresponds to a 3-term representation up to homotopy of A (see [51], see also next section, for more details on representations up to homotopy).

4 A 2-term representation up to homotopy from a plain representation

4.1 VB-groupoids

There is a nice relationship between VB-groupoids and representations up to homotopy [28]. Let $G \rightrightarrows M$ be a Lie groupoid. Recall that a *representation up to homotopy* of G [2] is a graded vector bundle $q_\mathcal{E} : \mathcal{E} = \bigoplus_k \mathcal{E}_k \rightarrow M$ together with a degree 1, $\mathbb{R}$ -linear operator

$$D : C^\bullet(G, \mathcal{E}) \rightarrow C^\bullet(G, \mathcal{E})\tag{4.1}$$

satisfying

$$D^2 = 0, \quad \text{and} \quad D(\omega \cdot \varepsilon) = \delta\omega \cdot \varepsilon + (-)^{\omega} \omega \cdot D\varepsilon,\tag{4.2}$$

for all $\omega \in C^\bullet(G)$, and all $\varepsilon \in C^\bullet(G, \mathcal{E})$. Here $(C^\bullet(G), \delta)$ is the Lie groupoid complex of G , and $C^p(G, \mathcal{E})$ consists of $\mathcal{E}$ -valued groupoid p -cochains, i.e. maps

$$\varepsilon : G^{(p)} \rightarrow \mathcal{E}, \quad (g_1, \dots, g_p) \mapsto \omega(g_1, \dots, g_p) \in \mathcal{E}_{t(g_1)}. \quad (4.3)$$

Additionally, D is required to preserve *normalized cochains*, i.e. cochains vanishing when one of their entry is 1. We stress that when we say that the degree of D is 1, we refer to the *total degree*, i.e. the sum of the cochain degree and the degree in $\mathcal{E}$. For more details see [2]. We are mainly interested in 2-term representations up to homotopy, i.e. to the case when $\mathcal{E} = \mathcal{E}_{-1} \oplus \mathcal{E}_0$ is concentrated in degree $-1, 0$. A 2-term representation up to homotopy $(\mathcal{E} = \mathcal{E}_{-1} \oplus \mathcal{E}_0, D)$ is actually equivalent to the following data

- a vector bundle map $\partial : \mathcal{E}_{-1} \rightarrow \mathcal{E}_0$,
- G -quasi-actions on $\mathcal{E}_{-1}$ and $\mathcal{E}_0$, i.e. vector bundle maps $\Delta_{-1} : G \times_{q_{\mathcal{E}}} \mathcal{E}_{-1} \rightarrow \mathcal{E}_{-1}$ and $\Delta_0 : G \times_{q_{\mathcal{E}}} \mathcal{E}_0 \rightarrow \mathcal{E}_0$ covering the target map $t : G \rightarrow M$,
- a transformation 2-cochain $\varpi \in C^2(G, \mathcal{E}_{-1} \rightarrow \mathcal{E}_0)$, i.e. a vector bundle map

$$\varpi : G^{(2)} \times_{\text{pr}_1 \times q_{\mathcal{E}}} \mathcal{E}_{-1} \rightarrow \mathcal{E}_0, \quad (g_1, g_2; v) \mapsto \varpi_{(g_1, g_2)} v \in E_{t(g_2)} \quad (4.4)$$

where $s \text{pr}_1 : G^{(2)} \rightarrow M$ is the composition of the first projection $\text{pr}_1 : G^{(2)} \rightarrow G$ followed by the source map.

Additionally data $\partial, \Delta_{-1}, \Delta_0, \varpi$ have to satisfy suitable compatibility conditions [28, Section 2.7].

The main remark here is that VB-groupoids $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ are equivalent to 2-term representations up to homotopy of the Lie groupoid $G \rightrightarrows M$ up to isomorphisms. From $(\mathcal{W} \rightrightarrows E; G \rightrightarrows M)$ one can construct the corresponding (isomorphism class of) 2-term representation(s) up to homotopy as follows. As usual, denote by C the core. There is a short exact sequence of vector bundles over G

$$0 \rightarrow t^*C \rightarrow \mathcal{W} \rightarrow s^*E \rightarrow 0. \quad (4.5)$$

where the inclusion $t^*C \rightarrow \mathcal{W}$ is defined by $(g, \chi) \mapsto \chi \cdot 0_g$, and the projection $\mathcal{W} \rightarrow s^*E$ is the obvious one. Next we need a *horizontal lift*, i.e. a splitting $h : s^*E \rightarrow \mathcal{W}$, $(g, e) \mapsto h_g e$ of (4.5) with the additional property of being compatible with the units, i.e. $h(1_x, e) = 1_e$ for all $e \in E_x$, $x \in M$. Horizontal lifts always exist. Now put $\mathcal{E}_{-1} = C[1]$ (where the “[1]” refers to the fact that sections of C should be thought as possessing degree -1), $\mathcal{E}_0 = E$ and finally

$$\begin{aligned} \partial c &= \tilde{t}(c) \\ \Delta_{-1}(g, c) &= h_g(\tilde{t}(c)) \cdot c \cdot 0_{g^{-1}} \\ \Delta_0(g, e) &= \tilde{t}(h_g(e)) \\ \varpi_{(g_1, g_2)} e' &= (h_{g_1 g_2}(e') - h_{g_1}(\tilde{t}(h_{g_2}(e')))) \cdot h_{g_2}(e') \cdot 0_{(g_1 g_2)^{-1}} \end{aligned} \quad (4.6)$$

for all $g \in G$, $c \in C_{s(g)}$, $e \in E_{s(g)}$, $(g_1, g_2) \in G^{(2)}$, and $e' \in E_{s(g_1)}$. Then data $(\partial, \Delta_{-1}, \Delta_0, \varpi)$ define a 2-term representation up to homotopy $(\mathcal{E} = \mathcal{E}_{-1} \oplus \mathcal{E}_0, D)$ which is actually independent of h up to isomorphisms (see [28] for more details).

In the last part of this subsection we describe the data $(\partial, \Delta_{-1}, \Delta_0, \varpi)$ for the 2-term representation up to homotopy corresponding to the VB-groupoid $(DE_G \rightrightarrows DE; G \rightrightarrows M)$ from Proposition 3.10. So let $G \rightrightarrows M$ be a Lie groupoid, and let E be a representation of it. Recall from Remark 3.12 that the core of $(DE_G \rightrightarrows DE; G \rightrightarrows M)$ is canonically isomorphic to the Lie algebroid A of G . The isomorphism identifies a to $\mathbb{D}_a$, where $\mathbb{D}$ is the flat $(\ker ds)$ -connection in E_G . So there is a short exact sequence of vector bundles over G

$$0 \rightarrow t^*A \rightarrow DE_G \rightarrow s^*DE \rightarrow 0, \quad (4.7)$$

where the inclusion $t_D^*A \rightarrow DE_G$ is given by

$$(g, a) = \mathbb{D}_a \cdot 0_g = \mathbb{D}_{\vec{a}_g}. \quad (4.8)$$

As G acts on E , then A acts on E as well, and we denote by $\nabla : A \rightarrow DE$ the corresponding flat A -connection. Now we choose a horizontal lift $h : s^*DE \rightarrow DE_G$, to get a 2-term representation up to homotopy $(\mathcal{E}, D)$ of G , with $\mathcal{E} = A[1] \oplus DE$. Specializing formulas (4.6) to the present situation it is easy to see that D corresponds to the data $(\partial, \Delta_{-1}, \Delta_0, \varpi)$ given by

$$\begin{aligned} \partial a &= \nabla_a \\ \Delta_{-1}(g, a) &= m_D(h_g(\nabla_a), \mathbb{D}_{\vec{a}_{g^{-1}}}) \\ \Delta_0(g, \square) &= t_D(h_g(\square)) \\ \varpi_{(g_1, g_2)} \square' &= DR_{(g_1 g_2)^{-1}}(h_{g_1 g_2}(\square') - m_D(h_{g_1}(t_D(h_{g_2}(\square'))), h_{g_2}(\square'))) \end{aligned} \quad (4.9)$$

for all $g : x \rightarrow y \in G$, $a \in A_{s(g)}$, $\square \in D_{s(g)}E$, $(g_1, g_2) \in G^{(2)}$, and $\square' \in D_{s(g_1)}E$, where $R_g : s^{-1}(y) \times E_y \rightarrow s^{-1}(x) \times E_x$ is the right translation.

Remark 4.1 Let $\pi : DE_G \rightarrow t^*A$, be the projection induced by the horizontal lift h . It is not hard to show that quasi-action Δ_{-1} is equivalently given by

$$\Delta_{-1}(g, a) = -\pi_g(DL_g \circ Di(\mathbb{D}_a)), \quad (4.10)$$

for all $g : x \rightarrow y \in G$, $a \in A_{s(g)}$, where $L_g : t^{-1}(x) \times E_x \rightarrow t^{-1}(y) \times E_y$ is the left translation, and $i : s^{-1}(x) \times E_x \rightarrow t^{-1}(x) \times E_x$ is the inversion. Formula (4.10) is the “derivation version” of the G -quasi-action on TM determined by an Ehresmann connection in G , entering the adjoint representation (up to homotopy) of G , and appearing in [2, Definition 2.11].

Summarizing, we proved the following

Proposition 4.2 *Let $G \rightrightarrows M$ be a Lie groupoid and let E be a representation of G . Additionally, let $h : s^*DE \rightarrow DE_G$ be a horizontal lift. These data define a 2-term representation up to homotopy $(\mathcal{E}, D)$ of G , where $\mathcal{E} = A[1] \oplus DE$, and D corresponds to data $(\partial, \Delta_{-1}, \Delta_0, \varpi)$ given by (4.9). The representation up to homotopy $(\mathcal{E}, D)$ does only depend on E , in particular it is independent of h , up to isomorphisms.*

Remark 4.3 Using the isomorphism $DE_G \cong TG_{ds \times \sigma} DE$, a horizontal lift $h : s^*DE \rightarrow DE_G$, can be thought of as a morphism $s^*DE \rightarrow TG$ of vector bundles over G , additionally covering $\sigma : DE \rightarrow TM$. The (4.9) can then be re-expressed in terms of the action $\mathcal{A}$ of TG on DE . We leave the details to the reader.

Remark 4.4 The symbol, relates the VB-groupoids $(DE_G \rightrightarrows DE; G \rightrightarrows M)$ and $(TG \rightrightarrows TM; G \rightrightarrows M)$ (Diagram 3.5). In turn, a horizontal lift $k : s^*TM \rightarrow TG$ of the tangent groupoid determines a 2-term representation up to homotopy $(\mathcal{E}^T, D^T)$ of G where $\mathcal{E}^T = A[1] \oplus TM$, the *adjoint representation* (see [2] for details, see also [28]). It is now natural to expect that the symbol relates $(\mathcal{E}, D)$ and $(\mathcal{E}^T, D^T)$. This is indeed the case if the horizontal lifts $h : s^*DE \rightarrow DE_G$ and $k : s^*TM \rightarrow TG$ are chosen in an appropriate way. Namely, let h, k be such that the diagram

$$\begin{array}{ccc} s^*DE & \xrightarrow{h} & DE_G \\ \downarrow & & \downarrow \sigma \\ s^*TM & \xrightarrow{k} & TG \end{array} \quad (4.11)$$

commutes (here the left vertical arrow is that induced by the symbol $DE \rightarrow TM$). Such h, k indeed exist. To see this, choose k and use the isomorphism $DE_G \cong TG_{ds \times \sigma} DE$ to put

$$h(g, \square) := (k(g, \sigma(\square)), \square).$$

The h, k have all the required properties. Finally, denote by $(\partial^T, \Delta_{-1}^T, \Delta_0^T, \varpi^T)$ the *structure data* of the adjoint representation. It immediately follows from Diagrams (3.5) and (4.11) that

$$\sigma \circ \partial = \partial^T, \quad \Delta_{-1} = \Delta_{-1}^T, \quad \sigma \circ \Delta_0 = \Delta_0^T \circ (\text{id}_G \times \sigma), \quad \text{and} \quad \sigma \circ \varpi = \varpi^T.$$

In other words, the homomorphism of $C^\bullet(G)$ -modules

$$C^\bullet(G, A[1] \oplus DE) \rightarrow C^\bullet(G, A[1] \oplus TM)$$

obtained from $\text{id}_A \oplus \sigma : A[1] \oplus DE \rightarrow A[1] \oplus TM$ by extension of scalars, is a DG-module homomorphism, i.e. it additionally intertwines the differentials D and D^T .

4.2 VB-algebroids

We recall again from [27] that there is a nice relationship between VB-algebroids and representations up to homotopy. Let $A \Rightarrow M$ be a Lie algebroid. Recall that a *representation up to homotopy* of A [1] is a graded vector bundle $\mathcal{E} \rightarrow M$ together with a degree 1, $\mathbb{R}$ -linear operator

$$D : \Omega(A, \mathcal{E}) \rightarrow \Omega(A, \mathcal{E}) \tag{4.12}$$

satisfying

$$D^2 = 0, \quad \text{and} \quad D(\omega \cdot \varepsilon) = d_A \omega \cdot \varepsilon + (-)^{\omega} \omega \cdot D\varepsilon, \tag{4.13}$$

for all $\omega \in \Omega(A)$, and all $\varepsilon \in \Omega(A, \mathcal{E})$ (i.e., D is an homological derivation of the graded $\Omega(A)$ -module $\Omega(A, \mathcal{E})$, with symbol d_A). Here $(\Omega(A) = \Gamma(\wedge^\bullet A^*), d_A)$ is the de Rham complex of A , and $\Omega(A, \mathcal{E}) = \Gamma(\wedge^\bullet A^* \otimes \mathcal{E})$. We are mainly interested in 2-term representations up to homotopy, i.e. to the case when $\mathcal{E} = \mathcal{E}_{-1} \oplus \mathcal{E}_0$ is concentrated in degree $-1, 0$. A 2-term representation up to homotopy $(\mathcal{E} = \mathcal{E}_{-1} \oplus \mathcal{E}_0, D)$ is equivalent to the following data

- a vector bundle map $\partial : \mathcal{E}_{-1} \rightarrow \mathcal{E}_0$,
- an A -connection ∇_{-1} in $\mathcal{E}_{-1}$, with curvature $R_{-1} \in \Omega^2(A, \text{End } \mathcal{E}_{-1})$,
- an A -connection ∇_0 in $\mathcal{E}_0$, with curvature $R_0 \in \Omega^2(A, \text{End } \mathcal{E}_0)$ and
- a $\text{Hom}(\mathcal{E}_0, \mathcal{E}_{-1})$ -valued 2-form $R \in \Omega^2(A, \text{Hom}(\mathcal{E}_0, \mathcal{E}_{-1}))$,

such that

$$\partial \circ (\nabla_{-1})_a = (\nabla_0)_a \circ \partial, \quad R_{-1}(a, b) = R(a, b) \circ \partial, \quad R_0(a, b) = \partial \circ R(a, b), \quad \text{and} \quad d_{\nabla^{\text{Hom}}} R = 0, \tag{4.14}$$

for all $a, b \in \Gamma(A)$, where ∇^{Hom} is the A -connection in $\text{Hom}(\mathcal{E}_0, \mathcal{E}_{-1})$ induced by ∇_{-1}, ∇_0 . The data $(\partial, \nabla_{-1}, \nabla_0, R)$ correspond to the representation up to homotopy whose differential D is

$$D = \partial + \nabla_{-1} + \nabla_0 + R \tag{4.15}$$

where we interpret $\partial, \nabla_{-1}, \nabla_0, R$ as degree 1 operators on $\Omega(A, \mathcal{E}_{-1} \oplus \mathcal{E}_0)$ in the obvious way (see [27] for more details).

The main remark here is that VB-algebroids $(W \Rightarrow E; A \Rightarrow M)$ are equivalent to 2-term representations up to homotopy of a Lie algebroid $A \Rightarrow M$ up to isomorphisms. From $(W \Rightarrow E; A \Rightarrow M)$

one can construct the corresponding (isomorphism class of) 2-term representation(s) up to homotopy as follows. As usual, denote by C the core and by $\hat{A}$ the fat algebroid, and consider the short exact sequence

$$0 \rightarrow \text{Hom}(E, C) \rightarrow \hat{A} \rightarrow A \rightarrow 0. \quad (4.16)$$

Choose a splitting $\sigma : A \rightarrow \hat{A}$. Now put $\mathcal{E}_{-1} = C[1]$, $\mathcal{E}_0 = E$, let $\partial : C \rightarrow E$ be the core-anchor, and additionally

$$\nabla_{-1} = \psi^c \circ \sigma, \quad \nabla_0 = \psi^s \circ \sigma, \quad (4.17)$$

where ψ^s, ψ^c are the side and the core representation respectively. Finally, we let R be given by

$$R(a, b) = \sigma[a, b] - [\sigma(a), \sigma(b)], \quad a, b \in \Gamma(A). \quad (4.18)$$

In the last part of this section we describe the 2-term representation up to homotopy corresponding to the VB-algebroid $(DE_A \rightrightarrows DE; A \rightrightarrows M)$ from Proposition 3.13. So let $A \rightrightarrows M$ be a Lie algebroid, and let E be a representation of it. We denote by ∇ the flat A -connection in E . We have already seen that the core of $(DE_A \rightrightarrows DE; A \rightrightarrows M)$ is (canonically isomorphic to) A . So there is a short exact sequence of vector bundles

$$0 \rightarrow \text{Hom}(DE, A) \rightarrow \hat{A} \rightarrow A \rightarrow 0 \quad (4.19)$$

where $\hat{A}$ is the fat algebroid. We begin describing $\hat{A}$. Denote by $p : A \rightarrow M$ the projection.

Lemma 4.5 *The fiber $\hat{A}_x$ of the fat algebroid $\hat{A} \rightrightarrows M$ of $(DE_A \rightrightarrows DE; A \rightrightarrows M)$ over a point $x \in M$ consists of pairs (a, h) , where $a \in A_x$ and $h : D_x E \rightarrow T_x A$ is a linear map such that $dp \circ h = \sigma$. The projection $\hat{A} \rightarrow A$ maps (a, h) to a .*

PROOF: The claim immediately follows from the isomorphism $DE_A \cong TA \mathbin{\mathcal{D}}_{\sigma} DE$. $\square$

Remark 4.6 The map $\Gamma(A) \rightarrow \Gamma(\hat{A})$, $a \mapsto Da$ splits the short exact sequence

$$0 \rightarrow \Gamma(\text{Hom}(DE, A)) \rightarrow \Gamma(\hat{A}) \rightarrow \Gamma(A) \rightarrow 0 \quad (4.20)$$

in the category of vector spaces. In particular, a section of $\hat{A} \rightarrow M$ can be seen as a pair (a, φ) where $a \in \Gamma(A)$ and $\varphi \in \Gamma(\text{Hom}(DE, A))$. We now take this point of view to describe a little bit more explicitly (the core-anchor and) the core, the side representations and the Lie bracket in $\Gamma(\hat{A})$. Actually, it easily follows from the definition of the Lie algebroid $DE_A \rightrightarrows DE$ that

- the core-anchor $\partial : A \rightarrow DE$ agrees with the A -connection ∇ in E ;
- the side representation is given by $\psi_{Da}^s = [\nabla_a, -]$ and $\psi_\varphi^s = -\nabla \circ \varphi$,
- the core representation is given by $\psi_{Da}^c = [a, -]$ and $\psi_\varphi^c = -\varphi \circ \nabla$;
- the Lie bracket in $\Gamma(\hat{A})$ is given by

$$[Da, Db] = D[a, b], \quad [Da, \varphi] = \mathcal{L}_a \varphi, \quad [\varphi, v] = v \circ \nabla \circ \varphi - \varphi \circ \nabla \circ v; \quad (4.21)$$

for all $a, b \in \Gamma(A)$, and $\varphi, v \in \Gamma(\text{Hom}(DE, A))$. Here $\mathcal{L}_a \varphi$ is the bundle map $DE \rightarrow A$ given by

$$(\mathcal{L}_a \varphi)(\Delta) := [a, \varphi(\Delta)] - \varphi([\nabla_a, \Delta]), \quad \Delta \in \Gamma(DE). \quad (4.22)$$

Now we choose a splitting of (4.19). It follows from Lemma 4.5 that such a splitting is equivalent to a DE -connection $\mathcal{D}$ in A . Namely, the connection $\mathcal{D}$ corresponds to the splitting $\sigma_{\mathcal{D}} : A \rightarrow \hat{A}$ given by

$$\sigma_{\mathcal{D}}(a) := Da - d_{\mathcal{D}}a, \quad a \in \Gamma(A), \quad (4.23)$$

where $d_{\mathcal{D}} : \Gamma(A) \rightarrow \Gamma(\text{Hom}(DE, A))$ is the connection differential. Hence, we get a 2-term representation up to homotopy $(\mathcal{E}, D)$ of A by putting $\mathcal{E} = A[1] \oplus DE$ and

$$D = \partial + \nabla_{-1} + \nabla_0 + R \quad (4.24)$$

with

$$\partial = \nabla, \quad \nabla_{-1} = \psi^s \circ \sigma_{\mathcal{D}}, \quad \nabla_0 = \psi^c \circ \sigma_{\mathcal{D}}, \quad (4.25)$$

and

$$R(a, b) = \sigma_{\mathcal{D}}[a, b] - [\sigma_{\mathcal{D}}(a), \sigma_{\mathcal{D}}(b)], \quad a, b \in \Gamma(A). \quad (4.26)$$

A straightforward computation using Remark 4.6 now shows that

$$(\nabla_{-1})_a \Delta = [\nabla_a, \Delta] + \nabla_{\mathcal{D}_\Delta a}, \quad (\nabla_0)_a b = [a, b] + \mathcal{D}_{\nabla_b} a \quad (4.27)$$

and

$$\begin{aligned} R(a, b)\Delta &= [\mathcal{D}_\Delta a, b] + [a, \mathcal{D}_\Delta b] - \mathcal{D}_\Delta[a, b] \\ &\quad - \mathcal{D}_{[\nabla_a, \Delta]}b + \mathcal{D}_{[\nabla_b, \Delta]}a + \mathcal{D}_{\nabla_{\mathcal{D}_\Delta b}}a - \mathcal{D}_{\nabla_{\mathcal{D}_\Delta a}}b, \end{aligned} \quad (4.28)$$

for all $a, b \in \Gamma(A)$, and all $\Delta \in \Gamma(DE)$.

Summarizing, we proved the following

Proposition 4.7 *Let $A \Rightarrow M$ be a Lie algebroid and let E be a representation of A with flat A -connection ∇ . Additionally, let $\mathcal{D}$ be a (not necessarily flat) DE -connection in A . These data define a 2-term representation up to homotopy $(\mathcal{E}, D)$ of A , where $\mathcal{E} = A[1] \oplus DE$, and $D = \nabla + \nabla_{-1} + \nabla_0 + R$, with ∇_{-1} , ∇_0 , and R being given by (4.27) and (4.28). The representation up to homotopy $(\mathcal{E}, D)$ does only depend on (E, ∇) , in particular it is independent of $\mathcal{D}$, up to isomorphisms.*

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