Stabilization of distributed time-delay systems: a Smoothed Spectral Abscissa optimization approach

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ABSTRACT

We address the stabilization problem of linear time-invariant systems with pointwise and distributed delays. We first introduce the concept of the smoothed spectral abscissa, a robust stability measure previously studied for [pointwise delay systems](#). We show that, as in the [pointwise](#) type case, the new stability measure for distributed systems is smooth with respect to the system parameters, and robust in the sense that it provides a trade-off between the decay rate of the solution and the $\mathcal{H}_2$ norm of the system. Next, we reformulate the stabilization problem as an unconstrained minimization problem considering the smoothed spectral abscissa as the objective function. We show that its properties enable solving the stabilization problem by using standard gradient-based optimization techniques. A distinctive feature of the proposed approach is that it allows adjusting the degree of robustness and smoothness of the stability measure by fixing a smoothing parameter. The smoothed spectral abscissa and its gradient are characterized by the so-called delay Lyapunov matrix. We show the potential of the proposed approach by using it for the stabilization of a fleet of vehicles, and an oscillator with a class of time-varying delays in the input.

KEYWORDS

Distributed delays; smoothed spectral abscissa; robust stabilization; $\mathcal{H}_2$ norm

1. Introduction

Stabilization of systems with distributed delays is of main interest since they naturally appear in the modelling of epidemics (De la Sen, Alonso-Quesada, & Ibeas, 2015; McCluskey, 2010), neural networks (Li, Wang, Li, Shen, & Lu, 2018; Zhang, Lam, & Xu, 2014), traffic flow processes (Michiels, Morărescu, & Niculescu, 2009), [design of resonators](#) (Pilbauer, Vyhřídál, & Michiels, 2019), among others.

In this paper, we consider distributed systems of the form

$$\dot{x}(t) = \sum_{j=0}^m A_j(p)x(t - h_j) + \int_{-h}^0 B(\theta, p)x(t + \theta)d\theta, \quad t \geq 0, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, delays are ordered as $0 = h_0 < h_1 < \dots < h_m = h$, $A_j(p) \in \mathbb{R}^{n \times n}$, $j = 0, \dots, m$, and $B(\cdot, p) : [-h, 0] \mapsto \mathbb{R}^{n \times n}$ is a piecewise continuous matrix function. Matrices $A_j(\cdot)$ and $B(\theta, \cdot)$ smoothly depend on the system parameters $p = (p_1, \dots, p_k)^T \in \mathbb{R}^k$. We denote by $x(t, \varphi)$ the solution of system (1) with the initial function φ that belongs to the space of piecewise continuous vector functions on $[-h, 0]$,

represented by $PC([-h, 0], \mathbb{R}^n)$.

We say that system (1) is exponentially stable if there exist $\beta \geq 1$ and $\sigma > 0$ such that for every $\varphi \in PC([-h, 0], \mathbb{R}^n)$ the solution $x(t, \varphi)$ satisfies

$$\|x(t, \varphi)\| \leq \beta e^{-\sigma t} \sup_{\theta \in [-h, 0]} \|\varphi(\theta)\|, \quad t \geq 0.$$

We note that system (1) may correspond to a *closed-loop* system with the vector of system parameters $p \in \mathbb{R}^k$ representing an adequate parametrization of the feedback controller. The addressed problem in this paper is concretely formulated as follows:

Problem 1. Find the parameters $p \in \mathbb{R}^k$ such that system (1) is exponentially stable.

Stabilization of systems with distributed delays can be carried out via both frequency and time-domain approaches. Within the first approach, one can find, for instance, the characterization of stability crossing curves, (Morărescu, Niculescu, & Gu, 2007), or the quasi-continuous pole placement methodology, (Michiels, Vyhldal, & Zitek, 2010), whereas within the second approach, one mostly finds results based on the solution of Linear Matrix Inequalities (LMIs), see, e.g., (Chen & Zheng, 2007; Feng, Nguang, & Perruquetti, 2020; Feng, Nguang, & Seuret, 2019; Liu, Fridman, Johansson, & Xia, 2016; Liu, Seuret, Xia, Gouaisbaut, & Ariba, 2019; Xie, Fridman, & Shaked, 2001; Zheng & Frank, 2002). Results based on the so-called delay Lyapunov matrix are also reported in (Juárez, Alexandrova, & Mondié, 2020), (Mondié, Ochoa, & Ochoa, 2011), (Egorov, Cuvas, & Mondié, 2017). It is important to mention that despite the stabilization algorithms based on LMIs are predominant in the literature, they lead to conservative results as they are based on sufficient conditions for stability.

The stabilization strategies of systems often go beyond achieving stability and aim at optimizing performance criteria that characterize robustness of the system, e.g. criteria expressed by means of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms, the decay rate of the system solution, or both. A stability measure that models the asymptotic growth/decay rate of the system solution is the spectral abscissa, the supremum of the real part of the spectrum of a system (Michiels & Niculescu, 2014), whereas a robust stability measure that provides a trade-off between the decay rate of solutions and the $\mathcal{H}_\infty$ norm of a system is the pseudo-spectral abscissa (Michiels, Green, Wagenknecht, & Niculescu, 2006). The spectral and pseudo-spectral abscissa have been used for stabilization and robust stabilization of systems by using numerical optimization techniques (Vanbiervliet, Verheyden, Michiels, & Vandewalle, 2008). However, since they are in general non-smooth functions of the system parameters, their optimization requires dedicated algorithms and software.

In this paper, we propose a new approach to address Problem 1 based on the minimization of a new robust stability measure called smoothed spectral abscissa. Inspired by the ideas tracing back to (Vanbiervliet, Vandereycken, Michiels, Vandewalle, & Diehl, 2009) for delay-free systems and recently developed in (Gomez & Michiels, 2019) for systems with pointwise delays, we extend the smoothed spectral abscissa concept to the case of systems with distributed delays. We show that it preserves the same properties as the smoothed spectral abscissa for [discrete delay systems](#), namely, smoothness with respect to the system parameters, and robustness in the sense that it provides a trade-off between the decay rate of solutions and the $\mathcal{H}_2$ norm of the system, where the input and output matrix can be determined by the designer. [In contrast to stabilization techniques based on either frequency or time domain techniques](#), the proposed approach is in between both, since the smoothed spectral abscissa, that ap-

proximates the spectral abscissa (a frequency domain concept), is characterized by the delay Lyapunov matrix (a time domain concept). Conservatism in obtaining stabilizing parameters p , as in time-domain approaches based on LMI, is not expected if the final value of the smoothing parameter in the novel stabilization procedure is chosen small enough and the optimization algorithm is properly initialized.

We reformulate the stabilization problem as an unconstrained minimization problem with the smoothed spectral abscissa as objective function, which enables using standard gradient-based techniques in order to obtain stabilizing parameters. A distinctive characteristic of the proposed approach is that, by fixing a smoothing parameter ϵ , it is possible to adjust both the decay rate of the system solution, and the robustness degree of the closed-loop system. The smoothed spectral abscissa and its gradient with respect to the parameters p are characterized by the delay Lyapunov matrix for distributed delay systems, and by the partial derivatives of the delay Lyapunov matrix with respect to the system parameters, respectively.

The organization of the manuscript is as follows. Section 2 is divided into two subsections. In subsection 2.1 we introduce the formal definition of the smoothed spectral abscissa and its properties. In Subsection 2.2 we present the characterization of the smoothed spectral abscissa and its gradient in terms of the delay Lyapunov matrix. The reformulation of the stabilization problem as an unconstrained minimization problem is presented in Section 3. There, we also discuss the trade-off between the decay rate of the system solution and the $\mathcal{H}_2$ norm of a transfer matrix related with the system. In Section 4, we show the potential of the proposed strategy by using it for the stabilization of systems of practical interest. We conclude the paper with some final remarks in Section 5.

Notation

The Euclidean norm is represented by $\|\cdot\|$. The set of real numbers is denoted by $\mathbb{R}$ and the set of complex numbers by $\mathbb{C}$. Let $s \in \mathbb{C}$, then $\text{Re}(s)$ denotes its real part. The transpose and trace of a matrix P are denoted by P^T and $\text{tr}(P)$, respectively. The vectorization of a matrix $A \in \mathbb{R}^{n \times p}$ is denoted by $\text{vec}(A) \in \mathbb{R}^{np}$, obtained by stacking up the columns of A . The symbol $\text{mat}(a, j)$ denotes the reshaping of the j th vector $a_j, j = 1, \dots, p$, of a block vector $a = (a_1^T \dots a_p^T)$, into a square matrix A_j such that $a_j = \text{vec}(A_j)$. The symbol $\otimes$ stands for the classical Kronecker product of two matrices A and B (Graham, 1981). The notation ∇f stands for the gradient of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\frac{\partial M(t, p)}{\partial p}$ denotes the partial derivative of M with respect to p .

2. Smoothed spectral abscissa for distributed time-delay systems

In this section, we introduce the concept of smoothed spectral abscissa for distributed delay systems of the form (1). In the first subsection we present the formal definition and its properties, whereas in the second subsection we provide a characterization of it and its gradient in terms of the delay Lyapunov matrix.

2.1. Definition and basic properties

Before introducing a formal definition of the smoothed spectral abscissa of system (1), we present some preliminary definitions and results.

Let us define the spectrum of system (1) as

$$\Delta := \{s \in \mathbb{C} : \det H(s) = 0\},$$

where

$$H(s) := sI - \sum_{j=0}^m A_j(p)e^{-sh_j} - \int_{-h}^0 B(\theta, p)e^{s\theta} d\theta,$$

and consider two matrices $Q_1 \in \mathbb{R}^{n \times n}$ and $Q_2 \in \mathbb{R}^{n \times n}$, used later on, satisfying

$$\text{rank} \begin{pmatrix} H(s_d) & Q_1 \end{pmatrix} = n, \quad (2)$$

and

$$\text{rank} \begin{pmatrix} H(s_d) \\ Q_2 \end{pmatrix} = n, \quad (3)$$

where s_d denotes the dominant root of the spectrum Δ of system (1). Equation (2) and (3) can be interpreted as spectral controllability and observability of the rightmost characteristic root of system (4), respectively, see (Manitius, 1976).

In the frequency domain, the exponential stability of (1) is equivalent to the strictly negativeness of its spectral abscissa, defined as (Michiels & Niculescu, 2014)

$$\alpha(p) := \sup\{\text{Re}(s) : s \in \Delta\}.$$

We introduce next an auxiliary system. For any $\gamma \in \mathbb{R}$, we consider the change of variable $x(t) = e^{\gamma t}z(t)$ into the dynamic equation (1), and obtain the system

$$\dot{z}(t) = (A_0(p) - \gamma I)z(t) + \sum_{j=1}^m A_j(p)e^{-\gamma h_j}z(t - h_j) + \int_{-h}^0 B(\theta, p)e^{\gamma\theta}z(t + \theta)d\theta, \quad t \geq 0, \quad (4)$$

whose spectrum is defined as

$$\Delta_\gamma := \{s \in \mathbb{C} : \det H(s + \gamma) = 0\}. \quad (5)$$

We observe that the spectrum Δ_γ corresponds to the spectrum of system (1) shifted by γ .

Let us introduce now the function

$$f(p, \gamma) := \int_0^\infty \text{tr} \{Q_1^T K_\gamma^T(t) Q_2^T Q_2 K_\gamma(t) Q_1\} dt. \quad (6)$$

Here, matrix $K_\gamma(t) \in \mathbb{R}^{n \times n}$ denotes the fundamental matrix of auxiliary system (4),

and satisfies

$$\begin{aligned} \frac{d}{dt}K_\gamma(t) = & K_\gamma(t) (A_0(p) - \gamma I) + \sum_{j=1}^m K_\gamma(t - h_j) A_j(p) e^{-\gamma h_j} + \\ & \int_{-h}^0 K_\gamma(t + \theta) B(\theta, p) e^{\gamma \theta} d\theta, \quad t \geq 0, \end{aligned} \quad (7)$$

with initial conditions

$$K_\gamma(0) = I, \text{ and } K_\gamma(t) = 0, \quad \forall t < 0. \quad (8)$$

Notice that $Q_2 K_\gamma(t) Q_1$ is the impulse response of system

$$\begin{aligned} \dot{z}(t) = & (A_0(p) - \gamma I) z(t) + \sum_{j=1}^m A_j(p) e^{-\gamma h_j} z(t - h_j) + \\ & \int_{-h}^0 B(\theta, p) e^{\gamma \theta} z(t + \theta) d\theta + Q_1 u(t), \quad t \geq 0, \\ y(t) = & Q_2 z(t). \end{aligned}$$

If system (4) is exponentially stable, by Parseval's Theorem, we have

$$f(p, \gamma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} (T_\gamma^*(p, i\omega) T_\gamma(p, i\omega)) d\omega =: \|T_\gamma(p, s)\|_{\mathcal{H}_2}^2, \quad (9)$$

where

$$T_\gamma(p, s) := Q_2 H_\gamma^{-1}(s) Q_1,$$

and

$$H_\gamma(s) = \left(sI - (A_0(p) - \gamma I) - \sum_{j=1}^m A_j(p) e^{-(s+\gamma)h_j} - \int_{-h}^0 B(\theta, p) e^{\theta(s+\gamma)} d\theta \right).$$

For $\gamma = 0$, it follows that

$$f(p, 0) = \|T(p, s)\|_{\mathcal{H}_2}^2,$$

with

$$T(p, s) := Q_2 H^{-1}(s) Q_1. \quad (10)$$

We show next that the function $f(p, \gamma)$ satisfies some properties that enable an extension of the definition of the smoothed spectral abscissa to systems of the form (1). Lemma 2.1 is a generalization of Lemma 1 in (Gomez & Michiels, 2019) in the sense that function $f(p, \gamma)$ in this paper is associated to the more general class of system (1).

Lemma 2.1. *The following assertions hold:*

1. If $\gamma > \alpha(p)$, then system (4) is exponentially stable.
2. $f(p, \gamma) < \infty$ for all $(p, \gamma) \in \mathbb{R}^k \times \mathbb{R}$ such that $\gamma > \alpha(p)$.
3. The function $(p, \gamma) \mapsto f(p, \gamma)$ is smooth for all $(p, \gamma) \in \mathbb{R}^k \times \mathbb{R}$ such that $\gamma > \alpha(p)$.
4. The function $\gamma \mapsto f(p, \gamma)$ satisfies $\frac{\partial f(p, \gamma)}{\partial \gamma} < 0$ for all $(p, \gamma) \in \mathbb{R}^k \times \mathbb{R}$ such that $\gamma > \alpha(p)$.
5. For any $p \in \mathbb{R}^k$,

$$\lim_{\gamma \rightarrow \infty} f(p, \gamma) = 0,$$

and

$$\lim_{\gamma \rightarrow \alpha(p)+} f(p, \gamma) = \infty.$$

Proof. The proof follows the same arguments as those in the proof of Lemma 1 in (Gomez & Michiels, 2019).

Statement 1. The assertion follows directly from the fact that Δ_γ in (5) is the spectrum Δ of system (1) shifted by γ .

Statement 2. From Statement 1, system (4) is exponentially stable, therefore the fundamental matrix $K_\gamma(t)$ satisfies

$$\|K_\gamma(t)\| \leq \mu e^{-\beta t}, \quad t \geq 0,$$

where both μ and β are positive real numbers. This exponential bound ensures the convergence of the integral in (6), and in turn that $f(p, \gamma) < \infty$.

Statement 3. The smoothness property of function $(p, \gamma) \mapsto f(p, \gamma)$ follows from the finiteness of $f(p, \gamma)$, and the fact that the matrices of system (4) smoothly depend on the vector parameters p and γ .

Statement 4. Consider the fundamental matrix of system (1), denoted by $K(t) \in \mathbb{R}^{n \times n}$, satisfying

$$\frac{d}{dt}K(t) = \sum_{j=0}^m K(t - h_j)A_j(p) + \int_{-h}^0 K(t + \theta)B(\theta, p)d\theta, \quad t \geq 0,$$

with initial conditions

$$K(0) = I, \text{ and } K(t) = 0, \quad \forall t < 0.$$

One can straightforwardly prove that

$$K_\gamma(t) = K(t)e^{-\gamma t}, \quad t \in \mathbb{R},$$

by showing that indeed it satisfies equation (7), with initial conditions (8). Then, by substituting the equality $K_\gamma(t) = K(t)e^{-\gamma t}$ into equation (6), it follows that

$$f(p, \gamma) = \int_0^\infty \text{tr} \{Q_1^T K^T(t) Q_2^T Q_2 K(t) Q_1\} e^{-2\gamma t} dt, \quad (11)$$

hence

$$\frac{\partial f(p, \gamma)}{\partial \gamma} = -2 \int_0^\infty \text{tr} \{ Q_1^T K^T(t) Q_2^T Q_2 K(t) Q_1 \} t e^{-2\gamma t} dt < 0.$$

Statement 5. Property

$$\lim_{\gamma \rightarrow \infty} f(p, \gamma) = 0, p \in \mathbb{R}^k$$

follows from the fact that $e^{-2\gamma t} \rightarrow 0$ when $\gamma \rightarrow \infty$ for any fixed $t \geq 0$ in equality (11). We now focus on proving that

$$\lim_{\gamma \rightarrow \alpha(p)+} f(p, \gamma) = \infty.$$

By using the same arguments used in the proof of Lemma 1 in (Gomez & Michiels, 2019) based on Keldysh's Theorem (Beyn, 2012), one can prove that under conditions (2) and (3) the following holds:

$$\frac{1}{2\pi} \int_{-\infty}^\infty \text{tr} (T^*(p, s_d + \eta + i\omega) T(p, s_d + \eta + i\omega)) d\omega \rightarrow \infty \text{ when } \eta \rightarrow 0+. \quad (12)$$

Now, by the change of variable $s = s + \gamma$, equation (10) becomes

$$T(p, s)|_{s=s+\gamma} = Q_2 H^{-1}(s + \gamma) Q_1,$$

hence, $T_\gamma(p, s) = T(p, s + \gamma)$ is fulfilled. Thus, we have that

$$\frac{1}{2\pi} \int_{-\infty}^\infty \text{tr} (T_\gamma^*(p, i\omega) T_\gamma(p, i\omega)) d\omega = \frac{1}{2\pi} \int_{-\infty}^\infty \text{tr} (T^*(p, \gamma + i\omega) T(p, \gamma + i\omega)) d\omega,$$

and by (12), we arrive at

$$\frac{1}{2\pi} \int_{-\infty}^\infty \text{tr} (T_\gamma^*(p, i\omega) T_\gamma(p, i\omega)) d\omega \rightarrow \infty \text{ when } \gamma \rightarrow \alpha(p) +.$$

By using this result in equation (9) the assertion $f(p, \gamma) \rightarrow \infty$ when $\gamma \rightarrow \alpha(p) +$ follows. $\square$

We are now in position to state the definition of the smoothed spectral abscissa of system (1). From Statements 4 and 5 of Lemma 2.1, one observes that the function $\gamma \mapsto f(p, \gamma)$ is monotonically decreasing on $(\alpha(p), +\infty)$ and has a unique intersection with the constant function ϵ^{-1} , where $\epsilon > 0$. Therefore, for any given $\epsilon > 0$, there exists a smooth function $\tilde{\alpha}_\epsilon : \mathbb{R}^k \rightarrow \mathbb{R}$, $p \mapsto \tilde{\alpha}_\epsilon(p)$, whose image is defined on $(\alpha(p), +\infty)$ such that equation (13) is satisfied. The function $\tilde{\alpha}_\epsilon$ is the smoothed spectral abscissa of function (1) and is formally defined as follows:

Definition 2.2. The smoothed spectral abscissa function, associated with smoothing parameter $\epsilon > 0$, is the mapping $\mathbb{R}^k \ni p \mapsto \tilde{\alpha}_\epsilon(p)$ that uniquely solves equation

$$f(p, \tilde{\alpha}_\epsilon(p)) = \frac{1}{\epsilon} \quad (13)$$

on the interval $(\alpha(p), +\infty)$.

Some properties of the smoothed spectral abscissa that play a key role in using it for stabilization of system (1) are given next.

Theorem 2.3. *The smoothed spectral abscissa $\tilde{\alpha}_\epsilon(p)$ of system (1) satisfies:*

1. *The function $p \mapsto \tilde{\alpha}_\epsilon(p)$ is smooth.*
2. *$\tilde{\alpha}_\epsilon(p) > \alpha(p)$ for all $\epsilon > 0$.*
3. *$\lim_{\epsilon \rightarrow +0} \tilde{\alpha}_\epsilon(p) = \alpha(p)$.*
4. *The function $\epsilon \mapsto \tilde{\alpha}_\epsilon(p)$ is increasing for all $p \in \mathbb{R}^k$.*

Proof. Statement 1. Since $f(p, \gamma) - \epsilon^{-1}$ is a smooth function of $p \in \mathbb{R}^k$, it follows from the implicit function theorem that function $p \mapsto \tilde{\alpha}_\epsilon(p)$ is also smooth; see, e.g. (Krantz & Parks, 2012).

Statement 2. From Definition 2.2, the smoothed spectral abscissa is defined on $(\alpha(p), \infty)$ for any $\epsilon > 0$, hence $\tilde{\alpha}_\epsilon(p) > \alpha(p)$.

Statement 3. Notice that, from Statement 2 and Statement 4 of Lemma 2.1, $\gamma \mapsto f(p, \gamma)$ is finite and decreasing with respect to γ for $\gamma > \alpha(p)$, but $f(p, \tilde{\alpha}_\epsilon(p)) \rightarrow \infty$ as $\epsilon \rightarrow 0+$. Thus, $\tilde{\alpha}_\epsilon(p) \rightarrow \alpha(p)$ as $\epsilon \rightarrow +0$.

Statement 4. To prove that $\epsilon \mapsto \tilde{\alpha}_\epsilon(p)$ is increasing, it is enough to prove that

$$\frac{\partial \tilde{\alpha}_\epsilon(p)}{\partial \epsilon} > 0$$

for any given $p \in \mathbb{R}^k$. This can be shown by differentiating (13) with respect to ϵ :

$$\frac{\partial f(p, \gamma)}{\partial \gamma} \frac{\partial \tilde{\alpha}_\epsilon(p)}{\partial \epsilon} = -\frac{1}{\epsilon^2}.$$

Then, by Statement 4 of Lemma 2.1, it follows that

$$\frac{\partial \tilde{\alpha}_\epsilon(p)}{\partial \epsilon} = -\left(\frac{\partial f(p, \gamma)}{\partial \gamma}\right)^{-1} \frac{1}{\epsilon^2} > 0.$$

□

In the next example, we illustrate the smoothed spectral abscissa of a scalar system.

Example 2.4. Consider the system

$$\dot{x}(t) = -5x(t) + p_1x(t-h) + p_2 \int_{-h}^0 x(t+\theta)d\theta. \quad (14)$$

We compute the smoothed spectral abscissa of this system for different values of ϵ by solving equation (13) with $Q_1 = 1$ and $Q_2 = 1$. Figure 1 depicts it, along with the spectral abscissa, for fixed $h = 0.5$, $p_1 = -1$, and $p_2 \in [-2, 8]$. We can observe the properties stated in Theorem 2.3.

Figure 1. Smoothed spectral abscissa and spectral abscissa of system (14).

2.2. The smoothed spectral abscissa and its gradient in terms of the delay Lyapunov matrix

In this subsection, we characterize the smoothed spectral abscissa, and its gradient with respect to p , in terms of the so-called delay Lyapunov matrix of system (4). This characterization is of particular importance for solving the optimization problem stated in Section 3, as it allows providing algebraic formulae that are useful in the implementation of the optimization algorithm.

The delay Lyapunov matrix of exponentially stable system (4) associated with a positive definite matrix $W \in \mathbb{R}^{n \times n}$ is the matrix function (Kharitonov, 2013)

$$U_\gamma(\tau, p) := \int_0^\infty K_\gamma^T(t) W K_\gamma(t + \tau) dt, \quad \tau \in [-h, h]. \quad (15)$$

Let us define $\hat{A}_0(p) := A_0(p) - \gamma I$ and $\hat{A}_j(p) := A_j(p)$, $j = 1, \dots, m$, the delay Lyapunov matrix satisfies the following three properties:

1. The dynamic property:

$$\begin{aligned} \frac{\partial}{\partial \tau} U_\gamma(\tau, p) &= \sum_{j=0}^m U_\gamma(\tau - h_j, p) \hat{A}_j(p) e^{-\gamma h_j} + \\ &\quad \int_{-h}^0 U_\gamma(\tau + \theta, p) B(\theta, p) e^{\gamma \theta} d\theta, \quad \tau \in [0, h]. \end{aligned} \quad (16)$$

2. The symmetry property:

$$U_\gamma(-\tau, p) = U_\gamma^T(\tau, p), \quad \tau \in [-h, h]. \quad (17)$$

3. The algebraic property:

$$\begin{aligned}
-W = & \sum_{j=0}^m \left[U_\gamma(-h_j, p) \hat{A}_j(p) e^{-\gamma h_j} + \hat{A}_j^T(p) e^{-\gamma h_j} U_\gamma(h_j, p) \right] + \\
& \int_{-h}^0 U_\gamma(\theta, p) B(\theta, p) e^{\gamma \theta} d\theta + \int_{-h}^0 B^T(\theta, p) e^{\gamma \theta} U_\gamma(-\theta, p) d\theta. \quad (18)
\end{aligned}$$

From (16) and (17), it also follows that

$$\begin{aligned}
\frac{\partial}{\partial \tau} U_\gamma(\tau, p) &= - \left[\frac{\partial}{\partial \tau} U_\gamma(-\tau, p) \right]^T \\
&= - \sum_{j=1}^m \hat{A}_j^T(p) e^{-\gamma h_j} U_\gamma(\tau + h_j, p) - \int_{-h}^0 B^T(\theta, p) e^{\gamma \theta} U_\gamma(\tau - \theta, p) d\theta \quad (19)
\end{aligned}$$

for $\tau \in [-h, 0]$.

In the next proposition, we state that the $\mathcal{H}_2$ norm of the transfer matrix T_γ can be written in terms of the delay Lyapunov matrix.

Proposition 2.5. *For all $(\gamma, p) \in \mathbb{R} \times \mathbb{R}^k$ such that $\gamma > \alpha(p)$, the equality*

$$\|T_\gamma(p, s)\|_{\mathcal{H}_2}^2 := f(p, \gamma) = \text{tr} (Q_1^T U_\gamma(0, p) Q_1)$$

is fulfilled, where $U_\gamma(0, p)$ is the delay Lyapunov matrix of system (4) associated with $Q_2^T Q_2$.

Proof. By Statement 1 of Lemma 2.1, $\gamma > \alpha(p)$ implies exponential stability of system (4). Thus, it follows from (15) that

$$\begin{aligned}
f(p, \gamma) &= \int_0^\infty \text{tr} (Q_1^T K_\gamma^T(t) Q_2^T Q_2 K_\gamma(t) Q_1) dt \\
&= \text{tr} \left(Q_1^T \int_0^\infty K_\gamma^T(t) Q_2^T Q_2 K_\gamma(t) dt Q_1 \right) = \text{tr} (Q_1^T U_\gamma(0, p) Q_1).
\end{aligned}$$

□

In view of Proposition 2.5, for any given value of $\epsilon > 0$ and $p \in \mathbb{R}^k$, the smoothed spectral abscissa can be computed by solving

$$1/f(p, \gamma) - \epsilon = 0 \quad (20)$$

for $\gamma \in (\alpha(p), \infty)$, where $f(p, \gamma) = \text{tr} (Q_1^T U_\gamma(0, p) Q_1)$. Notice that characterizing the function $f(p, \gamma)$ in terms of the Lyapunov matrix avoids computing an improper integral that depends on the fundamental matrix of solutions of system (4). If (20) is solved for γ with a Newton type method, the evaluation of $\frac{\partial f(p, \gamma)}{\partial \gamma}$ is required, which on its turn depends on $\frac{\partial U_\gamma(0, p)}{\partial \gamma}$. The latter can be characterized from (16)-(18). Indeed, by differentiating (16)-(18) with respect to γ , we get the following set of equations with inhomogeneous terms:

from the dynamic property for $\tau \in [0, h]$,

$$\begin{aligned} \frac{\partial}{\partial \gamma} \left(\frac{\partial}{\partial \tau} U_\gamma(\tau, p) \right) &= \frac{\partial U_\gamma(\tau, p)}{\partial \gamma} \hat{A}_0 - U_\gamma(\tau, p) + \\ &\sum_{j=1}^m \left(\frac{\partial U_\gamma(\tau - h_j, p)}{\partial \gamma} \hat{A}_j(p) - h_j U_\gamma(\tau - h_j, p) \hat{A}_j(p) \right) e^{-\gamma h_j} + \\ &\int_{-h}^0 \left(\frac{\partial U_\gamma(\tau + \theta, p)}{\partial \gamma} B(\theta, p) + \theta U_\gamma(\tau + \theta, p) B(\theta, p) \right) e^{\gamma \theta} d\theta; \end{aligned} \quad (21)$$

from the symmetry property,

$$\frac{\partial U_\gamma(-\tau, p)}{\partial \gamma} = \frac{\partial U_\gamma^T(\tau, p)}{\partial \gamma}, \quad \tau \in [-h, h]; \quad (22)$$

and from the algebraic property,

$$\begin{aligned} 0 &= \frac{\partial U_\gamma(0, p)}{\partial \gamma} \hat{A}_0(p) + \hat{A}_0^T(p) \frac{\partial U_\gamma(0, p)}{\partial \gamma} - 2U_\gamma(0, p) + \\ &\sum_{j=1}^m \left(\frac{\partial U_\gamma(-h_j, p)}{\partial \gamma} \hat{A}_j(p) - h_j U_\gamma(-h_j, p) \hat{A}_j(p) \right) e^{-\gamma h_j} + \\ &\sum_{j=1}^m \left(\hat{A}_j^T(p) \frac{\partial U_\gamma(h_j, p)}{\partial \gamma} - h_j \hat{A}_j^T(p) U_\gamma(h_j, p) \right) e^{-\gamma h_j} + \\ &\int_{-h}^0 \left(\frac{\partial U_\gamma(\theta, p)}{\partial \gamma} B(\theta, p) + \theta U_\gamma(\theta, p) B(\theta, p) \right) e^{\gamma \theta} d\theta + \\ &\int_{-h}^0 \left(\theta B^T(\theta, p) U_\gamma(-\theta, p) + B^T(\theta, p) \frac{\partial U_\gamma(-\theta, p)}{\partial \gamma} \right) e^{\gamma \theta} d\theta. \end{aligned} \quad (23)$$

In order to optimize the smoothed spectral abscissa it is very useful if its gradient is available. This is characterized by the following proposition

Proposition 2.6. *For fixed $\epsilon > 0$, the gradient of the smoothed spectral abscissa $\tilde{\alpha}_\epsilon(p)$ with respect to p is*

$$\nabla \tilde{\alpha}_\epsilon(p) = - \left(\text{tr} \left(Q_1^T \frac{\partial U_\gamma(0, p)}{\partial \gamma} Q_1 \right) \right)^{-1} \begin{pmatrix} \text{tr} \left(Q_1^T \frac{\partial U_\gamma(0, p)}{\partial p_1} Q_1 \right) \\ \vdots \\ \text{tr} \left(Q_1^T \frac{\partial U_\gamma(0, p)}{\partial p_k} Q_1 \right) \end{pmatrix}, \quad (24)$$

where $U_\gamma(0, p)$ is the delay Lyapunov matrix of system (4), associated with $Q_2^T Q_2$, where $\gamma = \tilde{\alpha}_\epsilon(p)$.

Proof. Notice first that for a given $\epsilon > 0$, the smoothed spectral abscissa satisfies $f(p, \tilde{\alpha}_\epsilon(p)) = \epsilon^{-1}$. Then, by using the chain rule, we have

$$\frac{\partial f(p, \gamma)}{\partial p_j} = - \frac{\partial f(p, \gamma)}{\partial \gamma} \frac{\partial \tilde{\alpha}_\epsilon(p)}{\partial p_j}, \quad j = 1, \dots, k.$$

Since $\frac{\partial f(p, \gamma)}{\partial \gamma}$ is different from zero by Lemma 2.1, we get

$$\frac{\partial \tilde{\alpha}_\epsilon(p)}{\partial p_j} = - \left(\frac{\partial f(p, \gamma)}{\partial \gamma} \right)^{-1} \frac{\partial f(p, \gamma)}{\partial p_j}, \quad j = 1, \dots, k,$$

and equation (24) follows from the equalities

$$\frac{\partial f(p, \gamma)}{\partial p_j} = \text{tr} \left(Q_1^T \frac{\partial U_\gamma(0, p)}{\partial p_j} Q_1 \right), \quad \frac{\partial f(p, \gamma)}{\partial \gamma} = \text{tr} \left(Q_1^T \frac{\partial U_\gamma(0, p)}{\partial \gamma} Q_1 \right),$$

and the gradient definition. $\square$

Note that the expression of the gradient depends on the derivatives $\frac{\partial U_\gamma(0, p)}{\partial p_j}$, $j = 1, \dots, k$, which can be obtained in the same way as the derivatives of $U_\gamma(0, p)$ w.r.t. γ .

To compute $U_\gamma(0, p)$, $\frac{\partial U_\gamma(0, p)}{\partial \gamma}$, and $\frac{\partial U_\gamma(0, p)}{\partial p_j}$, $j = 1, \dots, k$ one can solve (16)-(18), (21)-(23), and the resulting set of equations from taking the partial derivatives of (16)-(18) w.r.t. p_j , numerically, using a standard spectral collocation approach, which involves both a discretization of the differential equations and a numerical approximation of the integrals. However, if the delays are commensurate and the kernel piecewise constant, this can be avoided using a semi-analytic approach; see Appendix A.

3. Stabilization via the optimization of the smoothed spectral abscissa

We reformulate the stabilization problem of the parametrized system (1) as the following unconstrained optimization problem of the smoothed spectral abscissa:

$$\min_{p \in \mathbb{R}^k} \tilde{\alpha}_\epsilon(p), \quad (25)$$

with ϵ a given positive number. Since $\tilde{\alpha}_\epsilon(p)$ is larger than $\alpha(p)$ for any $\epsilon > 0$ and vector $p \in \mathbb{R}^k$, a solution of (25) such that

$$\alpha_\epsilon(p^*(\epsilon)) \leq 0,$$

where

$$p^*(\epsilon) := \arg \min_p \tilde{\alpha}_\epsilon(p),$$

denotes the minimizer of (25), provides a solution to Problem 1. Moreover, with stabilizing parameters $p^*(\epsilon)$, we have an exponentially stable system (1) with decay rate $-\sigma < \tilde{\alpha}_\epsilon(p^*)$.

Even though problem (25) is in general non-convex, the smoothness property of $\tilde{\alpha}_\epsilon(p)$ enables using standard gradient based techniques for its solution, in contrast to the minimization of the spectral abscissa $\alpha(p)$, which is in general a non-smooth function of the system parameters. [The following algorithm summarizes the proposed stabilization procedure of system \(1\):](#)

Algorithm 1

Require: $\epsilon > 0$, vector of initial parameters $p_0 \in \mathbb{R}^k$, matrices Q_1 and Q_2 satisfying (2) and (3).

```
1: while  $\tilde{\alpha}_\epsilon(p^*) > 0$  do :  
2:   Solve the minimization problem (25) with  $p_0$   
3:   Decrease  $\epsilon$   
4:   if  $\epsilon \approx 0$  then  
5:     Update  $p_0$  and  $\epsilon > 0$   
6:   end if  
7: end while
```

Return: stabilizing parameters p^* .

By Statement 4 of Theorem 2.3, the smoothed spectral abscissa is an increasing function of ϵ , hence, smaller values of $\tilde{\alpha}_\epsilon(p^*(\epsilon))$ are expected **when in the Step (3) one decreases ϵ** . Number ϵ can be set as small as desired, at the price of qualitatively losing smoothness of the smoothed spectral abscissa in the sense that it better approximates the possibly non-smooth spectral abscissa as ϵ goes to zero, **see Figure 1. If, for a given vector of initial parameters p_0 , condition $\tilde{\alpha}_\epsilon(p^*(\epsilon)) < 0$ is not fulfilled even for very small values of $\epsilon > 0$, then one can choose another vector of initial parameters $p_0 \in \mathbb{R}^k$ (Step (5)), set $\epsilon > 0$ and keep looking for a stabilizing vector parameter p by solving the minimization problem (25). If $\tilde{\alpha}_\epsilon(p^*(\epsilon)) \leq 0$ then the obtained parameters $p^*(\epsilon)$ stabilize system (1), since by Theorem 2.3 we have $\tilde{\alpha}_\epsilon(p) > \alpha(p)$.**

Consider now $\epsilon = \epsilon^*$ such that

$$\tilde{\alpha}_{\epsilon^*}(p^*(\epsilon^*)) = 0.$$

As the smoothed spectral abscissa is an increasing function of ϵ , it follows from the equality $f(p, 0) = \|T(p, s)\|_{\mathcal{H}_2}$ that $p^*(\epsilon^*)$ is an optimizer of the $\mathcal{H}_2$ norm of the transfer matrix (10), this is

$$p^*(\epsilon^*) = \arg \min_p \|T(p, s)\|_{\mathcal{H}_2}^2,$$

and

$$\|T(p^*, s)\|_{\mathcal{H}_2} = \sqrt{\frac{1}{\epsilon^*}}.$$

Thus, solving problem (25) for $\epsilon \in (0, \epsilon^*]$ provides a trade-off between the maximization of exponential decay rate of system (1) and the $\mathcal{H}_2$ norm of the transfer matrix (10). This is illustrated by the following example.

Example 3.1. We consider again system (14). We optimize the smoothed spectral abscissa of system (14) for several values of ϵ . The obtained results are displayed in Table 1. A comparison between third and fourth columns allows one to observe the trade-off, associated to the different values of ϵ , between the spectral abscissa and the $\mathcal{H}_2$ norm associated to system (14). The values on the third column increase (from the third row) when ϵ goes to zero, whereas the values on the fourth column decrease.

Table 1. Trade-off between the spectral abscissa and the $\mathcal{H}_2$ norm of system (14). The symbol e denotes Euler's number.

ϵ	$\tilde{\alpha}_\epsilon(p^*)$	$\ T(p^*, s)\ _{\mathcal{H}_2}$	$\alpha(p^*)$	p_1^*	p_2^*
18	2.4956		0.1147	0.2429	-52.7031
16	1.7815	0.3234	-1.0865	2.5154	-27.4206
8	-1.9383	0.2951	-4.2750	0.9601	-6.1796
4	-4.3030	0.3017	-6.0213	0.4283	-3.4738
1	-6.9028	0.3067	-7.8272	0.1881	-2.0321
e^{-2}	-8.6142	0.3091	-9.0243	0.1116	-1.4358
e^{-4}	-9.3851	0.3100	-9.5801	0.0886	-1.2268
e^{-6}	-9.7599	0.3104	-9.8558	0.0792	-1.1368

The second and third column illustrate Statement 3 of Theorem 2.3. The value of $\tilde{\alpha}_\epsilon(p^*(\epsilon))$ approximates $\alpha(p^*(\epsilon))$ as ϵ goes to zero, which means that for small ϵ , $\tilde{\alpha}_\epsilon(p)$ approaches to the minimum of $\alpha(p)$. This can be seen in Fig. 2 (a), where we depict the smoothed spectral abscissa and the spectral abscissa evaluated at the minimizers $p^*(\epsilon)$ for $\epsilon \in [e^{-6}, 16]$. Notice that $\tilde{\alpha}_{\epsilon^*}(p^*(\epsilon^*)) \approx 0$ when $\epsilon^* \approx 11.8768$. For this ϵ^* , we obtain the minimizer $p^*(\epsilon^*) = (1.8694 \quad -11.6427)^T$, which corresponds to a minimizer of the $\mathcal{H}_2$ norm of (14), and

$$\|T(p^*(\epsilon^*), s)\|_{\mathcal{H}_2} = \sqrt{\frac{1}{\epsilon^*}} \approx 0.2901.$$

Fig. 2 (b) depicts the $\mathcal{H}_2$ norm of system (14) evaluated at the minimizers $p^*(\epsilon)$ of $\tilde{\alpha}_\epsilon(p)$ for $\epsilon \in [e^{-6}, 16]$. We observe there the achieved minimum of the $\mathcal{H}_2$ norm for the minimizer $p^*(\epsilon^*)$.

(a) Smoothed spectral abscissa and spectral abscissa of system (14) evaluated at the minimizers of $\tilde{\alpha}_\epsilon(p)$. (b) $\mathcal{H}_2$ norm of system (14) evaluated at the minimizers of $\tilde{\alpha}_\epsilon(p)$.

Figure 2. Smoothed spectral abscissa, spectral abscissa and $\mathcal{H}_2$ norm of system (14) evaluated at the minimizers of $\tilde{\alpha}_\epsilon(p)$ for $\epsilon \in [e^{-6}, 16]$.

4. Application to controller design

In this section, we show the potential of the proposed stabilization strategy by two examples of practical interest. First, we address the stabilization of a vehicle platoon following model considering the human drivers's memory effects. Then, a second order linear system is stabilized by a controller with time-varying delay of triangular form.

The optimization of the smoothed spectral abscissa is carried out with MATLAB. We solve (20) by means of the Dekker-Brent algorithm implemented by the function `fzero`, and the optimization problem (25) by using the Broyden-Fletcher-Goldfarh-

Shanno (BFGS) quasi-Newton algorithm implemented by the optimization toolbox. For both examples, we consider the weighting matrices $Q_1 = I$ and $Q_2 = I$. Matrices Q_1 and Q_2 are chosen such that they always satisfy the rank conditions in (2) and (3). In order to do so one needs to compute the dominant root s_d of the spectrum Δ , which can be carried out by using, for instance, the Quasi-Polynomial Root Finder algorithm (Vyhlídal, Lafay, & Sipahi, 2013) or the methodology proposed in (Wu & Michiels, 2012).

It is worth mentioning that, if the design aims at stabilization and minimizing the spectral abscissa only, the choice of Q_1 and Q_2 is not relevant, and the rank condition in (2) and (3) can be guaranteed by choosing Q_1 and Q_2 as square full rank matrices. However, if the design aims at a compromise between stability and robustness/disturbance rejection, in the sense of having a small $\mathcal{H}_2$ norm of the transfer function between a particular input and output, then Q_1 and Q_2 are determined by the application under consideration. In this case, choosing Q_1 and/or Q_2 as matrices of lower rank (interpreted in terms of input and output matrices, and less than n inputs or outputs), the rank condition in (2) and (3) can be verified by computing the dominant root s_d .

4.1. Stabilization of a fleet of vehicles

We study the stabilization of a fleet of vehicles in closed-loop with the extended cooperative adaptive cruise control proposed in (Juárez & Mondié, 2019), which involves both a local and a network control action. We consider a scenario of three vehicles. We fixed the delays due to wireless communication as $h_{12}(t) = h_{21}(t) = h = 0.1$, the weights of the communication links as $k_{12} = k_{21} = 0.5$, and we consider $h_1 = h_2 = h$ as the length of time after which the drivers memory ($\xi_1 = h$ and $\xi_2 = 2h$) becomes effective.

In (Juárez & Mondié, 2019), it is shown that for the considered model the dynamic behavior in closed-loop of the fleet is described by a system of the form (1):

$$\dot{x}(t) = A_1 x(t - h) + B_1(p) \int_{-2h}^{-h} x(t + \theta) d\theta + B_2(p) \int_{-3h}^{-2h} x(t + \theta) d\theta,$$

where $x(t) = (e_1(t), e_2(t))^T$, $p = (p_1, p_2)^T$, and

$$A_1 = \begin{bmatrix} 0 & -0.5 \\ -0.5 & 0 \end{bmatrix}, B_1(p) = \begin{bmatrix} -\frac{p_1}{h} & 0 \\ 0 & -\frac{p_2}{2h} \end{bmatrix}, B_2(p) = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{p_2}{2h} \end{bmatrix}.$$

The states e_i , $i = 1, 2$ denote the tracking errors defined as $e_i(t) = d_i(t) - \bar{d}_i(t)$, where $d_i(t)$ and $\bar{d}_i(t)$ are, respectively, the inter vehicular distance and the desired spacing policy among vehicles. The pair of gains (p_1, p_2) represents a measure of the aggressiveness of the driver per unit of vehicle mass. We tuned these parameters by solving optimization problem (25). The obtained results for $\epsilon = e^{-9}$ and $\epsilon^* \approx 1.9127$ are displayed in Table 2.

Table 2. Trade-off between the decay rate of the system solution and the $\mathcal{H}_2$ norm.

ϵ	$\bar{\alpha}_\epsilon(p^*)$	$\ T(p^*, s)\ _{\mathcal{H}_2}$	$\alpha(p^*)$	p_1^*	p_2^*
e^{-9}	-5.0695	0.81746	-5.0726	2.6976	2.0917
ϵ^*	≈ 0	0.72306	-2.4029	5.1514	4.0393

The simulation results are depicted on Figure 3. We observe there that the objective of assigning a desired velocity to the platoon, that is the velocity of the first vehicle in the fleet denoted by v_0 , while ensuring a spacing policy among vehicles, is fully addressed.

Figure 3. Simulation results of the fleet of vehicles for $\epsilon = e^{-9}$ and $\epsilon^* \approx 1.9127$.

4.2. Controller with time-varying delay

We consider an oscillator

$$\ddot{y}(t) + y(t) = u(t), \quad (26)$$

with a control law of the form

$$u(t) = -k_r y(t - \tau_0 - \delta f(\Omega t)), \quad (27)$$

where $f(t)$ is given by

$$f(t) = \begin{cases} \frac{2}{\pi} \left(t - \frac{\pi}{2} \right), & t \in [0, \pi) \\ \frac{2}{\pi} \left(\frac{3\pi}{2} - t \right), & t \in [\pi, 2\pi) \end{cases}.$$

Results presented in (Michiels, Van Assche, & Niculescu, 2005) enable one to assess the stability of the closed-loop system (26)-(27) via the stability analysis of a system with distributed delays. More precisely, the closed-loop system (26)-(27) is asymptotically stable for large values of Ω if the system

$$\dot{x}(t) = A_0 x(t) + B_0(p) \int_{-\tau_0 - \delta}^{-\tau_0 + \delta} x(t + \theta) d\theta, \quad (28)$$

with $x(t) = \begin{pmatrix} y(t) & \dot{y}(t) \end{pmatrix}^T$, and

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_0(p) = \begin{bmatrix} 0 & 0 \\ -\frac{p_1}{2\delta} & 0 \end{bmatrix},$$

is asymptotically stable.

We fix parameters $(\tau_0, \delta) = (0.9, 0.3)$, and tuned the stabilizer parameter $p = k_r$ by optimizing the smoothed spectral abscissa of system (28). Table 3 displays the obtained results for $\epsilon = e^{-9}$ and $\epsilon^* \approx 0.1742$. For both cases, system (28) is stable, which implies asymptotic stability of (26)-(27).

Table 3. Trade-off between the decay rate of the system solution and the $\mathcal{H}_2$ norm.

ϵ	$\tilde{\alpha}_\epsilon(p^*)$	$\ T(p^*, s)\ _{\mathcal{H}_2}$	$\alpha(p^*)$	p_1^*
e^{-9}	-0.62642	2.4640	-0.64340	-0.78249
ϵ^*	≈ 0	2.3959	-0.4428	-0.68125

Fig. 4 shows the output $y(t)$ of the closed-loop system (26)-(27) using the parameters displayed in Table 3 and $\Omega = 50$. We observe that a fast decay rate for the system solution is achieved when $\epsilon = e^{-9}$.

Figure 4. Output of the closed-loop system (26)-(27) via simulation for $\epsilon = e^{-9}$ and ϵ^* .

5. Conclusions

We proposed a new approach for the stabilization of linear time-invariant systems involving multiple pointwise and distributed time-delays. The approach is based on the optimization of the smoothed spectral abscissa, a robust stability measure that provides a trade-off between the decay rate of the system solution and the $\mathcal{H}_2$ norm. For the case in which the kernel of the distributed delay is piecewise constant, we present a semi-analytical procedure in order to compute the smoothed spectral abscissa and its gradient. The potential of the proposed scheme was illustrated by two examples of practical interest.

The numerical implementation of the proposed approach depend on the computation of the delay Lyapunov matrix associated to the corresponding system. This is a

research problem that has been object of study in recent years, see e.g. Gumussoy and Abu-Khalaf (2019) and Aliseyko (2020).

Disclosure statement

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Appendix A. Semi-analytic approach for the computation of the delay Lyapunov matrix and its derivatives for distributed systems with piecewise constant kernel and commensurate delays

We consider distributed systems with commensurate delays and piecewise constant kernel of the form

$$\dot{x}(t) = \sum_{j=0}^m A_j(p)x(t-jh) + \sum_{j=0}^{m-1} B_j(p) \int_{-(j+1)h}^{-jh} x(t+\theta)d\theta, \quad t \geq 0, \quad (\text{A1})$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $h > 0$ is the basic delay and matrices $A_j(p)$, $B_j(p) \in \mathbb{R}^{n \times n}$, $j = 0, \dots, m$, smoothly depend on the system parameters $p = (p_1, \dots, p_k)^T \in \mathbb{R}^k$. Let us also introduce the auxiliary system

$$\begin{aligned} \dot{z}(t) = & (A_0(p) - \gamma I) z(t) + \sum_{j=1}^m A_j(p) e^{-\gamma jh} z(t-jh) + \\ & \sum_{j=0}^{m-1} B_j(p) \int_{-(j+1)h}^{-jh} e^{\gamma\theta} z(t+\theta)d\theta, \quad t \geq 0, \end{aligned} \quad (\text{A2})$$

obtained from the change of variable $x(t) = e^{\gamma t} z(t)$.

The delay Lyapunov matrix $U_\gamma(0, p)$ of system (A2) and its partial derivatives $\frac{\partial U_\gamma(0, p)}{\partial \gamma}$, $\frac{\partial U_\gamma(0, p)}{\partial p_j}$, $j = 1, \dots, k$, required for the computation of the smoothed spectral abscissa $\tilde{\alpha}_\epsilon(p)$ of system (A1), and its gradient $\nabla \tilde{\alpha}_\epsilon(p)$, can be computed semi-analytically. In the following subsections, we focus on obtaining the corresponding algebraic expressions.

A.1. Delay-free system for semi-analytic construction of the delay Lyapunov matrix

In this subsection we provide an auxiliary delay-free system required for the semi-analytic construction of the delay Lyapunov matrix for distributed systems of the form (A2). The delay Lyapunov matrix of system (A2) associated to $W \in \mathbb{R}^{n \times n}$ satisfies the following set of equations, called dynamic, symmetry and algebraic properties (Kharitonov, 2013):

$$\begin{aligned} \frac{\partial}{\partial \tau} U_\gamma(\tau, p) = & U_\gamma(\tau, p) (A_0(p) - \gamma I) + \sum_{j=1}^m U_\gamma(\tau-jh, p) A_j(p) e^{-\gamma jh} + \\ & \sum_{j=0}^{m-1} \int_{-(j+1)h}^{-jh} U_\gamma(\tau+\theta, p) B_j(p) e^{\gamma\theta} d\theta, \quad \tau \in [0, mh], \end{aligned} \quad (\text{A3})$$

$$U_\gamma(-\tau, p) = U_\gamma^T(\tau, p), \quad \tau \in [-mh, mh] \quad (\text{A4})$$

$$\begin{aligned}
-W = & U_\gamma(0, p) (A_0(p) - \gamma I) + (A_0(p) - \gamma I)^T U_\gamma(0, p) + \\
& \sum_{j=1}^m \left[U_\gamma(-jh, p) A_j(p) e^{-\gamma jh} + A_j^T(p) e^{-\gamma jh} U_\gamma(jh, p) \right] + \\
& \sum_{j=0}^{m-1} \left[\int_{-(j+1)h}^{-jh} U_\gamma(\theta, p) B_j(p) e^{\gamma \theta} d\theta + B_j^T(p) \int_{jh}^{(j+1)h} e^{-\gamma \theta} U_\gamma(\theta, p) d\theta \right]. \quad (\text{A5})
\end{aligned}$$

It is straightforward to prove that for $\tau \in [-mh, 0]$,

$$\begin{aligned}
\frac{\partial}{\partial \tau} U_\gamma(\tau, p) = & - (A_0(p) - \gamma I)^T U_\gamma(\tau, p) - \sum_{j=1}^m A_j^T(p) e^{-\gamma jh} U_\gamma(\tau + jh, p) - \\
& \sum_{j=0}^{m-1} B_j^T(p) \int_{jh}^{(j+1)h} e^{-\gamma \theta} U_\gamma(\tau + \theta, p) d\theta. \quad (\text{A6})
\end{aligned}$$

The proposed delay-free system to solve equations (A3)-(A6) follows the ideas in (Aliseyko, 2017). We define a set of $6m - 2$ auxiliary matrices as follows:

$$\begin{aligned}
Y_i(\tau, p) &= U_\gamma(\tau + ih, p), & -m \leq i \leq m-1, \\
Z_i(\tau, p) &= \int_{(i-1)h}^{ih} U_\gamma(\tau + \theta, p) e^{\gamma \theta} d\theta, & -m+1 \leq i \leq m-1, \\
Q_i(\tau, p) &= \int_{(i-1)h}^{ih} U_\gamma(\tau + \theta, p) e^{-\gamma \theta} d\theta, & -m+1 \leq i \leq m-1,
\end{aligned} \quad (\text{A7})$$

where $\tau \in [0, h]$.

Lemma A.1. *Let $U_\gamma(\tau, p)$ be a Lyapunov matrix of system (A2), associated with a symmetric matrix W . Then the auxiliary matrices (A7) satisfy the system of linear differential equations*

$$\begin{aligned}
Y_i'(\tau, p) &= Y_i(\tau, p) (A_0(p) - \gamma I) + \sum_{j=1}^m Y_{i-j}(\tau, p) A_j(p) e^{-\gamma jh} + \\
& \sum_{j=0}^{m-1} Z_{i-j}(\tau, p) B_j(p) e^{-\gamma ih} d\theta, \quad 0 \leq i \leq m-1, \\
Y_i'(\tau, p) &= - (A_0(p) - \gamma I)^T Y_i(\tau, p) - \sum_{j=1}^m A_j^T(p) e^{-\gamma jh} Y_{i+j}(\tau, p) - \\
& \sum_{j=0}^{m-1} B_j^T(p) e^{\gamma ih} Q_{i+j+1}(\tau, p) d\theta, \quad -m \leq i \leq -1, \\
Z_i'(\tau, p) &= Y_i(\tau, p) e^{\gamma ih} - Y_{i-1}(\tau, p) e^{\gamma(i-1)h} - \gamma Z_i(\tau, p), \quad -m+1 \leq i \leq m-1, \\
Q_i'(\tau, p) &= Y_i(\tau, p) e^{-\gamma ih} - Y_{i-1}(\tau, p) e^{-\gamma(i-1)h} + \gamma Q_i(\tau, p), \quad -m+1 \leq i \leq m-1, \quad (\text{A8})
\end{aligned}$$

and the boundary conditions

$$\begin{aligned}
Y_i(0, p) &= Y_{i-1}(h, p), \quad -m+1 \leq i \leq m-1 \\
Z_i(0, p) &= Z_{i-1}(h, p)e^{\gamma h}, \quad -m+2 \leq i \leq m-1 \\
Q_i(0, p) &= Q_{i-1}(h, p)e^{-\gamma h}, \quad -m+2 \leq i \leq m-1 \\
Z_0(0, p) &= \int_0^h Y_{-1}(\theta, p)e^{\gamma(\theta-h)}d\theta, \\
Q_0(0, p) &= \int_0^h Y_{-1}(\theta, p)e^{-\gamma(\theta-h)}d\theta, \\
-W &= Y_0(0, p)(A_0(p) - \gamma I) + (A_0(p) - \gamma I)^T Y_{-1}(h, p) + \\
&\quad \sum_{j=1}^m \left[Y_{-j}(0, p)A_j(p)e^{-\gamma jh} + A_j^T(p)e^{-\gamma jh}Y_{j-1}(h, p) \right] + \\
&\quad \sum_{j=0}^{m-1} \left[Z_{-j}(0, p)B_j(p) + B_j^T(p)Q_j(h, p)e^{-\gamma h} \right]. \tag{A9}
\end{aligned}$$

Proof. The system of equations (A8) follows from the dynamic properties (A3), (A6) and definition (A7). The boundary conditions (A9) are given by (A7) and the algebraic property (A5). $\square$

The following subsection shows how the set of equations (A8)-(A9) can be rewritten in vector form by means of properties of the Kronecker product.

A.1.1. Matrix representation

Matrices $L(p, \gamma)$, $M(p, \gamma)$, $N(p, \gamma)$ and $G(\theta, \gamma)$ used in the computation of the delay Lyapunov matrix of system (A2) and its partial derivatives with respect to system parameters are defined as follows.

Consider $\mathcal{A}_0 = (A_0 - \gamma I)^T \otimes I_n$, $\mathfrak{A}_0 = I_n \otimes (A_0 - \gamma I)^T$, $\mathcal{A}_l = A_l^T \otimes I_n e^{-\gamma lh}$, $\mathfrak{A}_l = I_n \otimes A_l^T e^{-\gamma lh}$, for $l = 1, \dots, m$, $\mathcal{B}_{i,j} = B_i^T \otimes I_n e^{-\gamma jh}$, $\mathfrak{B}_{i,j} = I_n \otimes B_i^T e^{-\gamma jh}$, $\mathcal{I}_i = I_n \otimes I_n e^{\gamma(i)h}$, $\mathfrak{I}_i = I_n \otimes I_n e^{-\gamma(i)h}$, $\mathbf{0} = 0_n \otimes 0_n$, where I_n , 0_n denote respectively the identity and zero matrices with dimension $n \times n$. Define $y_i(\tau, p) = \text{vec}(Y_i(\tau, p))$, $q_i(\tau, p) = \text{vec}(Q_i(\tau, p))$, $z_i(\tau, p) = \text{vec}(Z_i(\tau, p))$ and

$$\mathcal{Y}(\tau, p) = \begin{pmatrix} y_{m-1}(\tau, p) \\ \vdots \\ y_{-m}(\tau, p) \end{pmatrix}, \quad \mathcal{Z}(\tau, p) = \begin{pmatrix} z_{m-1}(\tau, p) \\ \vdots \\ z_{-m+1}(\tau, p) \end{pmatrix}, \quad \mathcal{Q}(\tau, p) = \begin{pmatrix} q_{m-1}(\tau, p) \\ \vdots \\ q_{-m+1}(\tau, p) \end{pmatrix},$$

$$\Omega_i = [\mathbf{0} \ \dots \ I_n^i \ \dots \ \mathbf{0}],$$

where Ω_i has $6m - 2$ blocks.

System equation (A8) can be rewritten as

$$\nu'_\gamma(\tau, p) = L(p, \gamma)\nu_\gamma(\tau, p), \quad L(p, \gamma) = \begin{pmatrix} L_1 & L_2 & 0 \\ L_3 & 0 & L_4 \\ L_5 & -\gamma I & 0 \\ L_6 & 0 & \gamma I \end{pmatrix}, \quad \nu_\gamma(\tau, p) = \begin{pmatrix} \mathcal{Y}(\tau, p) \\ \mathcal{Z}(\tau, p) \\ \mathcal{Q}(\tau, p) \end{pmatrix},$$

where

$$L_1 = \begin{pmatrix} \mathcal{A}_0 & \dots & \mathcal{A}_{m-1} & \mathcal{A}_m & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathcal{A}_0 & \mathcal{A}_1 & \dots & \mathcal{A}_m \end{pmatrix} \in \mathbb{R}^{mn^2 \times 2mn^2},$$

$$L_2 = \begin{pmatrix} \mathcal{B}_{0,m-1} & \dots & \mathcal{B}_{m-1,m-1} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathcal{B}_{0,0} & \dots & \mathcal{B}_{m-1,0} \end{pmatrix} \in \mathbb{R}^{mn^2 \times (2m-1)n^2},$$

$$L_3 = \begin{pmatrix} -\mathfrak{A}_m & \dots & -\mathfrak{A}_1 & -\mathfrak{A}_0 & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & -\mathfrak{A}_m & -\mathfrak{A}_{m-1} & \dots & -\mathfrak{A}_0 \end{pmatrix} \in \mathbb{R}^{mn^2 \times 2mn^2},$$

$$L_4 = \begin{pmatrix} -\mathfrak{B}_{m-1,-1} & \dots & -\mathfrak{B}_{0,-1} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & -\mathfrak{B}_{m-1,-m} & \dots & -\mathfrak{B}_{0,-m} \end{pmatrix} \in \mathbb{R}^{mn^2 \times (2m-1)n^2},$$

$$L_5 = \begin{pmatrix} \mathcal{I}_{m-1} & -\mathcal{I}_{m-2} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{I}_{m-2} & -\mathcal{I}_{m-3} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathcal{I}_0 & -\mathcal{I}_{-1} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & -\mathcal{I}_{-m+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathcal{I}_{-m+1} & -\mathcal{I}_{-m} \end{pmatrix},$$

$$L_6 = \begin{pmatrix} \mathfrak{I}_{m-1} & -\mathfrak{I}_{m-2} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathfrak{I}_{m-2} & -\mathfrak{I}_{m-3} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathfrak{I}_0 & -\mathfrak{I}_{-1} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & -\mathfrak{I}_{-m+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathfrak{I}_{-m+1} & -\mathfrak{I}_{-m} \end{pmatrix}.$$

with L_5 and $L_6 \in \mathbb{R}^{(2m-1)n^2 \times 2mn^2}$, and the boundary conditions (A9) have the form

$$M(p, \gamma)\nu_\gamma(0, p) + N(p, \gamma)\nu_\gamma(h, p) + \int_0^h G(\theta, \gamma)\nu_\gamma(\theta, p)d\theta = - \begin{pmatrix} 0 \\ \text{vec}(Q_2^T Q_2) \end{pmatrix},$$

where

$$M(p, \gamma) = \begin{pmatrix} y_{m-1} & \dots & y_0 & \dots & y_{-m+1} & y_{-m} & z_{m-1} & \dots & z_0 & \dots & z_{-m+2} & z_{-m+1} & q_{m-1} & \dots & q_0 & \dots & q_{-m+2} & q_{-m+1} \\ \mathcal{I}_0 & \vdots & 0 & \vdots & \mathcal{I}_0 & 0 & 0 & \vdots & 0 & \vdots & \vdots & 0 & 0 & \vdots & 0 & \vdots & 0 & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \mathcal{I}_0 & \vdots & 0 & 0 & \mathcal{I}_0 & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & \vdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & \vdots & \mathcal{I}_0 & \vdots & \vdots & \vdots & 0 & 0 & \vdots & \mathcal{I}_0 & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & 0 & 0 & 0 & \vdots & 0 & \vdots & \vdots & 0 & 0 & 0 & \vdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & \vdots & \mathcal{I}_0 & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 & \vdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & \vdots & \mathcal{I}_0 & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & 0 & 0 & 0 & \vdots & 0 & \vdots & \vdots & 0 & 0 & 0 & \vdots & 0 & \mathcal{I}_0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & \vdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 0 & \vdots & \mathcal{B}_{0,0} & \vdots & \vdots & \mathcal{B}_{m-2,0} & 0 & 0 & \vdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & \vdots & 0 & \vdots & \vdots & \vdots & \mathcal{B}_{m-1,0} & 0 & \vdots & 0 & 0 & 0 \\ 0 & 0 & \mathcal{A}_0 & \vdots & \mathcal{A}_{m-1} & \mathcal{A}_m & 0 & \vdots & \mathcal{B}_{0,0} & \vdots & \vdots & \vdots & 0 & 0 & \vdots & 0 & 0 & 0 \end{pmatrix},$$

[illegible]

and

$$G(\theta, \gamma) = -e^{\gamma(\theta-h)} \Omega_{6m-4}^T \Omega_{m+1} - e^{-\gamma(\theta-h)} \Omega_{6m-3}^T \Omega_{m+1}.$$

A.2. Computation of the delay Lyapunov matrix and its derivatives

In this subsection, we provide algebraic expressions in order to compute the delay Lyapunov matrix $U_\gamma(0, p)$ of system (A2) and its partial derivatives $\frac{\partial U_\gamma(0, p)}{\partial \gamma}$, $\frac{\partial U_\gamma(0, p)}{\partial p_j}$, $j = 1, \dots, k$.

For any fixed $p \in \mathbb{R}^k$, consider the linear differential equation

$$\nu'_\gamma(\tau, p) = L(p, \gamma) \nu_\gamma(\tau, p), \quad \tau \in [0, h], \quad (\text{A10})$$

with boundary conditions

$$M(p, \gamma) \nu_\gamma(0, p) + N(p, \gamma) \nu_\gamma(h, p) + \int_0^h G(\theta, \gamma) \nu_\gamma(\theta, p) d\theta = - \begin{pmatrix} 0 \\ \text{vec}(Q_2^T Q_2) \end{pmatrix}, \quad (\text{A11})$$

where $\nu_\gamma(\tau, p)$, $L(p, \gamma)$, $M(p, \gamma)$, $N(p, \gamma)$ and $G(\theta, \gamma)$ are given in section A.1.

From equation (A10), it follows that

$$\nu_\gamma(\tau, p) = e^{L(p, \gamma)\tau} \nu_\gamma(0, p).$$

Substituting the last expression into (A11) we have that

$$\nu_\gamma(0, p) = - \left(M(p, \gamma) + N(p, \gamma) e^{L(p, \gamma)h} + \int_0^h G(\theta, \gamma) e^{L(p, \gamma)\theta} d\theta \right)^{-1} \begin{pmatrix} 0 \\ \text{vec}(Q_2^T Q_2) \end{pmatrix}, \quad (\text{A12})$$

therefore, by devectorizing the m th element of $\nu_\gamma(0, p)$ the delay Lyapunov matrix $U_\gamma(0, p)$ of system (A2) is obtained as

$$U_\gamma(0, p) = \text{mat}(\nu_\gamma(0, p), m).$$

We focus now on characterizing the partial derivatives of the delay Lyapunov matrix with respect to the system parameters and γ .

Differentiating expressions (A10) and (A11) with respect to γ and p_j , $j = 1, \dots, k$, yields

$$\begin{aligned} \frac{\partial \nu'_\gamma(\tau, p)}{\partial \gamma} &= L(p, \gamma) \frac{\partial \nu_\gamma(\tau, p)}{\partial \gamma} + \frac{\partial L(p, \gamma)}{\partial \gamma} \nu_\gamma(\tau, p), \\ \frac{\partial \nu'_\gamma(\tau, p)}{\partial p_j} &= L(p, \gamma) \frac{\partial \nu_\gamma(\tau, p)}{\partial p_j} + \frac{\partial L(p, \gamma)}{\partial p_j} \nu_\gamma(\tau, p), \quad j = 1, \dots, k, \end{aligned} \quad (\text{A13})$$

for $\tau \in [0, h]$, and

$$\begin{aligned}
& M(p, \gamma) \frac{\partial \nu_\gamma(0, p)}{\partial \gamma} + \frac{\partial M(p, \gamma)}{\partial \gamma} \nu_\gamma(0, p) + N(p, \gamma) \frac{\partial \nu_\gamma(h, p)}{\partial \gamma} + \frac{\partial N(p, \gamma)}{\partial \gamma} \nu_\gamma(h, p) + \\
& \int_0^h \left[G(\theta, \gamma) \frac{\partial \nu_\gamma(\theta, p)}{\partial \gamma} + \frac{\partial G(\theta, \gamma)}{\partial \gamma} \nu_\gamma(\theta, p) \right] d\theta = 0, \\
& M(p, \gamma) \frac{\partial \nu_\gamma(0, p)}{\partial p_j} + \frac{\partial M(p, \gamma)}{\partial p_j} \nu_\gamma(0, p) + N(p, \gamma) \frac{\partial \nu_\gamma(h, p)}{\partial p_j} + \frac{\partial N(p, \gamma)}{\partial p_j} \nu_\gamma(h, p) + \\
& \int_0^h \left[G(\theta, \gamma) \frac{\partial \nu_\gamma(\theta, p)}{\partial p_j} \right] d\theta = 0, \quad j = 1, \dots, k.
\end{aligned} \tag{A14}$$

Rewriting (A10) and (A13) in matrix form we obtain

$$\chi'_\gamma(\tau, p) = \widehat{L}(p, \gamma) \chi_\gamma(\tau, p), \quad \tau \in [0, h], \tag{A15}$$

where

$$\chi_\gamma(\tau, p) = \left(\nu_\gamma^T(\tau, p), \frac{\partial \nu_\gamma^T(\tau, p)}{\partial \gamma}, \frac{\partial \nu_\gamma^T(\tau, p)}{\partial p_1}, \dots, \frac{\partial \nu_\gamma^T(\tau, p)}{\partial p_k} \right)^T,$$

and

$$\widehat{L}(p, \gamma) = \begin{pmatrix} L & 0 & 0 & \dots & 0 \\ \frac{\partial L}{\partial \gamma} & L & 0 & \dots & 0 \\ \frac{\partial L}{\partial p_1} & 0 & L & \dots & 0 \\ \vdots & & & \ddots & 0 \\ \frac{\partial L}{\partial p_k} & 0 & 0 & \dots & L \end{pmatrix}.$$

Boundary conditions (A11) and (A14) can be rewritten as

$$\widehat{M}(p, \gamma) \chi_\gamma(0, p) + \widehat{N}(p, \gamma) \chi_\gamma(h, p) + \int_0^h \widehat{G}(\theta, \gamma) \chi_\gamma(\theta, p) d\theta = \widehat{w}, \tag{A16}$$

where $\widehat{w}^T = (0_{1 \times (6m-3)n^2}, -\text{vec}(Q_2^T Q_2)^T, 0_{1 \times (k+1)(6m-2)n^2})$ and

$$\widehat{M}(p, \gamma) = \begin{pmatrix} M & 0 & 0 & \dots & 0 \\ \frac{\partial M}{\partial \gamma} & M & 0 & \dots & 0 \\ \frac{\partial M}{\partial p_1} & 0 & M & \dots & 0 \\ \vdots & & & \ddots & 0 \\ \frac{\partial M}{\partial p_k} & 0 & 0 & \dots & M \end{pmatrix},$$

$$\widehat{N}(p, \gamma) = \begin{pmatrix} N & 0 & 0 & \dots & 0 \\ \frac{\partial N}{\partial \gamma} & N & 0 & \dots & 0 \\ \frac{\partial N}{\partial p_1} & 0 & N & \dots & 0 \\ \vdots & & & \ddots & 0 \\ \frac{\partial N}{\partial p_k} & 0 & 0 & \dots & N \end{pmatrix},$$

$$\widehat{G}(\theta, \gamma) = \begin{pmatrix} G(\theta, \gamma) & 0 & 0 & \dots & 0 \\ \frac{\partial G(\theta, \gamma)}{\partial \gamma} & G(\theta, \gamma) & 0 & \dots & 0 \\ 0 & 0 & G(\theta, \gamma) & \dots & 0 \\ \vdots & & & \ddots & 0 \\ 0 & 0 & 0 & \dots & G(\theta, \gamma) \end{pmatrix}.$$

From (A15) it follows that

$$\chi_\gamma(\tau, p) = \widehat{L}(p, \gamma) \chi_\gamma(0, p).$$

Using the above equality into (A16), we obtain

$$\left(\widehat{M}(p, \gamma) + \widehat{N}(p, \gamma) e^{\widehat{L}(p, \gamma)h} + \int_0^h \widehat{G}(\theta, \gamma) e^{\widehat{L}(p, \gamma)\theta} d\theta \right) \chi_\gamma(0, p) = \widehat{w},$$

then

$$\chi_\gamma(0, p) = \left(\widehat{M}(p, \gamma) + \widehat{N}(p, \gamma) e^{\widehat{L}(p, \gamma)h} + \int_0^h \widehat{G}(\theta, \gamma) e^{\widehat{L}(p, \gamma)\theta} d\theta \right)^{-1} \widehat{w}. \quad (\text{A17})$$

Finally, we obtain from (A17) the matrices needed in the computation of the smoothed spectral abscissa and its gradient:

$$\begin{aligned} U_\gamma(0, p) &= \text{mat}(\nu_\gamma(0, p), m) = \text{mat}(\chi_\gamma(0, p), m), \\ \frac{\partial U_\gamma(0, p)}{\partial \gamma} &= \text{mat}\left(\frac{\partial \nu_\gamma(0, p)}{\partial \gamma}, m\right) = \text{mat}(\chi_\gamma(0, p), 7m - 2), \\ \frac{\partial U_\gamma(0, p)}{\partial p_j} &= \text{mat}\left(\frac{\partial \nu_\gamma(0, p)}{\partial p_j}, m\right) = \text{mat}(\chi_\gamma(0, p), (j+1)(6m-2) + m), \quad j = 1, \dots, k. \end{aligned}$$