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ECU SPOT RATES, INTEREST RATES AND YIELDS : ARE THERE ARBITRAGE POSSIBILITIES ?

BY

P. SERCU

Katholieke Universiteit Leuven

Dekenstraat 2, 3000 Leuven

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A widely held view is that, as the ECU is a linear combination of component currencies, ECU interest rates or bond yields ought to equal a linear weighted average of the component currencies' interest rates or bond yields. A corollary then would be that, as observed ECU rates are about 1% lower, the market is inefficient¹.

The message of the present paper is that such a conclusion may be premature. In section one we will argue that, even if at any point in time the ECU spot rate would equal the spot value of the basket, still ECU interest rates or bond yields should be highly non-linear functions of the component interest rates or bond yields; and these non-linearities produce arbitrage-free ECU interest rates or yields that are below a simple linear weighted average. We will also show, however, that the non-linearities cannot nearly account for all of the observed shortfall in ECU interest rates. This then brings us to a second tack: the value of the (private) ECU can deviate from the value of a fixed-composition basket without any implications about market inefficiency. We conclude from all this that rigorous efficiency tests are hard to devise, and that the nature of the ECU is rather vague.

1. Theoretical Interest Rates for a Basket

In this section we will demonstrate that, for any basket such as the official ECU or SDR, equilibrium (arbitrage-free) interest rates should be below the weighted average. We will derive the relation using numerical examples, with formal generalised proofs provided in the appendix. For ease of exposition, assume a

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1. See e.g. The Economist, 23 sept 1989, p 101.

simplified basket containing just DEM 3 ($=n_1$) and GBP 1 ($=n_2$). As the reference currency we pick the BEF, but that of course doesn't affect the outcome. Assume the following data:

spot rate ⁺ — $n_1 S_1$			interest rates* —		
	S_1		6 mo	12 mo	3 yr
DEM	20	60	6%	6.5%	7%
GBP	60	60	15%	15%	15%
$S_E = 120;$			avge 10.5%	10.75%	11%

⁺ spot rates are in units of BEF

^{*} interest rates are simple rates for 6 and 12 months, and compound rates for zero bonds in the 3-year case.

The one-year interest rate on a basket

We immediately notice that, if a banker accepts a 1-year basket deposit, he or she cannot just buy and invest that basket if all risk is to be avoided. Indeed, investing DEM 3 and GBP 1 at different interest rates will produce a portfolio containing

$$\text{DEM } 3 \times 1.065 = 3.195$$

$$\text{GBP } 1 \times 1.150 = 1.150$$

Clearly, the future ratio of DEM/GBP no longer is three to one; or conversely the portfolio does not contain a predictable number of basket units. The banker will have too many GBP, and too few DEM; in order to have a riskless position, a forward purchase of DEM against GBP would be necessary. With such a forward deal thrown in, the banker can construct an exact replica of a basket deposit, and by the Law of One Price the interest rate on the ECU deposit can be found.

Rather than pursuing this rather tortuous route with forward hedging, we will follow a simpler but equivalent approach². The

2. With bid-ask spreads in exchange rates and in interest rates, the two routes may lead to slightly different outcomes. In perfect markets, there is no distinction.

banker can easily buy initial amounts of spot DEM and GBP such that the final ratio of DEM/GBP is exactly three to one. For instance, in the one year case, the strategy would be to buy

$$\begin{aligned} \text{DEM } 3/1.065 &= 2.817 \text{ at a cost } 2.817 \times 20 = 56.338 \\ \text{GBP } 1/1.150 &= .8696 \text{ at a cost } .8696 \times 60 = 52.174 \\ &\text{total cost} = 108.512 \end{aligned}$$

By construction, this portfolio will contain DEM 3 and GBP 1 within one year -- one basket, in short. The immediate outlay needed to buy this portfolio is BEF 108.512, which corresponds to $108.512/120 = .9043$ basket units³. Therefore the one-year interest rate is to be set at $1/.9043 - 1 = 10.587\%$. This is somewhat less than the weighted average, $(.5 \times 15 + .5 \times 6.5) = 10.75\%$.

The direction of the deviation does not depend on our particular inputs. To see this, note that for a one-year deposit, the story can be summarised into the following formula:

$$\begin{aligned} 1 + R_{\text{basket}} &= \frac{1}{.5/(1+R_{\text{DEM}}) + .5/(1+R_{\text{GBP}})} \\ &= 1/.9042661 = 1.105869 \end{aligned}$$

In words: unity plus the riskfree rate of return for the basket is a weighted harmonic mean of unity plus the component riskfree rates of return⁴. Since a harmonic mean is always lower than a standard mean, the interest rate rate that is consistent with a pure basket-type ECU (with $S_E = \text{SUM } n_j S_j$ at all points in time) is always below the weighted average riskfree rate.

The above result holds for rates of return -- simple percentage

3. That is, for a unit ECU CD, the banker charges BEF 108.512 or, formally, ECU .9043. However, he or she buys not GBP .9043 and DEM .9043 $\times 3 = 2.713$, but rather GBP .896 and DEM 2.817.

4. This result is generalised in appendix.

differences between initial and final values. For a one-year deposit, such a rate of return is equal to the interest rate. For other maturities, rates of return are always annualised, using for instance linear or exponential intra- or extrapolation (leading to simple and compound annual rates, respectively). So we still have to check whether, for maturities other than one year, the customary annualisation process affects our conclusion. For coupon-bearing instruments, yields-to-maturity make the relation to simple returns even more distant. We will look at each of these three cases.

Simple interest rates

We can easily generalise the approach for maturities other than 12 months. First note that we can exactly repeat the above argument if we use rates of return, i.e. the (non-annualised) simple percentage difference between initial and final value. For example, on a 6-month deposit the rates of return are

$$\begin{aligned} 1 + R_{\text{DEM}} &= 1 + .06/2 = 1.03 \\ 1 + R_{\text{GBP}} &= 1 + .15/2 = 1.075 \\ &\text{(average: } 1.0525) \end{aligned}$$

Accordingly, the banker can buy DEM 3/1.03 and GBP 1/1.0750 at a total cost of BEF 114.066, which corresponds to $114.066/120 = .9505$ units of the basket, and implies a (non-annualised) rate of return of 5.202% in terms of the basket. Again, unity plus the basket rate of return is below the linear average, because we have implicitly used a harmonic average. It therefore still follows that the rate of return is below the linear average rate of return. And multiplying all returns by two (that is, going towards simple annualised interest rates) of course preserves that inequality. In short, we can still conclude that, in equilibrium, the simple interest rate on the official ECU or SDR should always be below the weighted average of the simple

interest rates for the components.

Compound interest rates on zero-bonds or T-bills

For longer maturities $T > 1$, where compound rates are most common, we can again use (non-annualised) rates of return⁵. For example, in the 3-year case, the rates of return are

$$1 + R_{\text{DEM}} = 1.07^3 = 1.225043$$

$$1 + R_{\text{GBP}} = 1.15^3 = 1.520875$$

$$\text{average } 1.372959$$

So the banker buys DEM $3/1.225043 = .8163$ and GBP $1/1.520875$ at a total cost of BEF 88.4288, corresponding to $88.4288/120 = .7369$ units of our basket, and therefore implying a 3-year rate of return of 35.57% in terms of that basket. For the same reason as before, this is below the average return. However, we still have to annualise. A total return of 35.57% over three years corresponds to a compound annual rate of $(1.3557)^{1/3} - 1 = 10.71\%$; and this is again below the weighted average, 11%.

Coupon Bond Yields

The above results were derived for zero-coupon type operations. For coupon bonds, the replication argument still works. To replicate a 10% basket bond with face value 10,000 units, the banker or investor can always buy 10% DEM bonds with a total face value of DEM 30,000, and 10% GBP bonds with a total face value of GBP 10,000 (⁶). As before, we can then compute the cash outlays

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5. The results do not depend on the example's assumption that T exceeds unity; the same conclusions hold for compound interest rates on say 3-month CDs.

6. If 10% bonds are not available, the same effect can be obtained by forming a portfolio containing $2xT$ zero bonds, paying off DEM 3000 and GBP 1000 at each coupon date, and DEM 33,000 and GBP 11,000 at the expiration date. Or, finally, by using stripped

needed to create the replicating portfolio, translate this value into basket units, and finally find the basket yield. There exists no general closed-form solution for the yield on coupon bonds, so numerical methods are called for. We will immediately illustrate this with genuine ECU data.

Results for real-world ECUs

The above results held for a highly simplified basket. To get an idea about the size of the effect for a real-world official ECU, we use the following inputs⁷:

	S_j^*	n_j	w_j	interest rates short	long
UK	.6350	.087840	.120	15.25	12.00
IRL	.6700	.008552	.011	18.25	15.00
NL	1.990	.219800	.096	8.25	8.25
B	37.05	3.43100	.080	10.00	9.70
DK	6.847	.197600	.025	11.50	12.00
D	1.765	.624200	.307	8.25	7.80
P	154.3	1.39300	.008	12.75	12.00
SP	114.1	6.88500	.052	17.25	14.00
I	1303.	151.800	.101	12.75	12.00
F	6.030	1.33200	.192	10.50	9.75
GR	162.75	1.44000	.008	20.50	15.00

*: value of the USD, in units of the home currencies

First look at the results for zero-coupon type operations (including deposits etc.). We assume that the above short term interest rates apply for 3, 6, 9, and 12 months (flat term structure), respectively, and we extract equilibrium ECU rates. Similarly, we assume that the long term rates are valid for 3, 5, 7, and 9 years zero-bonds, and we compute the corresponding

bonds one could always construct a replicating portfolio.

7. Most data are from The Financial Times, dec 7; the short-term drachma rate was derived from the forward premium. Some missing bond yields were taken from The Economist, dec 9; for Portugal, and Ireland, the bond rates are guesstimates; these countries have low weights, and the calculations are meant to be illustrative anyway.

equilibrium ECU rates. Of course, these assumptions are crude, but the results are only meant to give us an idea about the practical effect of the non-linearities rather than an exact value for the theoretical ECU yield. We also compare these to the weighted average rate. The results are summarised below:

short term rates		long term rates	
3 mo	10.888	3 yr	9.823
6 mo	10.868	5 yr	9.887
9 mo	10.849	7 yr	9.752
12 mo	10.831	9 yr	9.717
average	10.910	average	9.896

Notwithstanding all the non-linearities involved, we observe a close correspondence between the theoretical rates and the weighted average. As expected, the theoretical rates are always below that weighted average, but the differences are nowhere near the 1% deviation commonly observed in reality: discrepancies are less than 10 basis points for simple rates, and no more than 25 basis points for very long compound rates.

The results for bonds are very similar. Using the above technique, we form replicating portfolios for ECU bonds with coupons of 5, 7, 9, and 11%, and times to maturity of 3, 5, 7, and 9 years. The assumption here again is a flat term structure, but, not surprisingly, simulations (not shown here) have confirmed that a rising or falling term structure of spot rates hardly affects the results.

	3 yr	5 yr	7 yr	9 yr
coupon 5%	9.824	9.791	9.760	9.731
coupon 7%	9.825	9.792	9.762	9.734
coupon 5%	9.825	9.793	9.764	9.738
coupon 5%	9.825	9.794	9.766	9.740

weighted average yield: 9.896

Again, the theoretical yields are below the weighted average, but the difference is always less than 15 basis points, i.e. nowhere near the 1% deviations observed in reality.

As an overall conclusion thus far, the non-linearities seem to be a bit of a red herring. Let's therefore look at the crucial assumption that the ECU always is worth the basket.

2. Spot Exchange Rates of the Private ECU versus the Basket.

The "official" ECU, used for transactions between EMS Central Banks and EC institutions and as a unit of account by EC bodies, is defined as a basket of currencies. The private ECU isn't -- or no longer is in the strict sense.

In fact, when Kredietbank Luxembourgoise (KBL) first started the private ECU market, by accepting a ECU 100,000 term deposit from the EC Commission, it did have in mind the same basket as the official ECU. In the early days, the ECU money market was purely a deposit market. When an investor bought an "ECU" from KBL, he or she put up enough money for KBL to be able to buy the basket; and KBL then invested appropriate amounts in the respective money markets. Likewise, upon expiration, the investor got back the initial ECU deposit grossed up with the promised ECU interest, but also valued on the basis of the basket; that is, he or she got back the countervalue of the component currencies. This was roughly the mechanism described above.

When also borrowers started using the ECU, KBL and other banks of course shifted from investments in the national money markets towards investments in ECU loans or ECU commercial paper. Simultaneously, it became legally possible in most countries to sell ECUs among private investors. This saved transaction costs. Consider an investor A who steps out of ECU while B wants to buy ECUs; whereas in the beginning A would have to sell the basket,

and B to buy the basket, the cost and hassle of unbundling and re-bundling was now avoided. In this second stage the ECU traded at bid/ask rates that were in-between the bid and ask quotes for the basket. Still, the KB Group maintained its policy of accepting ECUs against the basket, and selling baskets against ECUs; in fact, KB acted as a self-styled Central Bank intervening in the spot ECU market.

One problem with the ECU however is its changing composition: under the EMS Treaty, the basket is to be revised every five years. In an interesting paper, S. Muller[1989] shows that even when the future weights of the next basket are fully known and markets are perfect, still the spot ECU could rationally deviate from its current "official" value. This then holds a fortiori when the future weights are unknown. Another potential problem could arise if ECU interest rates are not set fully correct: then in an efficient market the equilibrium would be upset, and it would be costly to go against the market. For these and maybe other reasons (including, possibly, market inefficiencies), KB soon found that its interventions in the ECU spot market became untenable, and stopped its policy; right now, KB merely offers two-way quotes of the spot ECU in terms of the basket, and no longer intervenes when both bid and ask are at the same side of unity⁸. In short, while the private ECU is still "defined" as a basket, there is no immediately obvious force that keeps the ECU within transaction-cost bounds around the basket itself -- neither now, nor in the foreseeable future.

What, then, is a private ECU? Can we still say anything about its current and future "normal" value, or about its interest rate? To see the full extent of the problem, we might imagine a bank-created currency, say the Thaler, which is "defined" to be equal

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8. At the time of writing, end december 1989, the private ECU traded at a .2-.3% premium as compared to the basket. This has been the case since a couple of months before the september 1989 redefinition of the official ECU; and, significantly, this phenomenon has not disappeared since.

to the USD. Clearly, if nobody intervenes to maintain this parity, the Thaler's value could substantially deviate from the USD spot rate. Also, if the Thaler does deviate from the USD and may deviate also in the future, its interest rate could be literally anything; and the question would be what this Thaler in fact stands for. In short, when the Thaler (or, for that matter, the ECU) is not convertible into its formally underlying asset(s), we have no reason to postulate a unit spot or forward parity, or a specific interest rate. If all this looks far-fetched, just look at what happened to the US dollar at the end of Bretton Woods. Between March 1968 and November 1973, the dollar was still "defined" in terms of gold. But nobody defended that theoretical value; no investor or borrower even believed that in the foreseeable future that parity would be defended again. On top of that, the interest rate on USD went out of line with the convenience yield on gold, which, if we look at futures prices, typically is quite low. Obviously, then, the market value of the dollar did lose all contact with its theoretical gold value, going from .0286 ounces to about .001 ounces per dollar.

Let's accordingly relax the link between the ECU and the basket. We will do so in two steps. First we build in the observed fact that the current ECU deviates from the basket, but we initially maintain the conventional (heroic) assumption that, in the future, the ECU will be close to the basket. Afterwards we'll also relax that second assumption.

Current Deviations from the Official Parity.

To isolate current deviations as an explanatory factor, we heroically assume that investors do believe in a one-to-one future parity for the ECU with a known composition. Most of the previous analysis still holds: if there will be parity in the future, investors can still build a replicating portfolio by buying n_j bonds with the chosen coupon or n_j CDs with a unit

expiration value; then they translate the resulting expense into current ECUs.

To illustrate this, return to our previous two-currency example. Assume the ECU trades at BEF 119 rather than its theoretical BEF 120 value, that is, at a .84% discount. For e.g. a 12-month deposit, the cost (in ECU) of forming the duplicating portfolio would be computed exactly as before:

$$\begin{aligned} \text{DEM } 3/1.065 &= 2.817 \text{ at a cost } 2.817 \times 20 = 56.338 \\ \text{GBP } 1/1.150 &= .8696 \text{ at a cost } .8696 \times 60 = 52.174 \\ &\text{total cost} = 108.512 \end{aligned}$$

The immediate outlay now corresponds to $108.512/119 = .9119$ ECU (rather than to .9043 ECU). Therefore the one-year interest rate is to be set at $1/.9119 - 1 = 9.66\%$ (rather than at 10.58%). So for a one-year deposit, a .84% discount in the ECU corresponds to a .92% fall in the theoretical one-year interest rate.

Using this type of calculations, we revise our previous computations as follows, for the case of a hypothetical 1% discount of the ECU vis-a-vis the basket; between parenthesis we show our old results (i.e. for an ECU trading without discount or premium):

short term rates			long term rates		
3 mo	6.779	(10.888)	3 yr	9.456	(9.823)
6 mo	8.759	(10.868)	5 yr	9.567	(9.887)
9 mo	9.407	(10.849)	7 yr	9.594	(9.752)
12 mo	9.723	(10.831)	9 yr	9.595	(9.717)
avge	10.910		avge	9.896	

We see that a 1% discount in the ECU's spot value does produce substantial drops in the equilibrium interest rates; the reason is that the assumed capital gain makes up for the yield

shortfall. Conversely, if ECU interest rates are "too low", the spot ECU should trade at a discount even if the market fully believes in a par value in the future. Yet we haven't explained the observed deviations. First and foremost, in reality the ECU trades at a premium, not at a discount. That is, if we want to maintain our combined assumption about parity in the future and rationality, we should observe higher rates if there currently is a premium. Second, the example shows that the interest rate effect of a 1% discount depends on time to maturity: the drop is about 4% for a quarter, about 1% for 1 year, and about .11% for 9 years; in short, for zero-coupon instruments the effect of a 1% discount is roughly equal to $1/T$ (9).

To be true, the Economist data bear on coupon bonds. But our results for bonds are very similar: for a 1% discount in the private ECU, the equilibrium yields consistent with a belief in future parity are as follows:

	3 yr	5 yr	7 yr	9 yr
coupon 5%	9.438	9.546	9.574	9.577
coupon 7%	9.431	9.539	9.568	9.572
coupon 9%	9.425	9.533	9.562	9.568
coupon 11	9.420	9.527	9.558	9.564

weighted average yield: 9.896

The implication of all this is that, if we want to blame the deviations from ECU rates and weighted average rates solely on (non-linearities and) current deviations in the ECU's value, then we again failed; if this were the explanation, interest rates should be somewhat above, not below, the theoretical value; and the difference should decrease with the horizon. This is not what the real world data show.

9. This, on reflection is obvious: if duration measures by howmany percent the price drops if we increase the interest rate by 1%, then duration inverse measures howmuch the interest drops if we rise the value (in ECU) of the replicating portfolio by one percent.

Potential Explanations Related to Future Premia.

The weighted average norm, possibly corrected for non-linearities and for the present premium or discount of the ECU, relies heavily on a belief about future parity between the private and official ECU. Given this assumption, the market is inefficient if rates are too low. We are quite willing to believe that the proverbial Swiss or Belgian dentist is not rationally analysing ECU bond yields in the way we do. If this lack of insight produces below-normal yields, more sophisticated investors should steer clear of ECU bonds. But what prevents large borrowers from exploiting the inefficiency? Can we seriously expect MBA- and PhD-staffed finance departments of large corporations or finance houses not to see as blatant an inefficiency as the proclaimed yield shortfall on ECU?

One obvious alternative explanation accordingly is the to relax the assumption of future parity. That hypothesis, on reflection, is very artificial: why should investors believe in future parity when there's no parity right now? We can't even invoke uncertainty about the new basket anymore, since the observed premia have persisted even after the new ECU was defined. And if nobody defends the parity right now, who would in the future?. Even if, one day, the official ECU become the EC's only currency, this doesn't necessarily mean that private ECUs will then trade at par. To return to the Thaler analogy: if a US commercial bank were to create Thalers "defined" as one USD, the FED has no incentive whatsoever to defend that parity. Likewise, there is no obvious legal obligation for a future European Central Bank to defend the parity of the private ECU: the only shared aspect with the official ECU is the name, ECU.

If we are willing to relax the assumption of future parity, then a 1% shortfall in observed yields as compared with the average is consistent with, among other things, a belief that the private

ECU will appreciate by about 1%/year as compared to the basket, and risk-neutrality. With risk-aversion, risk-premia may come into the picture; and any such risk premium is, as yet, beyond any empirical analysis: quite apart from the usual problem that the perceived distribution of the future ECU/basket rate is unobservable, we don't even have the benefit of lots of data from the past.

Another possible partial explanation may be an asymmetry in the distribution of the future ECU/basket rate. Casual empiricism shows that there is no obvious force which prevents a premium: apparently Central Banks do not feel compelled to intervene themselves, nor have they given hints to commercial banks that the latter should intervene in the market for their products. The situation may have been different if the ECU were trading at a discount: then Central Banks may become worried about the credibility of a possible future (official) ECU with legal tender status. In the case of a discount, the issuers of ECU instruments could even act without pressure from monetary authorities: commercial banks who have advocated the ECU as a safe investment may feel they have to offer the basket as a floor value if they want to maintain their credibility, or if they want to avoid any possible lawsuits from frustrated investors.

The message is that investors may, with some justification, see the value of the basket as a floor but not as a cap on private ECU values. This would create a put-option feature for any ECU investment vehicle, for which the investor pays via prima facie "abnormally low" interest rates. Again, testing the rationality of the implied option price is difficult: as the premium price is a relatively recent phenomenon, the few past data we have may be a very poor guide for the future distribution of ECU/basket exchange rates.

Conclusions

One general conclusion is that it is impossible to test efficiency in the case of ECU CD rates or bond yields. Even if the ECU turns out not to appreciate by 1%/year, this still doesn't demonstrate that the market was irrational. First, ex post realised values will always differ from ex ante expectations. Second, the interest-adjusted spot rate (or forward rate) is the risk-adjusted expectation, not the expectation itself. And third, with a perceived put-characteristic in the private ECU, the rationally expected degree of appreciation would be far less than 1%.

A related second conclusion is that, to this author at least, the true nature of the private ECU is a great mystery: what is a private ECU, if nobody defends the parity and if there seems to be no reason to expect future intervention?

Appendix

1. Relation for (non-annualised) returns

Denote

R_j = the risk-free rate of return for currency j over the chosen period -- i.e. the simple percentage difference between initial and final value, without any annualisation.

R_E = the riskfree return on ECU

S_j = the current spot value of currency j

n_j = the number of currency units j in the basket

$S_E = \sum S_j/n_j$ = theoretical current spot value of the basket

$w_j = S_j n_j / S_E$ = the current weight of currency j in the basket

Then, in order to obtain the basket at a chosen maturity point T ,

it suffices to buy $n_j/(1+R_j)$ units of each currency, at a total cost of $\text{SUM } S_j n_j/(1+R_j)$. To translate this into a number of current ECUs, we divide by $S_E = \text{SUM } S_j n_j$:

$$\begin{aligned} & \# \text{ of ECUs needed to buy one ECU at time } T \\ &= [\text{SUM } n_j S_j (1+R_j)^{-1}] / S_E \\ &= \text{SUM } (n_j S_j / S_E) (1+R_j)^{-1} \\ &= \text{SUM } w_j (1+R_j)^{-1} \end{aligned}$$

The implied ECU rate of return then is

$$\begin{aligned} 1 + R_E &= 1 / (\# \text{ of ECUs needed}) \\ &= \{\text{SUM } w_j (1+R_j)^{-1}\}^{-1} \quad \text{Q.E.D.} \end{aligned}$$

2. Simple interest rates.

From

$$1 + R_E < \text{SUM } w_j (1+R_j) = 1 + \text{SUM } w_j R_j$$

it follows that

$$R_E/T < \text{SUM } w_j R_j/T$$

where R/T is the simple interest rate on an annual basis. Q.E.D.

3. Compound annual rates

Denote a compound annual rate by i_j , Since $(1+i_j)^{-n}$ is a convex function, it follows that

$$\text{SUM } w_j (1+R_j)^{-1} = \text{SUM } w_j (1+i_j)^{-n} > (1+\text{SUM } w_j i_j)^{-n}$$

Therefore

$$\{\text{SUM } w_j (1+R_j)^{-n}\}^{-1} < (1+\text{SUM } w_j i_j)^n$$

and

$$1 + i_E = \{\text{SUM } w_j(1+R_j)^{-n}\}^{-1/n} < (1+\text{SUM } w_j i_j) \quad \text{Q.E.D.}$$