Comparison between the wave finite element and shift cell operator method for calculating dispersion diagrams of periodic metamaterial structures

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Abstract – In this paper, the accuracy of two popular finite element based methods for calculating dispersion diagrams of periodic metamaterials is compared. The study reveals that the wave finite element method (WFEM) tends to result in higher accuracy for more coarsely meshed models as compared to the shift cell operator method (SCOM) and that care should be taken when using the SCOM for accurate stop band predictions using dispersion diagrams.

I. INTRODUCTION

The dispersion band structure of metamaterials provides a basis for understanding the wave propagation in such media. In the case of periodic metamaterials, the band structure can be evaluated by calculating the eigenmodes of a single unit cell by applying the Floquet-Bloch theorem. For this calculation, two commonly used finite element (FE) based approaches are the WFEM [1] and the SCOM [2]. In the former approach, a dispersion eigenvalue problem can be formulated from the stiffness and mass matrices of the unit cell FE model by applying Floquet-Bloch periodicity boundary conditions. In the latter approach, the solution is a priori represented using the Floquet-Bloch decomposition. The transformed weak form produces a mass matrix and three stiffness matrices corresponding to the transformed strain energy, and regular periodic boundary conditions are eventually applied. On the one hand, the SCOM is generally favored for predicting complex wavenumber dispersion curves along arbitrary wave propagation directions, since the WFEM in this case results in large non-linear eigenvalue problems except along the main periodicity directions. On the other hand, the WFEM is generally applied in order to predict real frequency dispersion diagrams and to find stop bands along the irreducible Brillouin contour (IBC). However, for the latter case, the accuracy of both methods has not been thoroughly discussed yet. In this work, the features of the SCOM are compared with the WFEM for these so-called *inverse* computations where real wavenumbers are imposed and the corresponding real eigen-frequencies computed, leading to dispersion diagrams for free wave propagation. In what follows, two case studies are presented.

II. CASE STUDY A

In the first case study, an undamped metamaterial plate consisting of a 1 mm thick aluminum plate with periodically distributed beam-shaped PMMA resonators is considered (Fig. 1a). The geometric parameters and material properties for the host structure and resonator follow the data provided in the corresponding paper [3]. Two FE models are considered: the coarse FE model (Fig. 1c) consists of 2119 TET-10 quadratic solid elements, which leads to 12852 degrees of freedom (DOFs), and the refined FE model (Fig. 1d) consists of 6048 TET-10 elements, which results in 33627 DOFs. The dispersion diagrams are calculated for both FE models along the IBC of Fig. 1b and are shown in Fig. 2a. For the coarse FE model, the SCOM results in a higher frequency dispersion error along the IBC. In addition, the near flat curves at the lower edge of the bending wave stop band around 600 Hz are less well represented using the SCOM, in particular for the coarse FE model, which results in a narrower predicted stop band. To further investigate these differences, the wave propagation solutions along the x -direction are calculated in adjacent Brillouin zones (Fig. 2b). While the WFEM leads to periodicity in real wave space, this periodicity of the band structure is not preserved in case of the SCOM due to more highly oscillating Bloch waves in further Brillouin zones. Since those Bloch waves have smaller wavelengths, the resulting asymmetry

of the dispersion diagram is strongly mesh-dependent. The accuracy of the dispersion curves along the IBC is calculated for first 5 branches in a convergence study (Fig. 2c). The WFEM solution obtained on the mesh with 33627 DOFs serves as the reference solution. Starting with 2784 DOFs, the successive refinement steps involve meshes of 5169, 8916, 12852 and 19890 DOFs, respectively. The SCOM shows higher relative error but converges with similar rate. Below the curves, the dashed line shows the relative difference in dispersion curves along the IBC using both methods for the reference FE model, which justifies the WFEM solution as reference.

Fig. 1: Description of the metamaterial unit cell [3] (a), definition of the IBC in the real propagation constant space (b) and coarse (c) and refined (d) FE meshes.

Fig. 2: Dispersion diagram for free wave propagation obtained using the *inverse* approach along the IBC (a) and for adjacent Brillouin zones along the *x*-direction (b); —: WFEM using refined model (33627 DOFs). +: WFEM using coarse model (12852 DOFs). —: SCOM using refined model (33627 DOFs). •: SCOM using coarse model (12852 DOFs). Relative frequency error of free propagating waves along the IBC (c).

III. CASE STUDY B

In the second case study, an undamped periodic metamaterial plate made of a 1 mm thick aluminum plate with periodic cylindrical stubs is considered (Fig. 3a). The geometric and material parameters of the model follow the data in the corresponding paper [4]. The coarse FE model (Fig. 3c) is discretized using 1494 TET-10 quadratic solid elements, leading to 7653 DOFs, and the refined FE model (Fig. 3d) is composed of 8927 TET-10 elements, leading to 40533 DOFs. The dispersion diagrams along the IBC of Fig. 3b are calculated and shown in Fig. 4a. In addition, the wave propagation solutions along the *x*-direction are calculated in adjacent Brillouin zones (Fig. 4b). As for the first case study, the results reveal the loss of periodicity for band structures computed with the shift cell operator method as well as its slightly inferior accuracy in the first Brillouin zone as compared to the wave finite element method when applied to the same FE model. In the convergence study (Fig. 4c), the relative frequency error of the dispersion curves along the IBC is calculated for first 10 branches using both methods and the WFEM solution obtained on the finest mesh with 40533 DOFs serves as the reference solution. The refinement sequence involves models of meshes with: 696, 1611, 4431, 7653 and 22830 DOFs. Again, the WFEM shows lower relative error but the SCOM shows similar convergence rate. Below the curves, the dashed line shows the relative difference in dispersion curves along the IBC using both methods for the reference FE model, which justifies the WFEM solution as reference. For the considered cases, the WFEM hence allows to more accurately predict the stop band frequency range by means of dispersion diagram computations.

IV. CONCLUSION

In this work, the accuracy of the WFEM and the SCOM was compared in the context of band diagram computations carried out using the *inverse* solution method. Results show that the SCOM does not preserve the periodicity

Fig. 3: Description of the metamaterial unit cell [4] (a), definition of the IBC in the real propagation constant space (b) and coarse (c) and refined (d) FE meshes.

Fig. 4: Dispersion diagram for free wave propagation obtained using the *inverse* approach along the IBC (a) and for adjacent Brillouin zones along the x -direction (b); —: WFEM using refined model (40533 DOFs). +: WFEM using coarse model (7653 DOFs). —: SCOM using refined model (40533 DOFs). •: SCOM using coarse model (7653 DOFs). Relative frequency error of free propagating waves along the IBC (c).

of the band structure and generally provides lower accuracy as compared to the WFEM for this kind of computations. This issue is caused by the wavenumber ambiguity in the decomposition of Bloch-waves which may force the SCOM to represent artificially small wavelengths which the FE mesh of the unit cell cannot resolve. By contrast, the WFEM enforces Floquet-Bloch boundary conditions using the well-defined propagation constants and does not suffer from this problem making it more reliable and generally the more accurate method for band diagram computations.

ACKNOWLEDGEMENT

The work leading to this publication has been funded by the SBO projects “MOR4MDesign” and “DETECTION”, which fit in the MacroModelMat (M3) research program, coordinated by Siemens (Siemens Digital Industries Software, Belgium) and funded by SIM (Strategic Initiative Materials in Flanders) and VLAIO (Flemish government agency Flanders Innovation & Entrepreneurship). The Research Fund KU Leuven is gratefully acknowledged for its support. This research was partially supported by Flanders Make, the strategic research centre for the manufacturing industry. The research of L. Van Belle (fellowship no. 1271621N) is funded by a grant from the Research Foundation – Flanders (FWO).

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