The R_∞ -property for right-angled Artin groups

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Abstract

Given a group G and an automorphism φ of G , two elements $x, y \in G$ are said to be φ -conjugate if $x = gy\varphi(g)^{-1}$ for some $g \in G$. The number of equivalence classes is the Reidemeister number $R(\varphi)$ of φ , and if $R(\varphi) = \infty$ for all automorphisms of G , then G is said to have the R_∞ -property.

A finite simple graph Γ gives rise to the right-angled Artin group A_Γ , which has as generators the vertices of Γ and as relations $vw = wv$ if and only if v and w are joined by an edge in Γ . We conjecture that all non-abelian right-angled Artin groups have the R_∞ -property and prove this conjecture for several subclasses of right-angled Artin groups.

1 Twisted conjugacy and Reidemeister numbers

Let G be a group and $\varphi : G \rightarrow G$ be an automorphism. For $x, y \in G$, we say that x and y are φ -conjugate and write $x \sim_\varphi y$ if there exists a $g \in G$ such that $x = gy\varphi(g)^{-1}$. The equivalence class of x is denoted by $[x]$, or $[x]_\varphi$ for clarity if there are multiple automorphisms involved.

We define $\mathcal{R}[\varphi]$ to be the set of all $\sim_\varphi$ -equivalence classes and the *Reidemeister number* $R(\varphi)$ of φ as the cardinality of $\mathcal{R}[\varphi]$. Note that $R(\varphi) \in \mathbb{N}_0 \cup \{\infty\}$. Finally, we define the *Reidemeister spectrum* to be $\text{Spec}_R(G) := \{R(\varphi) \mid \varphi \in \text{Aut}(G)\}$. We say that G has the R_∞ -property, also denoted as $G \in R_\infty$, if $\text{Spec}_R(G) = \{\infty\}$. We say that G has *full Reidemeister spectrum* if $\text{Spec}_R(G) = \mathbb{N}_0 \cup \{\infty\}$.

The notion of Reidemeister number arises from Nielsen fixed-point theory, where its topological analog serves as a count of the number of fixed point classes of a continuous self-map, and is strongly related to the algebraic one defined above, see [15].

It has been proven for several (classes of) groups that they possess the R_∞ -property, e.g. Baumslag-Solitar groups [7] and their generalisations [19], extensions of $\text{SL}(n, \mathbb{Z})$ and $\text{GL}(n, \mathbb{Z})$ by a countable abelian group [20], and Thompson's group [1]. We refer the reader to [8] for a more exhaustive list of groups having the R_∞ -property.

In this article, we study the Reidemeister spectrum of right-angled Artin groups, RAAGs for short. Given a graph Γ with the set of vertices V , the RAAG associated to it is the group

$$A_\Gamma = \langle V \mid [v, w] \text{ if } v, w \in V \text{ are joined by an edge in } \Gamma \rangle.$$

Extreme cases of RAAGs include free groups and free abelian groups, coming from edgeless and complete graphs, respectively. From [6, Theorem 3] (see also [4]), it readily follows that all non-abelian free groups of finite rank have the R_∞ -property. On the other hand, $\text{Spec}_R(\mathbb{Z}) = \{2, \infty\}$ and $\text{Spec}_R(\mathbb{Z}^n) = \mathbb{N}_0 \cup \{\infty\}$ for $n \geq 2$ (see e.g. [22]). For groups closely related to right-angled Artin groups, several results regarding the R_∞ -property have been obtained, e.g. A. Juhász has proven that certain Artin groups which are not right-angled Artin groups possess the R_∞ -property [16] and recently, T. K. Naik, N. Nanda and M. Singh

¹Research supported by long term structural funding – Methusalem grant of the Flemish Government.

²Researcher funded by FWO PhD-fellowship fundamental research (file number: 1112520N).

have showed that twin groups, a subfamily of the right-angled Coxeter groups, all possess the R_∞ -property [21].

We suspect that, amongst all RAAGs, only the free abelian ones do not possess the R_∞ -property:

Conjecture. *Let $\Gamma(V, E)$ be a finite non-complete graph, i.e. V is finite and there are two (distinct) vertices not joined by an edge. Then A_Γ has the R_∞ -property.*

We first reduce the conjecture to graphs belonging to three specific classes, after which we prove the conjecture for one of these classes and for several subclasses of the other two.

We start by recalling two ways of proving that an automorphism has infinite Reidemeister number.

Definition 1.1.1. Let G, H be two groups. If $H \cong G/N$ for some characteristic subgroup N of G , we call H a *characteristic quotient* of G .

The following result is well-known, see e.g. [20, Lemma 2.1].

Lemma 1.1.2. *Let G, H be two groups. If H is a characteristic quotient of G and $H \in R_\infty$, then $G \in R_\infty$.*

Reidemeister numbers also behave nicely under conjugation. For elements a, b of a group G , we put $a^b := b^{-1}ab$.

Proposition 1.1.3. *Let G be a group and $\varphi, \psi \in \text{Aut}(G)$. Then $R(\varphi) = R(\varphi^\psi)$.*

Proof. Define $\rho : \mathcal{R}[\varphi] \rightarrow \mathcal{R}[\varphi^\psi] : [g]_\varphi \rightarrow [\psi^{-1}(g)]_{\varphi^\psi}$. As ψ^{-1} is surjective, so is ρ , and

$$\begin{aligned} x \sim_\varphi y &\iff \exists g \in G : x = gy\varphi(g)^{-1} \\ &\iff \exists g \in G : \psi^{-1}(x) = \psi^{-1}(g)\psi^{-1}(y)\psi^{-1}(\varphi(\psi(\psi^{-1}(g))))^{-1} \\ &\iff \exists g \in G : \psi^{-1}(x) = \psi^{-1}(g)\psi^{-1}(y)\varphi^\psi(\psi^{-1}(g))^{-1} \\ &\iff \psi^{-1}(x) \sim_{\varphi^\psi} \psi^{-1}(y), \end{aligned}$$

which shows well-definedness and injectivity of ρ . □

Next, we recall the definition and some properties of the lower central series of a group.

Definition 1.1.4. Let G be a group. The *lower central series* of G is defined as follows: put $\gamma_1(G) = G$ and inductively define $\gamma_{i+1}(G) = [\gamma_i(G), G]$ for $i \geq 1$.

Each term in the lower central series is a characteristic subgroup, the quotients $\gamma_i(G)/\gamma_{i+1}(G)$ are all abelian and we will refer to them as the *factors of the lower central series*. We can put these together to form the associated Lie ring of G :

Definition 1.1.5. Let G be a group. The *Lie ring associated to G* is the abelian group

$$L(G) := \bigoplus_{i=1}^{\infty} L_i(G), \quad \text{where } L_i(G) = \frac{\gamma_i(G)}{\gamma_{i+1}(G)},$$

and $L(G)$ is equipped with the following Lie bracket $[\cdot, \cdot]_L$: for $g \in \gamma_i(G)$ and $h \in \gamma_j(G)$, we define

$$[g\gamma_{i+1}(G), h\gamma_{j+1}(G)]_L := [g, h]\gamma_{i+j+1}(G),$$

and extend it by linearity to the whole of $L(G)$. Here, $[g, h]$ is the usual commutator bracket in G , i.e. $[g, h] = g^{-1}h^{-1}gh$.

Remark. For i -fold commutators, we work with left-normed commutators, i.e. $[x_1, x_2, x_3] := [[x_1, x_2], x_3]$ and inductively, $[x_1, x_2, \dots, x_n] := [[x_1, \dots, x_{n-1}], x_n]$.

Remark. We write cosets multiplicatively, i.e. $g\gamma_i(G)$ and operations with cosets additively, i.e. $g\gamma_i(G) + h\gamma_i(G)$.

Each automorphism φ of G induces an automorphism φ_* of $L(G)$ and if, moreover, each factor of the lower central series is finitely generated and torsion-free, we can talk about the eigenvalues of φ_* : these are simply the eigenvalues of each φ_i . The following result is then immediate from [4, Lemma 2.2]

Theorem 1.1.6. *Suppose G is a finitely generated group such that all factors of the lower central series are torsion-free. Let $\varphi \in \text{Aut}(G)$ be an automorphism. Denote by φ_* the induced automorphism on $L(G)$ and by φ_i the restriction of φ_* to $L_i(G)$.*

If φ_i has eigenvalue 1 for some i , then $R(\varphi) = \infty$.

2 Right-angled Artin groups

In this section, we briefly recall the necessary definitions and results regarding graphs and right-angled Artin groups, including isomorphisms between two related RAAGs and a generating set for the automorphism group of a RAAG.

2.1 Definitions and examples

By a *graph*, we mean a finite simple non-empty graph $\Gamma(V, E)$ with vertices V and edges E , although for technical reasons, we will sometimes mention the empty graph.

Definition 2.1.1. Let $\Gamma(V, E)$ be a graph. The *right-angled Artin group* A_Γ (or *RAAG*) associated to Γ is defined by the presentation

$$\langle V \mid [v, w], vw \in E \rangle.$$

The graph Γ is called the *defining graph* of A_Γ .

From the definition of a RAAG, it is clear that the complete graph K^n (i.e. every two vertices are connected by an edge) on n vertices corresponds to the free abelian group $\mathbb{Z}^n$ and that the edgeless graph $\overline{K}^n$ on n vertices corresponds to the free group F_n .

There are two operations on the level of graphs that correspond to natural operations on the level of groups and that will play an important role.

Definition 2.1.2. Let $\Gamma_i(V_i, E_i)$, $i = 1, 2$, be two graphs. The *disjoint union* of Γ_1 and Γ_2 is the graph

$$\Gamma_1 \sqcup \Gamma_2(V_1 \sqcup V_2, E_1 \sqcup E_2)$$

and the *simplicial join* of Γ_1 and Γ_2 is the graph $\Gamma_1 * \Gamma_2$ with vertices $V_1 \sqcup V_2$ and edges

$$E_{\Gamma_1 * \Gamma_2} = E_1 \sqcup E_2 \cup \{v_1 v_2 \mid v_i \in V_i\}.$$

We write $\sqcup_n \Gamma$ and $*_n \Gamma$ for the n -fold disjoint union, respectively, simplicial join of Γ with itself.

The following result follows readily from the definitions of a RAAG, direct product and free product.

Proposition 2.1.3. *Let Γ_1, Γ_2 be two graphs. Then $A_{\Gamma_1 \sqcup \Gamma_2} \cong A_{\Gamma_1} * A_{\Gamma_2}$ and $A_{\Gamma_1 * \Gamma_2} \cong A_{\Gamma_1} \times A_{\Gamma_2}$.*

2.2 General isomorphisms of RAAGs

In view of Lemma 1.1.2, it can be useful to transform one RAAG into another one by either deleting vertices or adding edges, and determining when this quotient is in fact characteristic. In this section, we make the first two claims more precise, in the next one, we discuss the characteristic quotients.

Definition 2.2.1. Let $\Gamma(V, E)$ be a graph. We say that a subgraph $\Gamma'(V', E') \subseteq \Gamma$ is a *full subgraph* or *induced subgraph*, if E' is given by $\{vw \in E \mid v, w \in V'\}$.

Similarly, for $V' \subseteq V$, the *subgraph induced on V'* is the graph $\Gamma'(V', E')$ where $E' = \{vw \in E \mid v, w \in V'\}$. We write $\Gamma(V')$.

The RAAG associated to an induced subgraph $\Gamma(V')$ can be viewed as a subgroup of A_Γ in a natural way.

Lemma 2.2.2 ([5, Proposition 3.1]). *Let Γ be a graph and $\Gamma' := \Gamma(V')$ an induced subgraph. The map*

$$i : A_{\Gamma'} \rightarrow A_\Gamma : v' \mapsto v', \quad \text{for } v' \in V'$$

is well-defined and injective.

We now formulate precisely how we can ‘delete vertices’. Given a subset S of a group G , we denote the *normal closure of S in G* by $\langle\langle S \rangle\rangle_G$ or simply $\langle\langle S \rangle\rangle$ if G is clear from the context.

Proposition 2.2.3. *Let Γ be a graph, Γ_1 an induced subgraph and view A_{Γ_1} as a subgroup of A_Γ . Let Γ_2 be the induced subgraph on $V_2 = V \setminus V_1$. We also write $\Gamma_2 = \Gamma \setminus \Gamma_1$. Define*

$$\varphi : A_\Gamma \rightarrow A_{\Gamma_2} : v \mapsto \begin{cases} 1 & \text{if } v \in V_1 \\ v & \text{otherwise} \end{cases}.$$

Then φ is a well-defined homomorphism, $\langle\langle A_{\Gamma_1} \rangle\rangle_{A_\Gamma} = \ker \varphi$ and

$$\frac{A_\Gamma}{\langle\langle A_{\Gamma_1} \rangle\rangle_{A_\Gamma}} \cong A_{\Gamma_2}.$$

For a proof, we refer the reader to Fontelles master thesis [9, Proposition 2.1] or the master thesis [23, Proposition 3.2.6] of one of the authors. To make notations less heavy, we introduce the following definition:

Definition 2.2.4. Let $\Gamma(V, E)$ be a graph and A_Γ its associated RAAG. For a subset $V' \subseteq V$, the normal subgroup $\langle\langle A_{\Gamma(V')} \rangle\rangle$ is called the *normal subgroup generated by V'* and we denote it by $N(V')$. Any subgroup $N(V')$ is called a *normal vertex-subgroup*.

As we will never consider the subgroups $A_{\Gamma(V')}$ but only their normal closures, we simply refer to $N(V')$ as *vertex-subgroups*.

Note that $N(V_1)N(V_2) = N(V_1 \cup V_2)$ for all $V_1, V_2 \subseteq V$. Indeed, both $N(V_1)$ and $N(V_2)$ lie in $N(V_1 \cup V_2)$, so their product does too. Conversely, it is clear that $A_{\Gamma(V_1 \cup V_2)}$ lies in $N(V_1)N(V_2)$, and as this product group is normal, we have that $N(V_1 \cup V_2) \leq N(V_1)N(V_2)$.

The second isomorphism result for RAAGs tells us how we can ‘add edges’.

Lemma 2.2.5. Let G, H be groups, $S \subseteq G$ a subset and $\varphi : G \rightarrow H$ a surjective homomorphism. Then $\varphi(\langle\langle S \rangle\rangle) = \langle\langle \varphi(S) \rangle\rangle$.

Proposition 2.2.6. Let $\Gamma_1 \subseteq \Gamma_2$ be two graphs on n vertices. Denote by R_i the relators in the presentation of A_{Γ_i} and put $N = \langle\langle R_2 \rangle\rangle_{A_{\Gamma_1}}$. Then

$$\frac{A_{\Gamma_1}}{N} \cong A_{\Gamma_2}.$$

Proof. Recall that $A_{\Gamma_1} \cong \frac{F_n}{\langle\langle R_1 \rangle\rangle_{F_n}}$. Let $\pi : F_n \rightarrow A_{\Gamma_1}$ be the natural projection and put $\tilde{N} = \pi^{-1}(N)$. We claim that $\tilde{N} = \langle\langle R_2 \rangle\rangle_{F_n}$. As $\langle\langle R_1 \rangle\rangle_{F_n}$ lies in both subgroups, it is sufficient to prove that $\pi(\tilde{N}) = N$, by the correspondence theorem. This follows from Lemma 2.2.5, as π is surjective. Hence, by the third isomorphism theorem, we have that

$$\frac{A_{\Gamma_1}}{N} = \frac{F_n / \langle\langle R_1 \rangle\rangle_{F_n}}{\langle\langle R_2 \rangle\rangle_{F_n} / \langle\langle R_1 \rangle\rangle_{F_n}} \cong \frac{F_n}{\langle\langle R_2 \rangle\rangle_{F_n}} \cong A_{\Gamma_2}. \quad \square$$

Corollary 2.2.7. Let Γ be a graph on n vertices and A_Γ its associated RAAG. Then $A_\Gamma / [A_\Gamma, A_\Gamma]$ is a free abelian group of rank n .

2.3 Generating set for $\text{Aut}(A_\Gamma)$

Naturally, to determine whether or not a RAAG has the R_∞ -property, we need to understand its automorphism group.

Definition 2.3.1. Let $\Gamma(V, E)$ be a graph and $v \in V$. A vertex w is called a *neighbour* of v , or *adjacent* to v , if $vw \in E$.

We then define the *link* of v as the set of all vertices adjacent to v , and it is denoted by $lk(v)$. The *star* of v is $lk(v) \cup \{v\}$, and it is denoted by $st(v)$.

If $v \neq w$ are vertices, we say that w *dominates* v if $lk(v) \subseteq st(w)$, and write $v \leq w$.

There are four basic types of automorphisms: *graph automorphisms*, *inversions*, *transvections* and *partial conjugations*.

- Graph automorphisms of Γ extend to automorphisms of A_Γ .
- Inversions ι_a send one generator a to its inverse and leave all other generators fixed.
- Transvections are maps τ_{ab} sending a generator a to ab and leaving all other generators fixed, where b is another (different) generator with $a \leq b$.

- Partial conjugations are maps $\gamma_{b,C}$ where b is a generator and C is a union of connected components of $\Gamma \setminus \Gamma(st(b))$, sending every generator a in C to a^b and leaving the other generators fixed.

We will refer to these as *automorphisms of basic type*. M. Laurence and H. Servatius have shown that the automorphisms of basic type generate $\text{Aut}(A_\Gamma)$:

Theorem 2.3.2 ([18, 24]). *For every graph Γ , $\text{Aut}(A_\Gamma)$ is generated by its graph automorphisms, inversions, transvections and partial conjugations.*

Lemma 2.3.3. *Let Γ be a graph and Γ_1 an induced subgraph. Let $\varphi \in \text{Aut}(A_\Gamma)$ be a composition of inversions and partial conjugations. Then $\varphi(\langle\langle A_{\Gamma_1} \rangle\rangle) \leq \langle\langle A_{\Gamma_1} \rangle\rangle$.*

Proof. Since $\varphi(N) \leq N$ and $\psi(N) \leq N$ together imply $(\varphi \circ \psi)(N) \leq N$ for any group G with normal subgroup N and automorphisms φ and ψ , it suffices to prove the statement for φ equal to an inversion or partial conjugation. If φ is an inversion, then $\varphi(A_{\Gamma_1}) = A_{\Gamma_1}$, hence $\varphi(\langle\langle A_{\Gamma_1} \rangle\rangle) \leq \langle\langle A_{\Gamma_1} \rangle\rangle$. If φ is a partial conjugation, then $\varphi(A_{\Gamma_1}) \subseteq \langle\langle A_{\Gamma_1} \rangle\rangle$ and thus $\varphi(\langle\langle A_{\Gamma_1} \rangle\rangle) \leq \langle\langle A_{\Gamma_1} \rangle\rangle$. $\square$

So, whenever we need to prove that $\langle\langle A_{\Gamma_1} \rangle\rangle$ is characteristic in A_Γ , we only need to prove it is preserved under graph automorphisms and transvections.

3 Characteristic vertex-subgroups

We begin the section by classifying all subsets of vertices that induce characteristic vertex-subgroups. After that, we treat two special cases, the vertices of maximal degree and the so-called transvection-free vertices, each of which implies a strong result regarding the R_∞ -property of RAAGs.

3.1 Classification of characteristic vertex-subgroups

For a graph $\Gamma(V, E)$ and a vertex $v \in V$, start with the set $V_0(v)$ containing only v . Add to $V_0(v)$ all vertices w such that τ_{vw} is a well-defined transvection, to obtain $V_1(v)$. In symbols,

$$V_1(v) = \{w \in V \mid v \leq w\}.$$

Inductively, given $V_i(v)$, put

$$V_{i+1}(v) = \{w \in V \mid \exists v' \in V_i(v) : v' \leq w\}.$$

Define $V_\omega(v) = \bigcup_{i \in \mathbb{N}} V_i(v)$. Finally, put

$$V_{\text{char}}(v) = \bigcup_{\varphi \in \text{Aut}(\Gamma)} \varphi(V_\omega(v)).$$

Definition 3.1.1. The set $V_{\text{char}}(v)$ defined above is called the *characteristic closure of v in V* .

With this terminology, we then have the following result (which also explains the name *characteristic closure*):

Theorem 3.1.2. *Let Γ be a graph and A_Γ its associated RAAG. For vertices $v_1, \dots, v_n \in V$, we have that*

$$N\left(\bigcup_{i=1}^n V_{\text{char}}(v_i)\right)$$

is characteristic in A_Γ .

Conversely, if $N(V')$ is a characteristic vertex-subgroup for some $V' \subseteq V$, then V' is a union of characteristic closures.

Proof. As

$$N\left(\bigcup_{i=1}^n V_{\text{char}}(v_i)\right) = N(V_{\text{char}}(v_1)) \dots N(V_{\text{char}}(v_n))$$

and the product of characteristic subgroups is characteristic, it is sufficient to prove the theorem for $n = 1$. For simplicity, we write V_i and V_ω for $V_i(v)$ and $V_\omega(v)$ as above.

Let $\varphi \in \text{Aut}(A_\Gamma)$ be of basic type. By Lemma 2.3.3, we only need to consider the case where φ is a graph automorphism or a transvection. If φ is a graph automorphism, then $\varphi(w) \in V_{\text{char}}(v)$ for all $w \in V_{\text{char}}(v)$, by construction, hence $\varphi(N(V_{\text{char}}(v))) \leq N(V_{\text{char}}(v))$. Similarly, if $\varphi = \tau_{v'w}$ is a transvection with $v' \in V_{\text{char}}(v)$, then $v' = \psi(v'')$ for some $\psi \in \text{Aut}(\Gamma)$ and $v'' \in V_\omega$. Note that $v'' \in V_i$ for some $i \in \mathbb{N}$. As $lk(v') \subseteq st(w)$, it follows that $lk(v'') \subseteq st(\psi^{-1}(w))$, hence $\psi^{-1}(w) \in V_{i+1} \subseteq V_\omega$, and thus $w \in \psi(V_\omega) \subseteq V_{\text{char}}(v)$. It follows that $\varphi(N(V_{\text{char}}(v))) \leq N(V_{\text{char}}(v))$.

Conversely, suppose $N(V')$ is characteristic in A_Γ for some $V' \subseteq V$. For each $v \in V'$, we will prove that $V_{\text{char}}(v) \subseteq V'$; the result then follows from the equality

$$V' = \bigcup_{v \in V'} V_{\text{char}}(v).$$

First, we prove by induction on i that for all $i \in \mathbb{N}$, V_i as defined above is a subset of V' . Clearly, $V_0 \subseteq V'$. So suppose $V_i \subseteq V'$. Let $w \in V_{i+1}$ be arbitrary. Then there is a $v' \in V_i$ such that $v' \leq w$. Then $\varphi = \tau_{v'w}$ is a well-defined transvection, and as $N(V')$ is characteristic, $\varphi(N(V')) \leq N(V')$. In particular, $v'w \in N(V')$ and as $v' \in V'$ by induction hypothesis, $w \in V'$. Consequently, $V_{i+1} \subseteq V'$.

From this, it follows that $V_\omega = \bigcup_{i \in \mathbb{N}} V_i \subseteq V'$. Now, let $\varphi \in \text{Aut}(\Gamma)$ and $v' \in V_\omega$. As $\varphi(N(V')) \leq N(V')$, we see that $\varphi(v') \in V'$. Hence, $\varphi(V_\omega) \subseteq V'$ for all $\varphi \in \text{Aut}(\Gamma)$. We conclude that

$$V_{\text{char}}(v) = \bigcup_{\varphi \in \text{Aut}(\Gamma)} \varphi(V_\omega) \subseteq V'. \quad \square$$

3.2 Vertices of maximal degree

In this section and the following, we will treat two particular instances of characteristic vertex-subgroups and each of them will have an important consequence in establishing the R_∞ -property of RAAGs. We begin by recalling the definition of the degree of a vertex.

Definition 3.2.1. Let $\Gamma(V, E)$ be a graph and $v \in V$ a vertex. The *degree* of v , denoted by $\deg(v)$, is the number of adjacent vertices of v . The *maximal degree* of Γ is

$$\Delta(\Gamma) := \max\{\deg(v) \mid v \in V\}.$$

Lemma 3.2.2. *Let Γ be a graph and $v \in V$. Then for all $w \in V_{\text{char}}(v)$, $\deg(w) \geq \deg(v)$.*

Proof. First, we prove by induction that for all $i \in \mathbb{N}$ and $w \in V_i(v)$, we have $\deg(w) \geq \deg(v)$. Again, put $V_i = V_i(v)$. It clearly holds for $i = 0$. Suppose it holds for i . Let $w \in V_{i+1}$. Then there is a $v' \in V_i$ such that $lk(v') \subseteq st(w)$. From this inclusion, it readily follows that $\deg(v') \leq \deg(w)$. As $\deg(v') \geq \deg(v)$ by induction hypothesis, we find that $\deg(w) \geq \deg(v)$.

By taking the union over all V_i , it follows that $\deg(w) \geq \deg(v)$ for all $w \in V_\omega$. To show that it holds for all $w \in V_{\text{char}}(v)$, recall that

$$V_{\text{char}}(v) = \bigcup_{\varphi \in \text{Aut}(\Gamma)} \varphi(V_\omega)$$

and note that every automorphism preserves the degree of a vertex, so if $w = \varphi(w')$ for some $\varphi \in \text{Aut}(\Gamma)$ and $w' \in V_\omega$, then $\deg(w) = \deg(w') \geq \deg(v)$. $\square$

Corollary 3.2.3. *Let Γ be a graph. Then $N(V_{\text{max}})$ is characteristic in A_Γ , where*

$$V_{\text{max}} = \{v \in V \mid \deg(v) = \Delta(\Gamma)\}.$$

Moreover, $A_\Gamma / N(V_{\text{max}}) \cong A_{\Gamma \setminus \Gamma(V_{\text{max}})}$.

Proof. The first claim follows from Theorem 3.1.2 together with the equality

$$V_{\text{max}} = \bigcup_{v \in V_{\text{max}}} V_{\text{char}}(v),$$

which holds by the previous lemma. The isomorphism is a direct application of Proposition 2.2.3. $\square$

The true power of the previous corollary lies in the fact that for ‘almost all’ graphs, V_{max} will be non-trivial. Indeed, as each graph considered is assumed to be finite, $V_{\text{max}} \neq \emptyset$, and $V_{\text{max}} = V$ if and only if Γ is *regular*, i.e. all vertices have the same degree. Therefore, every RAAG associated to a non-regular graph has a non-trivial characteristic vertex-subgroup, which provides a powerful tool in answering the conjecture. We elaborate further on this.

The next lemma is straightforward.

Lemma 3.2.4. *Let G be a group. If G_1 is a characteristic quotient of G and G_2 one of G_1 , then G_2 is a characteristic quotient of G .*

Lemma 3.2.5 (Simplification Lemma). *Let Γ be a non-complete graph. Then A_Γ has a RAAG $A_{\Gamma'}$ as a characteristic quotient where Γ' is either*

- *disconnected;*
- *connected, regular and non-complete;*
- *non-regular, connected and such that $\Gamma'(V \setminus V_{\text{max}})$ is complete.*

Proof. Put $\Gamma_0 = \Gamma$ and define inductively $\Gamma_{i+1} = \Gamma_i(V_i \setminus V_{i,\text{max}})$. Then each $A_{\Gamma_{i+1}}$ is a characteristic quotient of A_{Γ_i} , and by Lemma 3.2.4, also a characteristic quotient of A_Γ .

Since Γ is finite and for each i we have $|V_{i+1}| < |V_i|$, there is an m such that Γ_{m+1} is the empty graph. This implies that Γ_m is regular. If Γ_m is disconnected, or connected and non-complete, we are done, so suppose Γ_m is complete. Then Γ_{m-1} cannot be regular and $\Gamma_{m-1}(V_{m-1} \setminus V_{m-1,\text{max}})$ is complete. If Γ_{m-1} is disconnected, we are in the first case, otherwise we are in the third case. $\square$

We give a name to the third type of graph arising in the Simplification Lemma.

Definition 3.2.6. Let $\Gamma(V, E)$ be a connected non-regular graph. If the induced subgraph $\Gamma(V \setminus V_{\max})$ is complete, we call Γ a *(maximal degree)-by-(free abelian) graph*. We also say that Γ is *max-by-abelian*.

Remark. The inspiration for this terminology comes from group theory, where a group G is called $\mathcal{P}$ -by- $\mathcal{Q}$ if G has a normal subgroup N having property $\mathcal{P}$ such that G/N has property $\mathcal{Q}$. Here, if Γ is max-by-abelian, then the normal subgroup $N(V_{\max})$ of A_Γ is generated by the vertices of *maximal degree*, and the quotient $A_\Gamma/N(V_{\max})$ is a *free abelian* group.

The Simplification Lemma thus states that we only need to consider three types of graphs to establish the R_∞ -property for all non-abelian RAAGs. By demanding max-by-abelian graphs to be non-regular, there is no overlap between these three types of graphs. Note also that these types of graphs all have more structure than arbitrary graphs: although disconnectedness is not a very strong property from a graph theoretical point of view, it will be enough to ensure the R_∞ -property for the associated RAAG. For the second type, we have regularity, whereas the structure of max-by-abelian graphs is less obvious and less simple than that of the other two, as we will see later.

The Simplification Lemma proves the existence of at least one characteristic quotient of a particular form. It is of course possible for a RAAG to have multiple characteristic quotients, where for instance one quotient has a disconnected graph as defining graph and the other one a max-by-abelian graph.

3.3 Transvection-free vertices

Whereas $V_{\max}$ provided us with the Simplification Lemma, the set of all so-called transvection-free vertices will be in some sense more constructive towards establishing the R_∞ -property for RAAGs: if the aforementioned set contains all vertices, then A_Γ has the R_∞ -property.

3.3.1 Definition and statement of the main theorem

Let Γ be a graph and $v \in V$ a vertex. In the construction of $V_{\text{char}}(v)$, we started with looking at which vertices $w \in V$ induce a well-defined transvection τ_{vw} on A_Γ . We consider now the vertices for which no such w exist.

Definition 3.3.1. Let $\Gamma(V, E)$ be a graph and $v \in V$ a vertex. We call v *transvection-free* if the set $\{w \in V \mid v \leq w\}$ only contains v . The set of all transvection-free vertices is denoted by $V_{\overline{\tau}}$.

Remark. As $lk(v) \subseteq st(v)$, this definition makes sense.

The following result is quite straightforward.

Proposition 3.3.2. *For every graph Γ , the subgroup $N(V_{\overline{\tau}})$ is characteristic in A_Γ .*

Proof. Let $v \in V$ be transvection-free. It immediately follows that $V_\omega(v) = \{v\}$, thus

$$V_{\text{char}}(v) = \{\varphi(v) \mid \varphi \in \text{Aut}(\Gamma)\}.$$

If $\varphi \in \text{Aut}(\Gamma)$ and $lk(\varphi(v)) \subseteq st(w)$, then $lk(v) \subseteq st(\varphi^{-1}(w))$, hence if v is transvection-free, so is $\varphi(v)$. Consequently,

$$V_{\overline{\tau}} = \bigcup_{v \in V_{\overline{\tau}}} V_{\text{char}}(v)$$

and we conclude by Theorem 3.1.2. $\square$

Although the previous proposition provides us with a characteristic subgroup of A_Γ differing (in general) from $N(V_{\max})$, the most interesting situation occurs when $V_{\overline{\tau}} = V$. In that case, no generator of A_Γ admits transvections, so the group $\text{Aut}(A_\Gamma)$ will be significantly smaller. In fact, it turns out that in that case, A_Γ has the R_∞ -property, but even more is true:

Theorem 3.3.3. *Let Γ be a graph and denote by $\text{Aut}_{\overline{\tau}}(A_\Gamma)$ the subgroup of $\text{Aut}(A_\Gamma)$ generated by all graph automorphisms, inversions and partial conjugations. If Γ is not complete, then all $\varphi \in \text{Aut}_{\overline{\tau}}(A_\Gamma)$ have infinite Reidemeister number, i.e. $R(\varphi) = \infty$ for all $\varphi \in \text{Aut}_{\overline{\tau}}(A_\Gamma)$.*

To prove this theorem, we need to understand the quotient groups $\gamma_i(A_\Gamma)/\gamma_{i+1}(A_\Gamma)$ for $i = 1, 2, 3$. In order to do so, we study the associated Lie ring of a RAAG.

3.3.2 Lyndon elements

Lyndon elements can be used to show that the factors of the lower central series of a RAAG are free abelian groups, and they even provide us with a basis of these factors. This section is based on [25]. Throughout this section, $\Gamma(V, E)$ is a graph, $V = \{v_1, \dots, v_n\}$ and A_Γ is its associated RAAG.

Define $W(V)$ to be the set of all words in the letters $v_1, \dots, v_n$. The trivial word is denoted by 1. The length of a word $w \in W(V)$ is denoted by $\text{len}(w)$. For $w, w' \in W(V)$, we write $w \leftrightarrow w'$ if there are words $w_1, w_2 \in W(V)$ and vertices $v, v' \in V$ such that $vv' \in E$ and

$$w = w_1vv'w_2 \quad \text{and} \quad w' = w_1v'vw_2.$$

We then define an equivalence relation on $W(V)$: we write $w \sim w'$ if there are words $w_1, \dots, w_k \in W(V)$ such that

$$w = w_1 \leftrightarrow w_2 \leftrightarrow \dots \leftrightarrow w_k = w'.$$

It is clear that if $w \sim w'$ and $\tilde{w} \sim \tilde{w}'$, then $w\tilde{w} \sim w'\tilde{w}'$. This allows us to define a multiplication on the set $M := W(V)/\sim$. Also, the length of a word is preserved under the relation ' $\leftrightarrow$ ', hence under the equivalence relation ' $\sim$ '. Thus, we can define the length of an element $m \in M$, which we still denote by $\text{len}(m)$.

We put a total order on $W(V)$ as follows: for all $w \in W(V) \setminus \{1\}$, $1 < w$. Then, for $w \neq w' \in W(V) \setminus \{1\}$, write $w = v_iw_1$ and $w' = v_jw_2$ for $v_i, v_j \in V$ and $w_1, w_2 \in W(V)$. We put $w < w'$ if either $i < j$, or $i = j$ and $w_1 < w_2$.

We now proceed to define *Lyndon elements* in M . Denote by $\pi : W(V) \rightarrow M$ the projection map linked to the equivalence relation defining M .

Definition 3.3.4. For $m \in M$, we define the *standard representative of m in $W(V)$* to be the largest element of $\pi^{-1}(m)$ with respect to the total order. It is denoted by $\text{std}(m)$.

Note that this is indeed well-defined: every element in $\pi^{-1}(m)$ has the same length, and as the alphabet V is finite, so is the set of words of a fixed length. It follows that $\pi^{-1}(m)$ is finite, and as the order on $W(V)$ is total, this set has a largest element.

The standard representative of an element of M allows us to define an order on M : for $m \neq m' \in M$, we put $m < m'$ if and only if $\text{std}(m) < \text{std}(m')$. From the totality of the order on $W(V)$, it follows that this order on M is also total.

Definition 3.3.5. Let $m, m' \in M$. We say that m and m' are *transposed* if there exist $x, y \in M$ such that $m = xy$ and $m' = yx$.

In M , being transposed is not an equivalence relation: if we take Γ to be the graph with $V = \{v_1, v_2, v_3\}$ and $E = \{v_2v_3\}$, then putting $m_1 = v_2v_1v_3$, $m_2 = v_1v_3v_2 = v_1v_2v_3$ and $m_3 = v_3v_1v_2$, we find that m_1 and m_2 are transposed, as are m_2 and m_3 , but m_1 cannot be transformed into m_3 using only one transposition.

To circumvent this, we do a similar trick as in the definition of M .

Definition 3.3.6. Let $m, m' \in M$. We say that m and m' are *conjugate* if there exist $m_1, \dots, m_k$ such that $m = m_1$, $m' = m_k$, and m_i and m_{i+1} are transposed for $1 \leq i \leq k-1$.

It is clear that this does define an equivalence relation on M .

Definition 3.3.7. An element $m \in M$ is called *primitive* if for every $x, y \in M$, the equality $m = xy = yx$ implies that $x = 1$ or $y = 1$.

An element $m \in M$ is called a *Lyndon element* if it is non-trivial, primitive and minimal in its conjugacy class w.r.t. the order $<$.

Definition 3.3.8. Let $m \in M$. Then $\text{init}(m)$ is the set of all vertices in V that can occur as the initial letter of a word in $\pi^{-1}(m)$.

Lemma 3.3.9 ([17, Corollary 3.2]). *If $m \in M$ is a Lyndon element, then $\text{init}(m)$ consists of a single vertex.*

For a Lyndon element m , we consider $\text{init}(m)$ to be the single vertex it contains rather than the singleton.

Definition 3.3.10. For $m \in M$, we define $\zeta(m)$ to be

$$\text{supp}(m) \cup \{v_j \mid \exists v_i \in \text{supp}(m) : [v_i, v_j] \neq 1 \text{ in } A_\Gamma\}.$$

Remark. In the rest of this section, when writing $[v_i, v_j] \neq 1$, we mean $[v_i, v_j] \neq 1$ in A_Γ .

Checking if a given element is a Lyndon element by means of the definition is quite cumbersome. Luckily, D. Krob and P. Lalonde [17] proved the following equivalence.

Theorem 3.3.11 ([17, Propositions 3.5 and 3.7]). *For $m \in M$, the following are equivalent:*

- (i) *m is a Lyndon element.*
- (ii) *For all $x, y \in M \setminus \{1\}$ such that $m = xy$, we have $m < y$.*
- (iii) *Either $\text{len}(m) = 1$ or there exists Lyndon elements x, y with $x < y$ and $\text{init}(y) \in \zeta(x)$ such that $m = xy$.*

We finish by determining the Lyndon elements of length at most 3.

Proposition 3.3.12. *The sets*

$$\begin{aligned}
LE_1 &= \{v_i \mid 1 \leq i \leq n\} \\
LE_2 &= \{v_i v_j \mid 1 \leq i < j \leq n, [v_i, v_j] \neq 1\} \\
LE_3 &= \{v_i v_j v_k \mid 1 \leq i < k \leq n, [v_i, v_k] \neq 1\} \\
&\quad \cup \{v_i v_j v_k \mid 1 \leq i < j < k \leq n, [v_i, v_j] \neq 1 \neq [v_i, v_k]\} \\
&\quad \cup \{v_i v_j v_j \mid 1 \leq i < j \leq n, [v_i, v_j] \neq 1\} \\
&\quad \cup \{v_i v_j v_k \mid 1 \leq i < j \neq k \leq n, i < k, [v_i, v_j] \neq 1, ([v_i, v_k] \neq 1 \text{ or } [v_j, v_k] \neq 1)\}.
\end{aligned}$$

are precisely the Lyndon elements of length 1, 2 and 3, respectively.

Proof. For LE_1 , this is clear. For LE_2 , suppose m is a Lyndon element of length 2. Then $m = xy$ with $x < y$ both Lyndon elements and $\text{init}(y) \in \zeta(x)$. Note that $\text{len}(x) = \text{len}(y) = 1$, hence $x = v_i$ and $y = v_j$ for some i, j . As $x < y$, we have that $i < j$. Moreover, as $\zeta(x) = \{v_i\} \cup \{v_k \in V \mid [v_i, v_k] \neq 1\}$, we find that $[v_i, v_j] \neq 1$. Hence, $m = v_i v_j \in LE_2$. A similar argument shows that every element in LE_2 is a Lyndon element.

Suppose now that m is a Lyndon element of length 3. Write $m = v_i v_j v_k$. We distinguish two cases.

Case 1: $v_i < v_j v_k$ are both Lyndon elements. Then $v_j v_k \in LE_2$, hence $[v_j, v_k] \neq 1$ and $j < k$. As $v_i < v_j v_k$ and $\text{std}(v_i) = v_i$, $\text{std}(v_j v_k) = v_j v_k$, we find that $i \leq j$. Also, $v_j = \text{init}(v_j v_k) \in \zeta(v_i) = \{v_i\} \cup \{v_l \mid [v_i, v_l] \neq 1\}$. Hence, either $i = j$, or $i < j$ and $[v_i, v_j] \neq 1$.

Case 2: $v_i v_j < v_k$ are both Lyndon elements. Then $v_i v_j \in LE_2$, hence $i < j$ and $[v_i, v_j] \neq 1$. Similarly as in the previous case, $v_i v_j < v_k$ implies that $i < k$. Now, we have that $\text{init}(v_k) = v_k$ and

$$\zeta(v_i v_j) = \{v_i, v_j\} \cup \{v_l \mid [v_i, v_l] \neq 1 \text{ or } [v_j, v_l] \neq 1\}.$$

It follows that either

- $j = k$, or
- $j \neq k$ and $[v_i, v_k] \neq 1$, or
- $j \neq k$ and $[v_j, v_k] \neq 1$.

Conversely, suppose that $m = v_i v_j v_k \in LE_3$. In order to show that m is a Lyndon element, we use the third criterion of Theorem 3.3.11 and case distinction.

Case 1: $i = j < k$, $[v_i, v_k] \neq 1$: Then both v_i and $v_i v_k$ are Lyndon elements, $v_i < v_i v_k$ and $\text{init}(v_i v_k) = v_i \in \zeta(v_i)$, hence $m = v_i v_i v_k$ is a Lyndon element as well.

Case 2: $i < j < k$, $[v_i, v_j] \neq 1 \neq [v_i, v_k]$: In this case, v_i and $v_j v_k$ are Lyndon elements, $v_i < v_j v_k$ and $\text{init}(v_j v_k) = v_j \in \zeta(v_i)$, as $[v_i, v_j] \neq 1$. Hence, $m = v_i v_j v_k$ is a Lyndon element as well.

Case 3: $i < j = k$, $[v_i, v_j] \neq 1$: As $[v_i, v_j] \neq 1$ and $i < j$, $v_i v_j$ is a Lyndon element, as is v_j . Since $\text{std}(v_i v_j) = v_i v_j$, it follows that $v_i v_j < v_j$ in M . Lastly, note that $\text{init}(v_j) = v_j \in \zeta(v_i v_j)$, so $m = v_i v_j v_j$ is a Lyndon element.

Case 4: $i < j \neq k > i, [v_i, v_j] \neq 1$ and $([v_i, v_k] \neq 1 \text{ or } [v_j, v_k] \neq 1)$: Put $x = v_i v_j$ and $y = v_k$. As $i < j$ and $[v_i, v_j] \neq 1$, x is a Lyndon element, and clearly, so is y . Note that $\text{init}(y) = v_k$ and $\zeta(x) = \{v_i, v_j\} \cup \{v_l \mid [v_i, v_l] \neq 1 \text{ or } [v_j, v_l] \neq 1\}$. As either $[v_i, v_k] \neq 1$ or $[v_j, v_k] \neq 1$, it follows that $\text{init}(y) \in \zeta(x)$ and therefore, m is a Lyndon element. $\square$

This explicit description of LE_2 and LE_3 will be particularly useful to determine linearly independent subsets of $\gamma_2(A_\Gamma)/\gamma_3(A_\Gamma)$ and $\gamma_3(A_\Gamma)/\gamma_4(A_\Gamma)$.

Let m be a Lyndon element of length at least 2. By Theorem 3.3.11, there exist Lyndon elements $x < y$ such that $m = xy$ and $\text{init}(y) \in \zeta(x)$. However, there can be multiple factorizations of this form. The factorization of m where y is minimal is called the *standard factorization* of m and we write $S(m) = (x, y)$.

Using this standard factorization, we can define a bracketing procedure on the Lyndon elements: let m be a Lyndon element. If $\text{len}(m) = 1$, then $[m] = m$. If $\text{len}(m) \geq 2$, write $(x, y) = S(m)$. Then the bracketing of m is defined as $[[x], [y]]$, where $[x], [y]$ is the bracketing of x and y , respectively. This bracketing can be interpreted in A_Γ as the commutator bracket.

R. Wade [25] then essentially proved the following:

Theorem 3.3.13. *Let Γ be a graph and A_Γ its associated RAAG. Then the factors of the lower central series of A_Γ are finitely generated and torsion-free. Moreover, after bracketing, the Lyndon elements of length i form a $\mathbb{Z}$ -basis of $L_i(A_\Gamma)$ for all i .*

First of all, this theorem proves that we may use Theorem 1.1.6 for RAAGs. Secondly, combining this theorem with Proposition 3.3.12, we find explicit bases of $L_2(A_\Gamma)$ and $L_3(A_\Gamma)$. For $L_2(A_\Gamma)$, this is immediate:

Proposition 3.3.14. *The set*

$$\{[v_i, v_j] \gamma_3(A_\Gamma) \mid 1 \leq i < j \leq n, [v_i, v_j] \neq 1\}$$

is a basis of $L_2(A_\Gamma)$.

Turning the Lyndon elements of length 3 into an explicit basis of $L_3(A_\Gamma)$ gives a quite long and ugly set. However, in the proof of Theorem 3.3.3, we only need a linearly independent subset of this basis.

Proposition 3.3.15. *Let $1 \leq i < j \leq n$ be indices such that $[v_i, v_j] \neq 1$. Then the bracketing of the Lyndon elements $v_i v_i v_j$ and $v_i v_j v_j$ is given by $[v_i, [v_i, v_j]]$ and $[[v_i, v_j], v_j]$, respectively.*

In particular, the set

$$\begin{aligned} &\{[v_i, v_j, v_i] \gamma_4(A_\Gamma) \mid 1 \leq i < j \leq n, [v_i, v_j] \neq 1\} \cup \\ &\{[v_i, v_j, v_j] \gamma_4(A_\Gamma) \mid 1 \leq i < j \leq n, [v_i, v_j] \neq 1\} \end{aligned}$$

forms a linearly independent subset of $L_3(A_\Gamma)$.

Proof. Proposition 3.3.12 implies that $v_i v_i v_j$ and $v_i v_j v_j$ are indeed Lyndon elements. Now, as neither $v_i v_i$ nor $v_j v_j$ is a Lyndon element, the standard factorizations are given by

$$S(v_i v_i v_j) = (v_i, v_i v_j), \quad S(v_i v_j v_j) = (v_i v_j, v_j),$$

hence the bracketing procedure gives $[v_i, [v_i, v_j]]$ and $[[v_i, v_j], v_j]$.

The claim regarding the linearly independent subset follows from Theorem 3.3.13 and the fact that

$$[v_i, [v_i, v_j]] \gamma_4(A_\Gamma) = -[v_i, v_j, v_i] \gamma_4(A_\Gamma). \quad \square$$

3.3.3 Proof of Theorem 3.3.3

We now have all the background needed to give a proof of Theorem 3.3.3, which we restate here.

Theorem 3.3.3. *Let Γ be a graph and denote by $\text{Aut}_{\overline{\tau}}(A_\Gamma)$ the subgroup of $\text{Aut}(A_\Gamma)$ generated by all graph automorphisms, inversions and partial conjugations. If Γ is not complete, then all $\varphi \in \text{Aut}_{\overline{\tau}}(A_\Gamma)$ have infinite Reidemeister number, i.e. $R(\varphi) = \infty$ for all $\varphi \in \text{Aut}_{\overline{\tau}}(A_\Gamma)$.*

The main idea of the proof is to use Theorem 1.1.6, i.e. find for a given automorphism $\varphi \in \text{Aut}_{\overline{\tau}}(A_\Gamma)$, an index i such that the induced automorphisms φ_i on $L_i(A_\Gamma)$ has eigenvalue 1. More precisely, the proof consists of the following steps:

- Step 1. We argue that it is sufficient to only consider the automorphisms in $\text{Aut}_{\overline{\tau}}(A_\Gamma)$ generated by graph automorphisms and inversions, by showing that partial conjugations descend to the trivial automorphism of $L(A_\Gamma)$.
- Step 2. Denote by $\text{Aut}_{\iota, p}(A_\Gamma)$ the subgroup of $\text{Aut}_{\overline{\tau}}(A_\Gamma)$ generated by all graph automorphisms and inversions. We show that every element ψ in $\text{Aut}_{\iota, p}(A_\Gamma)$ is conjugate to an element ψ' of a certain ‘nice’ form. By Proposition 1.1.3, $R(\psi) = R(\psi')$.
- Step 3. Finally, we show that each such ‘nice’ ψ' has infinite Reidemeister number, by finding an eigenvalue 1 in either $L_1(A_\Gamma)$, $L_2(A_\Gamma)$ or $L_3(A_\Gamma)$.

Throughout this section, we fix a non-complete graph Γ on n vertices and its associated RAAG A_Γ . We start with the first of two technical lemmata, which can be proven by straightforward induction.

Lemma 3.3.16. *Let G be a group.*

(i) *If $x, y, z \in G$ are such that $x \equiv y \pmod{\gamma_i(G)}$, then $[x, z] \equiv [y, z] \pmod{\gamma_{i+1}(G)}$.*

(ii) *For every $k \geq 1$, $x_1, \dots, x_k \in G$ and $w_1, \dots, w_k \in G$, we have that*

$$[x_1^{w_1}, \dots, x_k^{w_k}] \equiv [x_1, \dots, x_k] \pmod{\gamma_{k+1}(G)}.$$

(iii) *For every $k \geq 1$, $x_1, \dots, x_k \in G$ and $n_1, \dots, n_k \in \mathbb{Z}$, we have that*

$$[x_1^{n_1}, \dots, x_k^{n_k}] \equiv [x_1, \dots, x_k]^{n_1 \dots n_k} \pmod{\gamma_{k+1}(G)}.$$

Corollary 3.3.17. *For every partial conjugation $\varphi \in \text{Aut}(A_\Gamma)$ and every $i \geq 1$, we have that the induced automorphism φ_i on $\gamma_i(A_\Gamma)/\gamma_{i+1}(A_\Gamma)$ is the identity map.*

Moreover, the induced automorphism φ_ on $L(A_\Gamma)$ is also the identity map.*

Proof. As $\gamma_i(A_\Gamma)/\gamma_{i+1}(A_\Gamma)$ is generated by the cosets of the i -fold commutators of the generators $a_1, \dots, a_n$ of A_Γ , it is sufficient to show that φ_i is the identity map on these commutators. But since φ is given by conjugation for each generator, Lemma 3.3.16 implies that

$$\varphi([a_{i_1}, \dots, a_{i_i}]) = [\varphi(a_{i_1}), \dots, \varphi(a_{i_i})] \equiv [a_{i_1}, \dots, a_{i_i}] \pmod{\gamma_{i+1}(A_\Gamma)}.$$

Hence, φ_i is the identity map. The ‘moreover’ statement is immediate. $\square$

The following lemma gives a more concrete description of a given element in $\text{Aut}_{\iota, \text{p}}(A_\Gamma)$.

Lemma 3.3.18. *For $\varphi \in \text{Aut}_{\iota, \text{p}}(A_\Gamma)$, there exist a permutation $\sigma \in S_n$ and integers $e_1, \dots, e_n \in \{-1, 1\}$ such that*

$$\varphi(a_i) = a_{\sigma(i)}^{e_i}$$

for all $i \in \{1, \dots, n\}$ and such that the map $a_i \mapsto a_{\sigma(i)}$ is a graph automorphism of Γ .

Proof. The result trivially holds for the identity map, graph automorphisms and inversions, and it easily seen that the given form is preserved under composition. Hence, the result follows. $\square$

The following lemma concerns the determinant of matrices of a specific form which will occur frequently in the proof of Theorem 3.3.3.

Lemma 3.3.19. *Let $k \geq 1$ be an integer and suppose $e_1, \dots, e_k$ are all either equal to 1 or -1 . Define*

$$P(e_1, \dots, e_k) := \begin{pmatrix} 0 & 0 & \dots & 0 & e_k \\ e_1 & 0 & \dots & 0 & 0 \\ 0 & e_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & e_{k-1} & 0 \end{pmatrix}$$

Note that $P(e_1) = (e_1)$. Then

$$\det(I_k - P(e_1, \dots, e_k)) = 1 - \prod_{i=1}^k e_i.$$

In particular, $P(e_1, \dots, e_k)$ has eigenvalue 1 if and only if an even number of the e_i equal -1 .

Proof. For $k = 1$, it is clear that $\det(I_1 - P(e_1)) = 1 - e_1$. For $k \geq 2$, expanding the determinant $\det(I_k - P(e_1, \dots, e_k))$ along the first row yields the desired result. $\square$

The last lemma (and the second technical one) makes more concrete what we meant by automorphisms in $\text{Aut}_{\iota, \text{p}}(A_\Gamma)$ of a ‘nice’ form.

Lemma 3.3.20. *Let $\varphi \in \text{Aut}_{\iota, \text{p}}(A_\Gamma)$. Then there is a graph automorphism ψ and an automorphism ι , which is the composition of (possibly zero) inversions, such that φ and $\psi \circ \iota$ are conjugate.*

Moreover, if $\sigma \in S_n$ is such that $\psi(a_i) = a_{\sigma(i)}$ and $\sigma = c_1 \circ \dots \circ c_k$ is the disjoint cycle decomposition of σ , we can choose ι such that each cycle $c_j = (c_{j,1} \dots c_{j,n_j})$ contains at most one number i_j with $\iota(a_{i_j}) = a_{i_j}^{-1}$. (Here, we also write cycles of length 1, e.g. the identity permutation is written as $(1)(2) \dots (n)$).

Proof. By Lemma 3.3.18, we know that there is a permutation $\sigma \in S_n$ and elements $e_1, \dots, e_n \in \{-1, 1\}$ such that $\varphi(a_i) = a_{\sigma(i)}^{e_i}$ and such that $\psi : \Gamma \rightarrow \Gamma : a_i \mapsto a_{\sigma(i)}$ is a well-defined graph automorphism of Γ . Therefore, ψ induces a graph automorphism of A_Γ , which we still denote by ψ .

Write $\sigma = c_1 \circ \dots \circ c_k$ in disjoint cycle decomposition. After renumbering the generators, we can assume that there are integers $1 = i_1 < i_2 < \dots < i_k$ such that $c_j = (i_j \dots i_{j+1} - 1)$, where we put $i_{k+1} - 1 = n$. If we consider $\varphi_1 : A_\Gamma / \gamma_2(A_\Gamma) \rightarrow A_\Gamma / \gamma_2(A_\Gamma)$, then the matrix of φ_1 with respect to the basis $a_1, \dots, a_n$ is of the form

$$M := \begin{pmatrix} P(e_{i_1}, \dots, e_{i_2-1}) & 0 & 0 & \dots & 0 \\ 0 & P(e_{i_2}, \dots, e_{i_3-1}) & 0 & \dots & 0 \\ 0 & 0 & P(e_{i_3}, \dots, e_{i_4-1}) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & P(e_{i_k}, \dots, e_n) \end{pmatrix}$$

with each $P(e_{i_j}, \dots, e_{i_{j+1}-1})$ as in Lemma 3.3.19.

If we can find a diagonal matrix D with only ± 1 on the diagonal such that DMD is of the form

$$\begin{pmatrix} P(\pm 1, 1, \dots, 1) & 0 & 0 & \dots & 0 \\ 0 & P(\pm 1, 1, \dots, 1) & 0 & \dots & 0 \\ 0 & 0 & P(\pm 1, 1, \dots, 1) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & P(\pm 1, 1, \dots, 1) \end{pmatrix}$$

with each block of the same dimension as in M , we are done. Indeed, the matrix D will then be the matrix of $j_1 : A_\Gamma / \gamma_2(A_\Gamma) \rightarrow A_\Gamma / \gamma_2(A_\Gamma)$ w.r.t. the basis $a_1, \dots, a_n$, with $j \in \text{Aut}(A_\Gamma)$ a composition of inversions and DMD corresponds to the automorphism $\psi \circ \iota$, where ι maps a_{i_j} to $a_{i_j}^{\pm 1}$ for all $1 \leq j \leq k$ according to the sign in the j -th block of DMD and leaves all other generators fixed. Then $\psi \circ \iota = j \circ \varphi \circ j$ and ι will satisfy the ‘moreover’-part of the lemma, since each P -block contains at most one -1 .

It is sufficient to find such a diagonal matrix D for each P -block and then putting all these blocks into one matrix. For ease of notation, we put $m = i_2 - 1$ and consider the block $P(e_1, \dots, e_m)$. If $m = 1$, there is nothing to prove, so assume $m \geq 2$. Denote by D_i the diagonal matrix with 1’s on the diagonal except for the i -th position, there we put -1 . Note that any product of the D_i ’s is a diagonal matrix with only ± 1 on the diagonal. It is not hard to see that

$$D_i P(e_1, \dots, e_m) D_i = P(e_1, \dots, e_{i-2}, -e_{i-1}, -e_i, e_{i+1}, \dots, e_m),$$

where $e_0 = e_m$. Note that the parity of the number of -1 ’s is left unchanged. By starting with the -1 with the highest index and moving to e_1 , we can clear out all -1 ’s, except for one if there were an odd number of them to begin with. Wherever this last -1 is situated, we can move it to the first position by conjugating with suitable D_i ’s. In the end, we end up with $P(\pm 1, 1, \dots, 1)$ and we are done. $\square$

At last, we can give the proof of Theorem 3.3.3.

Proof of Theorem 3.3.3. Let $\varphi \in A$. Since we will work with the induced morphism φ_* on $L(A_\Gamma)$, we may assume by Corollary 3.3.17 that $\varphi \in \text{Aut}_{\iota, p}(A_\Gamma)$. By Lemma 3.3.20, φ is conjugate to $\psi \circ \iota$ with ψ a graph automorphism and ι a composition of (possible zero) inversions satisfying the ‘moreover’-part of the statement of Lemma 3.3.20. As φ and $\psi \circ \iota$ are conjugate, $R(\varphi) = R(\psi \circ \iota)$ by Proposition 1.1.3. So, we can assume that $\varphi = \psi \circ \iota$.

Let $\sigma \in S_n$ be the permutation associated to ψ and write $\sigma = c_1 \circ \dots \circ c_k$ in disjoint cycle decomposition. Denote by φ_i the induced automorphism on $L_i(A_\Gamma)$. By Theorem 3.3.13, we can apply Theorem 1.1.6 to find that $R(\varphi) = \infty$ if φ_i has eigenvalue 1 for some i . We will distinguish several cases, and as each case will end with the sentence ‘ φ_i has eigenvalue 1’, we will not mention Theorem 1.1.6 each time.

Case 1: Suppose there is a cycle, say, c_1 , such that ι does not invert any of the generators a_i with $i \in c_1$. After renumbering, we can assume that $c_1 = (1 \dots m)$ for some $m \geq 1$. Then, on $L_1(A_\Gamma)$, we have that

$$\varphi_1(a_1 \dots a_m \gamma_2(A_\Gamma)) = a_2 a_3 \dots a_m a_1 \gamma_2(A_\Gamma),$$

so φ_1 has a fixed point, i.e. 1 is an eigenvalue of φ_1 , hence $R(\varphi) = \infty$.

From now on, assume that each cycle contains an index i_j such that $\iota(a_{i_j}) = a_{i_j}^{-1}$.

Case 2: $k = n$, i.e. each cycle in the decomposition of σ has length 1. Then $\varphi(a_i) = a_i^{-1}$ for all i . As Γ is not complete, there are $a_i \neq a_j$ with $a_i a_j \notin E$. Hence, $[a_i, a_j] \gamma_3(A_\Gamma)$ is non-trivial and

$$\varphi_2([a_i, a_j] \gamma_3(A_\Gamma)) = [a_i^{-1}, a_j^{-1}] \gamma_3(A_\Gamma) = [a_i, a_j] \gamma_3(A_\Gamma)$$

by Lemma 3.3.16(iii). Then φ_2 has eigenvalue 1, and thus $R(\varphi) = \infty$.

From now on, assume that $k < n$.

Case 3: There is a cycle, say, c_1 containing indices $i < j$ with $a_i a_j \notin E$. Again, we can assume that $c_1 = (1 \dots m)$. As σ induces a graph automorphism, we have that $a_{i+l} a_{j+l}$ is not an edge of E either for all $1 \leq l \leq m$ (here, we work with indices modulo m where we use 1 up to m as representatives, rather than 0 up to $m-1$). After renumbering the generators, we can assume that a_1 is mapped onto a_2^{-1} . Consider then the set

$$B := \{[a_{i+l}, a_{j+l}] \gamma_3(A_\Gamma) \mid 0 \leq l \leq m-1\}$$

Note that B does not necessarily form a linearly independent set: if $i \equiv j + l \pmod{m}$ and $j \equiv i + l \pmod{m}$ for some $1 \leq l \leq m-1$, then $[a_i, a_j] = -[a_{i+l}, a_{j+l}]$. These two congruences can be simultaneously fulfilled if and only if $2(j-i) \equiv 0 \pmod{m}$. If this condition is not satisfied, then B is indeed a linearly independent set: each element in B can be rewritten (up to sign) such that the first index is strictly less than the second. No two elements will have the same indices, and by Proposition 3.3.14, B is a subset of a basis of $L_2(A_\Gamma)$. We then can proceed as follows: first remark that

$$\varphi_2([a_{i+l}, a_{j+l}] \gamma_3(A_\Gamma)) = \begin{cases} -[a_{i+l+1}, a_{j+l+1}] \gamma_3(A_\Gamma) & \text{if } i+l \equiv 1 \pmod{m} \text{ or } \\ & j+l \equiv 1 \pmod{m} \\ [a_{i+l+1}, a_{j+l+1}] \gamma_3(A_\Gamma) & \text{otherwise.} \end{cases}$$

Note that $i+l$ and $j+l$ cannot both be congruent to 1 modulo m simultaneously, as otherwise $i \equiv j \pmod{m}$, which is impossible as $1 \leq i < j \leq m$ (and thus also $m \geq 2$). As all elements in B are distinct, there will be precisely two elements $[a_{i+l}, a_{j+l}] \gamma_3(A_\Gamma)$ that are mapped to $-[a_{i+l+1}, a_{j+l+1}] \gamma_3(A_\Gamma)$. Moreover, $\varphi_2(\text{span}_{\mathbb{Z}}(B)) = \text{span}_{\mathbb{Z}}(B) =: V$, hence we can consider the matrix of φ_2 restricted to V with respect to this basis and find a matrix $P(e_1, \dots, e_m)$ where precisely two of the e_i are equal to -1 and the rest equals 1. Hence, Lemma 3.3.19 implies that $P(e_1, \dots, e_m)$ has eigenvalue 1, so φ_2 restricted to V does too. This implies that φ_2 has eigenvalue 1, and consequently, $R(\varphi) = \infty$.

Now, suppose that $2(j-i) \equiv 0 \pmod m$. If $m = 2$, then B contains $[a_1, a_2]\gamma_3(A_\Gamma)$ and in that case,

$$\varphi_2([a_1, a_2]\gamma_3(A_\Gamma)) = -[a_2, a_1]\gamma_3(A_\Gamma) = [a_1, a_2]\gamma_3(A_\Gamma).$$

Consequently, φ_2 has eigenvalue 1 and therefore $R(\varphi) = \infty$.

Finally, if $m > 2$ and $2(j-i) \equiv 0 \pmod m$, note that $0 < j-i < m$, hence m is even and $j-i = m/2$. The set

$$\begin{aligned} \tilde{B} := & \{[a_{1+l}, a_{m/2+1+l}, a_{1+l}]\gamma_4(A_\Gamma) \mid 0 \leq l \leq m/2-1\} \\ & \cup \{[a_{1+l}, a_{m/2+1+l}, a_{m/2+1+l}]\gamma_4(A_\Gamma) \mid 0 \leq l \leq m/2-1\} \end{aligned}$$

will be linearly independent, as all elements in $\tilde{B}$ are distinct and $\tilde{B}$ forms a subset of a linearly independent set of $L_3(A_\Gamma)$, by Proposition 3.3.15. Also note that

$$\begin{aligned} \varphi_3([a_{1+l}, a_{m/2+1+l}, a_{1+l}]\gamma_4(A_\Gamma)) &= [a_{2+l}, a_{m/2+2+l}, a_{2+l}]\gamma_4(A_\Gamma) \\ \varphi_3([a_{1+l}, a_{m/2+1+l}, a_{m/2+1+l}]\gamma_4(A_\Gamma)) &= [a_{2+l}, a_{m/2+2+l}, a_{m/2+2+l}]\gamma_4(A_\Gamma) \end{aligned}$$

for all $0 \leq l \leq m/2-2$ and that

$$\begin{aligned} \varphi_3([a_{m/2}, a_m, a_{m/2}]\gamma_4(A_\Gamma)) &= [a_{m/2+1}, a_1, a_{m/2+1}]\gamma_4(A_\Gamma) \\ &= -[a_1, a_{m/2+1}, a_{m/2+1}]\gamma_4(A_\Gamma), \\ \varphi_3([a_{m/2}, a_m, a_m]\gamma_4(A_\Gamma)) &= [a_{m/2+1}, a_1, a_1]\gamma_4(A_\Gamma) \\ &= -[a_1, a_{m/2+1}, a_1]\gamma_4(A_\Gamma) \end{aligned}$$

Hence, putting $W = \text{span}_{\mathbb{Z}}(\tilde{B})$, we find that $\varphi_3(W) = W$ and that the matrix of φ_3 restricted to W with respect to $\tilde{B}$ is of the form $P(e_1, \dots, e_{2m})$ as in Lemma 3.3.19, where precisely two of the e_i 's are -1 . This matrix has eigenvalue 1, hence so does φ_3 . We conclude that $R(\varphi) = \infty$.

From now on, assume that each cycle c_j satisfies the following property: if $c_j = (i_1 \dots i_j)$, the induced subgraph $\Gamma(\{a_{i_1}, \dots, a_{i_j}\})$ is complete. As Γ itself is not complete, there are cycles, say $c_1 = (1 \dots m)$ and $c_2 = (m+1 \dots m+l)$, such that there are $i \in \{1, \dots, m\}$, $j \in \{1, \dots, l\}$ with $a_i a_{m+j} \notin E$. For notational convenience, we put $b_i = a_{i+m}$ for $i \in \{1, \dots, l\}$. Hence, $a_i b_j \notin E$. It follows that also $a_1 b_{j-i+1} \notin E$, as σ^{1-i} maps the non-edge $a_i b_j$ to the non-edge $a_{i+1-i} b_{j+1-i}$. Again, after renumbering, we can assume that $j-i+1 = 1$. However, then it does not necessarily hold that $b_1 \mapsto b_2^{-1}$. By use of conjugation, we can obtain this nonetheless: suppose $b_\alpha \mapsto b_{\alpha+1}^{-1}$ with $\alpha \neq 1$. Putting $j = \iota_2 \circ \iota_3 \circ \dots \circ \iota_\alpha$, where ι_k is the inversion b_k to b_k^{-1} , a similar argument as in Lemma 3.3.20 yields that $j \circ \varphi \circ j$ maps b_1 to b_2^{-1} , b_β to $b_{\beta+1}$ for $\beta \neq 1$ and coincides for the rest with φ . Thus, we can assume that $a_1 b_1 \notin E$.

Put $K = \text{lcm}(l, m)$ and

$$B = \{[a_1, b_1]\gamma_3(A_\Gamma), \dots, [a_K, b_K]\gamma_3(A_\Gamma)\}.$$

Again, we consider the indices modulo m (for a_i) and l (for b_j), respectively. Then B is a linearly independent subset of $L_2(A_\Gamma)$ and $V := \text{span}_{\mathbb{Z}}(B)$ satisfies $\varphi_2(V) = V$. We count the number of elements in B that are mapped to minus the next generator in B . For $1 \leq i \leq K$, we have that

$$\varphi_2([a_i, b_i]\gamma_3(A_\Gamma)) = \begin{cases} -[a_{i+1}, b_{i+1}]\gamma_3(A_\Gamma) & \text{if } i \equiv 1 \pmod m \text{ or } i \equiv 1 \pmod l, \text{ but} \\ & \text{not both} \\ [a_{i+1}, b_{i+1}]\gamma_3(A_\Gamma) & \text{otherwise.} \end{cases}$$

Note that $i \equiv 1 \pmod m$ and $i \equiv 1 \pmod l$ if and only if $i \equiv 1 \pmod K$, hence if and only if $i = 1$. The number of indices in $\{1, \dots, K\}$ that are congruent to 1 modulo m is K/m , and similarly there are K/l indices congruent 1 modulo l . As we have counted 1 twice, there are $K(m^{-1} + l^{-1}) - 1$ indices i that are congruent to 1 modulo m or l . It follows that there are $K' := K(m^{-1} + l^{-1}) - 2$ elements in B such that $\varphi_2([a_i, b_i]\gamma_3(A_\Gamma)) = -[a_{i+1}, b_{i+1}]\gamma_3(A_\Gamma)$.

Case 4: Suppose K' is even. Then the matrix of φ_2 restricted to V with respect to the basis B will be of the form $P(e_1, \dots, e_{K'})$ with an even number of e_i equal to -1 . Applying Lemma 3.3.19 gives that $P(e_1, \dots, e_{K'})$ and hence φ_2 has eigenvalue 1, so $R(\varphi) = \infty$.

Case 5: K' is odd. Then either K/l or K/m is even, say, K/m (the case K/l even is analogous). The set

$$B' := \{[a_1, b_1, b_1]\gamma_4(A_\Gamma), \dots, [a_K, b_K, b_K]\gamma_4(A_\Gamma)\}.$$

is a linearly independent subset of $L_3(A_\Gamma)$, by Proposition 3.3.15. Moreover, by Lemma 3.3.16(iii)

$$\varphi_3([a_i, b_i, b_i]\gamma_4(A_\Gamma)) = \begin{cases} -[a_{i+1}, b_{i+1}, b_{i+1}]\gamma_4(A_\Gamma) & \text{if } i \equiv 1 \pmod m \\ [a_{i+1}, b_{i+1}, b_{i+1}]\gamma_4(A_\Gamma) & \text{otherwise.} \end{cases}$$

Hence, precisely K/m elements of B' are mapped to minus the next element and $V' := \text{span}_{\mathbb{Z}}(B')$ satisfies $\varphi_3(V') = V'$. The matrix of φ_3 restricted to V' with respect to B' will then again be of the form as in Lemma 3.3.19 with an even number of e_i 's equal to -1 , hence φ_3 has eigenvalue 1 and we can conclude that $R(\varphi) = \infty$. $\square$

Definition 3.3.21. Let $\Gamma(V, E)$ be a graph. We call Γ *transvection-free* if Γ is not K^1 and $V_{\overline{\tau}} = V$.

Remark. For K^1 , the sole vertex v is, strictly speaking, transvection-free. Since $A_{K^1} = \mathbb{Z}$ is abelian, we exclude K^1 from the transvection-free graphs.

The following important corollary is immediate.

Corollary 3.3.22. *Let Γ be a transvection-free graph. Then $A_\Gamma \in R_\infty$.*

Proof. If A_Γ is non-abelian and does not admit any transvections, then $\text{Aut}_{\overline{\tau}}(A_\Gamma) = \text{Aut}(A_\Gamma)$ and Theorem 3.3.3 implies that every $\varphi \in \text{Aut}(A_\Gamma) = \text{Aut}_{\overline{\tau}}(A_\Gamma)$ has infinite Reidemeister number. $\square$

Of course, a theorem whose proof deserves a separate section should be of significant value. There are indeed a certain amount of transvection-free graphs, which we will discuss in the following section, but first we would like to make the following remark which explains to some extent the length of the proof: Theorem 3.3.3 holds for *every* non-abelian RAAG. The only assumption regarding Γ we needed in the proof was the fact that there are two vertices which are not connected by an edge. As only the complete graph does not meet this requirement, we consider a huge amount of very different graphs in the proof, which explains the number of cases.

The condition that Γ is non-complete is also crucial: for $\Gamma = K^n$, we have $A_\Gamma = \mathbb{Z}^n$ and the automorphism $-\text{Id}_{\mathbb{Z}^n}$ is a composition of graph automorphisms and inversions, but an easy calculation gives $R(-\text{Id}_{\mathbb{Z}^n}) = 2^n$.

3.3.4 Examples of transvection-free graphs

To illustrate the significance of Theorem 3.3.3 and Corollary 3.3.22, we give an example of a family of transvection-free graphs. Later, in the section on regular graphs, we will see that most of the strongly regular graphs are also transvection-free, providing yet another example.

Example 3.3.23 (Cycle graphs). Let $n \geq 3$ be an integer. The *cycle graph* C_n is the graph with vertex set $\{v_0, \dots, v_{n-1}\}$ and edge set $\{v_i v_{i+1} \mid i \in \{0, \dots, n-1\}\}$, where the indices are viewed modulo n . Note that C_n is connected and regular for each $n \geq 3$, and non-complete for $n \geq 4$.

Figure 1: Cycle graph C_9

We claim that C_n is transvection-free if and only if $n \geq 5$. Suppose $n \geq 5$. For $i \in \{0, \dots, n-1\}$, we have that $lk(v_i) = \{v_{i-1}, v_{i+1}\}$ and $st(v_i) = \{v_{i-1}, v_i, v_{i+1}\}$. If $lk(v_i) \subseteq st(v_j)$, then either $v_i = v_j$, or $v_{i-1} = v_{j+1}$ and $v_{i+1} = v_{j-1}$. The latter case implies that $i-1 \equiv j+1 \pmod n$ and $i+1 \equiv j-1 \pmod n$. Hence, $-2 \equiv i-j \equiv 2 \pmod n$. As $n \geq 5$, this is impossible, hence $v_i = v_j$. We conclude that C_n is indeed transvection-free. Consequently, A_{C_n} has the R_∞ -property for $n \geq 5$.

Conversely, if $n = 4$, note that $lk(v_1) = \{v_0, v_2\} = lk(v_3)$, so C_4 is not transvection-free. As $C_3 = K^3$, it is clear that also C_3 is not transvection-free.

3.3.5 Properties of transvection-free graphs

We end this section with some properties of transvection-free graphs.

Definition 3.3.24. Let $\Gamma(V, E)$ be a graph. The *complement graph* $\bar{\Gamma}$ is the graph with vertex set V and edge set $\bar{E} = \{vw \mid v \neq w, vw \notin E\}$. We also call $\bar{\Gamma}$ the *complement* of Γ .

For example, the complement of the complete graph K^n is the edgeless graph $\overline{K^n}$, which also explains the notation. Moreover, it is not hard to see that $\overline{\Gamma_1 * \Gamma_2} = \overline{\Gamma_1} \sqcup \overline{\Gamma_2}$ and that $\overline{\overline{\Gamma}} = \Gamma$.

The next result follows directly from the definitions.

Lemma 3.3.25. Let Γ be a graph and $v \in V$. Denote by $lk_\Gamma(v)$ the link of v seen as vertex of Γ and similarly for $st_\Gamma(v)$. Then $lk_{\bar{\Gamma}}(v) = V \setminus st_\Gamma(v)$ and $st_{\bar{\Gamma}}(v) = V \setminus lk_\Gamma(v)$.

Definition 3.3.26. Let $\Gamma(V, E)$ be a graph. A vertex $v \in V$ is called *isolated* if $lk(v) = \emptyset$.

Proposition 3.3.27. Let Γ_1, Γ_2 be transvection-free graphs. Then $\bar{\Gamma}_1$, $\Gamma_1 \sqcup \Gamma_2$ and $\Gamma_1 * \Gamma_2$ are all transvection-free as well.

Proof. A transvection-free graph cannot contain an isolated vertex v , since otherwise $lk(v) = \emptyset$ is contained in the star of every other vertex. Hence, the link of every vertex in Γ_1 and Γ_2 is non-empty, implying that any pair of dominating vertices in $\Gamma_1 \sqcup \Gamma_2$ arises in Γ_1 or Γ_2 . Consequently, $\Gamma_1 \sqcup \Gamma_2$ is transvection-free.

For the complement of $\Gamma := \Gamma_1$: note that $w \in V$ dominates $v \in V$ seen as vertices in $\bar{\Gamma}$ if and only if $lk_{\bar{\Gamma}}(v) \subseteq st_{\bar{\Gamma}}(w)$. By Lemma 3.3.25, this is equivalent to $lk_{\Gamma}(w) \subseteq st_{\Gamma}(v)$, i.e. v dominates w seen as vertices in Γ . As Γ is transvection-free, it follows that $\bar{\Gamma}$ is transvection-free as well.

Finally, consider the graph $\Gamma := \Gamma_1 * \Gamma_2$. Suppose $v, w \in V_1$ are vertices with $v \leq w$ (in Γ). Since $V_2 \subseteq lk(v)$, $V_2 \subseteq st(w)$, we have that $lk_{\Gamma}(v) = V_2 \cup lk_{\Gamma_1}(v)$ and similarly for $st_{\Gamma}(w)$. This implies that $lk_{\Gamma_1}(v) \subseteq st_{\Gamma_1}(w)$, which contradicts the transvection-freeness of Γ_1 . A similar argument holds if $v, w \in V_2$.

If $v \in V_1$ and $w \in V_2$ are such that $lk_{\Gamma}(v) \subseteq st_{\Gamma}(w)$, then $V_2 \subseteq lk_{\Gamma}(v) \subseteq st_{\Gamma}(w)$, hence, $st_{\Gamma_2}(w) = V_2$, which contradicts the fact that Γ_2 is transvection-free. We conclude that $\Gamma_1 * \Gamma_2$ is transvection-free as well. $\square$

4 Disconnected graphs

From this section onwards, we discuss right-angled Artin groups associated to the three types of graphs arising from the Simplification Lemma. We start with disconnected graphs, which correspond to free products of RAAGs.

Definition 4.1.1. A group G is called *freely indecomposable* if $G = G_1 * G_2$ implies $G_1 = 1$ or $G_2 = 1$, i.e. G cannot be written as a non-trivial free product.

Clearly, abelian groups are freely indecomposable. Recently, D. Gonçalves, P. Sankaran and P. Wong proved the following [13]:

Theorem 4.1.2. *Let $n \geq 2$ and suppose $G = G_1 * \dots * G_n$, where each G_i is freely indecomposable, and both G_1 and G_2 have a proper characteristic subgroup of finite index. Then $G \in R_{\infty}$.*

E. Green proved in her PhD-thesis [14, Lemma 4.7] that a RAAG is freely indecomposable if and only if its defining graph is connected. Next, we show that each RAAG admits a proper characteristic subgroup of finite index.

Lemma 4.1.3. *Let Γ be a graph and A_{Γ} its associated RAAG. Then A_{Γ} has a proper characteristic subgroup of finite index.*

Proof. Suppose Γ has n vertices. Let $p : A_{\Gamma} \rightarrow \frac{A_{\Gamma}}{[A_{\Gamma}, A_{\Gamma}]}$ be the natural projection. By Corollary 2.2.7, the abelianisation is isomorphic to $\mathbb{Z}^n$. Note that $2\mathbb{Z}^n$ is a proper characteristic subgroup of finite index in $\mathbb{Z}^n$. Then $p^{-1}(2\mathbb{Z}^n)$ will be the desired subgroup of A_{Γ} . $\square$

We can therefore apply Theorem 4.1.2 to find the following result.

Theorem 4.1.4. *Let Γ be a disconnected graph. Then A_{Γ} has the R_{∞} -property.*

Proof. Write $\Gamma = \bigsqcup_{i=1}^n \Gamma_i$, where each Γ_i is connected and where $n \geq 2$. By Proposition 2.1.3, $A_{\Gamma} \cong \bigstar_{i=1}^n A_{\Gamma_i}$. By [14, Lemma 4.7], each A_{Γ_i} is freely indecomposable and by Lemma 4.1.3, each A_{Γ_i} admits a proper finite index characteristic subgroup. Consequently, the conditions for Theorem 4.1.2 are fulfilled and we can conclude that $A_{\Gamma} \in R_{\infty}$. $\square$

Remark. In [23, Chapter 5], Theorem 4.1.4 was proved using the associated Lie ring of the RAAG.

5 Regular graphs

We start this section with some results regarding direct products of RAAGs, which will then be applied to RAAGs associated to regular graphs. In particular, we discuss strongly regular graphs, which provide examples of both transvection-free graphs and direct products of RAAGs.

5.1 Direct products of RAAGs

For groups $G_1, \dots, G_n$, the automorphism group of the direct product of the G_i 's always contains $\text{Aut}(G_1) \times \dots \times \text{Aut}(G_n)$, but the full automorphism group $\text{Aut}(G_1 \times \dots \times G_n)$ can be much bigger and much more difficult to describe. For RAAGs, however, a general description in terms of the factors is possible. We start with some general theory regarding automorphism groups of direct products before heading to direct products of RAAGs.

5.1.1 Subgroups of $\text{Aut}(G_1 \times \dots \times G_n)$

Proposition 5.1.1. *Let $G_1, \dots, G_n$ be groups and suppose (at least) one of them has the R_∞ -property. Then for each element φ of $\text{Aut}(G_1) \times \dots \times \text{Aut}(G_n) \leq \text{Aut}(G_1 \times \dots \times G_n)$, we have $R(\varphi) = \infty$.*

Proof. Write $\varphi = (\varphi_1, \dots, \varphi_n)$. It is clear that, for $(g_1, \dots, g_n), (h_1, \dots, h_n) \in G_1 \times \dots \times G_n$, we have

$$(g_1, \dots, g_n) \sim_\varphi (h_1, \dots, h_n) \iff \forall i \in \{1, \dots, n\} : g_i \sim_{\varphi_i} h_i.$$

Thus,

$$R(\varphi) = \prod_{i=1}^n R(\varphi_i),$$

from which the result immediately follows. $\square$

For an n -fold direct product of a group with itself, we can consider a bigger subgroup of the automorphism group than merely the direct product: there is an injective group homomorphism

$$\psi : \text{Aut}(G) \wr S_n \rightarrow \text{Aut}(G^n) : (\varphi, \sigma) = (\varphi_1, \dots, \varphi_n, \sigma) \mapsto \psi(\varphi, \sigma)$$

where

$$\psi(\varphi, \sigma)(g_1, \dots, g_n) = (\varphi_1(g_{\sigma^{-1}(1)}), \dots, \varphi_n(g_{\sigma^{-1}(n)})).$$

Here, $\text{Aut}(G) \wr S_n$ is the wreath product, i.e. the semidirect product $\text{Aut}(G)^n \rtimes S_n$, where the action is given by

$$\sigma \cdot (\varphi_1, \dots, \varphi_n) = (\varphi_{\sigma^{-1}(1)}, \dots, \varphi_{\sigma^{-1}(n)}).$$

We will write elements in $\text{Aut}(G) \wr S_n$ simply as (φ, σ) .

Proposition 5.1.2. *Let G be a group having the R_∞ -property and $n \geq 1$ an integer. Then for each $\chi \in \text{Aut}(G) \wr S_n \leq \text{Aut}(G^n)$, we have $R(\chi) = \infty$.*

Proof. Write $\chi = (\varphi, \sigma)$. Suppose $(g, 1, \dots, 1) \sim_\chi (h, 1, \dots, 1)$. Then there are $x_1, \dots, x_n \in G$ with

$$\begin{aligned} g &= x_1 h \varphi_1(x_{\sigma^{-1}(1)})^{-1} \\ 1 &= x_i \varphi_i(x_{\sigma^{-1}(i)})^{-1} \quad i \geq 2. \end{aligned} \tag{5.1}$$

First, suppose that $\sigma^{-1}(1) = 1$. Then $g \sim_{\varphi_1} h$, hence the map

$$F : \{[(g, 1, \dots, 1)]_\chi \mid g \in G\} \rightarrow \mathcal{R}[\varphi_1] : [(g, 1, \dots, 1)]_\chi \mapsto [g]_{\varphi_1}$$

is well-defined and surjective. The domain of F is a subset of $\mathcal{R}[\chi]$, hence $R(\chi) \geq R(\varphi_1) = \infty$, since $\varphi_1 \in \text{Aut}(G)$ and $G \in R_\infty$.

If $\sigma^{-1}(1) \neq 1$, let $m > 0$ be the smallest integer such that $\sigma^{-m}(1) = 1$. Repeatedly using (5.1) gives

$$g = x_1 h \varphi_1(x_{\sigma^{-1}(1)})^{-1} = x_1 h \varphi_1(\varphi_{\sigma^{-1}(1)}(x_{\sigma^{-2}(1)}))^{-1} = \dots = x_1 h \tilde{\varphi}(x_1)^{-1}$$

where

$$\tilde{\varphi} = \varphi_1 \circ \varphi_{\sigma^{-1}(1)} \circ \dots \circ \varphi_{\sigma^{1-m}(1)}.$$

Note that $\tilde{\varphi}$ only depends on χ . Thus, $g \sim_{\tilde{\varphi}} h$ and the map

$$F : \{[(g, 1, \dots, 1)]_\chi \mid g \in G\} \rightarrow \mathcal{R}[\tilde{\varphi}] : [(g, 1, \dots, 1)]_\chi \mapsto [g]_{\tilde{\varphi}}$$

is well-defined and surjective. Similarly as before, we find that $R(\chi) = \infty$. $\square$

Corollary 5.1.3. *If a group G has the R_∞ -property and $n \geq 1$ is an integer such that $\text{Aut}(G^n) = \text{Aut}(G) \wr S_n$, then $G^n \in R_\infty$.*

5.1.2 Automorphism group of direct product of RAAGs

The description of the automorphism group of a direct product of RAAGs is due to N. Fullarton [10], and G. Giovanni and N. Wahl [11], whose results we present in this section, together with its implications for the R_∞ -property for RAAGs. Recall that a direct product on group theoretical level corresponds to a simplicial join on graph theoretical level.

Proposition 5.1.4 ([11, Proposition 3.1]). *Let Γ be a graph. Then A_Γ admits a unique maximal decomposition as*

$$A_\Gamma = A_{\Gamma_1} \times \dots \times A_{\Gamma_k}$$

for induced subgraphs $\Gamma_1, \dots, \Gamma_k$. This decomposition is unique up to isomorphism and permutation of the factors.

Lemma 5.1.5 ([2, Proposition 2.2]). *Let Γ be a graph and A_Γ its associated RAAG. Then the centre of A_Γ is given by $Z(A_\Gamma) = \langle \{v \in V \mid \deg(v) = |V| - 1\} \rangle$.*

Proposition 5.1.6 ([11, Proposition 3.3]). *Let Γ be a graph with maximal decomposition*

$$\Gamma = K^d * \bigstar_{j=1}^k (*_{i_j} \Gamma_j) \tag{5.2}$$

where all Γ_j are non-isomorphic and non-complete graphs. Then

$$\text{Aut}(A_\Gamma) \cong \mathbb{Z}^{|V'|} \rtimes (\text{GL}_d(\mathbb{Z}) \times (\text{Aut}(A_{\Gamma_1}) \wr S_{i_1}) \times \dots \times (\text{Aut}(A_{\Gamma_k}) \wr S_{i_k})).$$

Using this description and the results from Section 5.1.1, we can describe what happens on the level of Reidemeister numbers.

Theorem 5.1.7. *Let Γ be a graph with maximal decomposition*

$$\Gamma = K^d * \bigstar_{j=1}^k (*_{i_j} \Gamma_j)$$

where all Γ_j are non-isomorphic and non-complete graphs. If any graph Γ_j is such that $A_{\Gamma_j} \in R_\infty$, then $A_\Gamma \in R_\infty$.

Proof. Since $Z(A_\Gamma) = A_{K^d}$ by Lemma 5.1.5 and the centre of a group is characteristic, we have by Proposition 2.2.3 the isomorphism

$$\frac{A_\Gamma}{A_{K^d}} \cong A_{\Gamma'}$$

where

$$\Gamma' = \bigstar_{j=1}^k (*_{i_j} \Gamma_j). \quad (5.3)$$

The maximality of the decomposition of Γ implies that (5.3) is the maximal decomposition of Γ' . Hence, Proposition 5.1.6 implies that

$$\text{Aut}(A_{\Gamma'}) \cong \bigtimes_{j=1}^k (\text{Aut}(A_{\Gamma_j}) \wr S_{i_j}).$$

Suppose that $A_{\Gamma_j} \in R_\infty$ for some $1 \leq j \leq k$. Then the isomorphism above combined with Propositions 5.1.1 and 5.1.2 implies that $A_{\Gamma'} \in R_\infty$ as well. Since $A_{\Gamma'}$ is a characteristic quotient of A_Γ , we consequently have that $A_\Gamma \in R_\infty$. $\square$

Example 5.1.8. As an application of the previous theorem, consider the smallest family $\mathcal{G}$ of groups satisfying the following two properties: $\mathcal{G}$ contains $\mathbb{Z}$ and $\mathcal{G}$ is closed under taking finite free products and finite direct products. We claim that any non-abelian group in $\mathcal{G}$ has the R_∞ -property. Note that $\mathcal{G}$ contains only RAAGs.

Let $G \in \mathcal{G}$ be a non-abelian group and write $G = A_\Gamma$ for some (non-complete) graph Γ . The group G splits as either a free or direct product of two groups in $\mathcal{G}$. In the former case, Γ is disconnected, hence $G \in R_\infty$ by Theorem 4.1.4. In the latter case, write $\Gamma = \Gamma_1 * \dots * \Gamma_k$ in maximal decomposition, with $k \geq 2$. Since G is non-abelian, one factor, say, Γ_1 , is non-complete. As the decomposition of Γ is maximal and $A_{\Gamma_1} \in \mathcal{G}$, we must have that Γ_1 is disconnected. Consequently, $A_{\Gamma_1} \in R_\infty$ by Theorem 4.1.4. We can then apply Theorem 5.1.7 to conclude.

In particular, any (non-abelian) finite direct product of free groups has the R_∞ -property.

We would like to remark that Theorem 5.1.7 does *not* state that any direct product of RAAGs has the R_∞ -property if (at least) one factor has it. It does state so for certain types of RAAGs, namely those for which their corresponding graph does not split as a simplicial join. Nonetheless, the result is strong enough for our purposes.

5.2 Regular graphs

We formally state the definition of a regular graph.

Definition 5.2.1. Let $\Gamma(V, E)$ be a graph and $k \geq 0$. We call Γ *k-regular* if $\deg(v) = k$ for all $v \in V$. We call Γ *regular* if there is some $k \geq 0$ such that Γ is *k-regular*.

For example, the complete graph K^n is $(n - 1)$ -regular, whereas all cycle graphs on at least 3 vertices are 2-regular. Clearly, there is (up to isomorphism) only one 0-regular graph on n vertices, namely $\overline{K}^n$. We start with some very basic properties of regular graphs. For a proof, we refer the reader to [3, Chapter 2].

Lemma 5.2.2 (Handshaking Lemma). *For a graph Γ the following equality holds:*

$$2|E| = \sum_{v \in V} \deg(v).$$

Lemma 5.2.3. *Let Γ be a k -regular graph on n vertices. Then*

- (i) $kn \equiv 0 \pmod{2}$.
- (ii) $\overline{\Gamma}$ is $(n - k - 1)$ -regular.

With this lemma, we classify all k -regular graphs on n vertices with $k \in \{1, 2, n - 2, n - 3\}$.

Proposition 5.2.4. *Suppose Γ is a 1-regular graph on n vertices. Then Γ is the disjoint union of $n/2$ copies of K^2 .*

Proof. Note that n is even, since $n \cdot 1 \equiv 0 \pmod{2}$ by Lemma 5.2.3(i). Write $\Gamma = \Gamma_1 \sqcup \dots \sqcup \Gamma_k$ with each Γ_i connected. Then each Γ_i is a connected 1-regular graph, so Γ_i contains at least 2 vertices, say v and w , that are connected by an edge. However, Γ_i cannot contain more than 2 vertices, since no additional vertex can be adjacent to either v or w . Hence, Γ_i is (isomorphic to) K^2 , implying that Γ is a disjoint union of $n/2$ copies of K^2 . $\square$

Taking complements, we obtain all $(n - 2)$ -regular graphs on n vertices.

Corollary 5.2.5. *Suppose Γ is an $(n - 2)$ -regular graph on n vertices. Then Γ is the simplicial join of $n/2$ copies of $\overline{K}^2$.*

Corollary 5.2.6. *Suppose Γ is a non-complete graph on n vertices that is either 1-regular or $(n - 2)$ -regular. Then $A_\Gamma \in R_\infty$.*

Proof. If Γ is 1-regular and non-complete, it is disconnected by Proposition 5.2.4, hence $A_\Gamma \in R_\infty$ by Theorem 4.1.4. If Γ is $(n - 2)$ -regular, it is a simplicial join of $n/2$ copies of $\overline{K}^2$. Then Example 5.1.8 implies that $A_\Gamma \in R_\infty$. $\square$

Proposition 5.2.7. *Suppose Γ is a 2-regular graph. Then Γ is the disjoint union of cycle graphs.*

Proof. It is sufficient to prove that every *connected* 2-regular graph is a cycle graph. So, suppose that Γ is connected. Denote by $v_1, \dots, v_n$ the vertices of Γ . Consider a path of maximal length k in Γ . Without loss of generality, this is the path $v_1, v_2, \dots, v_k$. For each $j \in \{2, \dots, k-1\}$, the link of v_j is given by $lk(v_j) = \{v_{j-1}, v_{j+1}\}$. Since $\deg(v_k) = 2$, there is a vertex besides v_{k-1} adjacent to v_k , say v_i . Note that $i \in \{1, k+1, \dots, n\}$. If $i \geq k+1$, then the path was not maximal. Hence, $i = 1$ and since Γ is connected, k must be equal to n ; otherwise, there would not exist a path from v_{k+1} to v_k . We conclude that Γ is indeed a cycle graph. $\square$

Again, taking complements gives us all $(n-3)$ -regular graphs on n vertices.

Corollary 5.2.8. *Suppose Γ is an $(n-3)$ -regular graph on n vertices. Then Γ is the simplicial join of complements of cycle graphs.*

Corollary 5.2.9. *Suppose Γ is a non-complete graph on n vertices that is either 2-regular or $(n-3)$ -regular. Then $A_\Gamma \in R_\infty$.*

Proof. If Γ is 2-regular, then Proposition 5.2.7 implies Γ is either disconnected or a cycle graph. In the former case, $A_\Gamma \in R_\infty$. In the latter, $n \geq 4$ since Γ is non-complete. If $n = 4$, then $\Gamma = \overline{K^2} * \overline{K^2}$. Consequently, Example 5.1.8 implies that $A_\Gamma \in R_\infty$. If $n \geq 5$, then Γ is transvection-free by Example 3.3.23, so Corollary 3.3.22 implies that $A_\Gamma \in R_\infty$.

Now suppose that Γ is $(n-3)$ -regular. Then $\Gamma = \overline{C}_{n_1} * \dots * \overline{C}_{n_k}$ for some $k \geq 1$ and $n_i \geq 3$. Note that the complement of the cycle graph C_3 is $\overline{K^3}$, the complement of C_4 is the disjoint union $K^2 \sqcup K^2$, whereas the complement of the cycle graph C_n for $n \geq 5$ is transvection-free by Example 3.3.23 and Proposition 3.3.27. Consequently, $A_{\overline{C}_m} \in R_\infty$ for all $m \geq 3$ by Theorem 4.1.4 and Corollary 3.3.22. Moreover, as each cycle graph is connected, their complements do not split as simplicial joins. Hence, the maximal decomposition of Γ is given by $\overline{C}_{n_1} * \dots * \overline{C}_{n_k}$. Applying Theorem 5.1.7 then gives the result. $\square$

5.3 Strongly regular graphs

We follow [12, Chapter 10] for the definition of strongly regular graphs. As the name suggests, these graphs have strong regularity conditions.

Definition 5.3.1. Let $\Gamma(V, E)$ be a graph. We say that Γ is *strongly regular with parameters* n, k, λ and μ if Γ is a k -regular graph on n vertices such that

- every two adjacent vertices have λ neighbours in common, i.e. if $vw \in E$, then $|lk(v) \cap lk(w)| = \lambda$,
- every two non-adjacent vertices have μ neighbours in common, i.e. if $vw \notin E$ and $v \neq w$, then $|lk(v) \cap lk(w)| = \mu$,
- $1 \leq k < n-1$.

We also say that Γ is an $sg(n, k, \lambda, \mu)$.

Remark. The only 0-regular graph on n vertices is the edgeless graph $\overline{K^n}$. If we would consider this graph to be strongly regular, then λ would be undefined. On the other hand, the only $(n-1)$ -regular graph on n vertices is the complete graph K^n , for which the parameter μ is

undefined. This explains why we require that $1 \leq k < n - 1$. The corresponding RAAGs of the aforementioned graphs are the free group and the free abelian group, respectively, and for both we already determined the Reidemeister spectrum. Hence, it is no loss to exclude these graphs from the strongly regular ones.

Note that we thus can assume that $n \geq 2$ and in that case, $\lambda \leq k - 1$ and $\mu \leq k$. For an example of a strongly regular graph, we first need a definition.

Definition 5.3.2. Let $\Gamma(V, E)$ be a graph and $p \geq 2$. We say that Γ is *p-partite* if V admits a partition $V_1, \dots, V_p$ such that no vertices in V_i are connected by an edge. If Γ is *p-partite* for some p , we say that Γ is *multipartite*.

Example 5.3.3. Let $p \geq 2$ and $n_1, \dots, n_p$ be strictly positive integers. The *complete p-partite graph of order $n_1, \dots, n_p$* is the graph

$$K(n_1, \dots, n_p) := \overline{K^{n_1}} * \overline{K^{n_2}} * \dots * \overline{K^{n_p}}.$$

If $n_1 = n_2 = \dots = n_p =: n$, we also write K_n^p . If $n = 1$, then K_1^p is the complete graph on p vertices, hence the notation coincides.

Now, we claim that for $n \geq 2$, K_n^p is an $srg(np, n(p-1), n(p-2), n(p-1))$. Clearly, the number of vertices is np . If we denote by $V_1, \dots, V_p$ the partition of the vertex set V , then $|V_i| = n$ for all i and each $v \in V_i$ is connected with every vertex in $V \setminus V_i$. Hence, $\deg(v) = (p-1)n$, so K_n^p is $(p-1)n$ -regular.

Two vertices that are not adjacent lie in the same V_i , hence they have $n(p-1)$ common neighbours. If two vertices v, w are adjacent, then $v \in V_i$ and $w \in V_j$ with $i \neq j$. A vertex $v' \in V$ is a neighbour of v if and only if $v' \notin V_i$ and it is a neighbour of w if and only if $v' \notin V_j$. Hence, the common neighbours of v and w are precisely all vertices in $V \setminus (V_i \cup V_j)$, which contains $n(p-2)$ elements.

The regularity conditions for a strongly regular graph Γ restrict the possibilities for characteristic vertex-subgroups: as Γ is regular, $V_{\max} = V$, and $V_{\overline{\tau}}$ will be either empty or the whole of V . In order to prove this last statement, it will be more convenient to work with the complement of $V_{\overline{\tau}}$, which we now give a name.

Definition 5.3.4. Let $\Gamma(V, E)$ be a graph. We call the complement of $V_{\overline{\tau}}$ the set of all *transvection-admitting vertices* and we denote it by V_{τ} . Note that

$$V_{\tau} = \{v \in V \mid \exists w \neq v \in V : v \leq w\}.$$

Lemma 5.3.5. Let Γ be a k -regular graph and $v, w \in V$ two vertices. Then

$$(i) \quad lk(v) \subseteq lk(w) \iff lk(w) \subseteq lk(v) \iff |lk(v) \cap lk(w)| = k.$$

$$(ii) \quad st(v) \subseteq st(w) \iff st(w) \subseteq st(v) \iff |st(v) \cap st(w)| = k + 1.$$

Proof. Both statements follow directly from the fact that $|lk(v)| = |lk(w)| = k$ and $|st(v)| = |st(w)| = k + 1$. $\square$

Proposition 5.3.6. Let Γ be an $srg(n, k, \lambda, \mu)$. Then

$$V_{\tau} = \begin{cases} \emptyset & \text{if } \lambda < k - 1 \text{ and } \mu < k \\ V & \text{otherwise.} \end{cases} \quad (5.4)$$

Proof. As mentioned before, we can assume that $n \geq 2$. Let $v, w \in V$ be distinct vertices. We look for equivalent conditions for $v \leq w$. We distinguish two cases.

Case 1: $vw \notin E$. Then $v \leq w$ is equivalent with $lk(v) \subseteq lk(w)$ and hence with $|lk(v) \cap lk(w)| = k$, by the previous lemma. As Γ is strongly regular, $|lk(v) \cap lk(w)| = \mu$. Hence, $lk(v) \subseteq lk(w)$ if and only if $k = \mu$.

Case 2: $vw \in E$. In this case, $v \leq w$ is equivalent with $st(v) \subseteq st(w)$ and hence with $|st(v) \cap st(w)| = k + 1$. Note that

$$|st(v) \cap st(w)| = |\{v, w\} \cup (lk(v) \cap lk(w))| = 2 + \lambda$$

Therefore, $st(v) \subseteq st(w)$ if and only if $\lambda = k - 1$.

With this information, we can prove (5.4): if $\lambda < k - 1$ and $\mu < k$, then $v \leq w$ can never happen, hence $V_\tau = \emptyset$.

If $\mu = k$, recall that Γ is a k -regular non-complete graph. Hence, for every vertex v there is a non-adjacent vertex w . Then $lk(v) \subseteq lk(w)$ as $\mu = k$. Consequently, $V_\tau = V$.

If $\lambda = k - 1$, take a vertex v and one of its neighbours w (which exists, as $k \geq 1$). As $\lambda = k - 1$, $st(v) \subseteq st(w)$. This holds for all $v \in V$, so $V_\tau = V$. $\square$

Corollary 5.3.7. *If Γ is an $srg(n, k, \lambda, \mu)$ with $\lambda < k - 1$ and $\mu < k$, then $A_\Gamma \in R_\infty$.*

Proof. From the previous proposition, it follows that $V_\tau = \emptyset$, hence Γ is transvection-free. Applying Corollary 3.3.22 now yields the result. $\square$

A natural question is to ask which strongly regular graphs have either $\lambda = k - 1$ or $\mu = k$. To answer this question, we first point out a relation between those two kinds of graphs.

Lemma 5.3.8 ([12, p. 218]). *Let Γ be an $srg(n, k, \lambda, \mu)$. Then $\bar{\Gamma}$ is an $srg(n, n - k - 1, n - 2k - 2 + \mu, n - 2k + \lambda)$.*

Lemma 5.3.9 ([12, p. 219]). *Let Γ be an $srg(n, k, \lambda, \mu)$. Then*

$$(n - k - 1)\mu = k(k - \lambda - 1).$$

In particular, $\mu = k$ if and only if $\lambda = 2k - n$, and $\lambda = k - 1$ if and only if $\mu = 0$.

Hence, if Γ is an $srg(n, k, k - 1, 0)$, then $\bar{\Gamma}$ is an $srg(n, k', \lambda', \mu')$ with

$$\begin{aligned} k' &= n - k - 1 \\ \lambda' &= n - 2k - 2 = 2k' - n \\ \mu' &= n - k - 1 = k'. \end{aligned}$$

So, we only have to determine the strongly regular graphs with $\mu = k$ in order to determine all srg 's with either $\mu = k$ or $\lambda = k - 1$.

Proposition 5.3.10 ([12, Lemma 10.1.1]). *Let Γ be an $srg(n, k, \lambda, \mu)$. If $\lambda = k - 1$, then Γ is the disjoint union of $\frac{n}{k+1}$ copies of K^{k+1} .*

Corollary 5.3.11. *Let Γ be an $srg(n, k, \lambda, \mu)$. If $\mu = k$, then Γ is the complete multipartite graph $K_{n-k}^{n/(n-k)}$.*

Proof. By Lemma 5.3.8 and Proposition 5.3.10, $\bar{\Gamma}$ is the disjoint union of $\frac{n}{n-k}$ copies of K^{n-k} . Taking complements again implies that Γ is the simplicial join of $\frac{n}{n-k}$ copies of $\overline{K^{n-k}}$, i.e. Γ is the complete multipartite graph $K_{n-k}^{n/(n-k)}$. $\square$

The RAAGs associated to srg's with $\lambda = k - 1$ and $\mu = k$ are

$$\bigstar_{i=1}^{\frac{n}{k+1}} \mathbb{Z}^{k+1} \quad \text{and} \quad \bigtimes_{i=1}^{\frac{n}{n-k}} F_{n-k},$$

respectively. The first group has the R_∞ -property by Theorem 4.1.4. For the second, we invoke Example 5.1.8. Combining with Corollary 5.3.7, we find the following result.

Theorem 5.3.12. *Let Γ be a strongly regular graph. Then $A_\Gamma \in R_\infty$.*

6 Max-by-abelian graphs

Max-by-abelian graphs possess more structure than arbitrary (connected) graphs, but this structure is much less apparent than the structure of regular graphs, for instance. We therefore start by discussing general graph theoretical properties of max-by-abelian graphs, and we also provide some examples. Thereafter, we move on to their associated RAAGs and use the aforementioned structural properties to prove the R_∞ -property in certain cases.

6.1 Examples and general properties

We extend the definition of a max-by-abelian graph with some parameters.

Definition 6.1.1. Let $\Gamma(V, E)$ be a graph. We say that Γ is (n, k, d) -max-by-abelian (or (n, k, d) -MBA) if

- Γ is non-regular and connected,
- $\Gamma(V \setminus V_{\max})$ is complete,
- $|V| = n$, $|V_{\max}| = k$ and $\Delta(\Gamma) = d$.

In particular, $k \leq n - 1$ and $d \leq n - 2$, since Γ is non-regular (and thus a fortiori non-complete). Figure 2 shows an example of a $(5, 4, 3)$ -MBA graph with $V_{\max} = \{v_1, v_2, v_3, v_4\}$.

Figure 2: A $(5, 4, 3)$ -max-by-abelian graph

Example 6.1.2. Let n, n_1, n_2 be strictly positive integers with $n \leq n_1, n \leq n_2$ and $n^2 < n_1 n_2$. We claim that $\Gamma := (K^n \sqcup K^{n_1}) * (K^n \sqcup K^{n_2})$ is an MBA-graph. It is clearly connected. Denote the vertices of the first copy of K^n by a_i , of the second copy by c_i , the vertices of K^{n_1} by b_i and those of K^{n_2} by d_i . Then

$$\begin{aligned}\deg(a_i) &= n - 1 + n + n_2 = 2n + n_2 - 1 \\ \deg(b_i) &= n_1 - 1 + n + n_2 = n + n_1 + n_2 - 1 \\ \deg(c_i) &= n - 1 + n + n_1 = 2n + n_1 - 1 \\ \deg(d_i) &= n_2 - 1 + n + n_1 = n + n_1 + n_2 - 1\end{aligned}$$

Since at least one of the inequalities $n < n_1$ and $n < n_2$ holds, Γ is not regular. We may assume that $n_1 \leq n_2$. If $n < n_1$, then

$$V_{\max} = \{b_i \mid 1 \leq i \leq n_1\} \cup \{d_i \mid 1 \leq i \leq n_2\}$$

and $\Gamma(V \setminus V_{\max}) = K^n * K^n = K^{2n}$ is complete. In this case, Γ is an $(2n + n_1 + n_2, n_1 + n_2, n + n_1 + n_2 - 1)$ -MBA graph.

If $n = n_1$, then $n < n_2$ and $V_{\max} = V \setminus \{c_i \mid 1 \leq i \leq n\}$. In this case, $\Gamma(V \setminus V_{\max}) = K^n$, so Γ is an $(2n + n_1 + n_2, n + n_1 + n_2, n + n_1 + n_2 - 1)$ -MBA graph.

Note that the corresponding RAAG is given by $(\mathbb{Z}^n * \mathbb{Z}^{n_1}) \times (\mathbb{Z}^n * \mathbb{Z}^{n_2})$, which has the R_∞ -property by Theorems 4.1.4 and 5.1.7.

The first properties we derive are inequalities providing relations amongst or restrictions on the parameters n, k and d .

Proposition 6.1.3. *Let Γ be an (n, k, d) -MBA graph. Then*

$$\binom{n}{2} - k(n - d - 1) \leq |E| \leq \frac{n(d - 1) + k}{2}. \quad (6.1)$$

Proof. The inequalities follow from a count of the minimal resp. maximal number of edges in Γ . First, we prove the upper bound. Every vertex in $V_{\max}$ has degree d , so every vertex in $V \setminus V_{\max}$ has at most degree $d - 1$. We thus have that

$$|E| = \frac{1}{2} \sum_{v \in V} \deg(v) \leq \frac{kd + (n - k)(d - 1)}{2} = \frac{n(d - 1) + k}{2}.$$

The lower bound is more involved. Since the induced subgraph $\Gamma(V \setminus V_{\max})$ is the complete graph on $n - k$ vertices, Γ has at least $\binom{n - k}{2}$ edges. We now count how many edges we deleted going from Γ to $\Gamma(V \setminus V_{\max})$. Put $V_{\max} = \{v_1, \dots, v_k\}$. Deleting v_1 results in deleting d edges, one for every neighbour of v_1 . Deleting v_i after deleting v_1 up to v_{i-1} results in deleting at least $d - i + 1$ edges. Indeed, the only edges with endpoint v_i that could already have been deleted are those with v_j with $1 \leq j \leq i - 1$ as (other) endpoint. There are at most $i - 1$ such vertices and since $\deg(v_i) = d$, there are at least $d - i + 1$ edges with one endpoint equal to v_i and one in $V \setminus V_{\max}$. Hence, deleting v_i after deleting v_1 up to v_{i-1} results in deleting at

least $d - i + 1$ edges. It follows that

$$\begin{aligned}
|E| &\geq \binom{n-k}{2} + \sum_{i=1}^k (d-i+1) \\
&= \binom{n-k}{2} + k(d+1) - \frac{k(k+1)}{2} \\
&= \frac{1}{2}((n-k)(n-k-1) + 2kd + 2k - k^2 - k) \\
&= \frac{n^2 - 2nk + k^2 - n + k + 2kd - k^2 + k}{2} \\
&= \frac{n^2 - n}{2} + kd + k - nk \\
&= \binom{n}{2} - k(n-d-1). \quad \square
\end{aligned}$$

Corollary 6.1.4. *Let Γ be an (n, k, d) -MBA graph. Then $n < 2k$ and $d \leq n - \frac{k}{2k-n}$.*

Proof. We transform the inequality in (6.1):

$$\begin{aligned}
&\binom{n}{2} - k(n-d-1) \leq \frac{n(d-1) + k}{2} \\
\iff n^2 - n - 2k(n-d-1) &\leq nd - n + k \\
\iff n^2 - 2kn + 2kd + 2k &\leq nd + k \\
\iff n^2 - nd &\leq k(2n - 2d - 1). \tag{6.2}
\end{aligned}$$

By dividing both sides of the last inequality by $2n - 2d - 1 \neq 0$, we have

$$\frac{n}{2} < \frac{n(n-d)}{2(n-d)-1} \leq k,$$

where the strict inequality is equivalent with $2n(n-d) - n < 2n(n-d)$.

Continuing from (6.2), we find

$$d(2k - n) \leq 2kn - k - n^2$$

and hence

$$d \leq \frac{2kn - n^2 - k}{2k - n} = n - \frac{k}{2k - n}. \quad \square$$

Note that since k is an integer, the condition $2k > n$ is equivalent to $k \geq \lceil \frac{n+1}{2} \rceil$.

Lemma 6.1.5. *Let Γ be an (n, k, d) -MBA graph. Then $k + d \geq n + 1$.*

Proof. Since $\Gamma(V \setminus V_{\max})$ is isomorphic to K^{n-k} , the degree of a vertex in $V \setminus V_{\max}$ is at least $n - k - 1$. Since Γ is connected, there is a $v \in V \setminus V_{\max}$ adjacent to a vertex $w \in V_{\max}$. For said vertex v , we have $d > \deg(v) \geq n - k$. Hence, $n - k \leq d - 1$, and consequently $n + 1 \leq k + d$. $\square$

All results combined give some non-existence results of certain max-by-abelian graphs.

Corollary 6.1.6. *Let Γ be an (n, k, d) -MBA graph. Then $n \geq 5$.*

Proof. Combining $k \leq n - 1$ with Lemma 6.1.5, we obtain

$$2 = n + 1 - (n - 1) \leq n + 1 - k \leq d \leq n - 2,$$

hence $n \geq 4$. If $n = 4$, then $d = 2$, and $\lceil \frac{n+1}{2} \rceil \leq k \leq n - 1$ implies $k = 3$. The sole vertex v of Γ not in $V_{\max}$ then has degree 1, since Γ is connected. The sum of the degrees is then $k \cdot d + 1 = 7$. This, however, contradicts the Handshaking Lemma. We conclude that $n \geq 5$. $\square$

Proposition 6.1.7. *Let Γ be an $(n, \lceil \frac{n+1}{2} \rceil, d)$ -MBA graph. Then n is even.*

Proof. Suppose n is odd. Then $k = \lceil \frac{n+1}{2} \rceil = \frac{n+1}{2}$. Applying Corollary 6.1.4 gives

$$d \leq n - \frac{(n+1)/2}{2(n+1)/2 - n} = n - \frac{n+1}{2} = \frac{n-1}{2}$$

and thus $k + d \leq n$, whereas Lemma 6.1.5 states that $k + d \geq n + 1$, a contradiction. Hence, n is even. $\square$

6.2 R_∞ -property for large values of k

Next, we focus on the associated RAAG of a max-by-abelian graph Γ . For large values of k , i.e. k close to the upper bound of $n - 1$, the graph is structured enough to find characteristic vertex-subgroups resulting in a non-trivial and non-abelian quotient.

We start with $(n, n - 1, d)$ -MBA graphs.

Proposition 6.2.1. *Let Γ be a graph. Put*

$$\tilde{V} = \bigcap_{v \notin V_{\max}} lk(v).$$

Then $N(\tilde{V})$ is characteristic in A_Γ .

Proof. If $\tilde{V}$ is empty, the claim is trivial. Since $v \notin lk(v)$ for any $v \in V$, we have that $(V \setminus V_{\max}) \cap \tilde{V} = \emptyset$, hence $\tilde{V} \subseteq V_{\max}$.

Let $v \in \tilde{V}$. By Theorem 3.1.2, it is sufficient to prove that $V_{\text{char}}(v) \subseteq \tilde{V}$. Put $V_0 = \{v\}$ and let V_i, V_ω be as in the construction of $V_{\text{char}}(v)$, i.e.

$$V_i = \{w \in V \mid \exists v' \in V_{i-1} : v' \leq w\}$$

and $V_\omega = \bigcup_{n \in \mathbb{N}} V_n$. We prove by induction that $V_i \subseteq \tilde{V}$. For $i = 0$, this is trivial. Suppose that $V_i \subseteq \tilde{V}$ and that $w \in V$ dominates $v' \in V_i$. As $v' \in \tilde{V} \subseteq V_{\max}$ by the induction hypothesis, also $w \in V_{\max}$ and $v'' \in lk(v') \subseteq st(w)$ for all $v'' \in V \setminus V_{\max}$.

Since $w \in V_{\max}$, this implies that $w \in \bigcap_{v'' \notin V_{\max}} lk(v'') = \tilde{V}$. Consequently, $V_{i+1} \subseteq \tilde{V}$. We conclude that $V_\omega \subseteq \tilde{V}$.

Now, for $\varphi \in \text{Aut}(\Gamma)$, note that $\varphi(V_{\max}) = V_{\max}$, hence $\varphi(V \setminus V_{\max}) = V \setminus V_{\max}$ and thus $\varphi(\tilde{V}) = \tilde{V}$. In particular is $\varphi(V_\omega) \subseteq \varphi(\tilde{V}) = \tilde{V}$, therefore, $V_{\text{char}}(v) \subseteq \tilde{V}$. $\square$

Proposition 6.2.2. *Let Γ be an $(n, n - 1, d)$ -MBA graph. Then $A_\Gamma \in R_\infty$.*

Proof. Let v be the sole vertex in $V \setminus V_{\max}$. By Proposition 6.2.1, $N(lk(v))$ is characteristic in A_Γ . Note that $lk(v) \neq V_{\max}$, since

$$|lk(v)| \leq d - 1 \leq n - 3 < n - 1 = |V_{\max}|.$$

Hence, $\tilde{\Gamma} := \Gamma(V \setminus lk(v))$ is equal to $\Gamma(\{v\}) \sqcup \Gamma(V_{\max} \setminus lk(v))$. Theorem 4.1.4 implies that $A_{\tilde{\Gamma}}$ has the R_∞ -property. Since $A_{\tilde{\Gamma}} \cong \frac{A_\Gamma}{N(lk(v))}$ (by Proposition 2.2.3) is a characteristic quotient of A_Γ , also A_Γ has the R_∞ -property. $\square$

Now we move on to $(n, n - 2, d)$ -MBA graphs.

Proposition 6.2.3. *Let Γ be a graph. Put*

$$\tilde{V} = V_{\max} \cap \left(\bigcup_{v \notin V_{\max}} lk(v) \right).$$

Then $N(\tilde{V})$ is characteristic in A_Γ .

Proof. Let $v \in \tilde{V}$. Again, by Theorem 3.1.2, it is sufficient to prove that $V_{\text{char}}(v) \subseteq \tilde{V}$. Put $V_0 = \{v\}$ and let V_i, V_ω be as usual. We prove by induction that $V_i \subseteq \tilde{V}$. For $i = 0$, this is trivial. Suppose $V_i \subseteq \tilde{V}$ and let $w \in V$ be a vertex dominating $v' \in V_i$. By the induction hypothesis, $v' \in V_{\max}$. Since w dominates v' , $\deg(w) \geq \deg(v') = \Delta(\Gamma)$, so $w \in V_{\max}$.

Also by the induction hypothesis, there exists a $w' \notin V_{\max}$ such that $v' \in lk(w')$. Consequently is $w' \in lk(v') \subseteq st(w)$. As $w' \notin V_{\max}$, we have that $w' \neq w$, so $w' \in lk(w)$. Therefore, $w \in lk(w')$ implying that $w \in \tilde{V}$. We conclude that $V_\omega \subseteq \tilde{V}$.

Next, let $\varphi \in \text{Aut}(\Gamma)$. As $\varphi(V_{\max}) = V_{\max}$, we have that $\varphi(V \setminus V_{\max}) = V \setminus V_{\max}$ and hence

$$\varphi(\tilde{V}) = V_{\max} \cap \left(\bigcup_{\varphi(v) \notin V_{\max}} lk(\varphi(v)) \right) = V_{\max} \cap \left(\bigcup_{v \notin V_{\max}} lk(v) \right) = \tilde{V}.$$

Consequently, $\varphi(V_\omega) \subseteq \tilde{V}$ and thus $V_{\text{char}}(v) \subseteq \tilde{V}$. $\square$

Before heading to the next result, we state three commutator identities, each of which can be easily checked by working out both sides of the equality.

Lemma 6.2.4. *Let G be a group, and $a, b, c \in G$. Then the following identities hold:*

- (i) $[ab, c] = [a, c]^b [b, c]$
- (ii) $[a^b, c] = [c, b]^{a^b} [a, c]^b [b, c]$
- (iii) $[a^b, c] = [b, a][a, c][a, b]^c$

Proposition 6.2.5. *Let Γ be an $(n, n - 2, d)$ -MBA graph. Write $\{v_1, v_2\} = V \setminus V_{\max}$. Suppose that V is the disjoint union of $lk(v_1)$ and $lk(v_2)$. Then*

- (i) $\Gamma(lk(v_i))$ is disconnected for $i = 1, 2$.
- (ii) The normal subgroup $N := \langle \langle \{[v, w] \mid v \in lk(v_1), w \in lk(v_2)\} \rangle \rangle_{A_\Gamma}$ is characteristic in A_Γ .

(iii) With N as above and $\tilde{\Gamma} := \Gamma(lk(v_1)) * \Gamma(lk(v_2))$, we have

$$\frac{A_\Gamma}{N} \cong A_{\tilde{\Gamma}}.$$

Proof. We give the proof for $i = 1$, the case $i = 2$ is analogous. Since $v_2 \notin lk(v_2)$ and $lk(v_1) \cup lk(v_2) = V$, we have $v_2 \in lk(v_1)$. Let $w \in lk(v_1) \setminus \{v_2\}$ be a vertex. Because $lk(v_1)$ and $lk(v_2)$ are disjoint, w does not lie in $lk(v_2)$. This means that v_2 is not connected to any other vertex in $lk(v_1)$, implying that v_2 is an isolated vertex in $\Gamma(lk(v_1))$. Hence, $\Gamma(lk(v_1))$ is disconnected.

Let $\varphi \in \text{Aut}(A_\Gamma)$ be an automorphism of basic type. If $\varphi = \iota_a$ is an inversion, then for vertices $b \neq c$ different from a we have

$$\iota_a([a, b]) = [a^{-1}, b] = [b, a]^{a^{-1}}$$

and $\iota_a([b, c]) = [b, c]$. Hence, if $[a, b] \in N$, then $\iota_a([a, b]) \in N$ as well.

Next, suppose that φ is a graph automorphism. As $\varphi(V_{\max}) = V_{\max}$, we have $\varphi(\{v_1, v_2\}) = \{v_1, v_2\}$. So either $\varphi(v_i) = v_i$ or $\varphi(v_i) = v_{3-i}$ for $i = 1, 2$. Let $v \in lk(v_1)$ and $w \in lk(v_2)$. In the first case, $\varphi(v) \in lk(v_1)$ and $\varphi(w) \in lk(v_2)$, so $\varphi([v, w]) = [\varphi(v), \varphi(w)] \in N$. In the second case, $\varphi(v) \in lk(v_2)$ and $\varphi(w) \in lk(v_1)$, and hence also $\varphi([v, w]) = [\varphi(v), \varphi(w)] \in N$.

Suppose $\varphi = \tau_{ab}$ is a transvection with $a \in lk(v_1)$; the case $a \in lk(v_2)$ is analogous and since $lk(v_1) \cup lk(v_2) = V$, this covers all cases. Let $w \in lk(v_2)$. We have to show that $\tau_{ab}([a, w]) = [ab, w]$ lies in N . As $a \in lk(v_1)$, we have $v_1 \in lk(a) \subseteq st(b)$, where the latter inclusion follows from $a \leq b$.

First, suppose that $b \neq v_1$. Then $v_1 \in lk(b)$, so $b \in lk(v_1)$, implying that $[b, w] \in N$. Thus,

$$\tau_{ab}([a, w]) = [ab, w] = [a, w]^b [b, w] \in N.$$

Now, if $b = v_1$, then $\deg(a) \leq \deg(v_1) < \Delta(\Gamma)$, implying that $a = v_2$, as $V \setminus V_{\max} = \{v_1, v_2\}$. Hence, $lk(v_2) \subseteq st(v_1)$. As $lk(v_1)$ and $lk(v_2)$ are disjoint, this implies that $lk(v_2) = \{v_1\}$. Consequently, $lk(v_1) = V \setminus \{v_1\}$ and thus $\deg(v_1) = n - 1$, which contradicts $v_1 \notin V_{\max}$.

Finally, suppose $\varphi = \gamma_{b,C}$ is a partial conjugation. Let $v \in lk(v_1)$ and $w \in lk(v_2)$. If v and w are both (not) conjugated by b via φ , we are done. So, suppose $\varphi(v) = v^b$ and $\varphi(w) = w$. By Lemma 6.2.4, we have

$$[v^b, w] = [w, b]^{v^b} [v, w]^b [b, w] \tag{6.3}$$

$$= [b, v][v, w][v, b]^w. \tag{6.4}$$

If $b \in lk(v_1)$, then (6.3) implies that $[v^b, w] \in N$. If $b \in lk(v_2)$, then the result follows from (6.4). This finishes the proof that N is characteristic in A_Γ .

The isomorphism is a direct application of Proposition 2.2.6. $\square$

Corollary 6.2.6. *Let Γ be an $(n, n - 2, d)$ -MBA graph satisfying the same conditions as in Proposition 6.2.5. Then $A_\Gamma \in R_\infty$.*

Proof. We use the same notation as before. Since $\Gamma(lk(v_1))$ is disconnected, Theorem 4.1.4 implies that $A_{\Gamma(lk(v_1))} \in R_\infty$. Putting $\tilde{\Gamma} = \Gamma(lk(v_1)) * \Gamma(lk(v_2))$, we note that this is the maximal decomposition of $\tilde{\Gamma}$, since $\Gamma(lk(v_1))$ and $\Gamma(lk(v_2))$ are both disconnected graphs. Therefore, Theorem 5.1.7 implies that $A_{\tilde{\Gamma}} \in R_\infty$ as well. Finally, since $A_{\tilde{\Gamma}}$ is a characteristic quotient of A_Γ , we conclude that A_Γ has the R_∞ -property. $\square$

Example 6.2.7. We provide two examples of $(n, n - 2, d)$ -MBA graphs: one satisfying the conditions of Proposition 6.2.5 and one not. Consider the graph $\Gamma_1 := (K^1 \sqcup K^2) * (K^1 \sqcup K^2)$ (see Figure 3a; one copy of $K^1 \sqcup K^2$ has solid vertices, the other one hollow). From Example 6.1.2, we know that Γ_1 is a $(6, 4, 4)$ -MBA graph.

Figure 3: Further examples of MBA-graphs

The vertices in $V \setminus V_{\max}$ are w_0 and w_1 , for which we have

$$lk(w_0) = \{w_1, v_0, v_1\}, \quad lk(w_1) = \{w_0, v_2, v_3\},$$

showing that Γ_1 satisfies the conditions of Proposition 6.2.5.

On the other hand, consider Γ_2 as in Figure 3b. Then $V \setminus V_{\max} = \{w_0, w_1\}$, but v_0 is adjacent to both w_0 and w_1 . Moreover, v_2 lies in neither $lk(w_0)$ nor $lk(w_1)$.

For $(n, n - 2, d)$ -MBA graphs not satisfying the conditions of Proposition 6.2.5, we can nonetheless find a suitable characteristic vertex subgroup.

Lemma 6.2.8. *Let Γ be a max-by-abelian graph. Then*

$$\bigcap_{v \notin V_{\max}} lk(v) \subsetneq V_{\max} \tag{6.5}$$

and

$$V_{\max} \cap \left(\bigcup_{v \notin V_{\max}} lk(v) \right) \neq \emptyset. \tag{6.6}$$

Proof. In Proposition 6.2.1, we have proven that the inclusion of (6.5) holds, albeit not necessarily strict. Suppose equality holds. Then for $v \in V \setminus V_{\max}$, we have $V_{\max} \subseteq lk(v)$. As $\Gamma(V \setminus V_{\max})$ is complete, we also have

$$V \setminus (V_{\max} \cup \{v\}) \subseteq lk(v).$$

Consequently, $V \setminus \{v\} \subseteq lk(v)$, contradicting $v \notin V_{\max}$. Hence, the inclusion in (6.5) is strict.

For (6.6), note that since Γ is connected, there is a $v \notin V_{\max}$ that is adjacent to a vertex in $V_{\max}$. Hence, the intersection in (6.6) is non-empty. $\square$

Proposition 6.2.9. *Let Γ be an $(n, n - 2, d)$ -MBA graph and put $V \setminus V_{\max} = \{v_1, v_2\}$.*

(i) *If $lk(v_1) \cap lk(v_2) \neq \emptyset$, then $\Gamma(V \setminus (lk(v_1) \cap lk(v_2)))$ is non-complete.*

(ii) If $lk(v_1) \cup lk(v_2) \neq V$, then $\Gamma(V \setminus (V_{\max} \cap (lk(v_1) \cup lk(v_2))))$ is non-complete.

In either case, A_Γ has a non-abelian RAAG $A_{\Gamma'}$ as characteristic quotient with $|V'| < |V|$.

Proof. Suppose $lk(v_1) \cap lk(v_2) \neq \emptyset$. Note that $v_1, v_2 \in V \setminus (lk(v_1) \cap lk(v_2))$ and because $lk(v_1) \cap lk(v_2) \subsetneq V_{\max}$ by Lemma 6.2.8, there is a vertex $v \in V_{\max} \setminus (lk(v_1) \cap lk(v_2))$. Then by definition of v , either $vv_1 \notin E$ or $vv_2 \notin E$, showing that $\Gamma(V \setminus (lk(v_1) \cap lk(v_2)))$ is non-complete.

On the other hand, suppose $lk(v_1) \cup lk(v_2) \neq V$. As both v_1 and v_2 lie in $lk(v_1) \cup lk(v_2)$, there is a $v \in V_{\max}$ such that $v \notin lk(v_1) \cup lk(v_2)$. Then v, v_1 and v_2 all lie in $V \setminus (V_{\max} \cap (lk(v_1) \cup lk(v_2)))$ and by definition of v is $vv_1 \notin E$. Hence, $\Gamma(V \setminus (V_{\max} \cap (lk(v_1) \cup lk(v_2))))$ is non-complete.

The claim regarding the characteristic quotients follows for (i) from Lemma 6.2.8 and Proposition 6.2.1. For (ii), it follows from Lemma 6.2.8 and Proposition 6.2.3. $\square$

7 Partial answer to conjecture

Combining all the obtained results, we are able to partially answer the conjecture.

Theorem 7.1.1. *Let Γ be a non-complete graph on at most 7 vertices. Then A_Γ has the R_∞ -property.*

The proof boils down to applying the Simplification Lemma and showing that the results from Sections 4 to 6 cover all remaining cases when Γ has at most 7 vertices, see [23, p. 155] for the details.

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