On the variational modeling of non-associative plasticity^{☆,☆☆}

Jacinto Ulloa^a, Roberto Alessi^b, Jef Wambacq^a, Geert Degrande^a, Stijn François^a

^a*KU Leuven, Department of Civil Engineering, Kasteelpark Arenberg 40, B-3001 Leuven, Belgium*

^b*Università di Pisa, Department of Civil and Industrial Engineering, Largo Lucio Lazzarino 2, 56122 Pisa, Italy*

Abstract

In this work, the energetic formulation for rate-independent dissipative materials is extended to consider non-associative plasticity models. In associative models, the fulfilment of the principle of maximum dissipation naturally leads to a variational formulation of the evolution problem. This is no longer true for non-associative models, which are generally presented in the literature in a non-variational form. However, recent studies have unveiled the possibility to recover a variational structure in non-associative plasticity by relying on a suitable state-dependent dissipation potential. Here, this idea is further elaborated in the framework of the energetic formulation, providing a systematic variational approach to non-associative plasticity. A clear link between the classical governing equations of non-associative plasticity and the energetic formulation is established, for which a state-dependent dissipation potential is derived from a generalization of the principle of maximum dissipation. The proposed methodology is then applied to recast specific non-associative plasticity models in variational form, highlighting the flexibility of the formulation. The examples include a Drucker-Prager model with combined isotropic-kinematic hardening and a ratcheting plasticity model. Several thermomechanical insights are provided for both examples. Moreover, exploiting the flexibility of the energetic formulation, extensions to gradient plasticity are devised, leading to representative finite element simulations.

Keywords: Non-associative plasticity, maximum dissipation principle, variational methods, energetic formulation, gradient plasticity

[☆]Postprint version.

^{☆☆}Published version: J. Ulloa, J. Wambacq, R. Alessi, G. Degrande, and S. François. On the variational modeling of non-associative plasticity. *International Journal of Solids and Structures*, 2021.
DOI: <https://doi.org/10.1016/j.ijsolstr.2020.10.027>

*Corresponding author: Jacinto Ulloa

Email addresses: jacintoisrael.ulloa@kuleuven.be (Jacinto Ulloa), roberto.alessi@unipi.it (Roberto Alessi), jef.wambacq@kuleuven.be (Jef Wambacq), geert.degrande@kuleuven.be (Geert Degrande), stijn.francois@kuleuven.be (Stijn François)

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1. Introduction

Continuum mechanics with internal variables provides a solid foundation to develop constitutive models that describe the inelastic behavior of different materials, despite the large differences in their microscopic structure [1]. In continuum elastoplasticity, to which this work is addressed, the goal is to describe deformations at the continuum scale caused by slips and dislocations in the microstructure of materials loaded beyond their resistance to plastic deformations. By now, a variety of well-established elastoplasticity models can be found in the literature [2–6]. Among these, *non-associative* models are particularly useful to describe a wide range of complex material behaviors. However, the mathematical formulation of non-associative models is plagued by difficulties related to the (apparent) absence of extremal energy principles.

To provide a realistic and physically meaningful representation of deforming solids, a material model must capture experimentally observed responses while obeying basic thermodynamic principles [7, 8]. In addition, an important feature of material models is the existence of a variational structure, which, among other benefits, allows for a natural derivation of the governing equations from energy minimization principles, allows to use the tools of calculus of variations to study the existence and uniqueness of solutions, and provides numerical insights for the solution of the associated (non-linear) boundary-value problem [9, 10]. The variational structure is inherent to *associative* plasticity models; however these models are applicable to a very limited range of material responses. Hence, more complex, non-associative models are often employed, which unfortunately lack a variational structure and its convenient properties. As a consequence, the mathematical treatment of non-associative plasticity is much less studied than its associative counterpart [11].

The simplest plasticity models have been developed in the context of ductile metals under monotonic loading, where plastic deformations are mainly driven by deviatoric strains. Models of this kind are typically associative, built upon the von Mises-based, perfectly plastic Prandtl-Reuss model, and endowed with hardening behaviors. On the other hand, geomaterials such as soils, rocks and concrete exhibit non-isochoric plastic deformations, for which a realistic representation of pressure-dependence and dilation requires the use of non-associative models. Models of this kind are usually rooted in the well-known Mohr-Coulomb and Drucker-Prager theories [12], or critical state soil mechanics [13].

Non-associative models are also essential to model the response of materials subjected to cyclic loading. Stress reversals exhibit the Bauschinger effect, where the yield strength in compression decreases with initial yielding in tension, and vice versa. Under cyclic loading, this effect is typically accompanied by a variation of the hardening properties. Moreover, under asymmetrical cycles in tension and compression, ratcheting effects and stress relaxation are commonly observed under force loading and displacement loading, respectively [1, 14]. While the Bauschinger effect can be reasonably captured by associative models with kinematic hardening, modeling ratcheting and stress relaxation requires the use of non-associative models that incorporate a strain accumulation measure, namely, a ratcheting strain. Models of this nature have been mainly developed in the context of metals [15–17]. However, ratcheting effects are also present in geomaterials, which has motivated the development of cyclic plasticity models to address different problems in geotechnical engineering [18–21]. Moreover, it is worth mentioning that the relevance of complex plastic behaviors goes far beyond these applications since practically all materials exhibit some degree of plasticity, ranging from electrode materials [22, 23] to biological tissues and cell aggregates [24–26].

The basic ingredients of elastoplasticity models consist of (i) a yield surface that bounds the elastic domain, commonly expressed in terms of a yield function and (ii) evolution equations for the internal variables, generally referred to as the flow rule. In associative models, the flow rule complies with the normality law, meaning that the rate of change of internal variables is normal to the yield surface. In non-associative mod-

els, normality is abandoned in favor of a more general flow rule, derived from a *plastic potential* different from the yield function.

A practical approach to the formulation of elastoplasticity models consists of postulating the above-mentioned ingredients from a purely phenomenological perspective, where the fulfilment of the second law of thermodynamics is not enforced a priori and a variational structure is not guaranteed. On the other hand, exhaustive research has focused on relating plasticity theories to physical principles embedded in a well-defined mathematical structure [27]. Undoubtedly, the most fruitful progress has taken place in the context of associative models, starting with the seminal work of Hill [28]. An important landmark is the theory of generalized standard materials [7, 29–31], an approach that satisfies the second law of thermodynamics a priori and postulates the existence of two basic energetic quantities: an energy storage potential and a dissipation potential. The governing equations can be derived from knowledge of these two quantities, and the evolution problem naturally admits a variational form. Indeed, the variational formulation of associative plasticity [32] is built upon generalized standard materials, where two versions can be distinguished: (i) the primal form, which makes direct use of the dissipation potential, with the displacements and internal variables as primary unknowns, and (ii) the dual form, which makes direct use of the yield function, with generalized stresses as independent variables.

The primal variational formulation of plasticity can be viewed as a particular case of the energetic formulation for rate-independent systems [33], which provides a more general variational framework for the evolution of various rate-independent processes using notions of energy balance and stability. Still rooted in generalized standard materials, the energetic formulation unifies and generalizes the more conventional variational theories, while offering a weak formulation that can naturally handle discontinuities in time and space [34].

Despite their advantages, theories based on generalized standard materials rely on the normality law and thus fail to account for non-associative models, for which a variational structure is seemingly precluded. Nevertheless, several efforts have been made to recast non-associative models in variational form. For instance, Polizzotto [35] considered a *reduced dissipation* principle using auxiliary variables for models with smooth yield functions and strictly convex, twice-differentiable plastic potentials. Another approach is the theory of *implicit standard materials* [36–41], where a *bi-potential* that depends on both internal variable rates and generalized stresses is introduced, leading to an implicit evolution problem. More recently, numerical schemes based on variational constitutive updates at finite strains [9, 42] were extended to specific non-associative models by Mosler and Bruhns [43] and Mosler [44], where auxiliary *pseudo-stress* variables were introduced to enforce the flow rule.

In close relation to the energetic formulation, recent mathematical studies have re-established an associative setting in non-associative models [11], recovering a variational structure to prove the existence of a quasi-static evolution. This is achieved by letting the set of admissible stresses depend on the current state. This line of research includes the works of Dal Maso et al. [45], Babadjian et al. [46] and Francfort and Stefanelli [47]. Apart from these works, very recent studies have presented variational principles that allow for a variety of phenomena including non-associative models. Specifically, we refer to the *quasi-extremal energy principle* by Petryk [48] and to the treatment based on *thermodynamic extremal principles* by Hackl et al. [49]. In the latter work, the authors introduce state-dependence by considering certain constraints depending on the thermodynamic driving force. The proposed framework was recently applied to non-associative plasticity in Hackl et al. [50].

Most of the aforementioned works on non-associativity somehow rely on letting the dissipation potential

depend on the current state. Interestingly, a similar idea was considered by Germain et al. [7] for non-associative viscoplasticity. Moreover, state-dependent dissipation potentials have been used in several studies involving non-associative plasticity [21, 51–56], as well as gradient-damage models [57] and fatigue [58, 59].

Inspired by these developments, and closely following the footsteps of Francfort [11], the present study presents a variational framework that allows for a general class of non-associative plasticity models. In contrast to previous works, a systematic approach based on the energetic formulation for rate-independent systems [33] is elaborated, where the evolution problem is written in variational form using notions of energy balance and stability. This approach is fundamentally aligned with previous studies on the variational modeling of damage and plasticity [60–68], all of which are restricted to associative models. Here, it is shown that the evolution of non-associative models can also be described using the energetic formulation. An important step is the derivation of a state-dependent dissipation potential which need not be viewed as an a priori assumption: it is shown how the dissipation potential of a non-associative model can be derived in agreement with the classical governing equations, providing a clear link between the classical and variational formulations. The key to establish this link is the definition of a *generalized principle of maximum dissipation*.

The proposed theoretical framework is used to establish the energetic formulation of two benchmark non-associative models: Drucker-Prager plasticity with combined isotropic-kinematic hardening, and a ratcheting plasticity model based on Houlsby et al. [21]. Mechanical insights are provided for the models as well as a description of the material response. Moreover, representative finite element simulations are performed, for which a straightforward extension to gradient plasticity is devised thanks to the flexibility of the energetic framework. This extension introduces an internal length scale that becomes crucial to overcome the spurious behavior of finite element simulations in cases where the boundary-value problem becomes ill-posed. While this occurrence is mainly attributed to softening responses, it has been recognized that non-associativity is also an underlying cause [69, 70], which has motivated the use of gradient-enhanced models in the present study.

The paper is organized as follows. Section 2 presents a general formulation for elastoplasticity in the context of convex analysis, where the notion of convex duality is restored for non-associative models. Section 3 presents the energetic formulation, which makes use of the results from section 2 to establish the link between classical and variational formulations. These theoretical developments are applied to Drucker-Prager plasticity in section 4, including a discussion on the thermodynamic interpretation of isotropic hardening, and to ratcheting plasticity in section 5. Finally, section 6 presents finite element simulations, where the flexibility of the formulation is further highlighted by an extension to gradient plasticity.

We employ the following notations. Using a compact tensor notation, a dot ($\cdot$) and a colon ($:$) denote products with simple and double contraction, respectively. The Euclidean norm is denoted as $\|\square\|$. The notation $\square_{\text{dev}} := \square - (1/3) \text{tr} \square \mathbf{1}$ is used to denote the deviatoric projection. Moreover, the notation $\|\bullet\|_{\square} := \sqrt{\bullet : \square : \bullet}$ for a second-order tensor $\bullet$ is used, where $\square$ is a fourth-order tensor. Space and time dependence are denoted with vector and scalar arguments, respectively, e.g., $\square(\mathbf{x})$ and $\square(t)$. Functions and function values are denoted using the same symbol: $\square$ is a space-time-dependent function evaluated as $\square(\mathbf{x}, t)$. The distinction between functions and values is made when required for clarity purposes, and is otherwise omitted and inferred from context. An over-dot $\dot{\square}$ is used to denote the derivative with respect to time, while the $\nabla \square$ is used to denote the spatial gradient. In the context of convex analysis, $\partial \square(\bullet)$ denotes the multivalued subdifferential of $\square$ at $\bullet$. Further notational specifications are made as required throughout the text.

2. Classical elastoplasticity in the framework of convex analysis

This section revisits the classical formulation of elastoplasticity in the framework of convex analysis, with focus on the generalization to non-associative models. After stating basic definitions, the concepts of elastic domain and yield function are introduced, in agreement with the classical engineering literature. For clarity and completeness, the main aspects of the formulation are then described in detail, starting with the well-known case of associative plasticity. The problem is presented in two alternative forms: (i) the *dual form*, which relates the internal variable rates to their conjugate generalized stresses by making direct use of the yield function, and (ii) the *primal form*, which relates the generalized stresses to the internal variable rates by means of the dissipation potential. The presentation is then extended to the non-associative case, where a generalization of the principle of maximum dissipation, hereafter referred to as *the generalized principle of maximum dissipation*, is established to recover convex duality and a variational structure. This generalization leads to dependence of the dissipation potential on the generalized-stress state.

Traditionally, the dual form, where the flow rule is explicitly defined, is employed as the basis for the statement of the classical evolution problem. Convex duality allows for a transition to the primal form, which instead is based on the definition of the dissipation potential, the key ingredient for the statement of the subsequent energetic formulation.

2.1. Kinematics, equilibrium and thermodynamic setting

Consider an arbitrary solid occupying an open domain $\Omega \subset \mathbb{R}^3$ with boundary $\Gamma \subset \mathbb{R}^3$, composed of a Dirichlet part Γ_D with imposed displacements $\bar{\mathbf{u}}(\mathbf{x}, t) \in \mathbb{R}^3$ and a Neumann part Γ_N with imposed external tractions $\bar{\mathbf{t}}(\mathbf{x}, t) \in \mathbb{R}^3$, such that $\Gamma_D \cup \Gamma_N = \Gamma$ and $\Gamma_D \cap \Gamma_N = \emptyset$. The solid can also be subjected to body forces $\rho \mathbf{b}(\mathbf{x}, t) \in \mathbb{R}^3$, where ρ is the mass density. The deformation process is assumed to be quasi-static, occurring in a pseudo-time (loading) interval $T := [0, t_{\max}]$.

The displacement field, given by $\mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$, is kinematically admissible, satisfying boundary conditions $\mathbf{u} = \bar{\mathbf{u}}$ on Γ_D . We restrict the formulation to the small-strain setting, with the compatible symmetric strain tensor $\boldsymbol{\varepsilon}(\mathbf{x}, t) \in \mathbb{R}_{\text{sym}}^{3 \times 3} := \{\mathbf{e} \in \mathbb{R}^{3 \times 3} \mid \mathbf{e} = \mathbf{e}^T\}$ given by

$$\boldsymbol{\varepsilon} := \nabla^s \mathbf{u}, \quad \text{with} \quad \nabla^s \square := \frac{1}{2}(\nabla \otimes \square + \square \otimes \nabla). \quad (1)$$

The Cauchy stress tensor is denoted by $\boldsymbol{\sigma}(\mathbf{x}, t) \in \mathbb{R}_{\text{sym}}^{3 \times 3}$, satisfying the equilibrium conditions for all $t \in T$:

$$\text{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \mathbf{0} \quad \text{in} \quad \Omega \quad \text{and} \quad \boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{on} \quad \Gamma_N. \quad (2)$$

The dissipative mechanisms that drive inelastic material behavior are characterized by a set of internal variables, here given by $\{\boldsymbol{\varepsilon}^P, \boldsymbol{\xi}\}$, where $\boldsymbol{\varepsilon}^P(\mathbf{x}, t) \in \mathbb{R}_{\text{sym}}^{3 \times 3}$ is the plastic strain tensor and $\boldsymbol{\xi}(\mathbf{x}, t) \in \mathbb{R}^m$ is a vector arrangement of m components, associated to both scalar- and tensor-valued variables. Specifically, $\boldsymbol{\xi}$ contains dependent variables that can be related to $\boldsymbol{\varepsilon}^P$ in rate form, such as hardening variables. In order to simplify the presentation and without losing generality, we consider in this general section the case of local plasticity. The primary variables are then given by the array $\mathbf{q} := \{\mathbf{u}, \boldsymbol{\varepsilon}^P, \boldsymbol{\xi}\}$, while the constitutive state reads $\mathbf{c} := \{\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^P, \boldsymbol{\xi}\}$. Examples with extensions to non-local models are considered in section 6.

The Helmholtz internal energy density is denoted by $\psi := \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^P, \boldsymbol{\xi})$. The second law of thermodynamics is then given for isothermal processes by the Clausius-Planck inequality

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^P, \boldsymbol{\xi}) \geq 0, \quad (3)$$

which must be satisfied by any admissible process. The constitutive stress-strain relation

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\xi}) \quad (4)$$

directly follows from equation (3). Further, we introduce the set of generalized stresses $\{\mathbf{s}^p, \boldsymbol{\eta}\}$ as the thermodynamic conjugate variables to $\{\boldsymbol{\varepsilon}^p, \boldsymbol{\xi}\}$, that is

$$\mathbf{s}^p = -\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^p}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\xi}) \quad \text{and} \quad \boldsymbol{\eta} = -\frac{\partial \psi}{\partial \boldsymbol{\xi}}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\xi}). \quad (5)$$

Replacing equations (4) and (5) in equation (3) yields the following *reduced dissipation inequality*, which inherits the role of the Clausius-Planck inequality (3) to ensure thermodynamic consistency:

$$\phi := \mathbf{s}^p : \dot{\boldsymbol{\varepsilon}}^p + \boldsymbol{\eta} \cdot \dot{\boldsymbol{\xi}} \geq 0. \quad (6)$$

At this point, a remark concerning the definition of generalized stresses is appropriate. In several works [8, 51, 71–74], a pertinent distinction is made between *non-dissipative stresses* obtained directly from the free energy in equation (5), such that $\{\mathbf{s}_{\text{nd}}^p, \boldsymbol{\eta}_{\text{nd}}\} = \{-\mathbf{s}^p, -\boldsymbol{\eta}\}$, and *dissipative stresses*, which appear in the dissipation inequality (6), that is, $\{\mathbf{s}_{\text{d}}^p, \boldsymbol{\eta}_{\text{d}}\} = \{\mathbf{s}^p, \boldsymbol{\eta}\}$. These relations imply that $\{\mathbf{s}^p, \boldsymbol{\eta}\} = \{-\mathbf{s}_{\text{nd}}^p, -\boldsymbol{\eta}_{\text{nd}}\} = \{\mathbf{s}_{\text{d}}^p, \boldsymbol{\eta}_{\text{d}}\}$. Therefore, to alleviate the notation, the formulation in this paper is only written in terms of $\{\mathbf{s}^p, \boldsymbol{\eta}\}$, although the distinction is occasionally made for clarity.

2.2. The elastic domain and the yield function

A common feature of plasticity models is the existence of a set $\mathbb{K}$, called the *elastic domain*, which bounds the admissible generalized stresses. In the dual form, $\mathbb{K}$ is a priori defined and assumed to be non-empty, closed, simply connected, containing the origin and convex. In associative models, the convexity of $\mathbb{K}$ can be shown to descend from the principle of maximum dissipation, allowing to take advantage of the framework of convex analysis [30, 32, 75, 76]. On the other hand, in the primal form, the existence of the elastic domain is a direct consequence of the formulation and the properties of $\mathbb{K}$ descend from the properties of the dissipation potential. In non-associative models, the relation between the elastic domain and the dissipation potential is not straightforward, as will be shown in section 2.4.

The elastic domain can be effectively represented in terms of a convex and continuous function $f: \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}^m \rightarrow \mathbb{R}$, called the *yield function*, such that

$$\mathbb{K} = \{ \{\mathbf{s}^p, \boldsymbol{\eta}\} \in \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}^m \mid f(\mathbf{s}^p, \boldsymbol{\eta}) \leq 0 \}. \quad (7)$$

Accordingly, the boundary of the elastic domain is referred to as the *yield surface*, characterized by $f(\mathbf{s}^p, \boldsymbol{\eta}) = 0$. In order for $\mathbb{K}$ to contain the origin, f must be such that $f(\mathbf{0}, \mathbf{0}) < 0$. A graphical interpretation of $\mathbb{K}$ and f in the space of generalized stresses is given in figure 1.

As will be shown in the model presented in section 4, the yield function may depend explicitly on the constitutive state $\mathbf{c}$, as well as on additional history parameters derived from the time history of $\mathbf{c}$. For notational simplicity, these cases are omitted in the formulation presented in this section, without losing generality. It is important to note that dependence on the constitutive state $\mathbf{c}$ and/or its time history should not be confused with the term *state-dependent*, used extensively hereinafter, which makes reference to dependence on the *generalized stresses* $\{\mathbf{s}^p, \boldsymbol{\eta}\}$ or, more rigorously, on the *non-dissipative stresses* $\{-\mathbf{s}_{\text{nd}}^p, -\boldsymbol{\eta}_{\text{nd}}\} \equiv \{\mathbf{s}^p, \boldsymbol{\eta}\}$. This remark is crucial for the developments of section 2.4.

2.3. The associative case

2.3.1. Dual form

The classical elastoplasticity problem in dual form is built upon the explicit definition of the elastic domain $\mathbb{K}$, which allows to introduce the generalized-stress constraint

$$\{\mathbf{s}^P, \boldsymbol{\eta}\} \in \mathbb{K}. \quad (8)$$

The problem is fully defined once the flow rule that characterizes the evolution of the internal variables is introduced. For the associative case, the flow rule is a consequence of the principle of maximum dissipation [2, 29, 30], which can be stated as follows. Given $\{\dot{\boldsymbol{\varepsilon}}^P, \dot{\boldsymbol{\xi}}\}$, the corresponding stresses $\{\mathbf{s}^P, \boldsymbol{\eta}\} \in \mathbb{K}$ are such that

$$\mathbf{s}^P : \dot{\boldsymbol{\varepsilon}}^P + \boldsymbol{\eta} \cdot \dot{\boldsymbol{\xi}} \geq \tilde{\mathbf{s}}^P : \dot{\boldsymbol{\varepsilon}}^P + \tilde{\boldsymbol{\eta}} \cdot \dot{\boldsymbol{\xi}}, \quad \forall \{\tilde{\mathbf{s}}^P, \tilde{\boldsymbol{\eta}}\} \in \mathbb{K}, \quad (9)$$

that is, $\{\mathbf{s}^P, \boldsymbol{\eta}\} \in \mathbb{K}$ maximize the dissipation (6) for all trial states $\{\tilde{\mathbf{s}}^P, \tilde{\boldsymbol{\eta}}\} \in \mathbb{K}$. It is worth mentioning that in this statement, the convexity of $\mathbb{K}$ has been assumed. However, the convexity of $\mathbb{K}$ can be shown to be a consequence of the principle of maximum dissipation. Figure 1(a) shows a geometric interpretation of this condition, where the projections $\overline{\text{OP}_1}$ and $\overline{\text{OP}_2}$ at the actual states are maximal for any other admissible state $\{\tilde{\mathbf{s}}^P, \tilde{\boldsymbol{\eta}}\} \in \mathbb{K}$. The stress state $\{\mathbf{s}^P, \boldsymbol{\eta}\}$ is then the solution of the problem

$$\sup \{ \tilde{\mathbf{s}}^P : \dot{\boldsymbol{\varepsilon}}^P + \tilde{\boldsymbol{\eta}} \cdot \dot{\boldsymbol{\xi}} \mid \{\tilde{\mathbf{s}}^P, \tilde{\boldsymbol{\eta}}\} \in \mathbb{K} \}. \quad (10)$$

In view of (7), the optimization problem (10) can be solved using the method of Lagrange multipliers [4]. To this end, let us define the *indicator function* of the elastic domain

$$I_{\mathbb{K}}(\mathbf{s}^P, \boldsymbol{\eta}) := \begin{cases} 0, & \text{if } \{\mathbf{s}^P, \boldsymbol{\eta}\} \in \mathbb{K}, \\ +\infty, & \text{otherwise.} \end{cases} \quad (11)$$

The optimization problem (10) is then equivalent to

$$\sup \{ \tilde{\mathbf{s}}^P : \dot{\boldsymbol{\varepsilon}}^P + \tilde{\boldsymbol{\eta}} \cdot \dot{\boldsymbol{\xi}} - I_{\mathbb{K}}(\tilde{\mathbf{s}}^P, \tilde{\boldsymbol{\eta}}) \} \equiv \sup \{ \tilde{\mathbf{s}}^P : \dot{\boldsymbol{\varepsilon}}^P + \tilde{\boldsymbol{\eta}} \cdot \dot{\boldsymbol{\xi}} - \lambda f(\tilde{\mathbf{s}}^P, \tilde{\boldsymbol{\eta}}) \mid \lambda \geq 0 \}. \quad (12)$$

Stationarity of $\tilde{\mathbf{s}}^P : \dot{\boldsymbol{\varepsilon}}^P + \tilde{\boldsymbol{\eta}} \cdot \dot{\boldsymbol{\xi}} - \lambda f(\tilde{\mathbf{s}}^P, \tilde{\boldsymbol{\eta}})$ with respect to the generalized stresses provides, as necessary conditions of the optimization problem (12), the associative flow rule according to the normality law:

$$\{\dot{\boldsymbol{\varepsilon}}^P, \dot{\boldsymbol{\xi}}\} \in \partial I_{\mathbb{K}}(\mathbf{s}^P, \boldsymbol{\eta}) \equiv N_{\mathbb{K}}(\mathbf{s}^P, \boldsymbol{\eta}), \quad (13)$$

along with primal feasibility, dual feasibility and the complementary condition:

$$\lambda \geq 0, \quad f(\mathbf{s}^P, \boldsymbol{\eta}) \leq 0, \quad \text{and} \quad \lambda f(\mathbf{s}^P, \boldsymbol{\eta}) = 0, \quad (14)$$

respectively. Conditions (14) are also known as Karush-Kuhn-Tucker conditions. In equation (13), $N_{\mathbb{K}}(\mathbf{s}^P, \boldsymbol{\eta})$ is the normal cone of $\mathbb{K}$ at $(\mathbf{s}^P, \boldsymbol{\eta})$, which in view of equations (12)–(14), takes the following expression:

$$N_{\mathbb{K}}(\mathbf{s}^P, \boldsymbol{\eta}) \equiv \{ \lambda \{ \hat{\mathbf{n}}, \hat{\boldsymbol{\nu}} \} \in \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}^m \mid \{ \hat{\mathbf{n}}, \hat{\boldsymbol{\nu}} \} \in \partial f(\mathbf{s}^P, \boldsymbol{\eta}); \lambda \geq 0, \lambda = 0 \text{ if } f(\mathbf{s}^P, \boldsymbol{\eta}) < 0 \}. \quad (15)$$

The generalized-stress constraint (8) and the associative flow rule (13) constitute the classical problem

Fig. 1. Elastic domain $\mathbb{K}$ (light-gray region) and plastic flow at a smooth point $S_1(\mathbf{s}_1^P, \boldsymbol{\eta}_1)$, where the flow rule is uniquely defined, and at a corner $S_2(\mathbf{s}_2^P, \boldsymbol{\eta}_2)$, where the flow rule is multivalued. Here, $\nabla \square(\mathbf{s}^P, \boldsymbol{\eta})$ denotes the derivative of $\square$ with respect to the stress state, evaluated at $(\mathbf{s}^P, \boldsymbol{\eta})$. The flow rule is shown in (a) for the associative case (blue), where the principle of maximum dissipation holds, and in (b) for the non-associative case (red), with g_1 and g_2 denoting level sets of g at states S_1 and S_2 , respectively, where dissipation is not always maximum (dark-gray regions). The dissipation inequality (6) is automatically satisfied in (a), while in (b), the cosine of the angle defined through the inner product between $(\dot{\boldsymbol{\epsilon}}^P, \dot{\boldsymbol{\xi}})$ and $(\mathbf{s}^P, \boldsymbol{\eta})$ must be non-negative, e.g., $\cos a_1 \geq 0$ and $\cos a_2 \geq 0$ at S_1 and S_2 , respectively.

of associative plasticity in dual form. This problem owns a clear variational structure imparted by the principle of maximum dissipation (10). Along with the weak form of the equilibrium equations, this form of the principle of maximum dissipation provides the basis for the dual variational formulation with $\{\mathbf{s}^P, \boldsymbol{\eta}\}$ as primary unknowns [32] and mixed variational formulations of elastoplasticity [4, 77, 78]. In the present work, the dual problem is kept in its classical, *subdifferential* form, given by equations (8) and (13).

2.3.2. Primal form

The primal conjugate form of the problem presented above directly follows from convex duality. In this case, the relation between the generalized stresses and the internal variable rates is given in terms of the dissipation potential [29–32, 79], which is the only ingredient required in the formulation. The existence of the elastic domain (7) and the generalized-stress constraint (8) follow as consequences.

Let us first introduce the *support function* of the convex set $\mathbb{K}$, given by the convex 1-homogeneous function of internal variables rates $\pi_{\mathbb{K}}(\dot{\boldsymbol{\epsilon}}^P, \dot{\boldsymbol{\xi}})$. From standard arguments of convex analysis [75], it follows that the indicator function (11) and the support function are Legendre-Fenchel conjugate pairs, namely

$$\left\{ [I_{\mathbb{K}}(\mathbf{s}^P, \boldsymbol{\eta})]^* = \sup\{\tilde{\mathbf{s}}^P : \tilde{\boldsymbol{\eta}} \cdot \dot{\boldsymbol{\xi}} - I_{\mathbb{K}}(\tilde{\mathbf{s}}^P, \tilde{\boldsymbol{\eta}})\} = \pi_{\mathbb{K}}(\dot{\boldsymbol{\epsilon}}^P, \dot{\boldsymbol{\xi}}), \right. \quad (16)$$

$$\left. [\pi_{\mathbb{K}}(\dot{\boldsymbol{\epsilon}}^P, \dot{\boldsymbol{\xi}})]^* = \sup\{\mathbf{s}^P : \tilde{\boldsymbol{\eta}} \cdot \dot{\boldsymbol{\xi}} - \pi_{\mathbb{K}}(\dot{\boldsymbol{\epsilon}}^P, \dot{\boldsymbol{\xi}})\} = I_{\mathbb{K}}(\mathbf{s}^P, \boldsymbol{\eta}), \right. \quad (17)$$

where $[\square]^*$ denotes the Legendre-Fenchel transform. Equation (16) is a restatement of the principle of maximum dissipation (10). On the other hand, the conjugate form (17) yields the necessary condition

$\{\mathbf{s}^P, \boldsymbol{\eta}\} \in \partial\pi_{\mathbb{K}}(\dot{\boldsymbol{\epsilon}}^P, \dot{\boldsymbol{\xi}})$. Plugging this relation in equation (6) gives

$$\phi = \partial_{\dot{\boldsymbol{\epsilon}}^P} \pi_{\mathbb{K}}(\dot{\boldsymbol{\epsilon}}^P, \dot{\boldsymbol{\xi}}) : \dot{\boldsymbol{\epsilon}}^P + \partial_{\dot{\boldsymbol{\xi}}} \pi_{\mathbb{K}}(\dot{\boldsymbol{\epsilon}}^P, \dot{\boldsymbol{\xi}}) \cdot \dot{\boldsymbol{\xi}} \quad \implies \quad \phi = \phi(\dot{\boldsymbol{\epsilon}}^P, \dot{\boldsymbol{\xi}}) \equiv \pi_{\mathbb{K}}(\dot{\boldsymbol{\epsilon}}^P, \dot{\boldsymbol{\xi}}), \quad (18)$$

which reveals the role of $\pi_{\mathbb{K}}$ as a potential that coincides with the dissipation rate due to the 1-homogeneity of $\pi_{\mathbb{K}}$. Hence, the dissipation rate may take the functional form $(\dot{\boldsymbol{\epsilon}}^P, \dot{\boldsymbol{\xi}}) \mapsto \phi(\dot{\boldsymbol{\epsilon}}^P, \dot{\boldsymbol{\xi}})$, which is generally referred to as the *dissipation potential*. As a characteristic of associative models, ϕ only depends on $(\dot{\boldsymbol{\epsilon}}^P, \dot{\boldsymbol{\xi}})$ and, like f , may also depend on the constitutive state $\mathbf{c}$ and other history parameters. This is no longer true in the non-associative case, treated in the following subsection, where ϕ also depends on the generalized stresses.

Using equation (18), equation (17) gives a primal version of the principle of maximum dissipation, where, given $\{\mathbf{s}^P, \boldsymbol{\eta}\} \in \mathbb{K}$, the corresponding set $\{\dot{\boldsymbol{\epsilon}}^P, \dot{\boldsymbol{\xi}}\}$ is such that dissipation is maximized for all other admissible rates $\{\tilde{\boldsymbol{\epsilon}}^P, \tilde{\boldsymbol{\xi}}\}$. The actual rates $\{\dot{\boldsymbol{\epsilon}}^P, \dot{\boldsymbol{\xi}}\}$ are then found as an optimal solution of the problem

$$\sup \{ \mathbf{s}^P : \tilde{\boldsymbol{\epsilon}}^P + \boldsymbol{\eta} \cdot \tilde{\boldsymbol{\xi}} - \phi(\tilde{\boldsymbol{\epsilon}}^P, \tilde{\boldsymbol{\xi}}) \}. \quad (19)$$

The necessary condition of (19) is a primal representation of the evolution equations (8) and (13):

$$\{\mathbf{s}^P, \boldsymbol{\eta}\} \in \partial\phi(\dot{\boldsymbol{\epsilon}}^P, \dot{\boldsymbol{\xi}}). \quad (20)$$

In addition to being convex and 1-homogeneous, the dissipation potential must be non-negative to comply with the dissipation inequality (6), and must vanish for null rates. Then, the existence of an elastic domain directly follows from the straightforward identification

$$\mathbb{K} \equiv \partial\phi(\mathbf{0}, \mathbf{0}). \quad (21)$$

Equation (20) represents the classical problem of associative plasticity in primal form, owning a variational structure imparted by equation (19). Equation (20) can be taken as the point of departure for the primal variational formulation with $\{\boldsymbol{\epsilon}^P, \boldsymbol{\xi}\}$ as primary unknowns [32].

The variational framework given in section 3 in terms of the energetic formulation is elaborated along these lines, after the following crucial extension to the non-associative case.

2.4. The non-associative case

The previous presentation is now extended to non-associative models. Given that the principle of maximum dissipation (9) and the resulting normality law (13) no longer hold, it seems impossible to recover a variational structure and the accompanying convex duality in non-associative models, which are indeed mostly presented in the literature from a purely phenomenological perspective, i.e., by postulating a flow rule that captures experimentally observed responses. However, as shown below, recovering a variational structure and a treatment based on convex analysis is possible by introducing a generalization of the principle of maximum dissipation, leading to a state-dependent dissipation potential.

2.4.1. Dual form

The dual problem of non-associative plasticity shares the elastic domain with the associative counterpart, i.e., equations (7) and (8), but employs a different flow rule for the internal variables. In particular, the evolution of $\{\boldsymbol{\epsilon}^P, \boldsymbol{\xi}\}$ no longer complies with the normality law (13) and is given by the more general expression

$$\{\dot{\mathbf{e}}^p, \dot{\boldsymbol{\xi}}\} \in \mathbb{Q}(\mathbf{s}^p, \boldsymbol{\eta}), \quad (22)$$

where

$$\mathbb{Q}(\mathbf{s}^p, \boldsymbol{\eta}) := \{\lambda \{\hat{\mathbf{n}}, \hat{\mathbf{v}}\} \in \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}^m \mid \{\hat{\mathbf{n}}, \hat{\mathbf{v}}\} \in \partial g(\mathbf{s}^p, \boldsymbol{\eta}); \lambda \geq 0, \lambda = 0 \text{ if } f(\mathbf{s}^p, \boldsymbol{\eta}) < 0\} \quad (23)$$

denotes a *plastic-flow set*, given in terms of the convex and continuous function $g: \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}^m \rightarrow \mathbb{R}$. The function g is generally called the *plastic potential*. Clearly, for $f = g$, the associative model with the normality rule (13) is recovered. As done for the yield function, the possible explicit dependence of the plastic potential on the constitutive state $\mathbf{c}$ and additional history parameters has been omitted for the sake of conciseness, without losing generality.

The immediate implication of the general flow rule (22) in the non-associative case, where $f \neq g$, is that the principle of maximum dissipation is no longer valid, seemingly precluding a variational structure for the formulation. Indeed, the lack of normality generally implies that the dissipation is no longer maximum during plastic flow, hindering the postulate of an extremum dissipation principle. This occurrence is illustrated in figure 1(b), where inequality (9) no longer holds for virtual generalized-stress states within the dark-gray regions of $\mathbb{K}$. Furthermore, thermodynamic consistency is no longer guaranteed by the convexity of $\mathbb{K}$: in order to comply with the dissipation inequality (6), the plastic potential g must be such that the direction of the plastic flow yields a non-negative dissipation power, that is,

$$\mathbf{s}^p : \partial_{\mathbf{s}^p} g(\mathbf{s}^p, \boldsymbol{\eta}) + \boldsymbol{\eta} \cdot \partial_{\boldsymbol{\eta}} g(\mathbf{s}^p, \boldsymbol{\eta}) \geq 0. \quad (24)$$

With reference to figure 1(b), this condition corresponds to ensuring that the cosine of the angle between the direction of the generalized-stress state at the yield surface and the direction of the plastic flow is always non-negative.

2.4.2. Recovering a variational structure

While the internal variable rates in figure 1(b) are not normal to the yield surface, a normality condition still exists with respect to level sets of the plastic potential g . Thus, it seems reasonable to define a state-dependent domain $\mathbb{L}$ in terms of g , such that any generalized-stress state on the yield surface also belongs to the boundary of a level set of $\mathbb{L}$. Then, $\mathbb{L}$ is expected to play, in the non-associative flow rule (22), the role of $\mathbb{K}$ in the associative flow rule (13), allowing to recover an associative-like setting (and hence a variational structure) in the non-associative case. To this end, Babadjian et al. [46] resorted to a formulation initially proposed by Laborde [80], which is adapted here as follows.

Consider a continuous function $r: \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}^m \rightarrow \mathbb{R}$, satisfying

$$\begin{cases} r(\mathbf{s}^p, \boldsymbol{\eta}) = g(\mathbf{s}^p, \boldsymbol{\eta}) & \text{if } f(\mathbf{s}^p, \boldsymbol{\eta}) = 0, \\ r(\mathbf{s}^p, \boldsymbol{\eta}) > g(\mathbf{s}^p, \boldsymbol{\eta}) & \text{if } f(\mathbf{s}^p, \boldsymbol{\eta}) < 0, \\ r(\mathbf{s}^p, \boldsymbol{\eta}) < g(\mathbf{s}^p, \boldsymbol{\eta}) & \text{if } f(\mathbf{s}^p, \boldsymbol{\eta}) > 0, \end{cases} \quad (25)$$

which allows to define a closed and convex set that depends on the generalized-stress state (figure 2):

$$\mathbb{L}(\mathbf{s}^p, \boldsymbol{\eta}) := \{\{\tilde{\mathbf{s}}^p, \tilde{\boldsymbol{\eta}}\} \in \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}^m \mid g(\tilde{\mathbf{s}}^p, \tilde{\boldsymbol{\eta}}) \leq r(\mathbf{s}^p, \boldsymbol{\eta})\}. \quad (26)$$

$$\{\dot{\epsilon}^p, \dot{\xi}\} \in \begin{cases} \lambda \partial f(\mathbf{s}^p, \boldsymbol{\eta}) & \text{Associative flow} \\ \lambda \partial g(\mathbf{s}^p, \boldsymbol{\eta}) & \text{Non-associative flow} \end{cases}$$

Fig. 2. Elastic domain $\mathbb{K}$ and level sets of the state-dependent domain $\mathbb{L}(\mathbf{s}^p, \boldsymbol{\eta})$ at four states $S_1^\alpha(\mathbf{s}_1^p, \boldsymbol{\eta}_1)$, $S_2^\beta(\mathbf{s}_2^p, \boldsymbol{\eta}_2)$, $S_3^\gamma(\mathbf{s}_3^p, \boldsymbol{\eta}_3)$ and $S_4^\delta(\mathbf{s}_4^p, \boldsymbol{\eta}_4)$. The labels α , β , γ and δ correspond, respectively, to flow conditions at four possible scenarios: smooth f and smooth g ; non-smooth f and non-smooth g ; smooth f and non-smooth g ; and non-smooth f and smooth g . The associative (blue) and non-associative (red) flow rules are normal to the boundaries of $\mathbb{K}$ and the level sets of $\mathbb{L}(\mathbf{s}^p, \boldsymbol{\eta})$, respectively, where $\nabla \square(\mathbf{s}^p, \boldsymbol{\eta})$ denotes the derivative of $\square$ with respect to the stress state, evaluated at $(\mathbf{s}^p, \boldsymbol{\eta})$.

By virtue of conditions (25), the following relations are straightforwardly deduced [80]:

$$\begin{cases} \{\mathbf{s}^p, \boldsymbol{\eta}\} \in \mathbb{K} \iff \{\mathbf{s}^p, \boldsymbol{\eta}\} \in \mathbb{L}(\mathbf{s}^p, \boldsymbol{\eta}), \\ \{\dot{\epsilon}^p, \dot{\xi}\} \in \mathbb{Q}(\mathbf{s}^p, \boldsymbol{\eta}) \iff \{\dot{\epsilon}^p, \dot{\xi}\} \in \partial I_{\mathbb{L}(\mathbf{s}^p, \boldsymbol{\eta})}(\mathbf{s}^p, \boldsymbol{\eta}). \end{cases} \quad (27)$$

Following the reasoning used to obtain the flow rule (13) from equation (12), the condition $\{\dot{\epsilon}^p, \dot{\xi}\} \in \partial I_{\mathbb{L}(\mathbf{s}^p, \boldsymbol{\eta})}(\mathbf{s}^p, \boldsymbol{\eta})$ appearing in (27) emerges as a necessary condition of the optimization problem

$$\sup \{ \tilde{\mathbf{s}}^p : \tilde{\epsilon}^p + \tilde{\boldsymbol{\eta}} \cdot \dot{\xi} \mid \{\tilde{\mathbf{s}}^p, \tilde{\boldsymbol{\eta}}\} \in \mathbb{L}(\mathbf{s}^p, \boldsymbol{\eta}) \}. \quad (28)$$

Then, (28) can be viewed as a *generalized principle of maximum dissipation*, which implies normality with respect to the *state-dependent domain* $\mathbb{L}(\mathbf{s}^p, \boldsymbol{\eta})$. The key point is that according to (27), this normality condition implies the non-associative flow rule (22). Moreover, the generalized-stress constraint (8) is recovered because $\mathbb{K}$ and $\mathbb{L}(\mathbf{s}^p, \boldsymbol{\eta})$ coincide at the actual state.

Interestingly, aside from Laborde [80] and the related references, a similar approach was conceived in Point and Erlicher [51], although with a different definition of the state-dependent domain $\mathbb{L}$. In the present

work, we adhere to the formulation of Laborde [80], where the elements of $\mathbb{L}$ can be interpreted as level sets. Moreover, we emphasize the thermomechanical interpretation of the normality condition (27) as a consequence of an extremal energy principle, namely, the generalized principle of maximum dissipation (28).

A graphical representation of these concepts is shown in figure 2, where all possible scenarios regarding the smoothness of f and g at the actual state are considered. It is clear that for all generalized-stress states, dissipation is maximum with respect to all other states belonging to the corresponding level set of $\mathbb{L}$. Thus, at this point, a variational structure has been imparted to the dual form of the elastoplastic problem in the non-associative case, at the expense of introducing a state-dependent set of generalized stresses.

2.4.3. Primal form

Having restored a variational structure in the dual form of non-associative models, we are now able to consistently derive the corresponding primal problem. This is possible because the dissipation potential can be defined as the (state-dependent) support function of $\mathbb{L}(\mathbf{s}^p, \boldsymbol{\eta})$:

$$\begin{aligned}\phi(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}}; \mathbf{s}^p, \boldsymbol{\eta}) &= \sup \{ \tilde{\mathbf{s}}^p : \dot{\boldsymbol{\varepsilon}}^p + \tilde{\boldsymbol{\eta}} \cdot \dot{\boldsymbol{\xi}} - I_{\mathbb{L}(\mathbf{s}^p, \boldsymbol{\eta})}(\tilde{\mathbf{s}}^p, \tilde{\boldsymbol{\eta}}) \} \\ &= \sup \{ \tilde{\mathbf{s}}^p : \dot{\boldsymbol{\varepsilon}}^p + \tilde{\boldsymbol{\eta}} \cdot \dot{\boldsymbol{\xi}} \mid g(\tilde{\mathbf{s}}^p, \tilde{\boldsymbol{\eta}}) \leq r(\mathbf{s}^p, \boldsymbol{\eta}) \}.\end{aligned}\tag{29}$$

This expression, which corresponds to the generalized principle of maximum dissipation (28), is central to the developments that follow. Specifically, equation (29) allows to compute the dissipation potential for non-associative models characterized by a yield function f and a plastic potential g . To this end, it suffices to define a suitable function r that complies with conditions (25). We will see that in the models explored in subsequent sections, it is possible to take $r(\mathbf{s}^p, \boldsymbol{\eta}) = g(\mathbf{s}^p, \boldsymbol{\eta}) - f(\mathbf{s}^p, \boldsymbol{\eta})$. Note that in this way, g may be defined up to a constant, without altering the state-dependent domain $\mathbb{L}$.

As in the associative case, the dissipation potential (29) may also depend on the constitutive state $\mathbf{c}$ and other history parameters, now following from f and g . Recall that this has been omitted from the notation without losing generality. In any case, the characterizing feature of non-associative models is the dependence of the dissipation potential (29) on the *generalized-stress state* $\{\mathbf{s}^p, \boldsymbol{\eta}\}$ or, more rigorously, on $\{-\mathbf{s}_{\text{nd}}^p, -\boldsymbol{\eta}_{\text{nd}}\} \equiv \{\mathbf{s}^p, \boldsymbol{\eta}\}$.

Given the 1-homogeneity of $(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}}) \mapsto \phi(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}}; \mathbf{s}^p, \boldsymbol{\eta})$, we may write the primal representation of the evolution equations (8) and (22) as

$$\{\mathbf{s}^p, \boldsymbol{\eta}\} \in \partial_1 \phi(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}}; \mathbf{s}^p, \boldsymbol{\eta}),\tag{30}$$

where $\partial_1 \phi(\square; \bullet)$ denotes the subdifferential of ϕ with respect to $\square$. Equation (30) provides a compact formulation with ϕ as the only dissipative function, whereas the dual form requires independent definitions for the yield function and the plastic potential. Moreover, the dissipation inequality (6) is ensured by demanding a priori that

$$\phi(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}}; \mathbf{s}^p, \boldsymbol{\eta}) \geq 0.\tag{31}$$

It is worth noting that the definition of a primal dissipation potential with state-dependence is not new. Dependence on the constitutive state was initially considered in Ziegler [71] and Germain et al. [7], while dependence on the stress state was later advocated [8, 52, 72, 81] as a way to construct thermodynamically admissible non-associative models. In these works, a primal form is established by postulating a state-dependent dissipation potential. In turn, in the present work, a clear link between the dual and primal forms has been established, showing, for instance, that it is possible to take the yield function and the plastic potential as a point of departure and to *derive* the corresponding dissipation potential via (29). In other

words, the notion of convex duality has been restored for non-associative models.

A similar approach to construct the dissipation potential in such a general form was considered in Point and Erlicher [51]. Therein, the authors discuss a link to the bi-potential theory [36], where the bi-potential is defined as the sum of the state-dependent dissipation potential and the indicator function $I_{\mathbb{L}(\mathbf{s}^P, \boldsymbol{\eta})}(\mathbf{s}^P, \boldsymbol{\eta}) = [\phi(\dot{\boldsymbol{\varepsilon}}^P, \dot{\boldsymbol{\xi}}; \mathbf{s}^P, \boldsymbol{\eta})]^*$. In the present work, focus is not put on such duality relations. Instead, the key implication of the results obtained in this section is that the state-dependent function $\phi(\dot{\boldsymbol{\varepsilon}}^P, \dot{\boldsymbol{\xi}}; \mathbf{s}^P, \boldsymbol{\eta})$ can be taken as a basic ingredient for the primal variational formulation of non-associative models in section 3.

2.5. Evolution problem

Recalling the kinematic and equilibrium requirements of section 2.1, the quasi-static evolution problem of an elastoplastic solid obeying a non-associative flow rule can be formally stated as follows.

Formulation 1 (Classical/subdifferential formulation). *Given the kinematic requirement (1), the free energy density with the constitutive relations (4) and (5), the elastic domain (7) and the plastic-flow set (23), the solution process $(\mathbf{x}, t) \mapsto \{\mathbf{u}(\mathbf{x}, t), \boldsymbol{\varepsilon}^P(\mathbf{x}, t), \boldsymbol{\xi}(\mathbf{x}, t)\}$ is such that for all $t \in T$, the following conditions hold:*

1. *Equilibrium and boundary conditions:*

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma}(\mathbf{x}, t) + \rho \mathbf{b}(\mathbf{x}, t) = \mathbf{0} & \text{in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) = \bar{\mathbf{t}}(\mathbf{x}, t) & \text{on } \Gamma_N, \\ \mathbf{u}(\mathbf{x}, t) = \bar{\mathbf{u}}(\mathbf{x}, t) & \text{on } \Gamma_D; \end{cases} \quad \begin{array}{l} (32a) \\ (32b) \\ (32c) \end{array}$$

2. *The generalized-stress constraint:*

$$\{\mathbf{s}^P(\mathbf{x}, t), \boldsymbol{\eta}(\mathbf{x}, t)\} \in \mathbb{K} \quad \text{in } \Omega; \quad (33)$$

3. *The flow rule:*

$$\{\dot{\boldsymbol{\varepsilon}}^P(\mathbf{x}, t), \dot{\boldsymbol{\xi}}(\mathbf{x}, t)\} \in \mathbb{Q}(\mathbf{s}^P(\mathbf{x}, t), \boldsymbol{\eta}(\mathbf{x}, t)) \quad \text{in } \Omega. \quad (34)$$

Note that in Formulation 1, we must assume that the state variables belong to function spaces that are sufficiently regular for the formulation to be well defined. Moreover, due to its familiarity in the engineering community, the dual form of the classical elastoplastic problem has been chosen for the statement of Formulation 1. In this case, the dissipation inequality (6) is enforced in $\Omega \times T$ if the plastic potential complies with inequality (24) for all $\{\mathbf{s}^P, \boldsymbol{\eta}\} \in \mathbb{K}$. On the other hand, the primal form can be established by computing the state-dependent dissipation potential (29), which must satisfy the dissipation inequality (31) a priori. The dissipation potential is one of the two basic ingredients (the other being the stored energy density ψ) of the primal variational formulation that follows.

3. Energetic formulation

In this section, the non-associative evolution problem presented in Formulation 1 is recast in (primal) variational form within the framework of generalized standard materials [29] and the energetic formulation for rate-independent systems [33]. Specifically, the setting of generalized standard materials provides the energetic quantities in terms of the stored energy density ψ and the dissipation potential ϕ . With these two

quantities as the only constitutive ingredients, the energetic formulation expresses the evolution problem in terms of two basic principles: energy balance and a stability condition. Given the state-dependent dissipation potential (29), the present approach follows similar steps as for the case of associative models [62–64], but allows to consider a much broader spectrum of material behaviors.

3.1. Energy functionals

The energetic formulation requires the definition of function spaces for the unknown variables together with their admissible test space. With applications to hardening plasticity in mind, we define the space of kinematically admissible displacement fields and the corresponding space of admissible variations by $\mathcal{U} := \{\mathbf{w} \in H^1(\Omega; \mathbb{R}^3) \mid \mathbf{w} = \bar{\mathbf{u}}(t) \text{ on } \Gamma_D\}$ and $\tilde{\mathcal{U}} := \{\mathbf{w} \in H^1(\Omega; \mathbb{R}^3) \mid \mathbf{w} = \mathbf{0} \text{ on } \Gamma_D\}$. The function space for the plastic strains [82] is then given by $\mathcal{B} := L^2(\Omega; \mathbb{R}_{\text{sym}}^{3 \times 3})$, with admissible variations in $\tilde{\mathcal{B}}$.¹ The function space for the dependent internal variables $\boldsymbol{\xi}$ is denoted by $\mathcal{X}$, and the corresponding space of admissible variations reads $\tilde{\mathcal{X}}$. As will be clear in subsequent sections, the specific form of these function spaces depends on the material model, where the choice of $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{X}}$ depends on the constraints imposed by the dissipation potential and the stability condition.

For notational convenience, we define the function spaces $\mathcal{Q} := \mathcal{U} \times \mathcal{B} \times \mathcal{X}$ and $\tilde{\mathcal{Q}} := \tilde{\mathcal{U}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{X}}$, such that $\mathbf{q}(t) = \{\mathbf{u}(t), \boldsymbol{\varepsilon}^p(t), \boldsymbol{\xi}(t)\} \in \mathcal{Q}$ and $\tilde{\mathbf{q}} := \{\tilde{\mathbf{u}}, \tilde{\boldsymbol{\varepsilon}}^p, \tilde{\boldsymbol{\xi}}\} \in \tilde{\mathcal{Q}}$. The stored internal energy functional $\mathcal{E}: \mathcal{Q} \rightarrow \mathbb{R} \cup \{+\infty\}$ is then given in terms of the free energy density ψ by the state function

$$\mathcal{E}(\mathbf{q}) := \int_{\Omega} \psi(\boldsymbol{\varepsilon}(\mathbf{x}), \boldsymbol{\varepsilon}^p(\mathbf{x}), \boldsymbol{\xi}(\mathbf{x})) \, d\mathbf{x}, \quad (35)$$

while the work of external actions is defined as the time-integral of the external power:

$$\mathcal{L}(\mathbf{u}; [0, t]) := \int_0^t \left[\int_{\Omega} \rho \mathbf{b}(\mathbf{x}, \tau) \cdot \dot{\mathbf{u}}(\mathbf{x}, \tau) \, d\mathbf{x} + \int_{\Gamma_N} \bar{\mathbf{t}}(\mathbf{x}, \tau) \cdot \dot{\mathbf{u}}(\mathbf{x}, \tau) \, dS + \int_{\Gamma_D} \mathbf{t}_r(\mathbf{x}, \tau) \cdot \dot{\mathbf{u}}(\mathbf{x}, \tau) \, dS \right] d\tau, \quad (36)$$

where $\mathbf{t}_r$ is the traction vector on Γ_D . To formulate a stability condition, we introduce the external work distance $\mathcal{T}: \mathbb{T} \times \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}$, representing the work done by the external forces at a given time t between two states $\{\mathbf{u}_0, \mathbf{u}_1\}$:

$$\mathcal{T}(t, \mathbf{u}_0, \mathbf{u}_1) = \int_{\Omega} \rho \mathbf{b}(\mathbf{x}, t) \cdot (\mathbf{u}_1(\mathbf{x}) - \mathbf{u}_0(\mathbf{x})) \, d\mathbf{x} + \int_{\Gamma_N} \bar{\mathbf{t}}(\mathbf{x}, t) \cdot (\mathbf{u}_1(\mathbf{x}) - \mathbf{u}_0(\mathbf{x})) \, dS. \quad (37)$$

Concerning dissipation, a global dissipative power functional $\mathcal{R}: \tilde{\mathcal{B}} \times \tilde{\mathcal{X}} \times \mathcal{B} \times \mathcal{X} \rightarrow [0, +\infty]$ is defined as

$$\mathcal{R}(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}}; \mathbf{s}^p, \boldsymbol{\eta}) := \int_{\Omega} \phi(\dot{\boldsymbol{\varepsilon}}^p(\mathbf{x}), \dot{\boldsymbol{\xi}}(\mathbf{x}); \mathbf{s}^p(\mathbf{x}), \boldsymbol{\eta}(\mathbf{x})) \, d\mathbf{x}. \quad (38)$$

In addition, we consider a dissipation distance $\mathcal{D}: \mathcal{Q} \times \mathcal{Q} \rightarrow [0, +\infty]$, which measures the minimum amount of energy dissipated in a transition from a state $\mathbf{q}_0$ to a state $\mathbf{q}_1$ along a continuous path at a given time. This central quantity is specified in the sequel.

¹In the case of perfect plasticity, the space of bounded deformations $\text{BD}(\Omega; \mathbb{R}^3)$ should be considered instead of the Sobolev space $H^1(\Omega; \mathbb{R}^3)$, where the plastic strains belong to a space of Radon measures $\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{R}_{\text{sym}}^{3 \times 3})$. We refer the reader to Suquet [83], Dal Maso et al. [84] and Bartels et al. [82] for mathematical treatments of perfect plasticity.

3.2. Evolution problem

With the energy functionals at hand, the evolution problem can be written in terms of energy balance and either a global or a local stability condition. The most general setting of the energetic formulation employs a global stability condition. This approach has the benefit of being completely derivative free, requiring minimum regularity assumptions. For sufficiently smooth evolutions with time-continuous displacements and well-defined internal variable rates, the energetic formulation based on global stability yields solutions that also comply with local stability as well as the classical equations of generalized standard materials. The converse implication, that is, a solution of the classical evolution equations is also a solution of the energetic formulation, also holds for local stability, but does not necessarily hold for global stability, as in the case of non-convex energy functionals [33]. In the present study, we assume time-continuous evolutions, and thus only consider the local stability condition. We refer the reader to Alessi [85] for a study on time-discontinuous evolutions and to Mielke and Roubíček [33] for a thorough review of the energetic formulation.

The energy balance is a particular statement of the first law of thermodynamics, expressed here as

$$\mathcal{E}(\mathbf{q}(t)) + \int_0^t \mathcal{R}(\dot{\varepsilon}^P(s), \dot{\xi}(s); \mathbf{s}^P(s), \boldsymbol{\eta}(s)) \, ds = \mathcal{E}(\mathbf{q}(0)) + \mathcal{L}(\mathbf{u}; [0, t]). \quad (39)$$

Provided that the energy functionals are sufficiently regular in $\mathbf{T}$, the time derivative of (39) yields the first-order energy balance, given by the power balance equation

$$\frac{d}{dt} \mathcal{E}(\mathbf{q}(t)) + \mathcal{R}(\dot{\varepsilon}^P(t), \dot{\xi}(t); \mathbf{s}^P(t), \boldsymbol{\eta}(t)) - \frac{d}{dt} \mathcal{L}(\mathbf{u}; [0, t]) = 0. \quad (40)$$

On the other hand, the local stability condition is enforced in directional form as follows. Let $h \in \mathbb{R}$ denote a variation parameter and $\tilde{\mathbf{q}} \in \tilde{\mathcal{Q}}$ be test directions in which admissible variations on $\mathbf{q}$ take place. Then, a process $\mathbf{q} : \mathbf{T} \rightarrow \mathcal{Q}$ is locally stable if for all $t \in \mathbf{T}$, there exists $\bar{h} > 0$ such that

$$\mathcal{E}(\mathbf{q}(t)) \leq \mathcal{E}(\mathbf{q}(t) + h\tilde{\mathbf{q}}) - \mathcal{T}(t, \mathbf{u}(t), \mathbf{u}(t) + h\tilde{\mathbf{u}}) + \mathcal{D}(\mathbf{q}(t), \mathbf{q}(t) + h\tilde{\mathbf{q}}), \quad \forall \tilde{\mathbf{q}} \in \tilde{\mathcal{Q}}, \quad \forall h \in [0, \bar{h}]. \quad (41)$$

In view of the regularity assumptions required for local stability, we further assume that variations occur in a small neighborhood of the current state, restricting the test directions to monotonic radial (straight) paths [86]. Then, noting that $\mathbf{s}^P$ and $\boldsymbol{\eta}$ depend on $\mathbf{q}$ (equation (5)) the dissipation distance reads

$$\mathcal{D}(\mathbf{q}_0, \mathbf{q}_1) = \int_{\Omega} \left\{ \int_0^1 \phi(\dot{\varepsilon}^P(\mathbf{x}, s), \dot{\xi}(\mathbf{x}, s); \mathbf{s}^P(\mathbf{x}, s), \boldsymbol{\eta}(\mathbf{x}, s)) \, ds \mid \mathbf{q}(s) = \mathbf{q}_0 + s(\mathbf{q}_1 - \mathbf{q}_0) \right\} d\mathbf{x}. \quad (42)$$

Assuming that the functionals are Gâteaux-differentiable up to a certain order, a differential stability condition can be defined by replacing the right-hand side of the directional stability condition (41) by a Taylor series expansion [86, 87]. A first-order condition can then be obtained as follows:

$$\left. \frac{d}{dh} \mathcal{E}(\mathbf{q}(t) + h\tilde{\mathbf{q}}) \right|_{h=0} - \left. \frac{d}{dh} \mathcal{T}(t, \mathbf{u}(t), \mathbf{u}(t) + h\tilde{\mathbf{u}}) \right|_{h=0} + \left. \frac{d}{dh} \mathcal{D}(\mathbf{q}(t), \mathbf{q}(t) + h\tilde{\mathbf{q}}) \right|_{h=0} \geq 0, \quad \forall \tilde{\mathbf{q}} \in \tilde{\mathcal{Q}}. \quad (43)$$

By virtue of the 1-homogeneity of $(\dot{\varepsilon}^P, \dot{\xi}) \mapsto \phi(\dot{\varepsilon}^P, \dot{\xi}; \mathbf{s}^P, \boldsymbol{\eta})$, and using equations (38) and (42), we evaluate the Gâteaux derivative of the dissipation distance as

$$\left. \frac{d}{dh} \mathcal{D}(\mathbf{q}(t), \mathbf{q}(t) + h\tilde{\mathbf{q}}) \right|_{h=0} = \lim_{h \rightarrow 0} \frac{\mathcal{D}(\mathbf{q}(t), \mathbf{q}(t) + h\tilde{\mathbf{q}})}{h} = \mathcal{R}(\dot{\varepsilon}^P, \dot{\xi}; \mathbf{s}^P(t), \boldsymbol{\eta}(t)). \quad (44)$$

By plugging this result in equation (43) and using equation (37), the first-order stability condition

$$\delta\mathcal{E}(\mathbf{q}(t))(\tilde{\mathbf{q}}) + \mathcal{R}(\tilde{\varepsilon}^p, \tilde{\boldsymbol{\xi}}; \mathbf{s}^p(t), \boldsymbol{\eta}(t)) - \int_{\Omega} \rho \mathbf{b}(\mathbf{x}, t) \cdot \tilde{\mathbf{u}}(\mathbf{x}, t) \, d\mathbf{x} - \int_{\Gamma_N} \bar{\mathbf{t}}(\mathbf{x}, t) \cdot \tilde{\mathbf{u}}(\mathbf{x}, t) \, dS \geq 0, \quad \forall \tilde{\mathbf{q}} \in \tilde{\mathcal{Q}} \quad (45)$$

is obtained, where $\delta\mathcal{E}(\mathbf{q}(t))(\tilde{\mathbf{q}})$ is the Gâteaux derivative of $\mathcal{E}$ in the direction $\tilde{\mathbf{q}}$. Equation (45) is a necessary condition for directional stability (41) that becomes sufficient if the inequality is strict. Note that the notion of directional stability allows for the study of higher-order stability conditions [60, 61, 86, 88]. For the sake of brevity, we will hereafter refer to equation (45) as the local stability condition.

Finally, to ensure physical soundness, the dissipation inequality (6) is enforced in $\Omega \times T$ by ensuring that the dissipation potential satisfies inequality (31) a priori. The variational evolution problem can now be formulated as follows.

Formulation 2 (Variational/energetic formulation). *Given the kinematic requirement (1), the free energy density with the constitutive relations (4) and (5), and the dissipation potential complying with the dissipation inequality (31), the solution process $\mathbf{q}: T \rightarrow \mathcal{Q}$ is such that for all $t \in T$, the following conditions hold:*

1. *The energy balance (39);*
2. *The local (first-order) stability condition (45).*

It is worth noting that Formulation 2 provides tools that are not embedded in Formulation 1, allowing to consider complex models and mechanical behaviours in a straightforward manner. For instance, gradient-extended internal variables, which have been omitted so far for explanatory simplicity, can be easily incorporated, as will be shown in section 6. In such cases, Formulation 2 automatically provides boundary conditions for the gradients of internal variables [59, 63]. Moreover, higher-order stability conditions can be considered in Formulation 2 by replacing the first-order condition (45) with the more general directional stability condition (41) [60, 86].

The cornerstone of the present study is the equivalence of Formulation 1 and Formulation 2. The crucial step to establish this equivalence is the derivation of the dissipation potential from equation (29), providing a direct link between the classical and variational formulations (figure 3). For the sake of completeness, we formally state this result as follows.

Proposition 1. *Given the dissipation potential (29) complying with the dissipation inequality (31), Formulation 1 is equivalent to Formulation 2.*

Proof. We first show that (32), (33) and (34) imply energy balance and local stability. Taking the inner product of the equilibrium equation (32a) with the displacement rate $\dot{\mathbf{u}}$, integrating in space and using the divergence theorem along with the boundary conditions (32b) and (32c) yields

$$\int_{\Omega} (\boldsymbol{\sigma} : \nabla^s \dot{\mathbf{u}} - \rho \mathbf{b} \cdot \dot{\mathbf{u}}) \, d\mathbf{x} - \int_{\Gamma_N} \bar{\mathbf{t}} \cdot \dot{\mathbf{u}} \, dS - \int_{\Gamma_D} \mathbf{t}_r \cdot \dot{\mathbf{u}} \, dS = 0. \quad (46)$$

Starting with (33) and (34), we have obtained the state-dependent dissipation potential via (29). We then write $\phi(\dot{\varepsilon}^p, \dot{\boldsymbol{\xi}}; \mathbf{s}^p, \boldsymbol{\eta}) - \mathbf{s}^p : \dot{\varepsilon}^p - \boldsymbol{\eta} \cdot \dot{\boldsymbol{\xi}} = 0$, which is integrated in space and added to (46) to obtain

$$\int_{\Omega} (\boldsymbol{\sigma} : \nabla^s \dot{\mathbf{u}} - \mathbf{s}^p : \dot{\varepsilon}^p - \boldsymbol{\eta} \cdot \dot{\boldsymbol{\xi}}) \, d\mathbf{x} + \int_{\Omega} \phi(\dot{\varepsilon}^p, \dot{\boldsymbol{\xi}}; \mathbf{s}^p, \boldsymbol{\eta}) \, d\mathbf{x} - \int_{\Omega} \rho \mathbf{b} \cdot \dot{\mathbf{u}} \, d\mathbf{x} - \int_{\Gamma_N} \bar{\mathbf{t}} \cdot \dot{\mathbf{u}} \, dS - \int_{\Gamma_D} \mathbf{t}_r \cdot \dot{\mathbf{u}} \, dS = 0. \quad (47)$$

In view of (35) and (38), equation (47) is the power balance (40), whose time integral gives the energy balance (39). To obtain local stability, we repeat the procedure used to obtain (46), but using a virtual field $\tilde{\mathbf{u}} \in \tilde{\mathcal{U}}$ instead of $\dot{\mathbf{u}}$. Hence, the third integral vanishes, yielding the weak form of equilibrium:

$$\int_{\Omega} (\boldsymbol{\sigma} : \nabla^s \tilde{\mathbf{u}} - \rho \mathbf{b} \cdot \tilde{\mathbf{u}}) \, dx - \int_{\Gamma_N} \bar{\mathbf{t}} \cdot \tilde{\mathbf{u}} \, dS = 0. \quad (48)$$

The subdifferential in (30) gives $\phi(\tilde{\boldsymbol{\varepsilon}}^p, \tilde{\boldsymbol{\xi}}; \mathbf{s}^p, \boldsymbol{\eta}) - \mathbf{s}^p : \tilde{\boldsymbol{\varepsilon}}^p - \boldsymbol{\eta} \cdot \tilde{\boldsymbol{\xi}} \geq 0$ for all $\{\tilde{\boldsymbol{\varepsilon}}^p, \tilde{\boldsymbol{\xi}}\} \in \tilde{\mathcal{B}} \times \tilde{\mathcal{X}}$. The space integral of this inequality is added to (48) to obtain

$$\int_{\Omega} (\boldsymbol{\sigma} : \nabla^s \tilde{\mathbf{u}} - \mathbf{s}^p : \tilde{\boldsymbol{\varepsilon}}^p - \boldsymbol{\eta} \cdot \tilde{\boldsymbol{\xi}}) \, dx + \int_{\Omega} \phi(\tilde{\boldsymbol{\varepsilon}}^p, \tilde{\boldsymbol{\xi}}; \mathbf{s}^p, \boldsymbol{\eta}) \, dx - \int_{\Omega} \rho \mathbf{b} \cdot \tilde{\mathbf{u}} \, dx - \int_{\Gamma_N} \bar{\mathbf{t}} \cdot \tilde{\mathbf{u}} \, dS \geq 0. \quad (49)$$

In view of (35) and (38), equation (49) corresponds to the local stability condition (45).

The converse implication, i.e., that Formulation 2 implies Formulation 1, follows directly from standard arguments of variational calculus. Letting $\{\tilde{\boldsymbol{\varepsilon}}^p, \tilde{\boldsymbol{\xi}}\} = \{\mathbf{0}, \mathbf{0}\}$, the stability condition (45) gives the weak form of the equilibrium equations (32). Letting $\tilde{\mathbf{u}} = \mathbf{0}$ in (45), and using the definition of the subdifferential, we obtain $\{\mathbf{s}^p, \boldsymbol{\eta}\} \in \partial_1 \phi(\mathbf{0}, \mathbf{0}; \mathbf{s}^p, \boldsymbol{\eta})$, which, by virtue of (27) and (29), implies the generalized-stress constraint (33). To obtain the flow rule, we note that for smooth evolutions, the energy balance (39) implies the power balance (40), which gives equation (30). By virtue of (27) and standard arguments of convex analysis, this expression is inverted as $\{\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}}\} \in \partial I_{\mathbb{L}(\mathbf{s}^p, \boldsymbol{\eta})}(\mathbf{s}^p, \boldsymbol{\eta})$, which, in view of (29), provides the flow rule (34). $\square$

Figure 3 summarizes the previously derived formulations (1 and 2) and their interplay. A possible approach to derive a non-associative model is adopt the classical formulation, where the yield function and the plastic potential are postulated in generalized-stress space, while the Helmholtz free energy is defined in terms of internal variables. Then, equations (4) and (5) provide the relation between the generalized stresses and the internal variables, while the plastic potential is enforced to comply with the dissipation inequality in terms of equation (24). The evolution problem can then be written in terms of Formulation 1. On the other hand, for the variational formulation, one may take the free energy and the dissipation potential as a point of departure, where the dissipation inequality is directly enforced via (31). Then, the evolution problem is given in terms of Formulation 2, which, by virtue of Proposition 1, is equivalent to Formulation 1. As highlighted in figure 3, the link between both formulations is the generalized principle of maximum dissipation (29), which allows to compute the dissipation potential that corresponds to the yield function and plastic potential. We note further that figure 3 equally applies to associative models where $f = g$.

The material models that follow will be presented according to the conceptual scheme of figure 3. As a common input of formulations 1 and 2, the free energy density is first defined, followed by the constitutive equations (4) and (5). The generalized-stress constraint and the flow rule are then postulated by defining the yield function and the plastic potential, taking Formulation 1 as a point of departure. The generalized principle of maximum dissipation is then used to derive the corresponding dissipation potential via (29). This function is used to formulate energy balance and the local stability condition (Formulation 2), from which the (originally postulated) yield function and flow rule are finally recovered as proof of concept. To the knowledge of the authors, even for the case of associative models, the steps required to establish these results are not widely available in the literature.

It is worth noting that in the present framework, little attention has been paid to advanced mathematical

Fig. 3. Formulations 1 and 2 and their interrelation provided by the generalized principle of maximum dissipation (GPMD). The gray blocks represent the constitutive ingredients of the formulations, the red blocks represent the thermodynamic conditions that are enforced a priori, and the blue blocks represent the formulations. As an equivalent condition to the generalized-stress constraint (33) and the flow rule (34) in Formulation 1, the generalized principle of maximum dissipation provides the state-dependent dissipation potential (29), which is then taken as a basic ingredient in Formulation 2.

aspects such as existence proofs. In case of softening and/or non-convex energies, which are here not a priori excluded, there are in general no solutions because of the formation of microstructure [42, 89]. Only in special cases, the existence of solutions can be guaranteed. Possible strategies to restore well-posedness are the application of condensed energies and relaxation theory [89–91], or the introduction of non-local energetic contributions [92–94], as exemplified in the numerical examples of section 6.

4. Frictional-dilational plasticity

As a first example, this section presents the classical non-associative Drucker-Prager model for frictional-dilational materials. We consider combined isotropic and kinematic hardening, including both volumetric and deviatoric kinematic hardening mechanisms. In addition, we provide a discussion regarding the energetic interpretation of isotropic hardening, a non-trivial topic in the present non-associative model.

4.1. The classical model ingredients

The primary set of internal variables is given by $\{\varepsilon^p, \kappa\}$, where $\kappa(\mathbf{x}, t) \in \mathbb{R}_+$ is an isotropic hardening variable. Note that this corresponds to setting $\xi := \{\kappa\}$. Based on micromechanical arguments [95,

96], the free energy density can be defined as the sum of elastic and plastic contributions:

$$\psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \kappa) := \underbrace{\frac{1}{2} \|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p\|_{\mathbf{C}}^2}_{\text{elastic}} + \underbrace{\frac{1}{2} \|\boldsymbol{\varepsilon}^p\|_{\mathbf{H}^{\text{kin}}}^2 + \frac{1}{2} (1-b) H^{\text{iso}} \kappa^2}_{\text{plastic}}. \quad (50)$$

Here, $\mathbf{C}$ is the fourth-order elasticity tensor, given for isotropic materials in terms of the bulk modulus K and the shear modulus μ :

$$\mathbf{C} = K \mathbf{1} \otimes \mathbf{1} + 2\mu \left(\mathbf{I} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right), \quad (51)$$

where $\mathbf{1}$ is the second-order identity tensor and $\mathbf{I}$ is the fourth-order symmetric identity tensor. $\mathbf{H}^{\text{kin}}$ is a fourth-order kinematic hardening tensor given by

$$\mathbf{H}^{\text{kin}} = H_K^{\text{kin}} \mathbf{1} \otimes \mathbf{1} + H_\mu^{\text{kin}} \left(\mathbf{I} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right), \quad (52)$$

where H_K^{kin} and H_μ^{kin} denote the bulk and shear kinematic hardening moduli, respectively [96]. Finally, H^{iso} is the isotropic hardening modulus, while $b \in [0, 1]$ is a parameter whose role is discussed in the sequel.

The Cauchy stress tensor, obtained from equations (4) and (50), reads

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p). \quad (53)$$

Denoting the generalized stress conjugate to κ by h , i.e., $\boldsymbol{\eta} := \{h\}$, equations (5), (50) and (53) give

$$\mathbf{s}^p = -\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^p} \equiv \boldsymbol{\sigma} - \mathbf{H}^{\text{kin}} : \boldsymbol{\varepsilon}^p \quad \text{and} \quad h = -\frac{\partial \psi}{\partial \kappa} \equiv -(1-b) H^{\text{iso}} \kappa. \quad (54)$$

The elastic domain is defined as

$$\mathbb{K} := \{ \{ \mathbf{s}^p, h \} \in \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}_- \mid f(\mathbf{s}^p, h) \leq 0 \}, \quad (55)$$

with the yield function of the Drucker-Prager type:

$$f(\mathbf{s}^p, h) := \|\mathbf{s}_{\text{dev}}^p\| + \sqrt{\frac{2}{3}} (A_\varphi \text{tr} \mathbf{s}^p + h - \bar{\sigma}^p(\kappa)) \leq 0, \quad (56)$$

where $\mathbf{s}_{\text{dev}}^p \in \mathbb{R}_{\text{dev}}^{3 \times 3} := \{ \mathbf{e} \in \mathbb{R}_{\text{sym}}^{3 \times 3} \mid \text{tr} \mathbf{e} = 0 \}$ and

$$\bar{\sigma}^p(\kappa) := \sigma^p + b H^{\text{iso}} \kappa. \quad (57)$$

We further define the plastic potential g introduced in equation (23) as

$$g(\mathbf{s}^p, h) := \|\mathbf{s}_{\text{dev}}^p\| + \sqrt{\frac{2}{3}} (A_\theta \text{tr} \mathbf{s}^p + h - \bar{\sigma}^p(\kappa)). \quad (58)$$

The parameters $A_\varphi \geq A_\theta \geq 0$ and $\sigma^p \geq 0$ are related to the friction angle φ , the dilation angle θ and the cohesion c as follows:

$$A_\varphi = \frac{2 \sin \varphi}{3 \pm \sin \varphi}, \quad A_\theta = \frac{2 \sin \theta}{3 \pm \sin \theta}, \quad \sigma^p = \frac{6c \cos \varphi}{3 \pm \sin \varphi}, \quad (59)$$

where $\pm$ corresponds to the inner- and outer-cone approximations of the Mohr-Coulomb model. It is worth

noting that g includes the constant term σ^p . This choice is arbitrary, since the flow rule only involves the derivative of g .

Note further that both the yield function (56) and the plastic potential (58) depend on the constitutive state through κ , i.e., $f(\mathbf{s}^p, h; \kappa)$ and $g(\mathbf{s}^p, h; \kappa)$ (this explicit dependence is omitted in the notation for the sake of simplicity). This can be viewed as a particular case of a more general formulation where instead of κ , a history-dependent variable $\bar{\kappa}$ is used. Then, the present model is recovered by prescribing $\dot{\bar{\kappa}} = \dot{\kappa}$.

At this point, it is convenient to provide some comments on the role of the parameter b . The term $(1/2)(1-b)H^{\text{iso}}\kappa^2$ in the free energy (50) can be viewed as a *defect energy* [97], which is consistent with the choice of κ as an internal variable, i.e., $\boldsymbol{\xi} := \{\kappa\}$. On the other hand, as will become clear in the sequel, the term $bH^{\text{iso}}\kappa$ in equation (57) contributes to the dissipation, where κ introduces dependence on the constitutive state. Let us now consider the limiting cases with $b = 1$ and $b = 0$. Setting $b = 1$ corresponds to the interpretation of isotropic hardening as a purely dissipative mechanism, where the corresponding term vanishes from the free energy. Contrarily, setting $b = 0$ corresponds to an alternative interpretation of isotropic hardening as an energetic mechanism intended to mimic the underlying dissipative process. As discussed by Gurtin and Reddy [97] for associative von Mises plasticity, both viewpoints yield equivalent models with thermodynamically admissible measures of dissipation in the isothermal case. Consequently, opting for either $b = 0$ or $b = 1$ seems to be an arbitrary choice. Different formulations of non-associative Drucker-Prager plasticity with isotropic hardening in the literature seem to conform to this idea. For instance, models corresponding to $b = 0$ and $b = 1$ in the present formulation can be found in Regueiro and Borja [98] and in Collins and Houlsby [52], respectively. However, as will be discussed in section 4.2, where the more general case of $0 \leq b \leq 1$ is considered, the choice of b plays a crucial role in ensuring the thermodynamic consistency of the present model.

The Drucker-Prager model is associative if $A_\varphi = A_\theta$. In that case, the model is unable to realistically capture the behavior of geomaterials [8, 54]. In particular, the dilation of granular materials is strongly overestimated, and a non-physical vanishing dissipation is obtained for cohesionless materials where $\bar{\sigma}^p(\kappa) = 0$. As a consequence, the non-associative version of the model with $0 \leq A_\theta < A_\varphi$ is commonly used in practice.

To characterize the flow rule (22), the subdifferential of g is evaluated as $\partial g(\mathbf{s}^p, h) = \{\hat{\mathbf{n}}, \sqrt{2/3}\}$, where $\hat{\mathbf{n}}$ denotes the direction of the plastic strain tensor:

$$\hat{\mathbf{n}} = \hat{\mathbf{n}}_{\text{dev}} + \sqrt{\frac{2}{3}}A_\theta \mathbf{1}, \quad \text{with} \quad \hat{\mathbf{n}}_{\text{dev}} \in \partial \|\mathbf{s}_{\text{dev}}^p\|. \quad (60)$$

Note that $\|\hat{\mathbf{n}}_{\text{dev}}\| = 1$ for $\|\mathbf{s}^p\| \neq 0$, while $\|\hat{\mathbf{n}}_{\text{dev}}\| \leq 1$ for $\|\mathbf{s}^p\| = 0$. Equations (22) and (23) then give the non-associative flow rule

$$\{\dot{\boldsymbol{\epsilon}}^p, \dot{\kappa}\} \in \mathbb{Q}(\mathbf{s}^p, h), \quad (61)$$

with

$$\mathbb{Q}(\mathbf{s}^p, h) = \{\lambda \{\hat{\mathbf{n}}, \sqrt{2/3}\} \in \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}_+ \mid \lambda \geq 0, \lambda = 0 \text{ if } f(\mathbf{s}^p, h) < 0\}. \quad (62)$$

In contrast with isochoric models, e.g., von Mises plasticity, the flow rule (61) allows for volumetric plastic strains, modulated by the dilation angle $0 \leq \theta \leq \varphi$. In addition, the term $A_\varphi \text{tr } \mathbf{s}^p$ in the yield function (56) introduces pressure-dependence. Clearly, $\theta = 0$ entails isochoric plastic strains, while $\varphi = \theta = 0$ recovers the von Mises yield criterion.

The present Drucker-Prager model results in a non-smooth yield surface, whose apex corresponds to

a tensile hydrostatic stress state (figure 4). At this state, the direction of the deviatoric component of the plastic strain rate $\hat{\mathbf{n}}_{\text{dev}}$ is given by the subdifferential of $\|\mathbf{s}_{\text{dev}}^{\text{p}}\|$ at $\mathbf{0}$. Noting from equation (60) that $\text{tr } \hat{\mathbf{n}} = \sqrt{6}A_\theta$, the flow rule (61) gives

$$\text{tr } \dot{\epsilon}^{\text{p}} \geq \sqrt{6} A_{\theta} \|\dot{\epsilon}_{\text{dev}}^{\text{p}}\|. \quad (63)$$

A strict inequality in equation (63) corresponds to the normal cone depicted in red in figure 4(b), while an equality is attained for $\|\mathbf{s}_{\text{dev}}^{\text{P}}\| > 0$.

The ingredients described above constitute the classical formulation of the non-associative Drucker-Prager model and allow to completely define the evolution problem in terms of Formulation 1.

$$\dot{\epsilon}^p \in \begin{cases} \lambda \left(\sqrt{\frac{2}{3}} A_{\varphi} \mathbf{1} + \partial \|\bar{\mathbf{s}}_{\text{dev}}^p\| \right) & \text{Associative flow} \\ \lambda \left(\sqrt{\frac{2}{3}} A_{\theta} \mathbf{1} + \partial \|\bar{\mathbf{s}}_{\text{dev}}^p\| \right) & \text{Non-associative flow} \end{cases}$$

Fig. 4. Elastic domain $\mathbb{K}$ (gray region) and state-dependent domain $\mathbb{L}(\mathbf{s}^p)$ (brown region) for the Drucker-Prager model at fixed h and κ , where $\tan a_\varphi = \sqrt{2}A_\varphi$ and $\tan a_\theta = \sqrt{2}A_\theta$. (a) shows the flow rule at states $\pm \mathbf{s}^p$, labeled A^α and B^α , corresponding to smooth points of $\mathbb{K}$ and $\mathbb{L}(\mathbf{s}^p)$. (b) shows the flow rule at the apex, labeled C^β . The labels α and β correspond to the scenarios described in figure 2. The tensors normal to the boundary of $\mathbb{L}(\mathbf{s}^p)$ (red) correspond to the non-associative flow rule (61), while the tensors normal to the boundary of $\mathbb{K}$ (blue) recover the associative case, where the volumetric plastic strain rate is larger.

4.2. Dissipation potential

We now aim at stating an equivalent variational formulation for the present model in terms of Formulation 2, such that the elastic domain (55) and the non-associative flow rule (61) are recovered. As depicted in figure 3, this task requires the derivation of the dissipation potential from the generalized principle of maximum dissipation (29). To this end, the state-dependent domain (26) is first determined by setting, in agreement with conditions (25), $r(\mathbf{s}^p) := g(\mathbf{s}^p, h) - f(\mathbf{s}^p, h) = \sqrt{2/3}(A_\theta - A_\varphi) \text{tr } \mathbf{s}^p$, leading to

$$\mathbb{L}(\mathbf{s}^p) = \left\{ \{\tilde{\mathbf{s}}^p, \tilde{h}\} \in \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}_- \mid \|\tilde{\mathbf{s}}_{\text{dev}}^p\| + \sqrt{\frac{2}{3}}(A_\theta \text{tr } \tilde{\mathbf{s}}^p + \tilde{h} - \bar{\sigma}^p(\kappa)) \leq \sqrt{\frac{2}{3}}(A_\theta - A_\varphi) \text{tr } \mathbf{s}^p \right\}. \quad (64)$$

Notice that the set (64) depends on the hydrostatic component of $\mathbf{s}^p$, which results from the non-associativity of the volumetric plastic strains. Consequently, the dissipation potential will also depend explicitly on $\text{tr } \mathbf{s}^p$.

For the subsequent analytical developments, it is useful to note that the elastic domain (55) only bounds $\text{tr } \mathbf{s}^P$ from above, by a positive limit value. Then, according to (64), $\text{tr } \tilde{\mathbf{s}}^P$ is bounded from above by a positive limit value and unbounded with respect to negative values.

Figure 4 provides an illustrative representation of the state-dependent domain $\mathbb{L}(\mathbf{s}^P)$ in $\mathbb{R}_{\text{dev}}^{3 \times 3} \times \mathbb{R}$ space, where the tensors normal to the boundaries of $\mathbb{K}$ and $\mathbb{L}(\mathbf{s}^P)$ comply with the associative and non-associative flow rules, respectively. In figure 4(a), the stress states $\pm \mathbf{s}^P$ are such that $\|\mathbf{s}_{\text{dev}}^P\| > 0$. As a consequence, $\mathbb{K}$ and $\mathbb{L}(\mathbf{s}^P)$ intersect at smooth points, such that, for the non-associative case, $\text{tr } \dot{\boldsymbol{\epsilon}}^P = \sqrt{6} A_\theta \|\dot{\boldsymbol{\epsilon}}_{\text{dev}}^P\|$. On the other hand, figure 4(b) shows a stress state with $\|\mathbf{s}_{\text{dev}}^P\| = 0$, where the deviatoric component of the plastic strain rate belongs to the normal cone of $\mathbb{L}(\mathbf{s}^P)$ at the apex. The flow rules in figures 4(a) and 4(b) correspond to the scenarios in figure 2 labeled with superscripts α and β , respectively.

Using the state-dependent domain (64), the state-dependent dissipation potential (29) is evaluated for all $\{\dot{\boldsymbol{\epsilon}}^P, \dot{\kappa}\} \in \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}_+$ as follows:

$$\begin{aligned} \phi(\dot{\boldsymbol{\epsilon}}^P, \dot{\kappa}; \mathbf{s}^P) &= \sup \left\{ \tilde{\mathbf{s}}^P : \dot{\boldsymbol{\epsilon}}^P + \tilde{h} \dot{\kappa} \mid \|\tilde{\mathbf{s}}_{\text{dev}}^P\| + \sqrt{\frac{2}{3}} (A_\theta \text{tr } \tilde{\mathbf{s}}^P + \tilde{h} - \bar{\sigma}^P(\kappa)) \leq \sqrt{\frac{2}{3}} (A_\theta - A_\varphi) \text{tr } \mathbf{s}^P, \tilde{h} \leq 0 \right\} \\ &= \sup \left\{ \tilde{\mathbf{s}}_{\text{dev}}^P : \dot{\boldsymbol{\epsilon}}_{\text{dev}}^P + \frac{1}{3} \text{tr } \tilde{\mathbf{s}}^P \text{tr } \dot{\boldsymbol{\epsilon}}^P + \tilde{h} \dot{\kappa} \right. \\ &\quad \left. \mid A_\theta \text{tr } \tilde{\mathbf{s}}^P \leq (A_\theta - A_\varphi) \text{tr } \mathbf{s}^P + \bar{\sigma}^P(\kappa) - \tilde{h} - \sqrt{\frac{3}{2}} \|\tilde{\mathbf{s}}_{\text{dev}}^P\|, \tilde{h} \leq 0 \right\} \\ &= \sup \left\{ \frac{1}{3A_\theta} (\bar{\sigma}^P(\kappa) + (A_\theta - A_\varphi) \text{tr } \mathbf{s}^P) \text{tr } \dot{\boldsymbol{\epsilon}}^P \right. \\ &\quad \left. + \tilde{h} \left(\dot{\kappa} - \frac{1}{3A_\theta} \text{tr } \dot{\boldsymbol{\epsilon}}^P \right) + \|\tilde{\mathbf{s}}_{\text{dev}}^P\| \left(\|\dot{\boldsymbol{\epsilon}}_{\text{dev}}^P\| - \frac{1}{\sqrt{6}A_\theta} \text{tr } \dot{\boldsymbol{\epsilon}}^P \right) \mid \tilde{h} \leq 0 \right\}. \end{aligned} \quad (65)$$

To obtain the last equality in (65), we have used the fact that $\phi = +\infty$ for $\text{tr } \dot{\boldsymbol{\epsilon}}^P < 0$, since $\text{tr } \tilde{\mathbf{s}}^P$ is allowed to assume arbitrarily large negative values. For $\text{tr } \dot{\boldsymbol{\epsilon}}^P \geq 0$, the supremum with respect to $\text{tr } \tilde{\mathbf{s}}^P \text{tr } \dot{\boldsymbol{\epsilon}}^P$ is finite and given by the largest admissible value of $\text{tr } \tilde{\mathbf{s}}^P$ allowed by the constraint, which has been replaced in the argument of the supremum. Note that while doing so, we have assumed that $A_\theta > 0$.² Then, noting that the expression inside the supremum is unbounded when the terms multiplying $\tilde{h}$ and $\|\tilde{\mathbf{s}}_{\text{dev}}^P\|$ are negative and positive, respectively, the following closed form of the dissipation potential is obtained:

$$\phi(\dot{\boldsymbol{\epsilon}}^P, \dot{\kappa}; \mathbf{s}^P) = \begin{cases} \frac{\text{tr } \dot{\boldsymbol{\epsilon}}^P}{3A_\theta} (\bar{\sigma}^P(\kappa) + (A_\theta - A_\varphi) \text{tr } \mathbf{s}^P) & \text{if } \dot{\kappa} \geq \frac{1}{3A_\theta} \text{tr } \dot{\boldsymbol{\epsilon}}^P \text{ and } \text{tr } \dot{\boldsymbol{\epsilon}}^P \geq \sqrt{6} A_\theta \|\dot{\boldsymbol{\epsilon}}_{\text{dev}}^P\|, \\ +\infty & \text{otherwise.} \end{cases} \quad (66)$$

It is noteworthy that the constraint on the volumetric plastic strains $\text{tr } \dot{\boldsymbol{\epsilon}}^P \geq \sqrt{6} A_\theta \|\dot{\boldsymbol{\epsilon}}_{\text{dev}}^P\|$ is a consequence of (65), and is therefore not required as an a priori assumption in the present framework. Using the notion of indicator functions, the dissipation potential takes the compact form

$$\phi(\dot{\boldsymbol{\epsilon}}^P, \dot{\kappa}; \mathbf{s}^P) = \frac{\text{tr } \dot{\boldsymbol{\epsilon}}^P}{3A_\theta} (\bar{\sigma}^P(\kappa) + (A_\theta - A_\varphi) \text{tr } \mathbf{s}^P) + I_{\mathbb{R}_+} \left(\dot{\kappa} - \frac{1}{3A_\theta} \text{tr } \dot{\boldsymbol{\epsilon}}^P \right) + I_{\mathbb{R}_+} (\text{tr } \dot{\boldsymbol{\epsilon}}^P - \sqrt{6} A_\theta \|\dot{\boldsymbol{\epsilon}}_{\text{dev}}^P\|). \quad (67)$$

In agreement with the yield function (56) and the plastic potential (58), the dissipation potential (67)

²An alternative evaluation of the dissipation potential consists of replacing $\tilde{\mathbf{s}}_{\text{dev}}^P$ by its upper bound in the argument of the supremum in equation (65). In this case, the assumption $A_\theta > 0$ is no longer required, and the dissipation potential is expressed in terms of $\|\dot{\boldsymbol{\epsilon}}_{\text{dev}}^P\|$ instead of $\text{tr } \dot{\boldsymbol{\epsilon}}^P$.

explicitly depends on κ , which is again omitted from the notation for simplicity. As shown below, it turns out that this dependence is crucial for the thermodynamic consistency of the dissipation potential, that is, the fulfilment of the dissipation inequality (31), where the choice of b plays a major role. For the discussion that follows, it is important to note that different choices of $b \in [0, 1]$ lead to the same model in terms of the governing equations. The difference lies in the interpretation of the isotropic hardening mechanism and the admissibility of the corresponding dissipation potential.

As previously mentioned, the parameter b in the free energy (50) allows to interpret isotropic hardening as free energy, where κ plays the role of an internal variable, while the presence of b in the yield function (56), the plastic potential (58) and, consequently, the dissipation potential (67), also allows to interpret isotropic hardening as dissipation. The reason for this hybrid formulation is that, while the value assigned to the constant b will not change the governing equations of the model, it will determine if the resulting dissipation potential (67) complies with the dissipation inequality (31). In particular, in order for the dissipation potential to attain non-negative values, equation (67) demands that

$$(A_\varphi - A_\theta) \operatorname{tr} \mathbf{s}^P \leq \sigma^P + bH^{\text{iso}}\kappa. \quad (68)$$

In the associative case, where $A_\varphi = A_\theta$, this condition is always satisfied, ensuring the thermodynamic consistency of the dissipation potential, regardless of the choice of $b \in [0, 1]$. In agreement with the discussion of Gurtin and Reddy [97] for von Mises plasticity, we can equivalently view isotropic hardening in the associative Drucker-Prager model as a purely energetic mechanism, with $b = 0$, or as a purely dissipative mechanism, with $b = 1$. On the other hand, for the non-associative case, equation (68) implies an upper bound on $\operatorname{tr} \mathbf{s}^P$, depending on the value of b .

As observed in figure 4, the yield function (56) implies, at a given generalized-stress state $\{\mathbf{s}^P, h\}$, a maximum allowable value for $\operatorname{tr} \mathbf{s}^P$ that corresponds to the position of the apex, such that

$$A_\varphi \operatorname{tr} \mathbf{s}^P \leq \sigma^P + bH^{\text{iso}}\kappa + h = \sigma^P + H^{\text{iso}}\kappa. \quad (69)$$

Thus, in order to guarantee that the dissipation potential is non-negative, the constraint in equation (69) must always be stricter than the constraint in equation (68), that is:

$$\frac{\sigma^P + H^{\text{iso}}\kappa}{A_\varphi} \leq \frac{\sigma^P + bH^{\text{iso}}\kappa}{A_\varphi - A_\theta}. \quad (70)$$

We immediately note that for $b = 0$, where the dissipative contribution of isotropic hardening vanishes in the dissipation potential (67), inequality (70) is not satisfied if $H^{\text{iso}}\kappa > \sigma^P A_\theta / (A_\varphi - A_\theta)$. Because κ can take arbitrarily large positive values, the model results in a dissipation potential that is not a priori non-negative if $b = 0$. Consequently, the interpretation of isotropic hardening as a purely energetic mechanism is *not thermodynamically admissible* in the non-associative Drucker-Prager model. On the other hand, if $b = 1$, the contribution of isotropic hardening to the free energy (50) vanishes, and inequality (70) gives $H^{\text{iso}}\kappa \geq -\sigma^P A_\theta$. Because this condition is always satisfied, the perspective of isotropic hardening as a purely dissipative mechanism results in a thermodynamically consistent model.

An alternative perspective is obtained by distributing the contribution of isotropic hardening to both the free energy and the dissipation, attained by intermediate values of b . In this case, the fulfilment of

inequality (70) is guaranteed if

$$\frac{A_\varphi - A_\theta}{A_\varphi} \leq b \leq 1. \quad (71)$$

An interesting option is to set $b := (A_\varphi - A_\theta)/A_\varphi$, such that the models corresponding to $A_\theta = 0$ and $A_\theta = A_\varphi$ are recovered with $b = 1$ and $b = 0$, respectively. Then, isotropic hardening is a purely dissipative mechanism in the non-dilatant case, and a purely energetic mechanism, contributing only to the free energy, in the associative case. In both cases, inequality (70) holds, and thus the model is thermodynamically admissible. However, it is important to note that $b = 0$ implies a vanishing dissipation if $\sigma^p = 0$, that is, for cohesionless materials. Indeed, the dissipation even vanishes for the case of von Mises plasticity, as previously obtained by Yang et al. [99], where isotropic hardening was only considered in the free energy.

This analysis above demonstrates that to ensure thermodynamic consistency, the energetic role of isotropic hardening, here tuned by b , should be carefully considered for the non-associative case. Moreover, a non-vanishing positive dissipation is only guaranteed in the most general case, including zero cohesion, if isotropic hardening is included in the dissipation, achieved in the present model by setting $0 < b \leq 1$.

4.3. Energetic formulation

To construct the energy functionals entering the variational principles of section 3, the function spaces for the internal variables are first defined. In particular, we set $\{\boldsymbol{\varepsilon}^p, \kappa\} \in \mathcal{B} \times \mathcal{K}$ and $\{\tilde{\boldsymbol{\varepsilon}}^p, \tilde{\kappa}\} \in \tilde{\mathcal{Q}} = \tilde{\mathcal{B}} \times \tilde{\mathcal{K}}$, where the specific forms of $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{K}}$ are chosen such that $\phi(\tilde{\boldsymbol{\varepsilon}}^p, \tilde{\kappa}; \mathbf{s}^p) < +\infty$. In this way, the global dissipative power entering the stability condition (45) remains finite, and we are thus able to account for the critical conditions under which (45) holds. With this in mind and in view of the dissipation potential (66), the function spaces are defined as

$$\mathcal{B} := L^2(\Omega; \mathbb{R}_{\text{sym}}^{3 \times 3}), \quad \tilde{\mathcal{B}} := \{\mathbf{b} \in \mathcal{B} \mid \text{tr } \mathbf{b} \geq \sqrt{6}A_\theta \|\mathbf{b}_{\text{dev}}\|\}, \quad (72)$$

$$\mathcal{K} := H^1(\Omega; \mathbb{R}_+), \quad \tilde{\mathcal{K}}(\mathbf{b}) := \{z \in \mathcal{K} \mid z \geq \text{tr } \mathbf{b}/(3A_\theta), \mathbf{b} \in \tilde{\mathcal{B}}\}. \quad (73)$$

Given the previous definitions, Proposition 1 establishes the equivalence between the subdifferential Formulation 1 and the variational Formulation 2 for the Drucker-Prager model. Recalling from Proposition 1 that the equilibrium conditions are satisfied, we derive here, for illustrative purposes, the elastic domain and the flow rule as consequences of local stability and energy balance, respectively.

The elastic domain as a consequence of local stability. Given the equilibrium conditions (48), the free energy (50) and the dissipation potential (66), the local stability condition (45) reads

$$\int_{\Omega} \left(\mathbf{s}_{\text{dev}}^p : \tilde{\boldsymbol{\varepsilon}}_{\text{dev}}^p + \frac{1}{3} \text{tr } \mathbf{s}^p \text{tr } \tilde{\boldsymbol{\varepsilon}}^p + h\tilde{\kappa} - \frac{1}{3A_\theta} (\bar{\sigma}^p(\kappa) + (A_\theta - A_\varphi) \text{tr } \mathbf{s}^p) \text{tr } \tilde{\boldsymbol{\varepsilon}}^p \right) d\mathbf{x} \leq 0, \quad \forall \{\tilde{\boldsymbol{\varepsilon}}^p, \tilde{\kappa}\} \in \tilde{\mathcal{B}} \times \tilde{\mathcal{K}}(\tilde{\boldsymbol{\varepsilon}}^p). \quad (74)$$

In agreement with the function spaces (72) and (73), we seek to maximize (74) by setting $\text{tr } \tilde{\boldsymbol{\varepsilon}}^p = \sqrt{6}A_\theta \|\tilde{\boldsymbol{\varepsilon}}_{\text{dev}}^p\|$ and $\tilde{\kappa} = \sqrt{2/3} \|\tilde{\boldsymbol{\varepsilon}}_{\text{dev}}^p\|$, while taking $\tilde{\boldsymbol{\varepsilon}}_{\text{dev}}^p$ collinear to $\mathbf{s}_{\text{dev}}^p$. Equation (74) then gives

$$\int_{\Omega} \left(\|\mathbf{s}_{\text{dev}}^p\| + \sqrt{\frac{2}{3}} (A_\varphi \text{tr } \mathbf{s}^p + h - \bar{\sigma}^p(\kappa)) \right) \|\tilde{\boldsymbol{\varepsilon}}_{\text{dev}}^p\| d\mathbf{x} \leq 0, \quad (75)$$

from where the yield function (56) and the elastic domain (55) directly follow due to the localization principle.

The flow rule as a consequence of energy balance. Using the equilibrium conditions (32), the free energy (50) and the dissipation potential (66) in the power balance (40) gives the bulk condition

$$\mathbf{s}^P : \dot{\boldsymbol{\varepsilon}}^P + h\dot{\kappa} - \phi(\dot{\boldsymbol{\varepsilon}}^P, \dot{\kappa}; \mathbf{s}^P) = 0. \quad (76)$$

Focusing on the hardening law, we take the subdifferential of both sides of (76) with respect to $\dot{\kappa}$. In view of equation (67), we obtain the relation

$$h \in \partial I_{\mathbb{R}_+}(\dot{\kappa} - \text{tr } \dot{\boldsymbol{\varepsilon}}^P / (3A_\theta)). \quad (77)$$

Recalling that $\partial I_{\mathbb{R}_+}(\square)$ gives $\{0\}$ for $\square > 0$, $\mathbb{R}_-$ for $\square = 0$, and $\emptyset$ otherwise, h vanishes for $\dot{\kappa} > \text{tr } \dot{\boldsymbol{\varepsilon}}^P / (3A_\theta)$. Hence, in order to comply with of the constitutive equations (54), it follows that $\dot{\kappa} = \text{tr } \dot{\boldsymbol{\varepsilon}}^P / (3A_\theta)$ is required for non-vanishing isotropic hardening. We now let $\dot{\boldsymbol{\varepsilon}}^P := \lambda \hat{\mathbf{n}}^\circ$, where, for convenience, $\hat{\mathbf{n}}^\circ \in \mathbb{R}_{\text{sym}}^{3 \times 3}$ is such that $\text{tr } \hat{\mathbf{n}}^\circ = \sqrt{6}A_\theta \iff \sqrt{6}A_\theta \lambda = \text{tr } \dot{\boldsymbol{\varepsilon}}^P$. As a consequence,

$$\dot{\kappa} = \sqrt{2/3} \lambda. \quad (78)$$

From this result, we may write the compact representation

$$\{\dot{\boldsymbol{\varepsilon}}^P, \dot{\kappa}\} = \lambda \{\hat{\mathbf{n}}^\circ, \sqrt{2/3}\}, \quad (79)$$

where neither the form of $\hat{\mathbf{n}}^\circ$ nor the evolution of λ have been fully specified. Replacing this expression in equation (76) gives

$$\lambda \left(\mathbf{s}_{\text{dev}}^P : \hat{\mathbf{n}}_{\text{dev}}^\circ + \sqrt{\frac{2}{3}} (A_\varphi \text{tr } \mathbf{s}^P + h - \bar{\sigma}^P(\kappa)) \right) = 0. \quad (80)$$

Moreover, the constraint $\text{tr } \dot{\boldsymbol{\varepsilon}}^P \geq \sqrt{6}A_\theta \|\dot{\boldsymbol{\varepsilon}}_{\text{dev}}^P\|$ implies that $\lambda \geq \|\dot{\boldsymbol{\varepsilon}}_{\text{dev}}^P\| \iff \|\hat{\mathbf{n}}_{\text{dev}}^\circ\| \leq 1$, such that the factor multiplying λ in equation (80) is always less than or equal to $f(\mathbf{s}^P, h)$. The complementary condition $\lambda f(\mathbf{s}^P, h) = 0$ is then recovered, such that $\lambda = 0$ if $f(\mathbf{s}^P, h) < 0$. Hence, in view of (62) and (79), it is only left to show that $\hat{\mathbf{n}}^\circ = \hat{\mathbf{n}}$ (equation (60)) when $f(\mathbf{s}^P, h) = 0$ to recover the flow rule (61). To this end, equation (56) can be used to solve for σ^P , which is replaced along with (79) in equation (76) to obtain

$$\lambda \mathbf{s}^P : \hat{\mathbf{n}}^\circ = \lambda \left(\|\mathbf{s}_{\text{dev}}^P\| + \sqrt{\frac{2}{3}} A_\theta \text{tr } \mathbf{s}^P \right). \quad (81)$$

The subdifferential of both sides with respect to $\mathbf{s}^P$ finally gives $\hat{\mathbf{n}}^\circ \equiv \hat{\mathbf{n}}$.

4.4. Material response

The material response of the Drucker-Prager plasticity model and its numerical treatment are topics that have been well studied in the literature [37, 54, 100]. A brief description of the response is provided here for the reader's convenience using a volume element subjected to a triaxial test with confining pressure. First, a hydrostatic confining stress is applied, with $\sigma_{xx} = \sigma_{yy} = \sigma_{zz}$ gradually varying from 0 to $-p_0$, with $p_0 \geq 0$. Then, a monotonically varying axial strain ε_{zz} is imposed such that $\dot{\varepsilon}_{zz} < 0$, while maintaining the lateral pressure $\sigma_{xx} = \sigma_{yy} = -p_0$. Fixed material parameters are chosen as follows: Young's modulus $E = 1$ MPa, Poisson's ratio $\nu = 0.3$, yield strength $\sigma^P = 0.025$ MPa, and friction constant $A_\varphi = 0.5$.

Figure 5 shows the response at different confining pressures for the case of perfect plasticity, where $H_\mu^{\text{kin}} = H_K^{\text{kin}} = H^{\text{iso}} = 0$ MPa, and dilation constant $A_\theta = 0.025$. The stress-strain response in figure 5(a)

Fig. 5. Response of Drucker-Prager model with perfect plasticity for $p_0 = 0.06$ MPa (gray), $p_0 = 0.03$ MPa (blue) and $p_0 = 0$ MPa (red): (a) axial stress ($-\sigma_{zz}$) vs. axial strain ($-\varepsilon_{zz}$); (b) stress state in $(\sqrt{3} \text{tr } \boldsymbol{\sigma} / 3, \|\boldsymbol{\sigma}_{\text{dev}}\|)$ space, showing the stress paths (solid lines), the elastic domain (dashed lines) and state-dependent domain at points C, C' and C'' (dotted lines). The plastic flow vectors at C, C' and C'' (arrows) are plotted in $(\sqrt{3} \text{tr } \boldsymbol{\varepsilon}^p / 3, \|\boldsymbol{\varepsilon}_{\text{dev}}^p\|)$ space.

Fig. 6. Response of the Drucker-Prager model with confining pressure $p_0 = 0.03$ MPa for $A_\theta = 0.025$ (red), $A_\theta = 0.25$ (blue) and the associative case $A_\theta = 0.5$ (gray), for $H_K^{\text{kin}} = 0$ MPa (solid lines) and $H_K^{\text{kin}} = 0.0926$ MPa (dashed lines): (a) axial stress ($-\sigma_{zz}$) vs. axial strain ($-\varepsilon_{zz}$) and (b) volumetric strain ($\text{tr } \boldsymbol{\varepsilon}$) vs. axial strain ($-\varepsilon_{zz}$).

shows the characteristic increase in strength as the confining pressure increases, with the corresponding stress state and plastic flow shown in figure 5(b).

Figure 6 shows the effect of the hardening mechanisms for different degrees of dilation at fixed confining pressure. The shear kinematic hardening modulus and the isotropic hardening modulus are fixed at $H_\mu^{\text{kin}} = 0.0741$ MPa and $H^{\text{iso}} = 0.0556$ MPa, respectively, while both the volumetric kinematic hardening modulus H_K^{kin} and the dilation constant A_θ are varied. Figure 6(b) shows the expected decrease in volumetric strains as A_θ decreases. Interestingly, the same effect is obtained by fixing A_θ and increasing H_K^{kin} . This result is consistent with previous studies combining Drucker-Prager plasticity with kinematic hardening [96].

In the present model, it is possible to obtain the same results shown in figure 6(b) by fixing H_K^{kin} and increasing H^{iso} . This is due to the effect of the hardening mechanisms in principal stress space: while the shear kinematic hardening mechanism shifts the yield surface on the deviatoric plane, the volumetric

kinematic hardening and isotropic hardening mechanisms simultaneously shift the yield surface along the hydrostatic axis. However, the difference between these two mechanisms is that the volumetric kinematic hardening effect depends on $\text{tr } \boldsymbol{\varepsilon}^p$ and is thus scaled by A_θ , vanishing for $A_\theta = 0$. In turn, the isotropic hardening mechanism is a function of κ and is still active under isochoric plastic strains.

5. Ratcheting plasticity

As a second application, this section presents a non-associative model for cyclic plasticity. The point of departure is the model proposed by Houlsby et al. [21] to describe the behavior of a monopile foundation as a one-dimensional mechanical system with a cyclic accumulation of plastic strains. However, when developed in a continuum setting, the model is applicable to a wide range of responses that exhibit ratcheting under cyclic loading [59].

To derive the governing equations, Houlsby et al. [21] relied on the framework of hyperplasticity [8], a theory strongly linked to generalized standard materials. Here, the model is revisited in the context of the formulation presented in section 2 and endowed with isotropic hardening. The reader is referred to Houlsby et al. [21] for the original model with multi-surface plasticity and to Ulloa et al. [59] for an application to fatigue in a phase-field model.

As a final note, the ratcheting model is extended in section 5.5 to consider pressure-dependence and dilation effects. This is achieved by combining the ratcheting mechanism with the Drucker-Prager model presented in the previous section, allowing for both volumetric and deviatoric ratcheting strains.

5.1. The classical model ingredients

Consider the set of internal variables $\{\boldsymbol{\varepsilon}^p, \kappa, \boldsymbol{\varepsilon}^r\}$, where, as in the previous section, $\kappa(\mathbf{x}, t) \in \mathbb{R}_+$ is an isotropic hardening variable. On the other hand, $\boldsymbol{\varepsilon}^r(\mathbf{x}, t) \in \mathbb{R}_{\text{sym}}^{3 \times 3}$ represents the ratcheting strain tensor. As will become clear in the sequel, this variable introduces inelastic strains that evolve in the direction of the deviatoric stress. This allows to capture an accumulation of plastic strains under cyclic loading [21, 101]. According to the notation adopted in section 2, this choice of internal variables corresponds to the array $\boldsymbol{\xi} := \{\kappa, \boldsymbol{\varepsilon}^r\}$. Then, the free energy density is defined as

$$\psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \kappa, \boldsymbol{\varepsilon}^r) := \underbrace{\frac{1}{2} \|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p - \boldsymbol{\varepsilon}^r\|_{\mathbf{C}}^2}_{\text{elastic}} + \underbrace{\frac{1}{2} H^{\text{kin}} \|\boldsymbol{\varepsilon}^p\|^2 + \frac{1}{2} H^{\text{iso}} \kappa^2}_{\text{plastic}}. \quad (82)$$

As in section 4, H^{iso} denotes the isotropic hardening modulus, while H^{kin} is now a scalar-valued kinematic hardening modulus. Moreover, the Cauchy stress tensor is obtained from equations (4) and (82) as

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p - \boldsymbol{\varepsilon}^r). \quad (83)$$

Similarly, equations (5), (82) and (83) provide the generalized stresses

$$\mathbf{s}^p = -\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^p} \equiv \boldsymbol{\sigma} - H^{\text{kin}} \boldsymbol{\varepsilon}^p, \quad h = -\frac{\partial \psi}{\partial \kappa} \equiv -H^{\text{iso}} \kappa \quad \text{and} \quad \mathbf{s}^r = -\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^r} \equiv \boldsymbol{\sigma}, \quad (84)$$

with the last two collected in the array $\boldsymbol{\eta} := \{h, \mathbf{s}^r\}$, conjugate to $\boldsymbol{\xi}$.

The elastic domain is defined as

$$\mathbb{K} := \left\{ \{\mathbf{s}^p, h, \mathbf{s}^r\} \in \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}_- \times \mathbb{R}_{\text{sym}}^{3 \times 3} \mid f(\mathbf{s}^p, h) \leq 0 \right\}, \quad (85)$$

where f is a von Mises-like yield function:

$$f(\mathbf{s}^p, h) := \|\mathbf{s}_{\text{dev}}^p\| + \sqrt{\frac{2}{3}}(h - \sigma^p) \leq 0. \quad (86)$$

Noting that f is independent of $\mathbf{s}^r$, it becomes clear that a non-associative flow rule with respect to $\dot{\boldsymbol{\varepsilon}}^r$ is required in order to introduce ratcheting effects. To this end, we define the plastic potential g as

$$g(\mathbf{s}^p, h, \mathbf{s}^r) := f(\mathbf{s}^p, h) + \beta \|\mathbf{s}_{\text{dev}}^r\|, \quad (87)$$

where $\beta \in [0, 1]$ is a material constant that determines the fraction of plastic strains that contributes to ratcheting. Note that we have chosen to include the isotropic hardening mechanism in the free energy (82) only, while both the yield function and the plastic potential do not explicitly depend on κ . In contrast with the Drucker-Prager model presented in section 4, this choice is thermodynamically admissible in the present model, leading to an a priori non-negative dissipation potential.

To characterize the flow rule (22), the subdifferential of g is evaluated as $\partial g(\mathbf{s}^p, h, \mathbf{s}^r) = \{\hat{\mathbf{n}}, \sqrt{2/3}, \beta \hat{\mathbf{r}}\}$, where

$$\hat{\mathbf{n}} = \frac{\mathbf{s}_{\text{dev}}^p}{\|\mathbf{s}_{\text{dev}}^p\|} \quad \text{and} \quad \hat{\mathbf{r}} \in \partial \|\mathbf{s}_{\text{dev}}^r\| \quad (88)$$

denote the direction of the plastic strain rate and the direction of the ratcheting strain rate, respectively. Equations (22) and (23) give the non-associative flow rule

$$\{\dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa}, \dot{\boldsymbol{\varepsilon}}^r\} \in \mathbb{Q}(\mathbf{s}^p, h, \mathbf{s}^r), \quad (89)$$

with

$$\mathbb{Q}(\mathbf{s}^p, h, \mathbf{s}^r) = \{\lambda \{\hat{\mathbf{n}}, \sqrt{2/3}, \beta \hat{\mathbf{r}}\} \in \mathbb{R}_{\text{dev}}^{3 \times 3} \times \mathbb{R}_+ \times \mathbb{R}_{\text{dev}}^{3 \times 3} \mid \lambda \geq 0, \lambda = 0 \text{ if } f(\mathbf{s}^p, h) < 0\}. \quad (90)$$

As a result of the specific choices for the yield function (86) and the plastic potential (87), the plastic strains and the ratcheting strains are purely deviatoric, i.e., $\boldsymbol{\varepsilon}^p(\mathbf{x}, t) \in \mathbb{R}_{\text{dev}}^{3 \times 3}$ and $\boldsymbol{\varepsilon}^r(\mathbf{x}, t) \in \mathbb{R}_{\text{dev}}^{3 \times 3}$. In equation (88), the direction of the plastic strain rate is always uniquely defined, since $\|\mathbf{s}_{\text{dev}}^p\| = 0$ implies that $\lambda = 0$. On the other hand, the direction of the ratcheting strain rate is given by the multivalued function $\partial \|\mathbf{s}_{\text{dev}}^r\|$, since $\|\mathbf{s}_{\text{dev}}^r\| = 0$ does not imply that $\lambda = 0$. This feature of the model is further discussed below.

5.2. Dissipation potential

In analogy with section 4, the first step for stating the variational formulation of the ratcheting model is the derivation of the dissipation potential from the generalized principle of maximum dissipation (29). To this end, the state-dependent domain (26) is determined by setting, in agreement with conditions (25), $r(\mathbf{s}^r) := g(\mathbf{s}^p, h, \mathbf{s}^r) - f(\mathbf{s}^p, h) = \beta \|\mathbf{s}_{\text{dev}}^r\|$, leading to

$$\mathbb{L}(\mathbf{s}^r) = \left\{ \{\tilde{\mathbf{s}}^p, \tilde{h}, \tilde{\mathbf{s}}^r\} \in \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}_- \times \mathbb{R}_{\text{sym}}^{3 \times 3} \mid \|\tilde{\mathbf{s}}_{\text{dev}}^p\| + \sqrt{\frac{2}{3}}(\tilde{h} - \sigma^p) + \beta \|\tilde{\mathbf{s}}_{\text{dev}}^r\| \leq \beta \|\mathbf{s}_{\text{dev}}^r\| \right\}. \quad (91)$$

In this case, the set (91) explicitly depends on the conjugate stress $\mathbf{s}^r$, due to the non-associativity of the ratcheting strains $\boldsymbol{\varepsilon}^r$. As a consequence, the dissipation potential will also depend explicitly on $\mathbf{s}^r$.

Figure 7 provides an illustrative representation of the set (91) in $\mathbb{R}_{\text{dev}}^{3 \times 3} \times \mathbb{R}_{\text{dev}}^{3 \times 3}$ space. $\mathbb{L}(\mathbf{s}^r)$ corresponds to the region that intersects the $\tilde{\mathbf{s}}_{\text{dev}}^p$ and $\tilde{\mathbf{s}}_{\text{dev}}^r$ axes at distances of $\sqrt{2/3}(\sigma^p - \tilde{h}) + \beta \|\mathbf{s}_{\text{dev}}^r\|$ and $\sqrt{2/3}(\sigma^p - \tilde{h})/\beta + \|\mathbf{s}_{\text{dev}}^r\|$ from the origin, respectively. In figure 7(a), consider states $\{\pm \mathbf{s}^r, \pm \mathbf{s}^p\}$, where $\|\mathbf{s}_{\text{dev}}^p\| = \sqrt{2/3}(\sigma^p - \tilde{h})$

and $\|\mathbf{s}_{\text{dev}}^r\| \neq 0$. Normality with respect to $\mathbb{K}$ yields a null ratcheting component, while the tensors normal to $\mathbb{L}(\mathbf{s}^r)$ comply with the non-associative flow rule (89). Moreover, the flow rule is uniquely defined because the boundaries of $\mathbb{K}$ and $\mathbb{L}(\mathbf{s}^r)$ intersect at smooth points of $\mathbb{L}(\mathbf{s}^r)$. We now turn to figure 7(b), where $\|\mathbf{s}_{\text{dev}}^r\| = 0$. The sets $\mathbb{K}$ and $\mathbb{L}(\mathbf{s}^r)$ intersect at corners of $\mathbb{L}(\mathbf{s}^r)$. As before, normality with respect to $\mathbb{K}$ yields a null ratcheting component, while the tensors normal to $\mathbb{L}(\mathbf{s}^r)$ comply with the non-associative flow rule (89). However, the flow rule is now multivalued in $\dot{\boldsymbol{\varepsilon}}^r$, such that $\{\dot{\boldsymbol{\varepsilon}}^r, \dot{\boldsymbol{\varepsilon}}^p\}$ belongs to normal cones of $\mathbb{L}(\mathbf{s}^r)$. This occurrence corresponds to the scenario labeled with a superscript γ in figure 2, where, at the current state, the yield surface is smooth but the plastic potential is not.

With the state-dependent set (91) at hand, the state-dependent dissipation potential (29) is evaluated for all $\{\dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa}, \dot{\boldsymbol{\varepsilon}}^r\} \in \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}_+ \times \mathbb{R}_{\text{sym}}^{3 \times 3}$ as

$$\begin{aligned} \phi(\dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa}, \dot{\boldsymbol{\varepsilon}}^r; \mathbf{s}^r) &= \sup \left\{ \tilde{\mathbf{s}}^p : \dot{\boldsymbol{\varepsilon}}^p + \tilde{h}\dot{\kappa} + \tilde{\mathbf{s}}^r : \dot{\boldsymbol{\varepsilon}}^r \mid \|\tilde{\mathbf{s}}_{\text{dev}}^p\| + \sqrt{\frac{2}{3}}(\tilde{h} - \sigma^p) + \beta\|\tilde{\mathbf{s}}_{\text{dev}}^r\| \leq \beta\|\mathbf{s}_{\text{dev}}^r\|, \tilde{h} \leq 0 \right\} \\ &= \sup \left\{ \tilde{\mathbf{s}}_{\text{dev}}^p : \dot{\boldsymbol{\varepsilon}}_{\text{dev}}^p + \tilde{\mathbf{s}}_{\text{dev}}^r : \dot{\boldsymbol{\varepsilon}}_{\text{dev}}^r + \frac{1}{3}(\text{tr } \tilde{\mathbf{s}}^p \text{tr } \dot{\boldsymbol{\varepsilon}}^p + \text{tr } \tilde{\mathbf{s}}^r \text{tr } \dot{\boldsymbol{\varepsilon}}^r) + \tilde{h}\dot{\kappa} \right. \\ &\quad \left. \mid \|\tilde{\mathbf{s}}_{\text{dev}}^p\| \leq \beta\|\mathbf{s}_{\text{dev}}^r\| - \sqrt{\frac{2}{3}}(\tilde{h} - \sigma^p) - \beta\|\tilde{\mathbf{s}}_{\text{dev}}^r\|, \tilde{h} \leq 0 \right\} \quad (92) \\ &= \sup \left\{ \left(\sqrt{\frac{2}{3}}\sigma^p + \beta\|\mathbf{s}_{\text{dev}}^r\| \right) \|\dot{\boldsymbol{\varepsilon}}_{\text{dev}}^p\| + \tilde{h} \left(\dot{\kappa} - \sqrt{\frac{2}{3}}\|\dot{\boldsymbol{\varepsilon}}_{\text{dev}}^p\| \right) + \|\tilde{\mathbf{s}}_{\text{dev}}^r\| (\|\dot{\boldsymbol{\varepsilon}}_{\text{dev}}^r\| - \beta\|\dot{\boldsymbol{\varepsilon}}_{\text{dev}}^p\|) \right. \\ &\quad \left. + \frac{1}{3}(\text{tr } \tilde{\mathbf{s}}^p \text{tr } \dot{\boldsymbol{\varepsilon}}^p + \text{tr } \tilde{\mathbf{s}}^r \text{tr } \dot{\boldsymbol{\varepsilon}}^r) \mid \tilde{h} \leq 0 \right\}. \end{aligned}$$

We note that the expression inside the supremum is unbounded when $\text{tr } \tilde{\mathbf{s}}^p \text{tr } \dot{\boldsymbol{\varepsilon}}^p$ or $\text{tr } \tilde{\mathbf{s}}^r \text{tr } \dot{\boldsymbol{\varepsilon}}^r$ is non-zero. This is also the case when the terms multiplying $\tilde{h}$ and $\|\tilde{\mathbf{s}}_{\text{dev}}^r\|$ are negative and positive, respectively. As a consequence, the following closed form of the dissipation potential is obtained

$$\phi(\dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa}, \dot{\boldsymbol{\varepsilon}}^r; \mathbf{s}^r) = \begin{cases} \left(\sqrt{\frac{2}{3}}\sigma^p + \beta\|\mathbf{s}_{\text{dev}}^r\| \right) \|\dot{\boldsymbol{\varepsilon}}_{\text{dev}}^p\| & \text{if } \dot{\kappa} \geq \sqrt{\frac{2}{3}}\|\dot{\boldsymbol{\varepsilon}}_{\text{dev}}^p\|, \|\dot{\boldsymbol{\varepsilon}}_{\text{dev}}^r\| \leq \beta\|\dot{\boldsymbol{\varepsilon}}_{\text{dev}}^p\| \text{ and } \text{tr } \dot{\boldsymbol{\varepsilon}}^p = \text{tr } \dot{\boldsymbol{\varepsilon}}^r = 0, \\ +\infty & \text{otherwise.} \end{cases} \quad (93)$$

We highlight that the isochoric nature of $\boldsymbol{\varepsilon}^p$ and $\boldsymbol{\varepsilon}^r$ is a consequence of equation (92), and is therefore not required as an a priori assumption in the present framework. In other words, we do not require to impose incompressibility as an a priori constraint. Equation (93) can be written in compact form as

$$\begin{aligned} \phi(\dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa}, \dot{\boldsymbol{\varepsilon}}^r; \mathbf{s}^r) &= \left(\sqrt{\frac{2}{3}}\sigma^p + \beta\|\mathbf{s}_{\text{dev}}^r\| \right) \|\dot{\boldsymbol{\varepsilon}}_{\text{dev}}^p\| + I_{\mathbb{R}_+} \left(\dot{\kappa} - \sqrt{\frac{2}{3}}\|\dot{\boldsymbol{\varepsilon}}_{\text{dev}}^p\| \right) + I_{\mathbb{R}_+} (\beta\|\dot{\boldsymbol{\varepsilon}}_{\text{dev}}^p\| - \|\dot{\boldsymbol{\varepsilon}}_{\text{dev}}^r\|) \\ &\quad + I_{\{0\}}(\text{tr } \dot{\boldsymbol{\varepsilon}}^p) + I_{\{0\}}(\text{tr } \dot{\boldsymbol{\varepsilon}}^r). \end{aligned} \quad (94)$$

It is easy to see that the dissipation potential obtained for the present model is a priori non-negative.

5.3. Energetic formulation

To construct the energy functionals entering the variational principles of section 3, we set $\{\boldsymbol{\varepsilon}^p, \kappa, \boldsymbol{\varepsilon}^r\} \in \mathcal{B} \times \mathcal{K} \times \mathcal{R}$ and $\{\tilde{\boldsymbol{\varepsilon}}^p, \tilde{\kappa}, \tilde{\boldsymbol{\varepsilon}}^r\} \in \tilde{\mathcal{Q}} = \tilde{\mathcal{B}} \times \tilde{\mathcal{K}}(\tilde{\boldsymbol{\varepsilon}}^p) \times \tilde{\mathcal{R}}(\tilde{\boldsymbol{\varepsilon}}^p)$. As discussed in section 4.3, the function spaces in $\tilde{\mathcal{Q}}$ are chosen such that the dissipation potential is finite, implying critical conditions for the stability

$$\{\dot{\epsilon}^r, \dot{\epsilon}^p\} \in \begin{cases} \lambda \left\{ \mathbf{0}, \frac{\tilde{\mathbf{s}}_{\text{dev}}^p}{\|\tilde{\mathbf{s}}_{\text{dev}}^p\|} \right\} & \text{Associative flow} \\ \lambda \left\{ \beta \partial \|\tilde{\mathbf{s}}_{\text{dev}}^r\|, \frac{\tilde{\mathbf{s}}_{\text{dev}}^p}{\|\tilde{\mathbf{s}}_{\text{dev}}^p\|} \right\} & \text{Non-associative flow} \end{cases}$$

Fig. 7. Elastic domain $\mathbb{K}$ (gray region) and state-dependent domain $\mathbb{L}(\mathbf{s}^r)$ (brown region) for the ratcheting model at fixed h . (a) shows the flow rule at states $\{\pm \mathbf{s}^r, \pm \mathbf{s}^p\}$, labeled A^α , B^α , C^α and D^α , corresponding to smooth points of $\mathbb{K}$ and $\mathbb{L}(\mathbf{s}^r)$. (b) shows the flow rule at states $\{\mathbf{0}, \pm \mathbf{s}^p\}$, corresponding to corners of $\mathbb{L}(\mathbf{s}^r)$, labeled E^γ and F^γ . The labels α and γ correspond to the scenarios described in figure 2. The tensors normal to the boundary of $\mathbb{K}$ (blue) yield a null ratcheting component, while the tensors normal to the boundary of $\mathbb{L}(\mathbf{s}^r)$ (red) are consistent with the non-associative flow rule (89).

condition (45). In view of equation (93), we set

$$\mathcal{B} := L^2(\Omega; \mathbb{R}_{\text{dev}}^{3 \times 3}), \quad \tilde{\mathcal{B}} \equiv \mathcal{B}, \quad (95)$$

$$\mathcal{K} := H^1(\Omega; \mathbb{R}_+), \quad \tilde{\mathcal{K}}(\mathbf{b}) := \{z \in \mathcal{K} \mid z \geq \sqrt{2/3} \|\mathbf{b}\|, \mathbf{b} \in \tilde{\mathcal{B}}\}, \quad (96)$$

$$\mathcal{R} := L^2(\Omega; \mathbb{R}_{\text{dev}}^{3 \times 3}), \quad \tilde{\mathcal{R}}(\mathbf{b}) := \{z \in \mathcal{R} \mid \|z\| \leq \beta \|\mathbf{b}\|, \mathbf{b} \in \tilde{\mathcal{B}}\}. \quad (97)$$

The isochoric nature of ϵ^p and ϵ^r is included in the function spaces (95) and (97), such that $\epsilon^p \equiv \epsilon_{\text{dev}}^p$ and $\epsilon^r \equiv \epsilon_{\text{dev}}^r$ hold in the following passages.

Given the previous definitions, Proposition 1 establishes the equivalence between the subdifferential Formulation 1 and the variational Formulation 2 for the ratcheting model. As done for the Drucker-Prager model in section 4.3, we derive here, for illustrative purposes, the elastic domain and the flow rule as consequences of local stability and energy balance.

The elastic domain as a consequence of local stability. Given the equilibrium conditions (48), the free energy (82) and the dissipation potential (93), the local stability condition (45) gives

$$\int_{\Omega} \left[\mathbf{s}^p : \tilde{\epsilon}^p + h \tilde{\kappa} + \mathbf{s}^r : \tilde{\epsilon}^r - \left(\sqrt{\frac{2}{3}} \sigma^p + \beta \|\tilde{\mathbf{s}}_{\text{dev}}^r\| \right) \|\tilde{\epsilon}^p\| \right] d\mathbf{x} \leq 0, \quad \forall \{\tilde{\epsilon}^p, \tilde{\kappa}, \tilde{\epsilon}^r\} \in \tilde{\mathcal{B}} \times \tilde{\mathcal{K}}(\tilde{\epsilon}^p) \times \tilde{\mathcal{R}}(\tilde{\epsilon}^p). \quad (98)$$

In view of the admissible arbitrariness of the virtual fields, we note from (95), (96) and (97) that the terms $\mathbf{s}^p : \tilde{\boldsymbol{\epsilon}}^p$, $h\dot{\kappa}$ and $\mathbf{s}^r : \tilde{\boldsymbol{\epsilon}}^r$ attain maximum values at $\|\mathbf{s}_{\text{dev}}^p\| \|\tilde{\boldsymbol{\epsilon}}^p\|$, $\sqrt{2/3}h\|\tilde{\boldsymbol{\epsilon}}^p\|$ and $\beta\|\mathbf{s}_{\text{dev}}^r\| \|\tilde{\boldsymbol{\epsilon}}^p\|$, respectively. Then, inequality (98) holds for any admissible test field if

$$\int_{\Omega} \left(\|\mathbf{s}_{\text{dev}}^p\| + \sqrt{\frac{2}{3}}(h - \sigma^p) \right) \|\tilde{\boldsymbol{\epsilon}}^p\| d\mathbf{x} \leq 0, \quad \forall \tilde{\boldsymbol{\epsilon}}^p \in \tilde{\mathcal{B}}, \quad (99)$$

from where the yield function (86) and the elastic domain (85) directly follow.

The flow rule as a consequence of energy balance. By using the equilibrium conditions (32), the free energy (82) and the dissipation potential (93) in the power balance (40), the following bulk condition is obtained:

$$\mathbf{s}^p : \dot{\boldsymbol{\epsilon}}^p + h\dot{\kappa} + \mathbf{s}^r : \dot{\boldsymbol{\epsilon}}^r - \phi(\dot{\boldsymbol{\epsilon}}^p, \dot{\kappa}, \dot{\boldsymbol{\epsilon}}^r; \mathbf{s}^r) = 0. \quad (100)$$

Let us first focus on the relations between $\dot{\boldsymbol{\epsilon}}^p$ and the dependent variables $\dot{\kappa}$ and $\dot{\boldsymbol{\epsilon}}^r$. Taking the subdifferential of both sides with respect to $\dot{\kappa}$ and $\dot{\boldsymbol{\epsilon}}^r$, equation (100) gives

$$h \in \partial I_{\mathbb{R}_+}(\dot{\kappa} - \sqrt{2/3}\|\dot{\boldsymbol{\epsilon}}^p\|) \quad \text{and} \quad \mathbf{s}^r \in \partial I_{\mathbb{R}_+}(\beta\|\dot{\boldsymbol{\epsilon}}^p\| - \|\dot{\boldsymbol{\epsilon}}^r\|). \quad (101)$$

From the subdifferential of the indicator function, it follows that h is undefined for $\dot{\kappa} < \sqrt{2/3}\|\dot{\boldsymbol{\epsilon}}^p\|$ and vanishes for $\dot{\kappa} > \sqrt{2/3}\|\dot{\boldsymbol{\epsilon}}^p\|$. Likewise, $\mathbf{s}^r$ is undefined for $\|\dot{\boldsymbol{\epsilon}}^r\| > \beta\|\dot{\boldsymbol{\epsilon}}^p\|$ and vanishes for $\|\dot{\boldsymbol{\epsilon}}^r\| < \beta\|\dot{\boldsymbol{\epsilon}}^p\|$. Then, to comply with the constitutive equations (84),

$$\dot{\kappa} = \sqrt{2/3}\|\dot{\boldsymbol{\epsilon}}^p\| \quad \text{and} \quad \|\dot{\boldsymbol{\epsilon}}^r\| = \beta\|\dot{\boldsymbol{\epsilon}}^p\| \quad (102)$$

must hold. We now let $\dot{\boldsymbol{\epsilon}}^p := \lambda \hat{\mathbf{n}}^\circ$, where, for convenience $\hat{\mathbf{n}}^\circ \in \mathbb{R}_{\text{dev}}^{3 \times 3}$ is such that $\|\hat{\mathbf{n}}^\circ\| = 1 \iff \lambda = \|\dot{\boldsymbol{\epsilon}}^p\|$. Letting $\dot{\boldsymbol{\epsilon}}^r := \lambda \beta \hat{\mathbf{r}}^\circ$, with $\hat{\mathbf{r}}^\circ \in \mathbb{R}_{\text{dev}}^{3 \times 3}$, and in view of (102), we may write the compact representation

$$\{\dot{\boldsymbol{\epsilon}}^p, \dot{\kappa}, \dot{\boldsymbol{\epsilon}}^r\} = \lambda \{\hat{\mathbf{n}}^\circ, \sqrt{2/3}, \beta \hat{\mathbf{r}}^\circ\}, \quad (103)$$

where neither the forms of $\hat{\mathbf{n}}^\circ$ and $\hat{\mathbf{r}}^\circ$ nor the evolution of λ have been fully specified. Replacing this expression in equation (100) directly yields the complementary condition $\lambda f(\mathbf{s}^p, h) = 0$, such that $\lambda = 0$ if $f(\mathbf{s}^p, h) < 0$. Hence, in view of (90) and (103), it is left to show that $\hat{\mathbf{n}}^\circ = \hat{\mathbf{n}}$ and $\hat{\mathbf{r}}^\circ = \hat{\mathbf{r}}$ (equation (88)) when $f(\mathbf{s}^p, h) = 0$ to recover the flow rule (89). To this end, equation (86) can be used to solve for σ^p , which is replaced along with (103) in equation (100) to obtain

$$\lambda(\mathbf{s}^p : \hat{\mathbf{n}}^\circ + \mathbf{s}^r : \hat{\mathbf{r}}^\circ) = \lambda(\|\mathbf{s}_{\text{dev}}^p\| + \beta\|\mathbf{s}_{\text{dev}}^r\|). \quad (104)$$

The subdifferential of both sides with respect to $\mathbf{s}^p$ and $\mathbf{s}^r$ finally gives $\hat{\mathbf{n}}^\circ \equiv \hat{\mathbf{n}}$ and $\hat{\mathbf{r}}^\circ \equiv \hat{\mathbf{r}}$, respectively.

5.4. Material response

A numerical example is presented to highlight the main features of the ratcheting model, considering a volume element under uniaxial cyclic loading. Specifically, a constant-amplitude, non-symmetric cyclic variation of the axial stress component σ_{xx} is imposed. Fixed material parameters are chosen as follows: Young's modulus $E = 1$ MPa, Poisson's ratio $\nu = 0.3$, yield strength $\sigma^p = 1$ MPa and ratcheting constant $\beta = 0.5$.

Figure 8(a) shows the response of the ratcheting model for the case of purely kinematic hardening, with $H^{\text{kin}} = 0.0741$ MPa. The stress-strain response shows the characteristic response of the ratcheting

Fig. 8. Response of the ratcheting model under cyclic stress control: deviatoric axial stress ($\sigma_{dev\,xx}$) vs. axial strain (ε_{xx}) with (a) purely kinematic hardening and (b) combined kinematic and isotropic hardening.

model, with a constant cyclic strain accumulation. On the other hand, figure 8(b) shows the response with combined isotropic and kinematic hardening, with $H^{\text{kin}} = 0.0737$ MPa and $H^{\text{iso}} = 5 \times 10^{-4}$ MPa. In this case, the ratcheting effect gradually decays, leading to a stabilized cyclic response that can be interpreted as *shakedown behavior*.

Fig. 9. Response of the ratcheting model for the first two cycles: (a) time variation of the imposed deviatoric axial stress ($\sigma_{dev\,xx}$, solid line) and the deviatoric relative axial stress ($s_{dev\,xx}^p$, dashed line); (b) time variation of the total axial strain (ε_{xx} , solid line), the axial plastic strain (ε_{xx}^p , dashed line) and the axial ratcheting strain (ε_{xx}^r , dotted line). The labels α and γ in points B and C correspond to the scenarios described in figure 2.

Figure 9(a) shows the time variation of the axial stress quantities, namely, the imposed deviatoric stress $\sigma_{dev\,xx}$ and the deviatoric relative stress $s_{dev\,xx}^p$. The response with purely kinematic hardening is shown for two consecutive cycles, with the first cycle between the points labeled A–D. Figure 9(b) shows the resulting strain response. While the axial plastic strain ε_{xx}^p shows a constant cyclic variation, the axial ratcheting strain ε_{xx}^r cyclically increases, resulting in an accumulation of the total strain ε_{xx} .

Figure 10(a) shows the corresponding stress-strain response. At point B^α , the specimen is yielding in tension, where $\mathbb{K}$ and $\mathbb{L}(\boldsymbol{\sigma})$ intersect at smooth points (figure 10(b)). On the other hand, at point C^γ , the

Fig. 10. Response of the ratcheting model for the first two cycles: (a) deviatoric axial stress ($\sigma_{\text{dev}xx}$) vs. axial strain (ε_{xx}); (b) stress state in $(\text{sign}(\sigma_{\text{dev}xx})\|\sigma_{\text{dev}}\|, \text{sign}(s_{\text{dev}xx}^p)\|s_{\text{dev}}^p\|)$ space, showing the stress path (solid black line), the elastic domain (dashed lines) and the state-dependent domain at points B^α and C^γ (solid green and red lines, respectively). The labels α and γ correspond to the scenarios described in figure 2. The plastic flow vectors at B^α and at instants before and after C^γ (gray arrows) are plotted in $(\text{sign}(\varepsilon_{xx}^r)\|\varepsilon^r\|, \text{sign}(\varepsilon_{xx}^p)\|\varepsilon^p\|)$ space.

specimen is reverse-yielding, where $\mathbb{K}$ intersects $\mathbb{L}(\sigma)$ at a corner. At this point, figure 9 shows that the imposed axial stress becomes zero, while the ratcheting strain rate changes sign. This results in the change of slope at point C^γ of the stress-strain curve in figure 10(a).

5.5. A frictional-dilational ratcheting model

To further highlight the flexibility of the proposed formulation, we briefly discuss in this section an extension of ratcheting plasticity to include pressure-dependence and dilation. Taking the present ratcheting model as the point of departure, this is achieved by introducing the mechanisms of the Drucker-Prager model presented in section 4 and allowing for both deviatoric and volumetric ratcheting strains.

The first ingredient of the model is a free energy density that includes the mechanisms of both the Drucker-Prager model and the ratcheting model, namely:

$$\psi(\varepsilon, \varepsilon^p, \kappa, \varepsilon^r) := \underbrace{\frac{1}{2}\|\varepsilon - \varepsilon^p - \varepsilon^r\|_{\mathbf{C}}^2}_{\text{elastic}} + \underbrace{\frac{1}{2}\|\varepsilon^p\|_{\mathbf{H}_{\text{kin}}}^2 + \frac{1}{2}(1-b)H^{\text{iso}}\kappa^2}_{\text{plastic}}. \quad (105)$$

The constitutive equation (4) gives the stress-strain relation (83). On the other hand, the generalized stresses s^p and h are given in equation (54), while s^r is given in equation (84).

To account for a pressure-dependent elastic domain, a possibility is to consider the yield function (56) from the Drucker-Prager model. Then, to attain a flow rule with both deviatoric and volumetric ratcheting strains, the plastic potential is defined as

$$g(s^p, h, s^r) := \|s_{\text{dev}}^p\| + \sqrt{\frac{2}{3}}(A_\theta(\text{tr } s^p + \beta_K \text{tr } s^r) + h - \bar{\sigma}^p(\kappa)) + \beta_\mu \|s_{\text{dev}}^r\|, \quad (106)$$

where β_μ and β_K denote the deviatoric and volumetric ratcheting constants, respectively. The flow rule defined through (22) and (23) is then characterized by $\partial g(s^p, h, s^r) = \{\hat{n}, \sqrt{2/3}, \hat{r}\}$, where the plastic strain

and ratcheting strain directions read

$$\hat{\mathbf{n}} = \hat{\mathbf{n}}_{\text{dev}} + \sqrt{\frac{2}{3}} A_\theta \mathbf{1} \quad \text{and} \quad \hat{\mathbf{r}} = \beta_\mu \hat{\mathbf{r}}_{\text{dev}} + \sqrt{\frac{2}{3}} \beta_K A_\theta \mathbf{1}, \quad (107)$$

with $\hat{\mathbf{n}}_{\text{dev}} \in \partial \|\mathbf{s}_{\text{dev}}^{\text{p}}\|$ and $\hat{\mathbf{r}}_{\text{dev}} \in \partial \|\mathbf{s}_{\text{dev}}^{\text{r}}\|$. In analogy with the procedure thoroughly described for the models presented in sections 4 and 5, it can be shown that the generalized principle of maximum dissipation (29) yields the dissipation potential

$$\phi(\dot{\boldsymbol{\varepsilon}}^{\text{p}}, \dot{\kappa}, \dot{\boldsymbol{\varepsilon}}^{\text{r}}; \mathbf{s}^{\text{p}}, \mathbf{s}^{\text{r}}) = \begin{cases} \frac{\text{tr } \dot{\boldsymbol{\varepsilon}}^{\text{p}}}{3A_\theta} \left(\bar{\sigma}^{\text{p}}(\kappa) + (A_\theta - A_\varphi) \text{tr } \mathbf{s}^{\text{p}} + \sqrt{\frac{3}{2}} \beta_\mu \|\mathbf{s}_{\text{dev}}^{\text{r}}\| + \beta_K A_\theta \text{tr } \mathbf{s}^{\text{r}} \right) \\ \quad \text{if } \begin{cases} \dot{\kappa} \geq \frac{1}{3A_\theta} \text{tr } \dot{\boldsymbol{\varepsilon}}^{\text{p}}, \quad \text{tr } \dot{\boldsymbol{\varepsilon}}^{\text{p}} \geq \sqrt{6} A_\theta \|\dot{\boldsymbol{\varepsilon}}_{\text{dev}}^{\text{p}}\|, \\ \|\dot{\boldsymbol{\varepsilon}}_{\text{dev}}^{\text{r}}\| \leq \frac{\beta_\mu}{\sqrt{6} A_\theta} \text{tr } \dot{\boldsymbol{\varepsilon}}^{\text{p}}, \quad \text{tr } \dot{\boldsymbol{\varepsilon}}^{\text{r}} = \beta_K \text{tr } \dot{\boldsymbol{\varepsilon}}^{\text{p}}, \end{cases} \\ +\infty \quad \text{otherwise.} \end{cases} \quad (108)$$

The dissipation potential (108) contains elements of the dissipation potentials derived for the ratcheting model (93) and the Drucker-Prager model (66), in addition to the term $\beta_K A_\theta \text{tr } \mathbf{s}^{\text{r}}$, which introduces dissipation due to the volumetric component of the ratcheting strains. The free energy density (105) and the dissipation potential (108) may now be taken as inputs for the variational formulation of the model in terms of Formulation 2 (figure 3).

As shown above, the proposed framework allows to treat complex non-associative models with several mechanisms. Models in the literature that account for cyclic plastic strain accumulation with both deviatoric and volumetric components are typically developed to tackle geotechnical problems [18–20]. These models usually include significant empirical components, hindering the analysis of plastic dissipation and a variational formulation. As an alternative, the present model can be taken as the point of departure, where extensions for applicability to such problems can be made. A crucial extension is the introduction of a compression cap that allows to model cyclic compaction. A thorough mechanical analysis of such a model is a topic worth considering for future studies.

6. Numerical applications with non-local effects

This section presents benchmark numerical simulations at the structural scale, where the material behavior is governed by the non-associative models thoroughly discussed in the previous sections. Moreover, thanks to the flexibility of the variational/energetic formulation presented in section 3, a straightforward extension of the non-associative models to gradient plasticity is devised. This extension endows the formulation with an internal length scale, which in turn allows for physically meaningful and mesh-objective computations.

Examples of a rather extensive list of initial works on gradient plasticity, mostly conceived as regularization for softening responses, include [92, 102–104]. General formulations can be found in references [74, 97, 105, 106].

6.1. Gradient-enhanced constitutive state

So far, the formulation has been restricted to local plasticity. However, the functional forms inherent to the energetic formulation of section 3 naturally allow for non-local models equipped with internal length

scales. Herein, we consider a case of first-order gradient models in which the constitutive state depends on the spatial gradient of the internal variables.

In agreement with classical gradient-plasticity theories [92, 102, 105], the first-order gradient of the equivalent plastic strain κ is included in the formulation. To this end, consider a unified model that includes both the Drucker-Prager model of section 4 and the ratcheting model of section 5. The constitutive state enriched with the non-local mechanism then reads $\mathbf{c} := \{\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\varepsilon}^r, \kappa, \nabla \kappa\}$. Consequently, the free energy density reads

$$\psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \kappa, \boldsymbol{\varepsilon}^r, \nabla \kappa) := \frac{1}{2} \|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p - \boldsymbol{\varepsilon}^r\|_{\mathbf{C}}^2 + \frac{1}{2} \|\boldsymbol{\varepsilon}^p\|_{\mathbf{H}^{\text{kin}}}^2 + \frac{1}{2} (1-b) H^{\text{iso}} \kappa^2 + \frac{1}{2} \eta_p^2 \|\nabla \kappa\|^2, \quad (109)$$

where η_p is the plastic internal length scale, which may be related to a *characteristic length* $\ell_p = \eta_p / \sqrt{\sigma^p}$ (cf. [107]). To compute the generalized stresses, the partial derivatives in equation (5) must be replaced by a spatial Euler-Lagrange operator [108, 109], denoted here as $\delta_{\square} := \partial_{\square} - \text{div}[\partial_{\nabla \square}]$. The generalized stresses then read

$$\mathbf{s}^p = -\delta_{\boldsymbol{\varepsilon}^p} \psi \equiv \boldsymbol{\sigma} - \mathbf{H}^{\text{kin}} : \boldsymbol{\varepsilon}^p, \quad h = -\delta_{\kappa} \psi \equiv -(1-b) H^{\text{iso}} \kappa + \eta_p^2 \text{div}(\nabla \kappa), \quad \mathbf{s}^r = -\delta_{\boldsymbol{\varepsilon}^r} \psi \equiv \boldsymbol{\sigma}. \quad (110)$$

With these modifications, the remainder of Formulation 2 is unaltered: it suffices to define the yield function and the flow rule, or, alternatively, the dissipation potential to derive the governing equations (figure 3). The essential feature of the gradient-extended model is that the evolution of the internal variables is now governed by a yield function that takes the form of a partial differential equation due to the Laplacian term in h . This results in a boundary-value problem that can be solved together with the equilibrium equation (48) using the finite element method, resulting in a convenient numerical strategy [92]. The underlying solution procedure can be cast as an incremental energy minimization problem, in line with Petryk [110] and Miehe [109], which can be shown to naturally emerge from the energetic formulation [33, 57, 111–113].

6.2. Frictional-dilational plasticity

Consider first the Drucker-Prager model for frictional-dilational plasticity presented in section 4, with the dissipation potential given in equation (65). Taking Formulation 2 as the point of departure, the evolution equations follow from energy balance (40) and local stability (45), yielding the equilibrium conditions (32), shown in weak form in equation (48). On the other hand, the fact that the dissipation potential is unaltered implies that the flow rule expressed in terms of the generalized stresses is also unaltered and thus given by equation (61), which was shown to descend from energy balance in section 4.3. Consequently, only the elastic domain is modified. In particular, the stability condition (45) now gives

$$\int_{\Omega} \left(\|\mathbf{s}_{\text{dev}}^p(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)\| + \sqrt{\frac{2}{3}} (A_{\varphi} \text{tr } \mathbf{s}^p(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p) - H^{\text{iso}} \kappa - \sigma^p + \eta_p^2 \text{div}(\nabla \kappa)) \right) \|\boldsymbol{\varepsilon}_{\text{dev}}^p\| \, d\mathbf{x} \leq 0, \quad (111)$$

which is the weak form of a non-local version of the yield function (56). Moreover, it can be shown that energy balance and stability imply the boundary condition

$$\nabla \kappa \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (112)$$

where $\mathbf{n}$ is the outward normal vector on Γ . Given all other fields, the last two expressions constitute a well-posed boundary-value problem in terms of the equivalent plastic strain κ that is suitable for finite

element implementation.

Fig. 11. Schematic representation of the perforated rectangular specimen for the example with Drucker-Prager plasticity. The central hole has a diameter of 6.8 mm.

Example. Consider a perforated rectangular specimen under plane strain conditions (figure 11) subjected to a biaxial compression test. Lateral pressure is first imposed while keeping the top and bottom borders fixed, until a uniform pressure of $p_0 = 5$ MPa is reached. Then, with fixed lateral pressure, an axial displacement loading $\bar{u}(t)$ is gradually imposed on the top border from 0 to 0.6 mm.

The example is adopted from Borja [6], where the following material parameters are employed: Young's modulus $E = 14000$ MPa, Poisson's ratio $\nu = 0.3$, cohesion $c = 10$ MPa, friction angle $\varphi = 30^\circ$, dilation angle $\theta = 25^\circ$, and perfect plasticity, such that $H_K^{\text{kin}} = H_\mu^{\text{iso}} = H^{\text{iso}} = 0$ MPa. With reference to equation (59), the inner-cone approximation is applied. We consider the case of gradient plasticity with $\ell_p = 5$ mm, and compare the results to the local plasticity counterpart. The numerical simulations are carried out using a finite element mesh composed of bilinear quadrilateral elements, with a minor local refinement near the hole and varying mesh sizes $\{h_c, h_c/2, h_c/4, h_c/6, h_c/8\}$, with $h_c = 2.85$ mm.

Figure 12(a) shows the responses obtained with local plasticity ($\ell_p = 0$), where significant mesh-dependence is observed. On the other hand, the simulations with a finite length scale in figure 12(b) suggest mesh convergence. This behavior is further illustrated in figure 12(c), where the local case results in blown-up maximum values for the equivalent plastic strain upon mesh refinement, while the non-local counterpart shows a much more stable behavior. In relation to this result, figure 13 shows contour plots for the equivalent plastic strain at the end of the simulation, where spurious mesh-sensitivity is only observed for local plasticity.

6.3. Ratcheting plasticity

We now turn to the ratcheting model presented in section 5. As in the previous subsection, it suffices to recall the dissipation potential, given for the ratcheting model in equation (92), and to enforce energy balance (40) and local stability (45) to derive the governing equations from Formulation 2. Once more, we

Fig. 12. Displacement at the top border vs. the corresponding reaction force for the perforated rectangular specimen with Drucker-Prager plasticity (figure 11), computed with different mesh sizes for (a) $\ell_p = 0$ mm and (b) $\ell_p = 5$ mm. The maximum equivalent plastic strain κ obtained with different mesh sizes is shown in (c).

Fig. 13. Equivalent plastic strain the rectangular perforated specimen with Drucker-Prager plasticity for different mesh sizes h_c (in mm), with (a1)-(a4) $\ell_p = 0$ mm and (b1)-(b4) $\ell_p = 5$ mm.

obtain the equilibrium conditions (32) given in weak form in equation (48), as well as the flow rule given for

the ratcheting model in equation (89). On the other hand, the stability condition (45) gives

$$\int_{\Omega} \left(\|\mathbf{s}_{\text{dev}}^{\text{p}}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{\text{p}}, \boldsymbol{\varepsilon}^{\text{r}})\| + \sqrt{\frac{2}{3}} (-H^{\text{iso}}\kappa - \sigma^{\text{p}} + \eta_{\text{p}}^2 \text{div}(\nabla\kappa)) \right) \|\boldsymbol{\varepsilon}^{\text{p}}\| \, d\mathbf{x} \leq 0, \quad (113)$$

which is the weak form of a non-local version of the yield function (86). As before, this result is accompanied by the boundary conditions (112) for the non-local variable κ .

Fig. 14. Schematic representation of hole-in-plate example for ratcheting plasticity.

Example. Consider a perforated rectangular specimen under uniaxial cyclic loading, for which a quarter of the domain may be considered due to symmetry (figure 14). A distributed load $\bar{f}(t)$ is imposed cyclically, linearly varying between -25 N/mm and 125 N/mm.

The following material parameters are employed: Young's modulus $E = 205000$ MPa, Poisson's ratio $\nu = 0.3$, yield strength $\sigma^{\text{p}} = 100$ MPa, kinematic hardening modulus $H^{\text{kin}} = 16348.75$ MPa, isotropic hardening modulus $H^{\text{iso}} = 165.14$ MPa and ratcheting constant $\beta = 0.1$. In this example, the mesh-sensitivity issues observed in the Drucker-Prager example were not encountered. Nevertheless, for the sake of completeness, we consider the case of gradient plasticity with $\ell_{\text{p}} = 0.8$ mm. As before, bilinear quadrilateral elements are employed, with mesh sizes $\{h_c, h_c/2, h_c/4\}$, where $h_c = 0.6$ mm in the region near the hole.

Figure 15(a) shows the time history of the displacements at location A (figure 14), where mesh-converging results are observed, while figure 15(b) shows the force-displacement response for $h_c/2$. The global behavior shows an initial ratcheting stage, which gradually stabilizes and falls into a *plastic shakedown* state due to the effect of isotropic hardening after a sufficient number of cycles. Moreover, the response shows pronounced asymmetry in tension and compression, capturing asymmetrical-loading effects. Finally, figure 16 shows the equivalent plastic strains at different stages of the cyclic load.

Fig. 15. Response of the perforated specimen with ratcheting plasticity under cyclic loading, showing (a) the time history of the displacements at location A (figure 14) computed with different mesh densities and (b) the force-displacement response for $h_c/2$.

Fig. 16. Equivalent plastic strain the perforated specimen with ratcheting plasticity under cyclic loading after (a) 15, (b) 30 and (c) 45 cycles.

7. Conclusions and perspectives

In this contribution, we have presented a systematic framework that allows for a variational formulation of non-associative plasticity models. An important step is the derivation of a state-dependent dissipation potential from a generalization of the principle of maximum dissipation, which we refer to as *the generalized principle of maximum dissipation*. This allowed to recover convex duality in the classical subdifferential formulation and to consistently link the (conventional) dual form to the primal form in the general case of non-associative models. Then, from the primal form, the state-dependent dissipation potential was taken as the point of departure for the variational formulation in terms of the energetic formulation for rate-independent systems. The equivalence of this variational approach with the classical formulation was formally established and thoroughly discussed.

The proposed framework offers a rigorous setting for material models with non-associative evolutions, allowing for a variationally consistent introduction of complex dissipative mechanisms. This was evidenced by the derivation of non-associative plasticity models that include ratcheting, pressure-dependence and

dilation effects, highlighting the flexibility of the framework. In the presentation of these models, several thermomechanical insights that naturally follow from the formulation were given. Finally, taking advantage of the adopted variational framework, a straightforward extension to gradient plasticity was devised, which in turn was applied to perform representative finite element simulations.

The proposed formulation provides means to construct a variety of material models in a thermodynamically and variationally consistent manner. For instance, a multi-surface plasticity model with ratcheting coupled to a phase-field approach to fatigue is presented in Ulloa et al. [59], while future work may focus on the introduction of damage in Drucker-Prager plasticity for the modeling of failure in geomaterials. Moreover, a model suitable for geotechnical problems involving cyclic loading can be conceived by taking the formulation in section 5.5 as a point of departure. Finally, the non-local treatment presented in section 6 may be generalized to higher-order gradients [112], dependence on the gradient of the plastic strain tensor [114–116], or micromorphic continua [107, 117]. Other possible extensions may consider viscoplasticity, provided that the evolution problem is written in variational form using the principle of virtual power [109, 118], or the construction of non-associative models for materials with complex microstructure.

Acknowledgements

R. Alessi acknowledges the Italian Ministry of Education, University and Research (MIUR) under the PRIN 2017 20177TTP3S grant.

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