Fully Decoupled Reliability-Based Optimization of Linear Structures Subject to Gaussian Dynamic Loading Considering Discrete Design Variables

Matthias G. R. Faes^{a,*}, Marcos A. Valdebenito^b

^a*KU Leuven, Department of Mechanical Engineering, Technology campus De Nayer, Jan De Nayerlaan 5,
St.-Katelijne-Waver, Belgium*

^b*Faculty of Engineering and Sciences, Universidad Adolfo Ibáñez, Av. Padre Hurtado 750, 2562340 Viña del
Mar, Chile*

Abstract

Reliability-based optimization (RBO) offers the possibility of finding an optimal design for a system according to a prescribed criterion while explicitly taking into account the effects of uncertainty. However, due to the necessity of solving simultaneously a reliability problem nested in an optimization procedure, the corresponding computational cost is usually high, impeding the applicability of the methods. This computational cost is even enlarged further when one or several design variables must belong to a discrete set, due to the requirement of resorting to integer programming optimization algorithms. To alleviate this issue, this contribution proposes a fully decoupled approach for a specific class of problems, namely minimization of the failure probability of a linear system subjected to an uncertain dynamic load of the Gaussian type, under the additional constraint that the design variables are integer-valued. Specifically, by using the operator norm framework, as developed by the authors in previous work, this paper shows that by reducing the RBO problem with discrete design variables to the solution of a single deterministic optimization problem followed by a single reliability analysis, a large gain in numerical efficiency can be obtained without compromising the accuracy of the resulting optimal design. The application and capabilities of the proposed approach are illustrated by means of two examples.

Keywords: Linear Structure, Gaussian loading, First excursion probability, Decoupling, Operator norm

*E-mail: matthias.faes@kuleuven.be

- Optimal design under uncertainty considering discrete design variables is addressed. 28
- Optimization and reliability analyses are fully decoupled. 29
- Application of standard integer programming algorithms becomes feasible. 30

1. Introduction 31

Reliability-based optimization (RBO) offers a convenient framework for decision making in structural and mechanical engineering, as it allows exploring different design solutions and then, selecting the best one according to a prescribed criterion [1]. Within such process, different sources of uncertainty that affect structural performance are described by means of probability theory and synthesized in terms of a probability of occurrence. In this way, it is possible to determine design solutions which are cost efficient, as available resources are used in an optimal way while ensuring an appropriate level of structural safety. 32 33 34 35 36 37 38

RBO offers several advantages when compared to traditional deterministic design procedures. Nonetheless, its practical application is usually quite challenging from a numerical viewpoint, as it involves a double-loop approach [2]. That is, different design solutions must be explored and for each of them, it is necessary to perform uncertainty quantification considering different realizations of the uncertain parameters affecting performance. In view of this challenge, several implementations of RBO that offer improved numerical efficiency have been proposed, which involve, for example, efficient integration of algorithms for optimization and reliability analysis (see, e.g. [3, 4, 5, 6, 7]), application of approximation concepts and surrogate models (see, e.g. [8, 9, 10, 11]) or specialized programming techniques that avoid double-loop implementation (see, e.g. [12, 13, 14, 15, 16]). For a more detailed overview on approaches for RBO, it is referred to, e.g. [1, 2, 17, 18]. 39 40 41 42 43 44 45 46 47 48 49

Most of the approaches developed for RBO consider design variables that can be modeled as real numbers lying within a prescribed interval. Nonetheless, in several practical cases, design variables must be chosen from an existing inventory offered by a manufacturer. For example, when designing steel structures, elements such as beams or columns must be selected out of a catalog of available profiles. Such issue poses a major challenge for RBO, as the associated optimization problem must be solved resorting to algorithms for integer programming, which are 50 51 52 53 54 55

generally much more demanding than their continuous counterparts [19, 20], and in some cases even NP-hard. Hence, specialized RBO strategies have been developed for solving problems involving discrete design variables. These strategies can be broadly classified into three classes. The first one involves direct coupling of algorithms for discrete optimization and reliability analysis, as proposed in, e.g. [21, 22, 23, 24, 25, 26]. The second class of approaches consists of constructing a surrogate model for reliability, thus rendering the application of the optimization algorithm trivial, see e.g. [10, 27, 28]. The third class of approaches consists of decoupling the discrete optimization and reliability analysis steps by creating local approximate representations of the reliability, e.g. [29, 30, 31, 32, 33, 34]. As an additional remark, it should be noted that RBO formulations addressing both discrete and continuous design variables (which are termed as mixed discrete-continuous) also constitute another important class of problems, see e.g. [31, 35, 36]. Mixed discrete-continuous RBO problems can be found, for example, in structural engineering [35], where one may consider discrete variables for selecting commercially available cross sections and continuous variables for dimensions that define the shape of the structure.

This contribution proposes an approach for RBO considering discrete design variables for a specific class of problems, that is, minimization of the first excursion probability of linear structures subject to stochastic Gaussian loading. The proposed approach exploits the specific nature of the problem considered and departs from other developments in the literature as it is formulated in a *fully decoupled* fashion. This means that the whole RBO problem is addressed by solving a single deterministic discrete optimization problem followed by a single reliability estimation. Such formulation is quite advantageous from a numerical viewpoint, as it effectively breaks the double-loop that arises in classical approaches for RBO. Furthermore, as the optimization problem to be solved is of deterministic nature, there is no need to deal with sampling estimators, which may become troublesome, see e.g. [6]. The theoretical basis of this fully decoupled formulation lies on the application of the operator norm theorem [37], which has already been explored by the authors within the context of imprecise reliability analysis [38] and optimization under uncertainty considering continuous design variables [39]. For practical implementation of the proposed approach, discrete optimization is carried out by means of Genetic Algorithms (see, e.g. [40, 41]) while reliability analysis is conducted by means of Directional Importance Sampling [42, 43].

The rest of this contribution is organized as follows. Section 2 provides a detailed formulation of the type of problems considered in this contribution. Then, Section 3 presents the fully decoupled

strategy to solve the RBO problem. The practical application of this strategy is analyzed by means of examples in Section 4. The paper concludes with discussions and outlook for future research in Section 5.

2. Formulation of the problem

2.1. General remarks

This section formulates the class of problems considered in this contribution, namely the minimization of the first excursion probability of a linear structural system subject to stochastic loading with design variables of discrete type. The discussion starts with the representation of the stochastic loading in Section 2.2. The characterization of the structural response is presented in Section 2.3. Then, the first excursion probability problem as well as the RBO problem involving discrete design variables are formulated in Sections 2.4 and 2.5, respectively.

2.2. Stochastic loading

It is assumed that dynamic loading f acting over the structural system can be characterized as a Gaussian stochastic process. Such model has been considered in several practical applications, see e.g. [44, 45]. The stochastic process possesses a duration T and is represented at discrete time instants $t_k = (k - 1)\Delta t$, $k = 1, \dots, n_T$, where Δt is the time step and n_T is the total number of time instants considered. The covariance of the discretized Gaussian process is denoted as matrix $\mathbf{\Gamma}$ of dimension $n_T \times n_T$ and it is assumed for the sake of simplicity that the mean of the process is zero. Following the above assumptions and applying the Karhunen-Loève expansion (see, e.g. [46]), the stochastic loading is described as:

$$\mathbf{f}(\mathbf{z}) = \mathbf{\Psi}\mathbf{\Lambda}^{1/2}\mathbf{z} \quad (1)$$

where $\mathbf{f}$ is a realization of the stochastic loading, which is a $n_T \times 1$ vector, whose k -th component f_k represents the loading at time t_k ; $\mathbf{\Psi}$ is a matrix of dimension $n_T \times n_{KL}$ containing the first n_{KL} eigenvectors of the covariance matrix $\mathbf{\Gamma}$; $\mathbf{\Lambda}$ is a matrix whose diagonal contains the first n_{KL} eigenvalues of the covariance matrix $\mathbf{\Gamma}$; n_{KL} is the number of terms retained for the Karhunen-Loève expansion ($n_{KL} \leq n_T$, see, e.g. [46]); and $\mathbf{z}$ is a realization of a standard Gaussian random variable vector $\mathbf{Z}$ of dimension $n_{KL} \times 1$ and whose probability density function is denoted as $p_{\mathbf{Z}}(\mathbf{z})$.

Gaussian stochastic processes offer the means for modeling uncertain dynamic loading in a number of problems, see e.g. [47]. Naturally, for some other problems, non-Gaussian stochastic process may be required. However, the latter is outside the scope of this contribution.

2.3. Structural model and its response

It is considered that the behavior of the structural system due to the stochastic loading can be modeled as linear elastic with classical damping. The associated equation of motion is [48, 49]:

$$\mathbf{M}(\mathbf{y})\ddot{\mathbf{x}}(t, \mathbf{y}, \mathbf{z}) + \mathbf{C}(\mathbf{y})\dot{\mathbf{x}}(t, \mathbf{y}, \mathbf{z}) + \mathbf{K}(\mathbf{y})\mathbf{x}(t, \mathbf{y}, \mathbf{z}) = \boldsymbol{\rho}f(t, \mathbf{z}) \quad (2)$$

where t denotes time; $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices of dimension $n_D \times n_D$ each; n_D is the number of degrees-of-freedom of the structural model; $\mathbf{y}$ denotes a vector of design variables, which may affect the mass, damping and stiffness matrices; $\boldsymbol{\rho}$ is a vector of dimension $n_D \times 1$ coupling the stochastic load $f(t, \mathbf{z})$ with the degrees-of-freedom of the structure; and where $\ddot{\mathbf{x}}$, $\dot{\mathbf{x}}$ and $\mathbf{x}$ denote the acceleration, velocity and displacement vectors, each of dimension $n_D \times 1$.

Practical design situations demand monitoring n_R responses of interest of the structural system. These responses are denoted in the following as $\eta_i(t)$, $i = 1, \dots, n_R$ and due to the linearity of the structure, are calculated by means of a convolution integral, that is:

$$\eta_i(t, \mathbf{y}, \mathbf{z}) = \int_0^t h_i(\mathbf{y}, t - \tau)f(\tau, \mathbf{z})d\tau, \quad i = 1, \dots, n_R \quad (3)$$

where h_i denotes the associated unit impulse response function (see Appendix A). In view of the discrete time representation considered in Section 2.2, the different responses of interest at different time instants can be calculated as:

$$\boldsymbol{\eta}_i(\mathbf{y}, \mathbf{z}) = \mathbf{A}_i(\mathbf{y})\mathbf{z}, \quad i = 1, \dots, n_R \quad (4)$$

where $\boldsymbol{\eta}_i$ is a vector of dimension $n_T \times 1$ containing the discrete-time representation of the i -th response along the duration T of the stochastic loading; and $\mathbf{A}_i$ is a matrix of dimension $n_T \times n_{KL}$ whose k -th row contains the discrete time representation of the convolution integral in eq. (3), as discussed in detail in Appendix B.

Eq. (4) provides a compact expression for calculating the i -th response of interest in time using

matrix and vector operations. From this equation, it is important to note that the response is a function of both the properties of the structural system and in particular, its design variables $\mathbf{y}$, as well as a function of the stochastic load, represented through $\mathbf{z}$. In other words, the responses of interest of the system are uncertain because they depend on $\mathbf{z}$, but can also be modified by changing the design variables $\mathbf{y}$.

2.4. First excursion probability

For practical design purposes, it is desirable that absolute value of the responses of interest remain below acceptable threshold levels b_i , $i = 1, \dots, n_R$ within the duration T of the stochastic load. For verifying such condition, it is useful to define the so-called normalized response function $r(\mathbf{y}, \mathbf{z})$ [50], which is defined as:

$$r(\mathbf{y}, \mathbf{z}) = \|\mathbf{A}(\mathbf{y})\mathbf{z}\|_\infty \quad (5)$$

where $\|\cdot\|_\infty$ denotes infinity norm; and $\mathbf{A}$ is a matrix that collects all matrices $\mathbf{A}_i$ associated with the calculation of the i -th response (see eq. (4)), normalized by their respective threshold levels. Under the assumption that the thresholds fulfill the condition $b_i > 0$, $i = 1, \dots, n_R$, matrix $\mathbf{A}$ is defined as:

$$\mathbf{A}(\mathbf{y}) = \begin{bmatrix} b_1^{-1} \mathbf{A}_1(\mathbf{y}) \\ \vdots \\ b_{n_R}^{-1} \mathbf{A}_{n_R}(\mathbf{y}) \end{bmatrix} \quad (6)$$

and its dimension is $(n_R n_T) \times n_{KL}$. The definition of matrix $\mathbf{A}$ as shown in eq. (6) ensures that all responses are calculated in a dimensionless fashion. In this way, it is possible to compare on a similar basis responses involving displacements, accelerations, etc. In addition, the infinity norm in eq. (5) allows retrieving the maximum normalized response. Naturally, in case $r < 1$, all n_R responses of interest of the structure remain below its prescribed threshold throughout the duration of the stochastic load. On the contrary, in case $r \geq 1$, one or more responses of interest exceed their prescribed threshold at one or more time instants and the structure exhibits undesirable behavior. The chances that such undesirable behavior occurs within the duration of the stochastic loading is termed as the first excursion probability (see, e.g. [47]), which is

calculated by means of the following classical integral:

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$$p_F(\mathbf{y}) = \int_{\mathbf{z} \in \mathbb{R}^{n_{KL}}} I_F(\mathbf{y}, \mathbf{z}) p_{\mathbf{z}}(\mathbf{z}) d\mathbf{z} \quad (7)$$

where p_F denotes the first excursion probability (or simply, the failure probability); and $I_F(\cdot, \cdot)$ is the indicator function, which is equal to one in case $r(\mathbf{y}, \mathbf{z}) \geq 1$ and zero, otherwise. Note that the first excursion probability as shown above corresponds to a series system formulation. Such probability is most useful for verifying, for example, serviceability conditions, as considered in this contribution. However, for analysis of redundant systems and collapse, such may not be the case. The practical estimation of the failure probability as shown in eq. (7) is far from trivial for cases of practical interest. This stems out of two issues. First, the number of terms n_{KL} to be retained for the Karhunen-Loève representation may be large (in the order of hundreds or thousands). Second, the normalized response function r (which is required for calculating the indicator function) is not known in closed form, and must be calculated for a specific structural setting (given by the design variable vector $\mathbf{y}$) and a specific realization of the stochastic loading (given by the realization $\mathbf{z}$). Thus, these two issues favor the application of advanced simulation methods for estimating the failure probability [51].

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2.5. Reliability-based optimization considering discrete design variables

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The structure of eq. (2) indicates that the structural response depends on the design variable vector $\mathbf{y}$. This can be understood on physical grounds considering that the components of this design variable vector represent, for example, the dimensions of cross section of structural members. Thus, it is evident that by modifying cross sections, it is possible to alter the structural response (assuming a fixed stochastic load $\mathbf{z}$) and in turn, the normalized response function r . Hence, the failure probability itself becomes a function of these design variables, as stated in eq. (7).

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The fact that the value of the failure probability can be changed depending on the value of $\mathbf{y}$ opens the possibility for performing structural design within the framework of optimization. In such context, several different formulations for the optimization problem can be devised [18], for example: minimization of structural weight subject to probability constraints (see, e.g. [52]); minimization of failure probability under a set of available resources (see, e.g. [5, 53]); or risk-optimization taking into account the costs of construction, failure and repair (see, e.g. [54]). For a comparison of such formulations, it is referred to, e.g. [55]. For the particular case of this

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contribution, the focus is on problems involving the minimization of the failure probability given 188
a set of available resources. Moreover, it is assumed that the design variables are of discrete type. 189
Such selection reflects the case where there is a finite number of choices for design variables due to, 190
e.g. fabrication constraints. The class of problems considered herein is quite useful when different 191
types of failure modes affect the reliability of a structural system. In this sense, minimization 192
of the failure probability constrained on available resources allows determining solutions which 193
may represent a trade-off between those failure modes [5, 53]. Mathematically, the problem to be 194
solved is formulated as follows. 195

$$\begin{aligned} & \min_{\mathbf{y}} p_F(\mathbf{y}) & (8) \\ & \text{subject to} \\ & c_l(\mathbf{y}) \leq 0, \quad l = 1, \dots, n_C \\ & y_j \in \Omega_j = \{y_{j,1}, y_{j,2}, \dots, y_{j,e_j}\}, \quad j = 1, \dots, n_y \end{aligned}$$

where $c_l(\cdot)$ represents a deterministic constraint; y_j is the j -th element of $\mathbf{y}$; n_y denotes the 196
dimension of the design variable vector; Ω_j denotes the set of available discrete values that the 197
design variable y_j may assume; e_j is the number of elements contained in set Ω_j ; and Ω is the set 198
of all values that $\mathbf{y}$ may assume, which is equal to the Cartesian product $\Omega_1 \times \dots \times \Omega_{n_y}$. Within 199
the context of this contribution, it is noted that the design variable y_j may represent directly a 200
certain structural property, for example, the thickness of an element, which assumes specific values 201
due to fabrication constraints. Or in the contrary, the design variables y_j may just be an integer 202
number that points out, for example, to a database of existing steel profiles. 203

The challenge with the solution of the optimal design problem in eq. (8) is threefold. First, the 204
solution of an optimization problem involving discrete design variables is usually quite demanding 205
on its own, as a large number of different structural configurations (that is, different values for 206
the design variables) must be considered. This implies in turn carrying out several structural 207
analyses. Second, the calculation of the failure probability associated with a given design vector by 208
means of simulation (as discussed in Section 2.4) demands performing repeated structural analyses 209
for different realizations of the stochastic loading vector $\mathbf{z}$. Third, the calculation of the failure 210
probability must be carried within the optimization algorithm, leading to the so-called double loop 211
implementation. These three issues impose a numerical burden which can be prohibitive, even 212

for relatively simple models. Hence, in the following, an approach for solving the optimization problem in eq. (8) is developed, which can break the double loop by applying the operator norm theorem.

3. Fully Decoupled Reliability-based Optimization

3.1. Operator norm theorem as a means for decoupling

Consider a matrix $\mathbf{B}$ of dimension $n_1 \times n_2$ and a vector $\boldsymbol{\xi}$ of dimension $n_2 \times 1$. The operator norm theorem (see, e.g. [37]) allows answering the following question: what is the largest value c by which matrix $\mathbf{B}$ stretches $\boldsymbol{\xi}$? Such question is posed in terms of a $p^{(1)}$ -norm applied over the vector $\mathbf{B}\boldsymbol{\xi}$ (that is, the *stretched* vector) with respect to another $p^{(2)}$ -norm applied over vector $\boldsymbol{\xi}$ (that is, the vector *before stretching*). Mathematically, the operator norm theorem is expressed as:

$$\|\mathbf{B}\|_{p^{(1)}, p^{(2)}} = \inf\{c \geq 0 : \|\mathbf{B}\boldsymbol{\xi}\|_{p^{(1)}} \leq c\|\boldsymbol{\xi}\|_{p^{(2)}}\} \quad (9)$$

where $\inf\{\cdot\}$ denotes infimum; $\|\cdot\|_{p^{(1)}, p^{(2)}}$ denotes operator norm with respect to p -norms $p^{(1)}$ and $p^{(2)}$; $\|\cdot\|_p$ denotes p -norm; and where c is a number that represents the maximum possible stretch. A comparison of eq. (9) with eq. (5) allows establishing the following connection between the operator norm theorem and the calculation of the normalized response function: matrix $\mathbf{B}$ corresponds to $\mathbf{A}$; vector $\boldsymbol{\xi}$ corresponds to $\mathbf{z}$; and $p^{(1)} \rightarrow \infty$. In addition and in order to apply the operator norm theorem as cast in eq. (9), it is assumed that $p^{(2)} = 2$, as this can be loosely associated with the energy content of the stochastic load [38]. Thus, under the previous observations and assumption, the following operator norm associated with matrix $\mathbf{A}$ is formulated.

$$\|\mathbf{A}(\mathbf{y})\|_{\infty, 2} = \inf\{c(\mathbf{y}) \geq 0 : \|\mathbf{A}(\mathbf{y})\mathbf{z}\|_{\infty} \leq c(\mathbf{y})\|\mathbf{z}\|_2\} \quad (10)$$

The expression above implies that the stochastic load vector can be *stretched* by matrix $\mathbf{A}$ by factor c at most. As discussed in [37], the solution to the operator norm problem in eq. (10) is equal to the row of matrix $\mathbf{A}$ with largest Euclidean norm, that is:

$$\|\mathbf{A}(\mathbf{y})\|_{\infty, 2} = \max_{\substack{i=1, \dots, n_R \\ k=1, \dots, n_T}} \left(\sqrt{\mathbf{a}_{i_k} \mathbf{a}_{i_k}^T} \right) \quad (11)$$

where $\mathbf{a}_{i_k}$ represents the row of matrix $\mathbf{A}$ associated with the calculation of the i -th response at time instant t_k (see Appendix B for more details).

An examination of eq. (10) reveals that as the *stretched* stochastic load is actually the normalized response function (see eq. (5)), factor c expresses the maximum amplification that the stochastic load may undergo when applied to the structure. Furthermore, as the normalized response function governs the calculation of the failure probability (see eq. (7)), it is postulated that minimizing the operator norm associated with $\|\mathbf{A}(\mathbf{y})\|_{\infty,2}$ is equivalent to minimizing the failure probability; in other words, by minimizing the maximum stretching factor of the load, one is actually causing the failure event to be less likely. Such reasoning, which combines physical and mathematical grounds, means that the operator norm of matrix $\|\mathbf{A}(\mathbf{y})\|_{\infty,2}$ offers a suitable proxy for determining the set of design variables which yields the minimum value of the failure probability as in eq. (8). The quality of this proxy has been verified and validated, e.g. for estimating imprecise probabilities in [38]. Therefore, the RBO problem formulated in eq. (8) can be replaced by the following problem:

$$\min_{\mathbf{y}} \|\mathbf{A}(\mathbf{y})\|_{\infty,2} \quad (12)$$

subject to

$$c_l(\mathbf{y}) \leq 0, \quad l = 1, \dots, n_C$$

$$y_j \in \Omega_j = \{y_{j,1}, y_{j,2}, \dots, y_{j,e_j}\}, \quad j = 1, \dots, n_y$$

The advantage of the problem formulated above is that it does not involve the stochastic load, as the operator norm $\|\mathbf{A}(\mathbf{y})\|_{\infty,2}$ depends purely on the information contained in matrix $\mathbf{A}$ and not on vector $\mathbf{z}$. In other words, the optimization problem in eq. (12) is actually a deterministic optimization problem. Once the optimum $\mathbf{y}^*$ has been found by solving eq. (12), it suffices to carry out a single reliability analysis for that optimum in order to determine the minimum failure probability $p_F(\mathbf{y}^*)$ associated with the RBO problem in eq. (8).

The framework for determining the optimum design $\mathbf{y}^*$ and its associated failure probability as described above has important implications from a practical viewpoint. First, the optimization problem in eq. (12) is purely deterministic, which means that its solution does not involve any double loop for conducting reliability assessment. This single fact can decrease numerical costs significantly. Furthermore, the optimization problem in eq. (12) can be solved by means of any suitable algorithm. Once the optimal design has been found, reliability analysis is conducted a

single time for that optimum. This means that the traditional double-loop encountered when solving RBO problems (that is, reliability assessment nested within an optimization procedure) is effectively broken and replaced by one deterministic optimization followed by a single reliability analysis. Thus, the proposed approach possesses the potential for decreasing the numerical cost associated with the solution of RBO problems significantly.

The approach for RBO as described above can be highly advantageous from a numerical viewpoint due to its decoupled nature. In this sense, the operator norm theorem takes full advantage of the structure of the problem under consideration, that is, minimization of the failure probability under certain constraints. However, the extension of the proposed approach to other classes of problems which are relevant in RBO may not be straightforward. For example, the solution of problems involving weight minimization under probability constraints or risk-based optimization [18, 55] cannot be directly solved following the proposed approach. This stems out of the fact that the operator norm as considered in the paper does not provide information on the actual value of the failure probability itself but just about the set of design variables that minimizes the failure probability.

3.2. Optimization considering discrete design variables

The optimization problem in eq. (12) involves design variables that belong to a discrete set Ω . Such class of problems may not be amenable for classical gradient-based algorithms (see, e.g. [56]) and hence, one must resort to alternatives algorithms such as branch-and-bound methods [57], annealing methods [58] or evolutionary algorithms [40, 59]. In this work, the optimization problem involving discrete design variables is solved by means of Genetic Algorithms, as its application for solving structural problems has been widely tested and validated in the literature, see e.g. [56]. Genetic Algorithms (GA) correspond to a class of optimization algorithms which rely on analogies to natural processes, that is, *natural evolution*. GA are one of three main streams of the so-called evolutionary algorithms for optimization, while the other two streams are Evolution Strategies and Evolutionary Programming, respectively (see, e.g. [60]). GA closely resembles the evolution of biological organisms, as it employs a *population* of possible solutions instead of a unique solution; this is a distinctive characteristic, as many other algorithms (such as gradient-based optimization algorithms) consider a single candidate solution. Each individual of the population of solutions has associated a certain *fitness*, which is a metric on how good a solution is from the point of view of the optimization problem. The individuals with highest fitness value can *reproduce* by means

of specific operations (namely, *crossover* and *mutation*) in order to generate a new population. This new population possesses properties of the individuals of highest fitness of the previous population. In this way, by generating a number of populations, it is possible to explore the space of design variables in order to determine the most promising locations [61]. It should be noted that GA is a gradient-free optimization algorithm. Moreover, design constraints can be included resorting to penalization, as described in [62].

From a numerical viewpoint, the application of GA may be quite demanding, as a single population may involve several individuals, which evolve through several generations. In turn, the evaluation of the fitness of each individual corresponds to calculating the operator norm associated with matrix $\mathbf{A}$ as well as the deterministic constraints c_l , $l = 1, \dots, n_C$, as noted from eq. (12). While numerical evaluation of the deterministic constraints is usually trivial, the computation of the operator norm demands performing structural analysis (that is, eigenvalue/eigenvector analysis). Eventually, the computation of the operator norm within the optimization problem in eq. (12) could be sped up by exploiting the linearity of the structure by means of substructuring concepts [63, 64]. Nonetheless, such possibility is not further explored in this contribution, as the focus is on the concept of decoupling and its practical application for RBO considering discrete design variables.

3.3. Estimation of first excursion probability

After identifying the optimal design $\mathbf{y}^*$ that solves eq. (12) by means of GA as described above, it is necessary to estimate the failure probability associated with that design. Such task could be carried out by applying, for example, Monte Carlo simulation (see, e.g. [65]), which would imply generating N random samples $\mathbf{z}^{(l)}$, $l = 1, \dots, N$ that follow a standard Gaussian distribution and then, evaluating the dynamic response (see eq. (5)) and the indicator function for each of those samples. Thus, the sought failure probability would be estimated as $p_F(\mathbf{y}^*) \approx \sum_{l=1}^N I_F(\mathbf{y}^*, \mathbf{z}^{(l)}) / N$. While such an approach is straightforward, it may be quite demanding from a numerical viewpoint, as a large number of samples N (proportional to $1/p_F$) are required to ensure a sufficiently accurate estimator. In order to reduce the number of simulations, one can resort to advanced variants of Monte Carlo simulation, as discussed in the sequence.

The structure of eq. (5) indicates that the normalized response function is actually the maximum among a series of individual functions which depend linearly on vector $\mathbf{z}$. Hence, the boundary separating the realizations $\mathbf{z}$ that cause failure or not in the standard normal space is a collection

of hyperplanes given by the limit state surface $r(\mathbf{y}, \mathbf{z}) = 1$ [66, 67], as depicted schematically in Figure 1. This fact brings important information which can be exploited for evaluating the failure probability in eq. (7). For example, Directional Importance Sampling (DIS) [42, 43] is an advanced simulation method that is built on such information and is capable of estimating small failure probabilities with high accuracy and a reduced number of samples. As the application of DIS is well documented [42, 43], only the essential aspects of this simulation method are described in the following.

Figure 1: Schematic representation of failure domain associated with probability integral.

DIS is built upon two key concepts. The first one is that vector $\mathbf{z}$ associated with the stochastic excitation is represented in a polar fashion. That is, $\mathbf{z} = R\mathbf{u}$, where R is a scalar representing the length of $\mathbf{z}$ (in terms of Euclidean norm) and $\mathbf{u}$ is a direction vector (with unit Euclidean norm) pointing towards $\mathbf{z}$. The advantage of such representation is that all structural responses of interest associated with realizations lying along the ray that originates at $\mathbf{0}$ and that points in the direction of vector $\mathbf{u}$ can be calculated with a single system analysis, by proper scaling according to the length R . Such property can be understood due to the linearity of the structural response with respect to $\mathbf{z}$, as noted from eq. (4) and Appendix B. In other words, if the structural response is calculated for a given realization $\mathbf{u}$ of the direction vector, the responses for all values of $R \in (0, \infty)$ associated with that direction can be determined by simply amplifying by factor R . Such concept is illustrated schematically in Figure 1, where two direction vectors $\mathbf{u}^{(l)}$, $l = 1, 2$ are illustrated as well as their corresponding rays. Provided that the structural response is evaluated for each of those direction vectors, it is then possible to calculate the structural response at any point along the respective ray by amplification according to R . The second concept behind

DIS is that samples of the direction vector $\mathbf{u}$ are generated with a specially design Importance Sampling density function. Such density function ensures that those direction vectors which are highly relevant for calculating the failure probability are sampled frequently. For example, from the schematic representation in Figure 1, it is noted that the portion of the ray associated with sample $\mathbf{u}^{(1)}$ that falls within the failure domain is longer than that associated with ray 2. Thus, the Importance Sampling density function gives more importance to sampling direction $\mathbf{u}^{(1)}$ and less relevance to sampling direction $\mathbf{u}^{(2)}$. In order to ensure that all directions that cause failure are properly accounted for, the Importance Sampling density function is constructed considering that failure may take place at any time within the duration of the stochastic excitation as well as for any of the responses of interest. This means that the Importance Sampling density function comprises the occurrence of failure at n_T times instants over n_R responses. Numerical experience as reported in [43] indicates that DIS is applicable to small and large scale structural models and that it provides highly accurate estimates of the failure probability with a modest number of simulations (in the order of few hundreds).

3.4. Summary of proposed approach

The proposed approach for minimizing the first excursion probability of a linear structure subject to a stochastic Gaussian loading considering design variables belonging to a discrete set can be summarized as follows.

1. Characterize the stochastic input load by means of the Karhunen-Loève expansion, as described in eq. (1).
2. Select the structural responses to be controlled as well as their corresponding threshold levels.
3. Define the set of discrete values Ω_j , $j = 1, \dots, n_y$ that the design variables of the problem y_j , $j = 1, \dots, n_y$ may assume.
4. Solve the (deterministic) optimization problem in eq. (12) by means of Genetic Algorithms (see Section 3.2) and determine the optimal design $\mathbf{y}^*$. Recall that for each candidate solution evaluated by the optimization algorithm, it is necessary to carry out structural analysis in order to assemble matrix $\mathbf{A}$ (see eq. (5) and Appendix B) and calculate the associated operator norm (see eq. (11)).
5. Estimate the first excursion probability associated with the optimal design $\mathbf{y}^*$ by means of Directional Importance Sampling (see Section 3.3).

4. Examples

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4.1. Test Example 1: Single-degree-of-freedom oscillator subject to discrete white noise excitation

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The first example consists of a single-degree-of-freedom oscillator subject to a stochastic ground acceleration modeled as a discrete white noise, as depicted schematically in Figure 2. The objective of the problem is selecting the value of the stiffness such that the first excursion probability of the oscillator is minimized.

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Figure 2: Example 1 – Single-degree-of-freedom oscillator subject to discrete white noise excitation.

The white noise excitation possesses a duration of $T = 10$ [s], a spectral density $S = 5 \times 10^{-4}$ [m²/s³] and is represented considering a time discretization of $\Delta t = 0.01$ [s]. Thus, the total number of time steps is equal to $n_T = 1001$. As the covariance matrix associated with a discrete white noise process is of diagonal type, the Karhunen-Loève representation involves a total of $n_{KL} = 1001$ terms.

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The oscillator possesses a mass $m = 10^4$ [kg] and damping ratio $\zeta = 5\%$, while its stiffness k is a design variable to be selected within the discrete set $\Omega_k = \{1.0, 1.1, 1.2, \dots, 4.9, 5.0\}$ [MN/m].

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The responses of interest are the relative displacement of the oscillator with respect to the ground and the absolute acceleration (that is, $n_R = 2$); the respective threshold levels are set as $b_1 = 0.01$ [m] for the displacement and $b_2 = 3$ [m/s²] for the acceleration.

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The probability that any of these two responses exceed their respective threshold levels within $t \in [0, 10]$ [s] is calculated using Directional Importance Sampling, considering a total of $N = 500$ samples of the stochastic ground acceleration.

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It should be noted that although this example involves a relatively simple model, the calculation of the associated failure probability is already quite challenging, as it comprises a total of $n_{KL} = 1001$ dimensions and $n_R n_T = 2002$ responses to be controlled.

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The RBO problem to be solved is the following in mathematical terms.

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$$\min_k p_F(k) \tag{13}$$

subject to:

$$k \in \Omega_k = \{1.0, 1.1, 1.2, \dots, 4.9, 5.0\} \text{ [MN/m]}$$

As the discrete set Ω_k involves 41 elements, it is decided to assess the probability associated with each of those possible designs. Hence, for this first example only, no optimization is considered. Figure 3 contains the results associated with each element of set Ω_k : the top plot contains the operator norm of matrix $\mathbf{A}$ as a function of the stiffness while the bottom plot contains the failure probability as a function of k . It is observed that the sets of points associated with the operator norm and failure probability possess a similar behavior: that is, first decreasing with respect to the stiffness up to the point $k = 3$ [MN/m]; and then, increasing from that point onward. Such similar behavior reinforces the concept of applying the operator norm theorem for replacing the RBO problem in eq. (8) with the deterministic problem in eq. (12), as both curves $\|\mathbf{A}(k)\|_{\infty,2}$ vs. k and $p_F(k)$ vs. k reach their minimum for the same value $k = 3$ [MN/m].

Figure 3: Example 1 – Operator norm ($\|\mathbf{A}(k)\|_{\infty,2}$) and failure probability ($p_F(k)$) versus discrete-valued stiffness (k).

Figure 4 presents the failure probability ($p_F(k)$) as a function of the operator norm of matrix $\mathbf{A}$ (that is, $\|\mathbf{A}(k)\|_{\infty,2}$). For this set of points, it is observed that the minimum value of $\|\mathbf{A}(k)\|_{\infty,2}$ corresponds to the minimum value of $p_F(k)$, as already noted from Figure 3.

From a physical viewpoint, the results shown in the bottom plot of Figure 3 possess a relevant interpretation. For small stiffness values, the oscillator is quite flexible and thus, it is more likely to fail because of the displacement condition rather than by the acceleration condition. As the stiffness increases (up to 3 [MN/m]), the oscillator is less flexible, thus decreasing its maximum

Figure 4: Example 1 – Operator norm ($\|\mathbf{A}(k)\|_{\infty,2}$) versus failure probability ($p_F(k)$).

displacement and consequently, its failure probability. However, for stiffness values larger than 3 [MN/m], the oscillator becomes too stiff and the failure criterion associated with acceleration becomes more relevant, thus explaining the increase in failure probability. Such behavior had already been observed within the context of RBO problems involving continuous design variables in [5].

4.2. Test Example 2: Three-story shear frame subject to discrete white noise excitation

The second example involves the three-story shear beam model subject to a stochastic ground acceleration illustrated in Figure 5. The objective of the problem is selecting the cross sections of the columns of each floor such that the first excursion probability of the structure is minimized.

Figure 5: Example 2 – Three-story shear frame subject to discrete white noise excitation.

The ground acceleration is modeled by means of the same discrete white noise process considered in Section 4.1. Concerning the structural model, it is assumed that the mass of the model is concentrated at the infinitely stiff floor beams and is equal to $m_i = 50 \times 10^3$ [kg], $i = 1, 2, 3$. A classical modal damping of $\zeta_i = 2\%$, $i = 1, 2, 3$ is considered for each mode. Each floor of the shear frame is supported by two identical, perfectly clamped steel columns, whose Young's modulus is

equal to $E = 2 \times 10^{11}$ [N/m]. The cross section of the columns of the three stories are the design variables of the problem, which must be selected from the database of HEM profiles belonging to the Euronorm 53-62. As the considered database comprises a total of 24 profiles and there are $n_y = 3$ design variables, the set Ω of discrete elements for design comprises a total of 13824 entries. Because of constructive reasons, it is imposed that the depth of the column's cross section of floor $i + 1$ should be equal or smaller than the depth of the column's cross section of floor i . Such condition imposes then $n_C = 2$ deterministic constraints to the optimization problem. The responses of interest of the system are the interstory drifts and the absolute acceleration of each floor ($n_R = 6$), which should remain below prescribed thresholds of 7×10^{-3} [m] and 5 [m/s²], respectively. The probability that any of these responses exceeds its prescribed threshold is calculated by means of Directional Importance Sampling, considering a total of $N = 500$ samples. The RBO problem to be solved is:

$$\begin{aligned} & \min_{\mathbf{y}} p_F(\mathbf{y}) \\ & \text{subject to:} \\ & d_p(y_{l+1}) - d_p(y_l) \leq 0, \quad l = 1, 2 \\ & y_j \in \Omega_{\text{HEM}}, \quad j = 1, 2, 3 \end{aligned} \tag{14}$$

where $d_p(y_l)$ denotes the cross section's depth associated with the l -th design variable; and Ω_{HEM} denotes the set of 24 HEM admissible steel profiles.

As a means for exploring the problem at hand in a preliminary fashion, a total of 1000 designs are generated at random (each of them containing three cross sections chosen from the HEM profile database). The operator norms and failure probabilities associated with each of these designs are shown in Figure 6 with blue dots. It is observed that these random designs span about three orders of magnitude of failure probabilities, indicating that the performance of the three-story shear beam model is sensitive to the selection of the column's cross sections. Furthermore, the cloud of points suggests that finding the minimum value of the operator norm indeed corresponds to finding the minimum value of the failure probability.

In a next step, the RBO problem is addressed by means of the decoupling approach proposed in this work. For this purpose, the optimization problem in eq. (12) is addressed by means of Genetic Algorithms, considering a population size of 50 individuals. The optimum is found after 60

Figure 6: Example 2 – Operator norm ($\|\mathbf{A}(\mathbf{y})\|_{\infty,2}$) versus failure probability ($p_F(\mathbf{y})$). ON: operator norm; DL: double loop.

generations and is equal to $\mathbf{y}^* = [\text{HEM320}, \text{HEM300}, \text{HEM300}]^T$. Then, a single reliability analysis is conducted for estimating the failure probability associated with that optimal design, and it is found that $p_F(\mathbf{y}^*) = 1.3 \times 10^{-3}$. The results obtained with such procedure are plotted in Figure 6 with a red x mark. In addition and as a means to validate the results described previously, the RBO problem is solved once more, this time using a classical double loop approach. This means that the optimization problem in eq. (8) is solved by means of Genetic Algorithms and that for each candidate design, the failure probability is evaluated by means of Directional Importance Sampling. The optimum design obtained with such procedure is shown with a black circle in Figure 6. As it can be noticed from that Figure, the results produced with both the proposed decoupling approach employing operator norm (ON) and double loop (DL) are identical. That is, for both cases the optimal design is $\mathbf{y}^* = [\text{HEM320}, \text{HEM300}, \text{HEM300}]^T$ with failure probability $p_F(\mathbf{y}^*) = 1.3 \times 10^{-3}$. However, the numerical efforts associated with each approach are quite different, as shown in Figure 7. This Figure illustrates the total execution times associated with the optimization considering the operator norm and the classical double loop approach. These execution times are normalized by the execution time associated with the operator norm. It is noted that the operator norm offers a significant reduction in computation time.

4.3. Example 3: Ten-story moment frame subject to filtered white noise excitation

The third example involves the ten-story moment frame of Figure 8, which is subjected to a stochastic ground acceleration. The objective of the problem is selecting the cross sections of the columns of each floor such that the first excursion probability of the frame is minimized. The stochastic ground acceleration is characterized as a modulated white noise excitation, filtered

Figure 7: Example 2 – relative execution time.

Figure 8: Example 3 – Ten-story moment frame subject to filtered white noise excitation.

by means of the Clough-Penzien model (see, e.g. [68]). The discrete white noise process possesses 473
a duration of $T = 15$ [s], a spectral density $S = 3 \times 10^{-4}$ [m²/s³] and is represented considering a 474
time discretization of $\Delta t = 0.01$ [s]. Thus, the total number of time steps is equal to $n_T = 1501$. 475
This discrete white noise process is modulated by the following function $m(t)$: 476

$$m(t) = \begin{cases} t^2/16 & \text{if } 0 \text{ [s]} \leq t \leq 4 \text{ [s]} \\ 1 & \text{if } 4 \text{ [s]} < t \leq 10 \text{ [s]} \\ e^{(t-10)^2} & \text{if } 10 \text{ [s]} < t \leq 15 \text{ [s]} \end{cases} \quad (15)$$

The Clough-Penzien filter possesses circular natural frequencies $\omega_{g,1} = 15.6$ [rad/s] and $\omega_{g,2} = 1.0$ 477
[rad/s] and corresponding damping ratios $\zeta_{g,1} = 0.6$ and $\zeta_{g,2} = 0.9$. The stochastic process is 478

represented by means of the Karhunen-Loève expansion considering a total of $n_{KL} = 1501$ terms. 479
The structural model considers the moment resistance provided by the columns and beams of the 480
structure plus the contribution of the reinforced concrete floor slab. It is assumed that the mass 481
of the model is concentrated at slab level and is equal to $m_i = 100 \times 10^3$ [kg], $i = 1, \dots, 10$. A 482
classical modal damping of 2% is considered for all modes. Each floor of the frame is supported 483
by six identical, perfectly clamped steel columns, whose Young's modulus is equal to $E = 2 \times 10^{11}$ 484
[N/m]. The cross section of the columns of the ten stories are the design variables of the problem, 485
which must be selected from the database of HEM profiles belonging to the Euronorm 53-62. 486
Given that $n_y = 10$ and that the HEM database contains 24 profiles, the set Ω possesses about 487
 6×10^{13} entries. Following an approach similar to the one in Section 4.2, it is imposed that the 488
depth of the column's cross section of floor $i + 1$ should be equal or smaller than the depth of 489
the column's cross section of floor i , leading to a total of $n_C = 9$ deterministic constraints for the 490
optimization problem. 491

The responses of interest of the system are the interstory drifts and the absolute acceleration of 492
each floor ($n_R = 20$), which should remain below prescribed thresholds of 5×10^{-3} [m] and 3 493
[m/s²], respectively. The probability that any of these responses exceeds its prescribed threshold 494
is calculated by means of Directional Importance Sampling, considering a total of $N = 1000$ 495
samples. 496

The RBO problem to be solved is:. 497

$$\min_{\mathbf{y}} p_F(\mathbf{y}) \quad (16)$$

subject to:

$$d_p(y_{l+1}) - d_p(y_l) \leq 0, \quad l = 1, \dots, 9$$

$$y_j \in \Omega_{\text{HEM}}, \quad j = 1, \dots, 10$$

where $d_p(y_l)$ and Ω_{HEM} have been defined previously. It should be noted that while the structural 498
model associated with this RBO problem is relatively simple (that is, a 10-story moment frame), it 499
serves the purposes of this contribution. This point becomes clearer when analyzing the numerical 500
performance, as discussed at the end of this Section. 501

Before solving the optimization problem, 6400 random designs are explored. Each of these random 502
designs contains 10 HEM profiles. Figure 9 illustrates the values of the operator norm and failure 503

probability associated with each of those designs, represented with blue dots. A pattern similar to the one observed in in Section 4.2 is found in this Figure, that is, the random samples span a wide range of failure probabilities and the minimum operator norm value coincides with the minimum failure probability.

Figure 9: Example 3 – Operator norm ($\|\mathbf{A}(\mathbf{y})\|_{\infty,2}$) versus failure probability ($p_F(\mathbf{y})$). ON: operator norm; DL: double loop.

After the preliminary analysis carried out by means of random samples, the problem of finding the column's sections that minimize the first excursion probability associated with the 10 story frame model is solved by means of Genetic Algorithms. For comparison purposes, both the classical double loop approach and the decoupling approach based on operator norm are carried out. For each case, a population size of 1280 individuals is considered in the implementation of Genetic Algorithms. This number is larger than the one considered in Section 4.2 in order to cope with the increased dimension of the design variable vector. The optimal designs found for each case are reported in Table 1 in terms of the selected cross sections, the associated operator norms and the failure probabilities; these optimal designs are also plotted in Figure 9 in terms of the corresponding operator norm and failure probability. It is seen that while the optimum designs found by each procedure are not exactly the same, they are quite similar and possess almost identical values for the operator norm and failure probability. In this sense, it should be kept in mind that the optimization problem itself is quite challenging and that Genetic Algorithms corresponds to a stochastic search procedure, so it is not surprising that the optimum designs found in each case are slightly different.

Figure 10 illustrates the execution time associated with the optimization considering decoupling by means of operator norm and the classical double loop (normalized by the execution time of

Floor	Optimal design	
	ON	DL
1	HEM500	HEM700
2	HEM450	HEM450
3	HEM360	HEM340
4	HEM340	HEM340
5	HEM320	HEM300
6	HEM300	HEM300
7	HEM300	HEM280
8	HEM300	HEM260
9	HEM300	HEM260
10	HEM260	HEM260
$\ \mathbf{A}(\mathbf{y}^*)\ _{\infty,2}$	0.2672	0.2671
$p_F(\mathbf{y}^*)$	5.47×10^{-3}	5.45×10^{-3}

Table 1: Example 3 – Optimal designs found applying decoupling approach by means of operator norm (ON) and classical double loop (DL).

optimization considering decoupling). As already noticed from the example in Section 4.2, the application of the proposed approach leads to a drastic reduction of computation times. As an additional reference, all computations were performed on a server equipped with 512 GB of RAM and two 64-thread AMD EPYC 6701 CPU @ 2.65 GHz. The solution of the RBO problem by means of the operator norm framework took 30.5 [s] while the classical double-loop demanded 2600 [s], thus highlighting the advantages of the proposed framework. This is quite remarkable, particularly taking into account that the structural model involved in the problem is relatively simple, as already stated above. In this sense, the numerical challenge of solving the problem with a double loop comes into the exhaustive exploration of the design space which must be carried out combined with the reliability analysis conducted at each design explored. Hence, it is expected that the proposed decoupling approach may offer considerable benefits in problems involving large structural models where, in addition, one may take advantage of substructuring, as discussed in, e.g. [63].

5. Conclusions and Outlook

This contribution presents a decoupling approach based on operator norm theory for the solution of a class of RBO problems, namely minimization of the first excursion of a linear structure subject to a stochastic loading that involves design variables belonging to a discrete set. The core of the proposed approach lies in the concept of decoupling, as the solution of the RBO problem is

Figure 10: Example 3 – relative execution time.

replaced by one deterministic optimization problem (which involves no uncertainty propagation at all) followed by one reliability analysis (which does not involve any optimization). For practical implementation, this decoupling approach is applied in conjunction with Genetic Algorithms, which guarantees that the space of available discrete designs is examined exhaustively. Numerical examples as considered in this contribution show that the proposed approach is capable of determining the same optimal design as that obtained by means of a standard, double loop technique, but with a reduction of computation times close to two orders of magnitude.

While the approach presented in this contribution shows an excellent performance, it should be kept in mind that as of yet, its range of application is quite specific: minimization of the failure probability of a linear stochastic system with discrete design variables. Hence, future research efforts will aim at extending the range of application considering, for example, weight minimization under reliability constraints and risk-based optimization of linear stochastic systems. It is envisioned that such extension could be achieved by relating the operator norm and failure probability by means of an appropriate surrogate model. Another possible extension of the proposed approach consists of extending its range of applications to RBO problems involving mixed discrete-continuous design variables and non-Gaussian stochastic load. Yet another path for future development involves the extension of the approach to the nonlinear case. However, this last goal seems more challenging, as the linearity of the response with respect to the stochastic excitation does no longer hold. Probably, methods based on equivalent linearization may be of help in such situation.

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Appendix A. Unit impulse response function

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Unit impulse response functions associated with different responses of interest can be conveniently calculated by means of modal analysis. For example, under the assumption that the i -th response of interest η_i is a linear combination of the displacement vector $\mathbf{x}$, it can be shown that the unit impulse response function $h_i(t)$ is given by (see, e.g. [49]):

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$$h_i(t, \mathbf{y}) = \sum_{v=1}^{n_D} \frac{\alpha_{i,v}(\mathbf{y}) e^{-\zeta_v(\mathbf{y}) \omega_v(\mathbf{y}) t}}{\omega_{d,v}(\mathbf{y})} \sin(\omega_{d,v}(\mathbf{y}) t), \quad i = 1, \dots, n_R \quad (\text{A.1})$$

where ω_v , $v = 1, \dots, n_D$ are the natural frequencies; ζ_v , $v = 1, \dots, n_D$ are the damping ratios; $\omega_{d,v} = \omega_v \sqrt{1 - \zeta_v^2}$, $v = 1, \dots, n_D$ are the damped frequencies; $\alpha_{i,v}$, $i = 1, \dots, n_R$, $v = 1, \dots, n_D$ are the modal participation factors, defined as:

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$$\alpha_{i,v}(\mathbf{y}) = \frac{\mathbf{q}_i^T \phi_v(\mathbf{y}) \phi_v(\mathbf{y})^T \boldsymbol{\rho}}{\phi_v(\mathbf{y})^T \mathbf{M}(\mathbf{y}) \phi_v(\mathbf{y})}, \quad i = 1, \dots, n_R, \quad v = 1, \dots, n_D \quad (\text{A.2})$$

in addition, ϕ_v , $v = 1, \dots, n_D$ are the eigenvectors associated with the eigenproblem of the undamped equation of motion; and $\mathbf{q}_i$ is a vector such that $\eta_i = \mathbf{q}_i^T \mathbf{x}$. The contribution of higher order modes to the unit impulse response function in eq. (A.1) can be neglected for several cases of practical interest [49].

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The expression in eq. (A.1) presents a formula for calculating an impulse response function associated with displacements. In case the response of interest is a velocity or acceleration, the associated unit impulse response function is expressed in terms of the corresponding time deriva-

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tive of eq. (A.1), that is:

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$$\frac{\partial h_i(t, \mathbf{y})}{\partial t} = \sum_{v=1}^{n_D} \frac{\alpha_{i,v}(\mathbf{y}) e^{-\zeta_v(\mathbf{y}) \omega_v(\mathbf{y}) t}}{\omega_{d,v}(\mathbf{y})} (\omega_{d,v}(\mathbf{y}) \cos(\omega_{d,v}(\mathbf{y}) t) - \zeta_v(\mathbf{y}) \omega_v(\mathbf{y}) \sin(\omega_{d,v}(\mathbf{y}) t)),$$

$$i = 1, \dots, n_R \quad (\text{A.3})$$

$$\frac{\partial^2 h_i(t, \mathbf{y})}{\partial t^2} = \sum_{v=1}^{n_D} \frac{\alpha_{i,v}(\mathbf{y}) e^{-\zeta_v(\mathbf{y}) \omega_v(\mathbf{y}) t}}{\omega_{d,v}(\mathbf{y})} (-2\zeta_v(\mathbf{y}) \omega_v(\mathbf{y}) \omega_{d,v}(\mathbf{y}) \cos(\omega_{d,v}(\mathbf{y}) t) +$$

$$\zeta_v(\mathbf{y})^2 \omega_v(\mathbf{y})^2 \sin(\omega_{d,v}(\mathbf{y}) t) - \omega_{d,v}(\mathbf{y})^2 \sin(\omega_{d,v}(\mathbf{y}) t)), \quad i = 1, \dots, n_R \quad (\text{A.4})$$

Appendix B. Discretization of convolution integral

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The convolution integral in eq. (3) can be approximated taking into account the discrete representation of the loading in eq. (1). Thus, the i -th response at time instant t_k is equal to:

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$$\eta_i(t_k, \mathbf{y}, \mathbf{z}) = \sum_{l_1=1}^k \epsilon_{l_1} h_i(\mathbf{y}, t_k - t_{l_1}) f(t_{l_1}, \mathbf{z}) \Delta t$$

$$= \sum_{l_1=1}^k \epsilon_{l_1} h_i(t_k - t_{l_1}) \left(\sum_{l_2=1}^{n_{KL}} \psi_{l_1, l_2} \sqrt{\lambda_{l_2}} z_{l_2} \right) \Delta t$$

$$= \mathbf{a}_{i_k}(\mathbf{y}) \mathbf{z}, \quad i = 1, \dots, n_\eta, \quad k = 1, \dots, n_T \quad (\text{B.1})$$

where ψ_{l_1, l_2} is the (l_1, l_2) -th element of matrix $\mathbf{\Psi}$; ϵ_{l_1} is a coefficient depending on the quadrature scheme used to approximate the convolution integral; and $\mathbf{a}_{i_k}$ is a row vector of dimension $n_{KL} \times 1$ defined as:

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$$\mathbf{a}_{i_k}(\mathbf{y}) = \left[\sum_{l_1=1}^k \Delta t \epsilon_{l_1} h_i(\mathbf{y}, t_k - t_{l_1}) \psi_{l_1, 1} \sqrt{\lambda_1}, \dots, \sum_{l_1=1}^k \Delta t \epsilon_{l_1} h_i(\mathbf{y}, t_k - t_{l_1}) \psi_{l_1, n_{KL}} \sqrt{\lambda_{n_{KL}}} \right]$$

$$i_k = (i - 1)n_T + k, \quad i = 1, \dots, n_R, \quad k = 1, \dots, n_T \quad (\text{B.2})$$

It is straightforward to verify that matrix $\mathbf{A}_i$ in eq. (4) collects all row vectors $\mathbf{a}_{i_k}$ associated with the i -th response, that is:

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$$\mathbf{A}_i(\mathbf{y}) = \begin{bmatrix} \mathbf{a}_{i_1}(\mathbf{y}) \\ \vdots \\ \mathbf{a}_{i_{n_T}}(\mathbf{y}) \end{bmatrix}, \quad i = 1, \dots, n_R \quad (\text{B.3})$$

In this work, the trapezoidal integration rule is chosen for approximating the convolution integral (see, e.g. [69]). Thus, ϵ_{l_1} is selected as $\epsilon_{l_1} = 1/2$ if $l_1 = 1$ or $l_1 = k$; otherwise, $\epsilon_{l_1} = 1$.

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