Fixed-time stabilization of linear delay systems by smooth periodic delayed feedback

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***Abstract*—This paper studies *fixed-time stabilization (FTS)* of a general controllable linear system with an input delay τ. It is shown that such a problem is not solvable if the prescribed convergence time T_τ is smaller than 2τ. For $T_\tau \geq 3\tau$, a solution based on linear periodic delayed feedback (PDF) without any distributed delay is established. For $T_\tau > 2\tau$, a solution based on linear predictor-based PDF containing a distributed delay is proposed. For both cases, the gains of the PDF can be chosen as continuous, continuously differentiable, and even smooth, in the sense of infinitely many times differentiable. If only an output signal is available for feedback, two classes of linear observers with periodic coefficients are designed so that their states converge to the current and future states of the system at a prescribed finite time, respectively. With the observed current and future states, FTS can also be achieved by using respectively the PDF and observer-based PDF. A linear periodic feedback (without delay) is also established to solve the FTS problem of linear systems with both instantaneous and delayed controls, which cannot be stabilized by any constant instantaneous feedback in certain cases. Two numerical examples verify the effectiveness of the proposed approaches.**

***Index Terms*—Fixed-time stabilization; Prescribed finite-time stabilization; Periodic delayed feedback; Input time-delay; Linear time-varying feedback.**

I. Introduction

Time delay poses a significant challenge to the analysis and design of feedback control systems. One well-known design method in the control of delay systems is that via the so-called *Smith predictor* [40]. The Smith predictor received considerable attention in the 1970s and onwards [1], [25], [28], which led to the emergence of the predictor feedback method [24], [29], [45], more generally referred to as the Artstein transformation/model reduction [1], [32]. Using this method, a linear controller containing distributed delays can be designed such that the closed-loop system possesses only a finite spectrum, and for that reason has often been referred to as finite-spectrum assignment methods [28]. The mechanism underlying this method is delay compensation, as the closed-loop system acts like a delay-free system (having only a finite spectrum). One noteworthy feature of the delay compensation approach lies in the fact that by modelling a control system with input delay as a coupled ordinary differential equation and a partial differential equation, the approach is applicable to a wide variety of systems with sophisticated delay dynamics [2], [24]. Applications of this approach can be found in, e.g., implementation of distributed delays in the controller [31], [43], stability analysis under predictor feedback [29], finite-dimensional predictor feedback [7], [45], observer-based predictor feedback [3], and predictor feedback for systems with both state and input delays [46].

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The design of control systems having a finite convergence time, known as finite-time stabilization, is a longstanding problem that has received increased interest in the last two decades. Differing from traditional stabilization methods, by which the closed-loop system converges to zero asymptotically, the finite-time stabilizing control drives the closed-loop system state to rest in finite time [4], [37], [41]. Problems of this nature are relevant to such tasks as precision tracking and localization, and are reminiscent of the traditional deadbeat or “ripple-free” control (see, e.g., [13], [14]). By now several efficient methods have been developed for achieving finite-time stabilization, including those by sliding mode control [37], by a Lyapunov differential inequality-based approach [4], [42], that by a homogeneous approach [5], [11], [17] (see also [38], [39] for its use to design distributed finite-time observers), and by an implicit Lyapunov function approach [35]. It is worth noting that in these methods the convergence time generally depends on and hence varies with the initial condition. On the other hand, it is possible to achieve finite-time stability independent of initial conditions (referred to as fixed-time stability), albeit unable to ascertain the convergence time *a priori*. Such a problem was solved in [34], which likewise, adopts an approach based on Lyapunov differential inequalities. Further advances on finite-time stabilization were made in [41] by using time-varying feedback, for nonlinear systems in the normal form, where the convergence time can be prescribed in advance. Moreover, finite- and fixed-time stabilization designs were also sought after in [47], [48] for general linear systems, by solving certain parametric Lyapunov equations or nonlinear differential equations, resulting in bounded linear time-varying high-gain feedback. This latter series of works [41], [47], [48], however, suffer from the fact that they either result in non-smooth feedback, or when reaching the prescribed terminal time, the control becomes singular in the sense that the feedback gain will grow unbounded [36], [44].

In this paper, we provide a novel approach to solve the *fixed-time stabilization (FTS)* problem for general linear systems with input delay, by smooth feedback independent of initial conditions and free of terminal singularities. In a significant departure, we employ *periodic delayed feedback* (PDF) to achieve the objective. Our approach is motivated by the so-called act-and-wait control [19], which uses piecewise constant feedback gains, and by the observation that more generally, the stability of a (special) continuous-time periodic time-delay system is determined by the spectrum of a (finite-dimensional) *monodromy matrix* (for the definition of monodromy matrix, see, for example, [6]). Hence, unlike using an act-and-wait control strategy, we may use continuous, continuously differentiable, and even smooth feedback gains, and yet the monodromy matrix of the corresponding periodic system can still be made nilpotent, hence achieving FTS. It is useful to point out that, except for scalar systems, it appears highly nontrivial, if ever possible, to find solutions to guarantee finite-time/fixed-time stability by the act-and-wait control [20]. It is also worth noting that periodic feedback has been used to stabilize systems with uncertain, possibly arbitrarily large delays [30]. Additionally, periodic controllers are known to offer a number of distinct advantages, such as availing the capability of closed-loop zero placement [26] and hence providing an infinite gain margin [23].

Our present work reveals yet another advantage of periodic feedback.

The main contributions of this paper can be summarized as follows. First, we observe that for a general linear system subject to an input delay $\tau > 0$, the FTS problem with a prescribed convergence time T_τ cannot be solved if $T_\tau < 2\tau$, and we show that with state-feedback PDF, the problem is solvable whenever $T_\tau \geq 3\tau$. In the case $T_\tau > 2\tau$, a linear predictor-based state-feedback PDF achieving the FTS is designed accordingly. The results are then extended to output feedback, by means of designing a fixed-time observer. As a by-product, we also develop PDF controllers for achieving FTS of general linear system free of delay, with both instantaneous and delayed control actions. Furthermore, we show that the periodic controllers can stabilize (in finite-time) certain systems that otherwise cannot be stabilized by any constant instantaneous feedback.

The PDF design method offers a number of distinctive advantages over the previous approaches for finite-time/fixed-time stabilization. In its essential difference, the method enables us to achieve FTS with more desirable controllers, notably smooth, bounded linear controllers that are independent of initial conditions and can prevent terminal singularities. From a technical standpoint, the method also differ significantly. In this vein, we note that previous finite-time/fixed-time stabilization approaches for delay systems typically consist of two steps: In the first step, by applying the Artstein transformation, the delayed linear system is transformed into a linear system without delay, and next a nonlinear controller is designed for the resultant linear system so as to achieve finite-time/fixed-time stability [5], [32]. As a result, a nonlinear control design is generally required, which in turn requires the construction of Lyapunov-Razumikhin (LR) functions or the Lyapunov-Krasovskii (LK) functionals to ascertain finite-time stability [10], [27], [32], [35]. The FTS design method developed here remains a linear control design, and as such appears conceptually more straightforward and intuitively more appealing.

The remainder of this paper is organized as follows. The problem formulation and the core idea are proposed in Section II. The PDF approach and the observer based PDF approach are then respectively established in Sections III and IV. An extension to a linear system with both an instantaneous and a delayed controls is then discussed in Section V. Two numerical examples are given in Section VI to illustrate the proposed theory and Section VII concludes the paper.

Notation: For a matrix A, we use $\|A\|, A^{\mathrm{T}}, \lambda(A), \lambda_i(A)$ and $\rho(A)$ to denote its norm, transpose, eigenvalue set, i-th eigenvalue, and spectral radius (when A is square). For $a \leq b$, denote $x_{[a,b]} = x(s), s \in [a,b]$.

II. Problem Formulation, Motivation and the Core Idea

A. Problem Formulation

We first give the following definition.

Definition 1: [32] Consider the time-delay system

$$\dot{x}(t) = f(t, x_{[t-\tau,t]}),\ t \geq t_0,$$

where $f(t,0) = 0$. Assume that the system has a unique solution for any given initial condition $x_{[t_0-\tau,t_0]} = \phi(t_0 + s), s \in [-\tau, 0]$. The system is said to be T_τ-fixed-time stable if it is stable in the usual sense and, additionally, there exists a $T_\tau > 0$ such that $x(t) = 0, \forall t \geq T_\tau + t_0$. Moreover, T_τ is referred to as the convergence time.

Consider the input-delayed linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t-\tau),\ t \geq 0, \tag{1}$$

where $(A,B) \in (\mathbf{R}^{n\times n}, \mathbf{R}^{n\times m})$ is controllable and $\tau \geq 0$ is a known constant denoting the input delay. The initial condition for system (1) is $x(0) = x_0$ and $u(s) = \upsilon(s), s \in [-\tau, 0)$, which is an arbitrary (piecewise continuous and bounded) function but is not accessible/assignable, herein by assignability of a signal, we mean that it cannot be designed a priori.

In this paper, we are interested in the design of controllers such that the closed-loop system is T_τ-fixed-time stable with the convergence time T_τ being *prescribed* in advance, as stated below.

Problem 1: (FTS of input-delayed linear systems) Let $T_\tau > 0$ be a given constant. Design a feedback control law $u(t) = u(t, x_{[0,t]}, u_{[0,t]})$ such that the closed-loop system consisting of (1) and $u(t)$ is T_τ-fixed-time stable.

Problem 1 is not solvable for all $T_\tau > 0$. Notice that

$$\dot{x}(t) = Ax(t) + B\upsilon(t-\tau),\ t \in [0,\tau), \tag{2}$$

which means that system (1) operates in "open-loop" in the first τ seconds, and hence the state $x(t)$ cannot be driven to zero by any $u(t)$ in less than τ seconds. Thus, we must have $T_\tau > \tau$. A necessary condition of solvability in this spirit is established in the next lemma, whose proof can be found in Appendix A1.

Lemma 1: Suppose that the initial condition $\upsilon(s), s \in [-\tau, 0)$ is not assignable. Then Problem 1 is not solvable if $T_\tau < 2\tau$.

B. Predictor-Based Feedback

As shown in [32], Problem 1 can be transformed into a problem of FTS free of delay, by employing predictor-based feedback. To this end, denote

$$w(t) = \mathrm{e}^{A\tau}x(t) + \int_t^{t+\tau} \mathrm{e}^{A(t+\tau-s)}B\overline{u}(s-\tau)\,\mathrm{d}s, \tag{3}$$

where $t \geq 0$ and

$$\overline{u}(t) = \begin{cases} u(t),\ \forall t \geq 0, \\ \overline{\upsilon}(t),\ \forall t \in [-\tau, 0), \end{cases} \tag{4}$$

with $\overline{\upsilon}(t)$ being an arbitrary bounded function. The following lemma states that Problem 1 is equivalent to one of FTS in the absence of delay. The proof is relegated to Appendix A2.

Lemma 2: Let $\Delta\upsilon(s) = \upsilon(s) - \overline{\upsilon}(s), s \in [-\tau, 0)$. Then

$$\dot{w}(t) = Aw(t) + Bu(t) + \mathrm{e}^{A\tau}B\Delta\upsilon(t-\tau),\ t \in [0,\tau), \tag{5}$$

and

$$\dot{w}(t) = Aw(t) + Bu(t),\ \forall t \geq \tau. \tag{6}$$

Moreover, $w(t)$ and $x(t+\tau)$ are related with

$$w(t) = \begin{cases} x(t+\tau) - \int_{t-\tau}^{0} \mathrm{e}^{A(t-\theta)}B\Delta\upsilon(\theta)\mathrm{d}\theta,\ t \in [0,\tau), \\ x(t+\tau),\ t \geq \tau. \end{cases} \tag{7}$$

It follows from (7) that, for $t \geq \tau$, $w(t)$ is simply the future state $x(t+\tau)$ [1], [24]. Note from (5) that $w(t)$ is not accessible in the first τ seconds since $\upsilon(t-\tau) - \overline{\upsilon}(t-\tau), t \in [0,\tau)$ is not assignable. From (3) and (6) we can see that if $(w(t), u(t))$ converges to zero in finite-time, then so does $x(t)$. Hence, to enforce $x(t) = 0$ for $t \geq 2\tau$, that is, to achieve FTS of the system (1), it suffices to achieve the condition $w(t) = 0$ for $t \geq \tau$. This leads to the following problem of FTS of delay free systems.

Problem 2: (FTS of delay-free linear systems) Let $T > 0$ be a given constant. Design a feedback controller $u(t) = u(t, w_{[0,t]}, u_{[0,t]}), t \geq \tau$ such that the closed-loop system is T-fixed-time stable in the sense of Definition 1 where $t_0 = \tau$.

Note that in the problem above the fixed-time stability is independent of τ and hence said to be uniformly fixed-time stable, since both the convergence time T and the system model (6) are independent of τ. With this, a solution to Problem 1 can be obtained immediately if a solution to Problem 2 is available, as stated below.

Lemma 3: Assume that there exists a controller

$$u(t) = u(t, w_{[0,t]}, u_{[0,t]}),\ t \geq \tau \tag{8}$$

that solves Problem 2, namely,

$$w(t)=0,\ \forall t\geq\tau+T. \tag{9}$$

Then the controller (8) also solves Problem 1 with $T_\tau=2\tau+T$.

Proof. It follows from (9) and the second equation of (7) that $x(t+\tau)=w(t)=0,\forall t\geq\tau+T$, which is just $x(t)=0,\forall t\geq 2\tau+T=T_\tau$. The proof is finished. ■

Remark 1: The controller (8) is the predictor-based feedback for the original input-delayed system (1) since $w(t)$ is the prediction of the future state $x(t+\tau),t\geq\tau$. This predictor-based feedback scheme can be easily extended to linear systems with multiple input delays. Consider for example

$$\dot{x}(t)=Ax(t)+\sum_{i=0}^{N}B_iu\left(t-\tau_i\right), \tag{10}$$

where $\tau_i\geq 0$ are known constant delays and $0=\tau_0<\tau_1<\cdots<\tau_N=\tau$. By constructing [1]

$$w_N(t)=x(t)+\sum_{i=1}^{N}\int_{t-\tau_i}^{t}\mathrm{e}^{A(t-\tau_i-s)}B_i\overline{u}(s)\mathrm{d}s,\ t\geq 0,$$
$$\mathscr{B}_N=B_0+\mathrm{e}^{-A\tau_1}B_1+\cdots+\mathrm{e}^{-A\tau_N}B_N,$$

where $\overline{u}(t)$ is defined in (4), we obtain at once

$$\dot{w}_N(t)=Aw_N(t)+\mathscr{B}_Nu(t),\ t\geq\tau,$$

which is exactly in the form of (6). As such, much of our subsequence in this paper can be extended readily to system (10), and hence will be omitted.

C. Motivation and Core Ideas

If T is not prescribed, the finite-time stabilization of system (6) has been solved; for example, *nonlinear controllers* are designed in [5], [17] and the convergence time T was shown to depend on the system's initial condition. If T is prescribed, linear time-varying high-gain feedback was designed recently in [47] and [48], where, however, the controllers have a singularity at $t=\tau+T$, due to the fact that the feedback gains are not defined for $t\geq\tau+T$.

In a significant departure, in this paper we design a class of linear periodic controllers achieving the FTS. For this purpose, we first note from [47] that the instantaneous, memoryless linear controller $u(t)=K(t)x(t)$ may achieve the finite-time stabilization only if $K(t)$ is time-varying. That being the case, the linear time-varying gain $K(t)$, as constructed in [47], [48], will approach to infinity as t approaches to T [36], [41], [44], thus resulting in a singular state. Unlike in [47], [48], the linear periodic controller to be developed in this paper prevents this singularity from occurring, and further, can be more readily implemented in a manner of generalized sampled-data feedback. Essentially, the approach utilizes the fact that the stability of a class of continuous-time periodic time-delay systems is completely determined by the eigenvalue set of a finite-dimensional monodromy operator (matrix) that is easy to compute, and fixed-time stability can be guaranteed if its monodromy matrix is nilpotent. To the best of our knowledge, the existence of a finite-dimensional monodromy matrix for a class of periodic delay systems was first utilized for stabilization of linear systems in [19] by the act-and-wait control strategy. To expound further this idea, we present the following proposition.

Proposition 1: Consider the linear periodic time-delay system

$$\dot{x}(t)=F(t)x(t)+G(t)x\left(t-h\right),\ t\geq 0, \tag{11}$$

where $h>0$ is a constant, $x(t)=\varphi(t),t\in[-h,0]$, and $(F(t),G(t)):\mathbf{R}\rightarrow(\mathbf{R}^{n\times n},\mathbf{R}^{n\times n})$ is a pair of piecewise-continuous and bounded $2h$-periodic matrices (namely, $F(t+2h)=F(t),G(t+2h)=G(t),\forall t\in\mathbf{R}$) and is such that

$$G(t)=\begin{cases}0, & t\in[2kh,(2k+1)\,h),\\ G_0(t), & t\in[(2k+1)\,h,2\,(k+1)\,h),\end{cases} \tag{12}$$

where $k=0,1,2,\ldots$. Denote the state transition matrix for the linear periodic system

$$\dot{x}(t)=F(t)x(t), \tag{13}$$

by $\varPhi(t,s),\forall t,s\in\mathbf{R}$. Then

1) The state of system (11) satisfies

$$x\left(2\left(k+1\right)h\right)=\Delta(h)x\left(2kh\right),\ k=0,1,\ldots, \tag{14}$$

where $\Delta(h)$ is referred to as the monodromy matrix of system (11) and is given by

$$\begin{aligned}\Delta(h)&=\varPhi\left(2h,0\right)+\int_{h}^{2h}\varPhi\left(2h,s\right)G_0(s)\varPhi\left(s-h,0\right)\mathrm{d}s\\&=\varPhi\left(2h,0\right)+\int_{h}^{2h}\varPhi\left(2h,s\right)G(s)\varPhi\left(s-h,0\right)\mathrm{d}s.\end{aligned} \tag{15}$$

2) System (11) is exponentially stable if and only if

$$\rho\left(\Delta(h)\right)<1. \tag{16}$$

3) System (11) is fixed-time stable if and only if

$$\rho\left(\Delta(h)\right)=0, \tag{17}$$

namely, $\Delta(h)$ is a nilpotent matrix. Moreover, if $\nu\geq 1$ is the minimal integer such that $(\Delta(h))^\nu=0$, then

$$x(t)=0,\ \forall t\geq 2\nu h, \tag{18}$$

i.e., the system is $2\nu h$-fixed-time stable in the sense of Definition 1.

Proof. *Proof of 1).* We first consider $t\in[2kh,(2k+1)h),k=0,1,2,\ldots$. By (12) we have $G(t)=0$, which means that the system (11) reduces to (13), whose solution is given by

$$x(t)=\varPhi\left(t,2kh\right)x\left(2kh\right),\ \forall t\in[2kh,(2k+1)\,h). \tag{19}$$

For $t\in[(2k+1)\,h,2\,(k+1)\,h)$, we have from (19) that

$$x\left(t-h\right)=\varPhi\left(t-h,2kh\right)x\left(2kh\right),$$

by which the system (11) can be rewritten as

$$\dot{x}(t)=F(t)x(t)+G_0(t)\varPhi\left(t-h,2kh\right)x\left(2kh\right),$$

where $t\in[(2k+1)\,h,2\,(k+1)\,h)$. With the variant of constant formula, we obtain, for all $t\in[(2k+1)h,2(k+1)h)$,

$$\begin{aligned}x(t)=&\varPhi\left(t,(2k+1)\,h\right)x\left((2k+1)\,h\right)\\&+\int_{(2k+1)h}^{t}\varPhi\left(t,\theta\right)G_0(\theta)\varPhi\left(\theta-h,2kh\right)\mathrm{d}\theta x\left(2kh\right)\\=&\varPhi\left(t,(2k+1)\,h\right)\varPhi\left((2k+1)\,h,2kh\right)x\left(2kh\right)\\&+\int_{(2k+1)h}^{t}\varPhi\left(t,\theta\right)G_0(\theta)\varPhi\left(\theta-h,2kh\right)\mathrm{d}\theta x\left(2kh\right)\\=&\left(\varPhi\left(t,2kh\right)+\varPi(t)\right)x\left(2kh\right),\end{aligned} \tag{20}$$

where

$$\varPi(t)\triangleq\int_{(2k+1)h}^{t}\varPhi\left(t,\theta\right)G_0(\theta)\varPhi\left(\theta-h,2kh\right)\mathrm{d}\theta.$$

Since $G(t)$ is a $2h$-periodic function, by setting $s=\theta-2kh$, we have, for all $t\in[(2k+1)h,2(k+1)h)$,

$$\begin{aligned}\Pi(t)&=\int_{h}^{t-2kh}\Phi(t,\theta)G_0(s+2kh)\Phi(\theta-h,2kh)\mathrm{d}s\\&=\int_{h}^{t-2kh}\Phi(t-2kh,s)G_0(s)\Phi(s-h,0)\mathrm{d}s,\end{aligned}$$

where we have noticed that $\Phi(t+2kh,s+2kh)=\Phi(t,s),\forall t,s\in\mathbf{R},k=0,\pm1,\pm2,\ldots$. Letting $t=2(k+1)h,k=0,1,\ldots$, in equation (20) gives (14) where

$$\begin{aligned}\Delta(h)&=\Phi(2(k+1)h,2kh)+\Pi(2(k+1)h)\\&=\Phi(2h,0)+\int_{h}^{2h}\Phi(2h,s)G_0(s)\Phi(s-h,0)\mathrm{d}s,\end{aligned}$$

which is exactly in the form of (15).

Proof of 2). By using (14) recursively we obtain

$$x(2(k+1)h)=\Delta^{k+1}(h)x(0),\ k=0,1,\ldots.\tag{21}$$

We first show that system (11) is exponentially stable if (16) is satisfied. In fact, it follows from (21) that $x(2kh)$ converges to zero exponentially as $k\to\infty$, which, by (19) and noting that $\|\Pi(t)\|$ is bounded for all $t\in[0,2h]$, implies that $x(t)$ converges to zero exponentially as $t\to\infty$. To show the converse, we assume that system (11) is exponentially stable but (16) is not satisfied. Then, for some $x_*(0)\neq0$, we get from (21) that

$$\lim_{k\to\infty}\|x(2(k+1)h)\|=\lim_{k\to\infty}\left\|\Delta^{k+1}(h)x_*(0)\right\|\neq0,$$

which contradicts with the exponential stability of system (11).

Proof of 3). If (17) holds true, there must exist an integer $\nu\geq1$ such that $\Delta^{\nu}(h)=0$, which, by (21), implies that $x(2\nu h)=0$, namely, (18) is satisfied. The converse is similar to the proof of 2) and is thus omitted. The proof is thus finished. ■

The essential property behind Proposition 1 is that by the *method of steps* the solutions of system (11) can be constructed solely from $x(0)$, as the dynamics on $[0,h]$ are determined by (13). Hence, even though there is a delay in system (11), it is inherently *finite-dimensional*. The finite dimension of the monodromy operator has been utilized in [19] to stabilize a linear system by delayed feedback, referred to as act-and-wait control. The main procedure is as follows. Consider system (1) without input delay (namely, $\tau=0$), as well as the piecewise constant delayed feedback [19]

$$u(t)=g(t)Dx(t-h),\tag{22}$$

where $h>0$ is a constant, $D\in\mathbf{R}^{m\times n}$ is a constant matrix, and $g(t)$ is the $2h$-periodic function satisfying [19]

$$g(t)=\begin{cases}0, & t\in[0,h),\\1, & t\in[h,2h).\end{cases}\tag{23}$$

It follows that (22) is switched off for a period of length h and is switched on for a period of length h. Thus it is a special case of periodic controllers. The closed-loop system can be written as

$$\dot{x}(t)=Ax(t)+g(t)BDx(t-h),\ t\geq0,\tag{24}$$

which is exactly in the form of (11). Similar to (15), the associated monodromy matrix can be computed as [19]

$$\Delta_{\mathrm{aaw}}(h)=\mathrm{e}^{2Ah}+\int_{h}^{2h}\mathrm{e}^{A(2h-s)}BD\mathrm{e}^{A(s-h)}\mathrm{d}s.$$

Thus, the fixed-time stability of the closed-loop system (24) is guaranteed if and only if [19], [20]

$$\rho(\Delta_{\mathrm{aaw}}(h))=0.\tag{25}$$

In this case, the controller (22)-(23) is called a deadbeat controller [19]. However, as a matrix exponential is involved in the monodromy matrix $\Delta_{\mathrm{aaw}}(h)$, except for scalar systems (namely, $m=n=1$), it is not easy (and might be impossible) to find the feedback gain D such that (25) is satisfied in general [20].

In this paper, we will establish a new PDF that is different from (22) and (23). In addition to being continuous, continuously differentiable, or even smooth, our new PDF law can make the associated monodromy matrix $\Delta(h)$ be 0, which is more strict than the condition in (17) and (25). Both state feedback and observer-based output feedback are considered. As a by-product, the FTS problem (Problem 2) for the delay-free linear system (6) is also solved, based on which and Lemma 3 an alternative PDF is also established to solve Problem 1.

We emphasize that the fixed-time stability of periodic time-delay systems as shown in Proposition 1, referred to as the deadbeat property, is widely recognized in the literature, such as [15] (pp. 87-88) and [22] where time-varying feedback by using distributed delay was designed for global finite-time stabilization of triangular control systems.

III. State-Based Periodic Delayed Feedback

In this section, based on the ideas behind Proposition 1, we will establish a new PDF approach to solve Problem 1. To present our results, we need some notations. For an integer $r\geq0$, a function f is said to be of class $\mathbf{C}^r$ if the derivatives $f^{(k)},k=0,1,\ldots,r$ exist and are continuous. Let $\mathbf{C}^r_{n\times m}$ denote the set of $n\times m$ matrices whose elements are of class $\mathbf{C}^r$.

Definition 2: Let $h>0$ be a given constant and $r\geq0$ be a given integer (can be infinity). A symmetric matrix function $R_h(\cdot):\mathbf{R}\to\mathbf{R}^{m\times m}$ is said to be an $\mathbf{S}_m^{(r)}(h)$ function if

- $R_h(\cdot)$ is $2h$-periodic and $R_h(t)=0,\forall t\in[0,h]$.
- $R_h(t)\geq0,\forall t\in[h,2h]$ and there is an $h^*\in(h,2h)$ such that $R_h(h^*)>0$.
- $R_h(\cdot)\in\mathbf{C}^r_{m\times m}$, which implies that $R_h^{(i)}(h)=R_h^{(i)}(2h)=0,i=0,1,\ldots,r$.

It follows that $\mathbf{S}_m^{(r)}(h)\subset\mathbf{C}^r_{m\times m}$. It is easy to construct an $\mathbf{S}_m^{(r)}(h)$ function. For example, for any integer $r\in[1,\infty)$, an $\mathbf{S}_m^{(r)}(h)$ function $R_h(\cdot)$ can be chosen as (we only consider $t\in[0,2h]$)

$$R_h(t)=\begin{cases}0,t\in[0,h],\\\left(\left(\mathrm{e}^{a_1(t-h)}-1\right)\left(1-\mathrm{e}^{a_2(t-2h)}\right)\right)^{r+1}I_m,t\in(h,2h),\end{cases}\tag{26}$$

where $a_1>0$ and $a_2>0$ are constants, and an $\mathbf{S}_m^{(\infty)}(h)$ function $R_h(\cdot)$ can be chosen as

$$R_h(t)=\begin{cases}0,\ t\in[0,h],\\\exp\left(-\frac{1}{(t-h)(2h-t)}\right)I_m,\ t\in(h,2h).\end{cases}$$

A. Solution to Problem 1 by State Feedback

We now apply the core idea behind Proposition 1 to solve Problem 2 by using state feedback. As we have mentioned before, the difficulty is the design of the feedback gain (which appears in $G(t)$ in (11)) such that the associated monodromy matrix (15) is nilpotent (zero). We will show that a simple solution can be obtained if we use a time-varying feedback gain instead of the piecewise constant feedback gain in the act-and-wait control (22).

Let $K_{(A,h)}(t):\mathbf{R}\to\mathbf{R}^{m\times n}$ be a $2h$-periodic function defined by

$$K_{(A,h)}(t)=R_h(t)B^{\mathrm{T}}\mathrm{e}^{-A^{\mathrm{T}}t}W\mathrm{e}^{A(h-t)},\tag{27}$$

where $h>0$ is a constant, $R_h(t)$ is an $\mathbf{S}_m^{(r)}(h)$ function for some integer $r\geq 0$, and $W\in\mathbf{R}^{n\times n}$ is some constant matrix. It follows that $K_{(A,h)}(t)\in\mathbf{C}_{m\times n}^r$. Denote

$$W_{\rm c}(A,h)=\int_h^{2h}{\rm e}^{-As}BR_h(s)B^{\rm T}{\rm e}^{-A^{\rm T}s}{\rm d}s.$$

Consider for system (1) the state-based PDF

$$u(t)=-K_{(A,h)}(t)x\left(t-(h-\tau)\right), \tag{28}$$

where $h\geq\tau$ is a constant. Clearly, this controller is well defined for all $t\geq h-\tau$ since the state $x(t-(h-\tau))$ is accessible for all $t\geq h-\tau$. Notice that, for all $t\in[0,h-\tau)$, we have $K_{(A,h)}(t)=0$. Thus $u(t)$ is well defined for all $t\geq 0$ (we may set simply $x(s)=0,s\in[-(h-\tau),0)$). Then we can state the following result.

Theorem 1: Let $T_\tau>0$ be a prescribed number and satisfy

$$T_\tau\geq 3\tau, \tag{29}$$

and h be a constant defined by

$$h\triangleq\frac{1}{2}\left(T_\tau-\tau\right)\geq\tau. \tag{30}$$

Then the closed-loop system consisting of (1) and (28) is exponentially stable if and only if

$$\rho\left({\rm e}^{2hA}\left(I_n-W_{\rm c}(A,h)W\right)\right)<1. \tag{31}$$

Moreover, the matrix $W_{\rm c}(A,h)$ is nonsingular and the control law (28) solves Problem 1 if and only if

$$W=W_{\rm c}^{-1}(A,h). \tag{32}$$

In this case, the control input satisfies the condition

$$u(t)=0,\ \forall t\geq 2h.$$

Proof. Since $u(t)$ is well defined for all $t\geq 0$, the closed-loop system can be written as (2) and

$$\dot{x}(t)=Ax(t)-BK_{(A,h)}\left(t-\tau\right)x\left(t-h\right),\ t\geq\tau. \tag{33}$$

Let $K=K_{(A,h)}$. Define $s=t-\tau$ and $\chi(s)=x(s+\tau)=x(t)$. Then system (33) can be written as

$$\dot{\chi}(s)=A\chi(s)-BK(s)\chi\left(s-h\right),\ s\geq 0, \tag{34}$$

whose initial condition is given by $\chi(\theta)=x(\theta+\tau)=\xi(\theta+\tau),\theta\in[-h,0]$, and $\dot{\chi}(s)={\rm d}\chi(s)/{\rm d}s$.

We notice that system (34) is exactly in the form of (11) with $F(t)=A$ and $G(t)=-BK(t)$. Thus, by Proposition 1, the associated monodromy matrix is given by (see (15))

$$\begin{aligned}\Delta_K(h)&={\rm e}^{2Ah}-\int_h^{2h}{\rm e}^{A(2h-s)}BK(s){\rm e}^{A(s-h)}{\rm d}s\\&={\rm e}^{2Ah}-\int_h^{2h}{\rm e}^{A(2h-s)}BR_h(s)B^{\rm T}{\rm e}^{-A^{\rm T}s}{\rm d}sW\\&={\rm e}^{2Ah}\left(I_n-W_{\rm c}(A,h)W\right).\end{aligned} \tag{35}$$

According to Proposition 1, the closed-loop system is exponentially stable if and only if (31) is satisfied, and is fixed-time stable if and only if $\rho(\Delta_K(h))=0$.

Next we show that $W_{\rm c}(A,h)$ is nonsingular. As (A,B) is controllable, its controllability Grammian

$$G_{\rm c}\left(A,B,t\right)\triangleq\int_0^t{\rm e}^{-As}BB^{\rm T}{\rm e}^{-A^{\rm T}s}{\rm d}s,$$

is nonsingular for any $t>0$. As $R_h(s)$ is an $\mathbf{S}_m^{(r)}(h)$ function, there exists an $\varepsilon>0$ and an interval $[h_1,h_2]\subset[h,2h]$ with $h_1<h_2$ such that $R_h(s)\geq\varepsilon I_m>0,\forall s\in[h_1,h_2]$. Therefore, we obtain

$$\begin{aligned}W_{\rm c}(A,h)&\geq\int_{h_1}^{h_2}{\rm e}^{-As}BR_h(s)B^{\rm T}{\rm e}^{-A^{\rm T}s}{\rm d}s\\&\geq\varepsilon\int_{h_1}^{h_2}{\rm e}^{-As}BB^{\rm T}{\rm e}^{-A^{\rm T}s}{\rm d}s\\&=\varepsilon{\rm e}^{-Ah_1}G_{\rm c}\left(A,B,h_2-h_1\right){\rm e}^{-A^{\rm T}h_1}\\&>0.\end{aligned}$$

Finally we prove the assertions on fixed-time stability. If matrix W is chosen as (32), then we have $\Delta_K(h)=0$. Then by (18) we have $\chi(s)=0,\forall s\geq 2h$, which corresponds to

$$x(t)=0,\ \forall t\geq\tau+2h=T_\tau. \tag{36}$$

Hence, Problem 1 is solved. Since $K_{(A,h)}(t)=0,\forall t\in[2h,3h]$, we have from (28) and (36) that $u(t)=0,\forall t\in[2h,3h]$. For $t\geq 3h$, we have $x(t-(h-\tau))=0$, which, in view of (28), implies $u(t)=0$.

With a different choice of W it might still be possible that $\Delta_K^\nu=0$ for some $\nu>1$ but $\Delta_K^{\nu-1}\neq 0$. However, the latter implies that the convergence time must be strictly larger than $2(\nu-1)h+\tau>T_\tau$. The proof is finished. ■

In the following, we make some explanations on Theorem 1.

Remark 2: Condition (32) can be satisfied if and only if (A,B) is controllable. In fact, the controllability of (A,B) is necessary and sufficient for Problem 1 to be solvable. On the one hand, if (A,B) is controllable, according to Theorem 1, $W_{\rm c}(A,\tau)$ is nonsingular and condition (32) can be satisfied, resulting a solution to Problem 1 as given in (28). On the other hand, if Problem 1 is solvable, then (A,B) must be controllable since otherwise the uncontrollable mode of A either diverges to infinity or converges to zero asymptotically.

Remark 3: Theorem 1 shows that under conditions (29) and (30), linear periodic controllers (28) of a wide range of gains can be employed to stabilize asymptotically system (1). On the other hand, the requirement for fixed-time stability is much more stringent; in fact, condition (32) indicates that such a controller is unique whenever the function $R_h(t)$ is selected. As a further illustration, consider a Hurwitz matrix A. It is evident from (31) that exponentially stability can be maintained if $W=0$, or equivalently $K_{(A,h)}(t)=0$. In other words, no control is required to stabilize asymptotically a stable system. This, of course, is consistent with one's intuition.

Remark 4: The PDF (28) is robust to parameter perturbations in the sense that, if (32) is not satisfied accurately due to small perturbations of the system parameter (A,B,τ), say $\left\|W-W_{\rm c}^{-1}(A,h)\right\|\leq\varepsilon$, then exponential stability of the closed-loop system can still be guaranteed provided $\varepsilon<1/\left\|{\rm e}^{2hA}W_{\rm c}(A,h)\right\|$ since

$$\rho\left({\rm e}^{2hA}\left(I_n-W_{\rm c}(A,h)W\right)\right)\leq\left\|{\rm e}^{2hA}W_{\rm c}(A,h)\right\|\varepsilon<1.$$

In particular, when the perturbation is small enough, $\rho(\Delta_K(h))$ can still be quite small (close to 0), making the closed-loop system behave like a system that is fixed-time stable.

Remark 5: To examine properties of the gain $K_{(A,h)}(t)$ corresponding to FTS as a function of h, we consider a scalar system where A is a scalar and $B=1$. Let $R_h(t)\in\mathbf{S}_1^{(1)}(h)$ be chosen as

$$R_h(t)=\left\{\begin{array}{ll}0, & t\in[0,h],\\ \sin^2\left(\frac{\pi t}{h}\right), & t\in[h,2h].\end{array}\right.$$

According to (27) and (32), the $2h$-periodic feedback gain for achieving FTS is found as

$$K_{(A,h)}\left(t\right)=\left\{\begin{array}{ll}\frac{4A\left(A^2h^2+\pi^2\right)}{\pi^2\left({\rm e}^{2Ah}-1\right)}\sin^2\left(\frac{\pi t}{h}\right){\rm e}^{A(5h-2t)}, & A\neq 0,\\ \frac{2}{h}\sin^2\left(\frac{\pi t}{h}\right), & A=0,\end{array}\right.$$

where $t \in [h, 2h]$. It is interesting to notice that $K_{(A,h)}(t) \geq 0$ for all $t \geq 0$ even if $A < 0$, namely, *negative feedback* is necessary for achieving FTS even if the open-loop system is exponentially stable. Note that

$$K_{(A,h)}\left(\frac{3h}{2}\right) = \begin{cases} \frac{4A\mathrm{e}^{2Ah}\left(A^2h^2+\pi^2\right)}{\pi^2\left(\mathrm{e}^{2Ah}-1\right)}, & A \neq 0, \\ \frac{2}{h}, & A = 0, \end{cases}$$

from which it follows that

$$\lim_{h\downarrow 0}\sup_{t\geq 0}\left\{\left|K_{(A,h)}(t)\right|\right\} \geq \lim_{h\downarrow 0}\left|K_{(A,h)}\left(\frac{3h}{2}\right)\right| = \infty.$$

Therefore, the feedback gain $K_{(A,h)}(t)$ will approach to infinity as h approaches to zero (under the condition that $\tau = 0$). This is also intuitively clear, since to achieve FTS with the convergence time $T = 3h$ approaching to zero, an infinitely high gain feedback is necessary. Note however that this high gain is different from those in [18], [36], [41], [44], [47], [48] where the gains become unbounded.

Remark 6: In case h in (30) satisfies $h > \tau$, induced by $T_\tau > 3\tau$, then an additional input delay $h - \tau$ is used in such a way that the (actual) convergence time equals T_τ. If h would be alternatively chosen equal to τ, then the convergence time would be reduced to 3τ, at the price of an increased control effort, observed in simulations (e.g., Section VI). Hence, the choice of h satisfying (30) induces a trade-off between the convergence time and control efforts.

Remark 7: As noticed on page 563 of [32], "the use of a discrete delayed feedback control is inadequate to solve the finite-time stabilization problem". However, Theorem 1 shows that discrete delayed feedback control is indeed adequate to solve even the FTS problem if the feedback gain is periodic (and well designed).

The design of the periodic gain $K_{(A,h)}(t)$ (in the nonzero interval, say, $t \in (h, 2h)$) has been motivated by the generalized sampled-data hold functions that were frequently used in the sampled-data control [9], [21]. The merit of the design is the using of a time-varying matrix function instead of a constant matrix as used in the act-and-wait control (22). However, this will need the integration in the expression for $W_{\mathrm{c}}(A, h)$, which may be difficult to compute. We present an alternative computation method by solving a linear differential equation *off-line*, as in the following lemma whose proof is given in Appendix A3.

Lemma 4: Let $S(\sigma), \sigma \in [0, h]$ solve the following linear Lyapunov differential equation

$$\dot{S}(\sigma) = AS(\sigma) + S(\sigma)A^{\mathrm{T}} + BR_h(h+\sigma)B^{\mathrm{T}},\ S(0) = 0. \quad (37)$$

Then the feedback gain in (27) can be represented as

$$K_{(A,h)}(t) = R_h(t)B^{\mathrm{T}}\mathrm{e}^{A^{\mathrm{T}}(2h-t)}S^{-1}(h)\mathrm{e}^{A(3h-t)}. \quad (38)$$

B. Solution to Problem 2 by State Feedback

Let $K_{(A_{\mathrm{c}},h)}(t) : \mathbf{R} \to \mathbf{R}^{m\times n}$ be a $2h$-periodic function defined by

$$K_{(A_{\mathrm{c}},h)}(t) = R_h(t)B^{\mathrm{T}}\mathrm{e}^{-A_{\mathrm{c}}^{\mathrm{T}}t}W\mathrm{e}^{A_{\mathrm{c}}(h-t)}, \quad (39)$$

where $A_{\mathrm{c}} = A - BK_0$ with $K_0 \in \mathbf{R}^{m\times n}$ being a given constant matrix, $W \in \mathbf{R}^{n\times n}$ is a constant matrix to be determined, $R_h(t)$ is an $\mathbf{S}_m^{(r)}(h)$ function for some integer $r \geq 1$, and $h > 0$ is a given constant. Consider the PDF

$$u(t) = -K_0w(t) - K_{(A_{\mathrm{c}},h)}(t-\tau)\,w(t-h)\,, \quad (40)$$

which is clearly well defined for all $t \geq h$ since $w(t)$ defined in (3) is accessible for all $t \geq 0$. For $t \in [0, h)$, we may simply set $w(t-h) = 0$ in (40) (this is however not necessary). Different from (28), the PDF (40) contains an additional instantaneous feedback $-K_0w(t)$ which changes the system matrix from A to $A - BK_0 = A_{\mathrm{c}}$.

We then can state the following result, which is similar to Theorem 1, regarding a solution to Problem 2.

Lemma 5: Let $T > 0$ be a prescribed number and $h = T/2$. Then the closed-loop system consisting of (6) and (40) is exponentially stable if and only if

$$\rho\left(\mathrm{e}^{2hA_{\mathrm{c}}}\left(I_n - W_{\mathrm{c}}(A_{\mathrm{c}}, h)W\right)\right) < 1. \quad (41)$$

Moreover, the PDF (40) solves Problem 2 if and only if

$$W = W_{\mathrm{c}}^{-1}(A_{\mathrm{c}}, h). \quad (42)$$

In this case, the PDF (40) satisfies

$$u(t) = 0,\ \forall t \geq \tau + 2h = \tau + T. \quad (43)$$

Proof. For simplicity, we denote $K_{\mathrm{c}}(t) = K_{(A_{\mathrm{c}},h)}(t)$. If $h < \tau$, then, for all $t \geq \tau$, the closed-loop system is given by

$$\begin{aligned}\dot{w}(t) &= (A - BK_0)\,w(t) - BK_{\mathrm{c}}(t-\tau)\,w(t-h) \\ &= A_{\mathrm{c}}w(t) - BK_{\mathrm{c}}(t-\tau)\,w(t-h)\,,\ \forall t \geq \tau.\end{aligned} \quad (44)$$

If $h \geq \tau$, then, for any $t \in [\tau, h)$, we have $t-\tau \in [0, h-\tau) \subset [0, h]$, which implies $K_{\mathrm{c}}(t-\tau) = 0$. Thus the closed-loop system can also be written as (44) which is exactly in the form of (33) where (A, K) is replaced by $(A_{\mathrm{c}}, K_{\mathrm{c}})$. Similarly to (the proof of) Theorem 1, the monodromy matrix for system (44) can be computed as (see (35))

$$\Delta_{K_{\mathrm{c}}}(h) = \mathrm{e}^{2A_{\mathrm{c}}h}\left(I_n - W_{\mathrm{c}}(A_{\mathrm{c}}, h)W\right).$$

Hence system (44) is exponentially stable if and only if (41) is satisfied, and is fixed-time stable if and only if $\rho\left(\Delta_{K_{\mathrm{c}}}(h)\right) = 0$. Similarly to the proof of Theorem 1, we can show that this is further equivalent to (42) as $h = T/2$. Moreover, it holds that (see (36), but notice that the parameter h here is different from the one in Theorem 1)

$$w(t) = 0,\ \forall t \geq \tau + 2h = \tau + T. \quad (45)$$

As $K_{\mathrm{c}}(t-\tau) = 0, \forall t \in [\tau + 2h, \tau + 3h]$, we know from (45) that (43) holds true. The proof is finished. ■

Based on Lemmas 3 and 5, we have immediately the following result regarding a new solution to Problem 1.

Theorem 2: Let T_τ be a constant satisfying

$$T_\tau > 2\tau, \quad (46)$$

and h be chosen as

$$h = \frac{1}{2}(T_\tau - 2\tau). \quad (47)$$

Let $(K_0, K_{(A_{\mathrm{c}},h)}(t))$ be the same as that in Lemma 5 where $T = 2h$. Then the closed-loop system consisting of (1) and (40) is exponentially stable if and only if (41) is satisfied. Moreover, the PDF (40) solves Problem 2 if and only if W satisfies (42), which is possible if and only if (A, B) is controllable. In this case, the PDF (40) satisfies (43).

Proof. We only prove the fixed-time stability of the closed-loop system. By Lemma 5 we know that the state $w(t)$ satisfies (45) if and only if (42) is satisfied. Since (45) is in the form of (9), it follows from Lemma 3 that the same controller solves Problem 1 with $T_\tau = 2\tau + T = 2\tau + 2h$, which is (47). Finally, the condition (42) can be satisfied if and only if (A_{c}, B) is controllable, or, equivalently, (A, B) is controllable. The proof is finished. ■

Remark 8: Combining Lemma 1 and Theorem 2 shows that condition (46) is almost necessary and sufficient for the solvability of Problem 2, as a necessary condition is given by $T_\tau \geq 2\tau$.

Remark 9: Condition (29) is stronger than the necessary and sufficient condition (46). This is reasonable since only the current (delayed) state was used for feedback in the PDF (28).

The periodic gain $K_{(A_{\mathrm{c}},h)}(t)$ in (39) depends on the closed-loop system matrix A_{c}. We next establish a method where only the open-loop system matrix A is involved. Consider the parametric Lyapunov equation (PLE)

$$A^{\mathrm{T}}P+PA-PBB^{\mathrm{T}}P=-\gamma P, \tag{48}$$

which has a unique positive definite solution if and only if [45]

$$\gamma>-2\min_{i=1,2,\ldots,n}\{\mathrm{Re}\{\lambda_i(A)\}\}. \tag{49}$$

Let the instantaneous feedback gain K_0 be chosen as

$$K_0=B^{\mathrm{T}}P(\gamma). \tag{50}$$

Then we have the following result, whose proof is also given in Appendix A3.

Lemma 6: Let γ be a constant satisfying (49), $P=P(\gamma)>0$ be the unique positive definite solution to the PLE (48), K_0 be designed as (50) and $S_\gamma(\sigma),\sigma\in[0,h]$ solve the following linear Lyapunov differential equation

$$\begin{aligned}\dot{S}_\gamma(\sigma)=&-(A+\gamma I_n)^{\mathrm{T}}S_\gamma(\sigma)-S_\gamma(\sigma)(A+\gamma I_n)\\&+PBR_h(h+\sigma)B^{\mathrm{T}}P,\ S_\gamma(0)=0.\end{aligned} \tag{51}$$

Then the $2h$-periodic function $K_{\mathrm{c}}(t)=K_{(A_{\mathrm{c}},h)}(t)$ defined in (39) takes the form

$$K_{\mathrm{c}}(t)=\mathrm{e}^{\gamma(2t-5h)}R_h(t)B^{\mathrm{T}}P\mathrm{e}^{A(t-2h)}S_\gamma^{-1}(\sigma)\mathrm{e}^{A^{\mathrm{T}}(t-3h)}P. \tag{52}$$

Similarly to (37), the linear Lyapunov differential equation (51) can be solved *off-line.*

C. A Comparison of Theorems 1 and 2

It is clear to see that both Theorems 1 and 2 can be used to solve Problem 1 if $T_\tau\geq 3\tau$. Thus it is expected to carry out a comparison of these two approaches. To this end, we give the following lemma whose proof is moved to Appendix A4.

Lemma 7: Suppose T_τ satisfies (29) and h satisfies (30). Then the closed-loop system consisting of (1) and (28) can be written as

$$\dot{x}=\begin{cases}Ax(t)+B\upsilon(t-\tau),\ t\in[0,\tau),\\Ax(t),\ t\in[\tau,2\tau),\\Ax(t)-BK(t-\tau)x(t-h),t\in[2\tau,3\tau),\end{cases} \tag{53}$$

where $K(t-\tau)=K_{(A,h)}(t-\tau)$. Suppose that T_τ satisfies (46) and h satisfies (47). Then the closed-loop system consisting of (1) and (40) can be written as

$$\dot{x}=\begin{cases}Ax(t)+B\upsilon(t-\tau),\ t\in[0,\tau),\\A_{\mathrm{c}}x(t),\ t\in[2\tau,2\tau+h),\\A_{\mathrm{c}}x(t)-BK_{\mathrm{c}}x(t-h),t\in[2\tau+h,2\tau+2h),\end{cases} \tag{54}$$

and for $t\in[\tau,2\tau)$,

$$\dot{x}=\begin{cases}A_{\mathrm{c}}x(t)-BK_0\delta(t-\tau),t\in[\tau,\tau+\eta),\\A_{\mathrm{c}}x(t)-BK_{\mathrm{c}}x(t-h)-\delta_1(t),t\in[\tau+\eta,2\tau),\end{cases} \tag{55}$$

where $\eta=\min\{\tau,h\},A_{\mathrm{c}}=A-BK_0,K_{\mathrm{c}}=K_{(A_{\mathrm{c}},h)}(t-2\tau)$, and

$$\begin{aligned}\delta(t)&=-\int_{t-\tau}^{0}\mathrm{e}^{A(t-\theta)}B\Delta\upsilon(\theta)\mathrm{d}\theta,t\in[0,\tau),\\\delta_1(t)&=B(K_0\delta(t-\tau)+K_{\mathrm{c}}\delta(t-\tau-h)),t\in[\tau+\eta,2\tau).\end{aligned}$$

Based on the above lemma, we give a comparison of Theorems 1 and 2 by assuming that A is not Hurwitz but $A-BK_0$ is Hurwitz.

- For Theorem 1, it follows from (53) that the closed-loop system operates in "open-loop" in the first $\tau+h\geq 2\tau$ seconds, and then converges to zero in the next h seconds. Since A is not Hurwitz, the state will become large in the first $\tau+h\geq 2\tau$ seconds, which in turn implies that the control should be large enough so that the state can be driven to zero in the next h seconds.
- For Theorem 2, it follows from (54) that the closed-loop system also operates in "open-loop" in the first τ seconds. However, for $t\in[\tau,2\tau)$, since A_{c} is Hurwitz and the third equation in (54) is exactly in the form of (44) and is thus exponentially stable (by Lemma 5), the closed-loop system (55) (with a bounded external inputs $BK_0\delta(t-\tau)$ and $\delta_1(t)$) is exponentially stable, which is different from (53). Thus the state will converge on the time interval $[\tau,2\tau+h)$. As a result, it requires less control effort to drive the state to zero in the next h seconds.
- In case $T_\tau\in(2\tau,3\tau)$, Theorem 2 is applicable while Theorem 1 is not. Of course, in this case, as h is very small, we may have (see Remark 5) $\left\|W_{\mathrm{c}}^{-1}(A_{\mathrm{c}},h)\right\|\overset{h\to 0}{\to}\infty$ and $\left\|K_{(A_{\mathrm{c}},h)}\right\|\overset{h\to 0}{\to}\infty$, which implies that the controller will be rather aggressive.
- Controller (40) in Theorem 2 involves the memory term

$$u_1(t)=\int_t^{t+\tau}\mathrm{e}^{A(t+\tau-s)}Bu(s-\tau)\mathrm{d}s, \tag{56}$$

that may be expensive/unsafe in the implementation [31], [43], [45]. In contrast, controller (28) in Theorem 1 *does not* involve the memory term $u_1(t)$ and is thus easy to be implemented.

To summarize, the controller in Theorem 2 might be better than the controller in Theorem 1 in terms of control performance/ability, but the price to pay is that the former involves distributed delays that are difficult/expensive in the implementation.

Remark 10: From (54) we can see that the periodic term $K_{(A_{\mathrm{c}},h)}(t-\tau)w(t-h)$ in the controller (40) only works with $t\in[2\tau+h,2\tau+2h)$. Thus we may drop out such a periodic term after $t=2\tau+2h$. Such an action is safe (say, will not lead to instability) if A_{c} is Hurwitz, since the closed-loop system becomes $\dot{x}(t)=A_{\mathrm{c}}x(t),\forall t\geq 2\tau+2h$ with $x(2\tau+2h)=0$.

IV. Observer-Based Periodic Delayed Feedback

In this section, based on the results in the previous one, we will consider observer-based PDF, by assuming that only an output is available for feedback. For easy reference, we rewrite system (1) as

$$\begin{cases}\dot{x}(t)=Ax(t)+Bu(t-\tau),\\y(t)=Cx(t),\end{cases} \tag{57}$$

where $(A,C)\in(\mathbf{R}^{n\times n},\mathbf{R}^{p\times n})$ is observable. Observers whose states converge to some expected signals related to system (1) in prescribed finite time will be designed first, and then the observed signals are used for feedback. We will consider three cases:

1) A state observer is constructed to observe $x(t)$ and then the method in Theorem 1, where $x(t)$ is replaced by the observed state $\widehat{x}(t)$, is adopted (Section IV-A).
2) A state observer is constructed to observe $x(t)$ and then the method in Theorem 2 is adopted by replacing $x(t)$ in $w(t)$ with the observed state $\widehat{x}(t)$ (Section IV-B).
3) A state observer is constructed to observe the future state $x(t+\tau)$ and then the method in Theorem 2, where the predicted state $w(t)$ is replaced by the observed future state $\widehat{x}(t+\tau)$, is adopted (Section IV-C).

A. State-Observer-Based Periodic Delayed Feedback

As mentioned above, we will first design a state observer to observe the state of system (57) at prescribed finite time. This problem, of cause, is the "dual" of Problem 1.

Let $L_0 \in \mathbf{R}^{n\times p}$ be a given constant matrix, $l > 0$ be a constant, $H_l(t)$ be an $\mathbf{S}_p^{(r)}(l)$ function for some integer $r \geq 1$, and $L_{(A_\mathrm{o},l)}(t) : \mathbf{R} \to \mathbf{R}^{n\times p}$ be a $2l$-periodic function defined by

$$L_{(A_\mathrm{o},l)}(t) = \mathrm{e}^{A_\mathrm{o}t} M \mathrm{e}^{A_\mathrm{o}^\mathrm{T}(t-l)} C^\mathrm{T} H_l(t), \tag{58}$$

where $A_\mathrm{o} = A - L_0 C$ and $M \in \mathbf{R}^{n\times n}$ is a matrix to be determined. A state observer is constructed as follows:

$$\begin{aligned}\dot{\widehat{x}}(t) =& A\widehat{x}(t) + B\widetilde{u}(t-\tau) + L_0\left(y(t) - C\widehat{x}(t)\right) \\ &+ L_{(A_\mathrm{o},l)}(t-\tau)\left(\widetilde{y}(t-l) - C\widehat{x}(t-l)\right),\end{aligned} \tag{59}$$

where

$$\widetilde{u}(t) = \left\{\begin{array}{l} u(t),\ \forall t \geq 0 \\ \widetilde{v}(t),\ \forall t \in [-\tau, 0), \end{array}\right. \quad \widetilde{y}(t) = \left\{\begin{array}{l} y(t),\ \forall t \geq 0 \\ \widetilde{\varphi}(t),\ \forall t \in [-l, 0), \end{array}\right.$$

in which $\widetilde{v}(t)$ and $\widetilde{\varphi}(t)$ are arbitrarily (piecewise continuous) functions. The initial condition for $\widehat{x}(t)$ is set as $\widehat{x}(s) = \widetilde{\psi}(s), s \in [-l, 0]$.

Denote the observer error as $e(t) = x(t) - \widehat{x}(t), t \geq 0$. It follows from (57) and (59) that

$$\begin{aligned}\dot{e}(t) =& \left(A - L_0 C\right) e(t) + B\left(u(t-\tau) - \widetilde{u}(t-\tau)\right) \\ &- L_{(A_\mathrm{o},l)}(t-\tau)\left(\widetilde{y}(t-l) - C\widehat{x}(t-l)\right).\end{aligned} \tag{60}$$

Denote also

$$W_\mathrm{o}(A_\mathrm{o}, l) = \int_0^l \mathrm{e}^{A_\mathrm{o}^\mathrm{T} s} C^\mathrm{T} H_l\left(s + l\right) C \mathrm{e}^{A_\mathrm{o} s} \mathrm{d}s,$$

which is invertible if and only if (A, C) is observable (similar to the discussion in Remark 2, we know that the observability of (A, C) is necessary for solving the FTS problem). Then we can state the following result whose proof is given in Appendix A5.

Lemma 8: The observer-error system (60) is exponentially stable if and only if

$$\rho\left(\mathrm{e}^{2lA_\mathrm{o}}\left(I_n - M W_\mathrm{o}(A_\mathrm{o}, l)\right)\right) < 1.$$

Moreover, $e(t) = 0, \forall t \geq \tau + 2l$, if and only if

$$M = W_\mathrm{o}^{-1}(A_\mathrm{o}, l). \tag{61}$$

We point out that the prescribed finite-time observer (59) is completely different from the one in [12] where two normal linear observers are needed, and is also different from the prescribed finite-time observer in [18] which exhibits a high-gain nature as [41].

The prescribed finite-time observer (59) and the finite-time stabilizing controller (28) can be combined together to achieve both FTS and exponential stabilization of the input-delayed linear system (1) by output feedback. For simplicity, we only consider the FTS problem.

Consider the observer-based PDF

$$u(t) = -K_{(A,h)}\left(t - 2l\right) \widehat{x}\left(t - (h - \tau)\right), \tag{62}$$

where $\widehat{x}(t)$ is the state of the observer (59) whose parameters satisfy (58) and (61), $K_{(A,h)}(t)$ satisfies (27) and (32), and (h, l) is a pair of constants to be determined (in this controller we will set simply $\widehat{x}(s) = 0$ for any $s < 0$).

Theorem 3: Let T_τ be a constant satisfying

$$T_\tau > 3\tau, \tag{63}$$

and (h, l) be a pair of positive constants satisfying

$$0 < l \leq \frac{1}{2}\left(T_\tau - 3\tau\right), \quad h = \frac{1}{2}\left(T_\tau - (\tau + 2l)\right) \geq \tau. \tag{64}$$

Then the closed-loop system consisting of (1) and (62) is T_τ-fixed-time stable, namely, (62) solves Problem 1.

Proof. For simplicity, we denote $K(t) = K_{(A,h)}(t)$. The closed-loop system consisting of (1) and (62) can be written as, for $t \geq h$,

$$\begin{aligned}\dot{x}(t) =& Ax(t) - BK\left(t - (\tau + 2l)\right) \widehat{x}\left(t - h\right) \\ =& Ax(t) - BK\left(t - (\tau + 2l)\right) x(t - h) \\ &+ BK\left(t - (\tau + 2l)\right) e\left(t - h\right),\end{aligned} \tag{65}$$

where $e(t)$ satisfies (60). By Lemma 8 we have $e(t) = 0, \forall t \geq \tau + 2l$. Thus $e(t - h) = 0, \forall t \geq \tau + 2l + h$. As $K(t - (\tau + 2l)) = 0, t \in [\tau + 2l, \tau + 2l + h]$, system (65) can be further rewritten as

$$\dot{x}(t) = Ax(t) - BK\left(t - (\tau + 2l)\right) x(t - h),\ t \geq \tau + 2l, \tag{66}$$

which is exactly in the form of (33) with τ being replaced by $\tau + 2l$. Thus, it follows from Theorem 1 that the state of (66) satisfies (see (36)) $x(t) = 0, \forall t \geq \tau + 2l + 2h = T_\tau$, where we have used (64). This completes the proof. ■

B. State-Observer and Predictor-Based Periodic Delayed Feedback

In this subsection, we will use the observed state $\widehat{x}(t)$ to construct the predicted state $w(t)$, denoted $\widehat{w}(t)$, and then use $\widehat{w}(t)$ to establish a solution to Problem 1. Similarly to Theorem 3, we only consider the FTS problem here for simplicity.

According to (3), we denote

$$\widehat{w}(t) = \mathrm{e}^{A\tau}\widehat{x}(t) + \int_t^{t+\tau} \mathrm{e}^{A(t+\tau-s)} B\overline{u}\left(s - \tau\right) \mathrm{d}s,\ t \geq 0, \tag{67}$$

where $\widehat{x}(t)$ is the state of the observer (59) and $\overline{u}(t)$ is defined in (4). According to (40), the observer-based PDF can be designed as

$$u(t) = -K_0\widehat{w}(t) - K_{(A_\mathrm{c},h)}\left(t - (\tau + 2l)\right) \widehat{w}\left(t - h\right), \tag{68}$$

where $(K_0, K_{(A_\mathrm{c},h)}(t))$ takes the same form as that in Lemma 5 and (h, l) is a pair of constants to be determined (in (68) we will set simply $\widehat{w}(s) = 0$ for any $s < 0$). Then we have the following result.

Theorem 4: Let T_τ be a constant satisfying (46) and (h, l) be a pair of positive constants satisfying

$$0 < l < \frac{1}{2}(T_\tau - 2\tau), \quad h = \frac{1}{2}(T_\tau - (2\tau + 2l)). \tag{69}$$

Then the closed-loop system consisting of (1) and (68) is T_τ-fixed-time stable, namely, (68) solves Problem 1.

Proof. For simplicity, we denote $K_\mathrm{c}(t) = K_{(A_\mathrm{c},h)}(t)$. By $e(t) = x(t) - \widehat{x}(t)$ and the definition of $w(t)$ in (3) we have $\widehat{w}(t) = w(t) - \mathrm{e}^{A\tau}e(t), \forall t \geq 0$, by which the controller (68) can be rewritten as

$$\begin{aligned}u(t) =& -K_0 w(t) - K_\mathrm{c}\left(t - (\tau + 2l)\right) w\left(t - h\right) \\ &+ K_0\mathrm{e}^{A\tau}e(t) + K_\mathrm{c}\left(t - (\tau + 2l)\right) \mathrm{e}^{A\tau} e\left(t - h\right),\end{aligned}$$

where we set $e(s) = 0, \forall s < 0$. By (6), the closed-loop system is given by, for all $t \geq \tau$,

$$\begin{aligned}\dot{w}(t) =& A_\mathrm{c} w(t) - BK_\mathrm{c}\left(t - (\tau + 2l)\right) w\left(t - h\right) + BK_0\mathrm{e}^{A\tau}e(t) \\ &+ BK_\mathrm{c}\left(t - (\tau + 2l)\right)\mathrm{e}^{A\tau} e\left(t - h\right),\ \forall t \geq \tau.\end{aligned}$$

By Lemma 8 we have $e(t) = 0, \forall t \geq \tau + 2l$. Thus $e(t - h) = 0, \forall t \geq \tau + 2l + h$. For $t \in [\tau + 2l, \tau + 2l + h]$, we have $K_\mathrm{c}(t - (\tau + 2l)) = 0$, which implies that the above system can be simplified as

$$\dot{w}(t) = A_\mathrm{c} w(t) - BK_\mathrm{c}\left(t - (\tau + 2l)\right) w\left(t - h\right),\ \forall t \geq \tau + 2l. \tag{70}$$

Similarly to (66), this system is also in the form of (33) where (A, K, τ) is replaced by $(A_\mathrm{c}, K_\mathrm{c}, \tau + 2l)$. Thus, it follows from Theorem 1 that the state of (70) satisfies (see (36))

$$w(t) = 0, \forall t \geq \tau + 2l + 2h. \tag{71}$$

As (71) is in the form of (9), we have from Lemma 3 that $x(t) = 0, \forall t \geq (\tau + 2l + 2h) + \tau = T_\tau$, where we have used (69). ■

C. Future-State Observer-Based Periodic Delayed Feedback

In our previous two schemes, the observer (59) is constructed to observe the current state $x(t)$. In this subsection, we provide a new observer which observes the future state $x(t+\tau)$ directly [3]. This approach was also named as "anticipating synchronization" (see, for example, [16]).

Let $L_{(A,\tau)}(t):\mathbf{R}\to\mathbf{R}^{n\times p}$ be a 2τ-periodic function defined as in (58). We construct the following observer

$$\dot{z}(t)=Az(t)+Bu(t)-L_{(A,\tau)}(t)\left(Cz(t-\tau)-y(t)\right), \tag{72}$$

where $t\geq 0$ and $z(s)=\zeta(s), s\in[-\tau,0]$. Since the observer in (72) aims to observe the future state $x(t+\tau)$, we denote the observer-error as $d(t)=x(t+\tau)-z(t), t\geq 0$. It then follows from (57) and (72) that, for all $t\geq 0$,

$$\begin{aligned}\dot{d}(t)=&Ax(t+\tau)+Bu(t)-(Az(t)+Bu(t))\\&-L_{(A,\tau)}(t)\left(Cx(t)-Cz(t-\tau)\right)\\=&Ad(t)-L_{(A,\tau)}(t)Cd\left(t-\tau\right),\end{aligned} \tag{73}$$

which is in the dual form of (33) (or exactly in the form of (118) in Appendix A5 where τ is replaced by 0 and l is replaced by τ). Thus the following result can be obtained immediately from Lemma 8.

Corollary 1: The observer-error system (73) is exponentially stable if and only if

$$\rho\left(\mathrm{e}^{2\tau A}\left(I_n-MW_{\mathrm{o}}\left(A,\tau\right)\right)\right)<1.$$

Moreover, $d(t)=0,\forall t\geq 2\tau$, if and only if

$$M=W_{\mathrm{o}}^{-1}\left(A,\tau\right). \tag{74}$$

We now consider the following observer-based PDF

$$u(t)=-K_0z(t)-K_{(A_{\mathrm{c}},h)}\left(t-2\tau\right)z\left(t-h\right)\ \ t\geq 0, \tag{75}$$

where K_0 and $K_{(A_{\mathrm{c}},h)}(t)$ are the same as that in (40) and (39), and $h>0$ is some constant. The initial condition $z(s), s\leq 0$ can be arbitrarily chosen.

Theorem 5: Consider the observer (72) where M satisfies (74). Let T_τ satisfy (63) and h be chosen as

$$h=\frac{1}{2}(T_\tau-3\tau).$$

Then the closed-loop system consisting of (1) and (75) is exponentially stable if and only if (41) is satisfied. Moreover, the PDF (75) solves Problem 1 if and only if W is chosen as (42).

Proof. We only prove the fixed-time stability of the closed-loop system. For $t\in[2\tau,2\tau+h]$, we have $t-2\tau\in[0,h]$ and thus

$$\begin{aligned}u(t)&=-K_0z(t)=-K_0x\left(t+\tau\right)+K_0d(t)=-K_0x\left(t+\tau\right)\\&=-K_0x(t+\tau)-K_{(A_{\mathrm{c}},h)}\left(t-2\tau\right)x\left(t+\tau-h\right).\end{aligned}$$

For $t\geq 2\tau+h$, we have $t-h\geq 2\tau>0$ and thus

$$\begin{aligned}u(t)=&-K_0z(t)-K_{(A_{\mathrm{c}},h)}\left(t-2\tau\right)z\left(t-h\right)\\=&-K_0x(t+\tau)-K_{(A_{\mathrm{c}},h)}\left(t-2\tau\right)x\left(t+\tau-h\right)\\&+K_0d(t)+K_{(A_{\mathrm{c}},h)}\left(t-2\tau\right)d\left(t-h\right)\\=&-K_0x(t+\tau)-K_{(A_{\mathrm{c}},h)}\left(t-2\tau\right)x\left(t+\tau-h\right).\end{aligned}$$

As a result, the closed-loop system can be written as

$$\dot{x}(t)=A_{\mathrm{c}}x(t)-BK_{(A_{\mathrm{c}},h)}\left(t-3\tau\right)x\left(t-h\right),\ t\geq 3\tau,$$

which is exactly in the form of (33) where τ is replaced by 3τ and A is replaced by A_{c}. Therefore, it follows from (36) that $x(t)=0,\forall t\geq 3\tau+2h=T_\tau$ if and only if (42) is satisfied. ■

At the end of this section, for the purpose of comparison, we list the five theorems proposed in this paper so far in Table 1.

Table 1: A comparison of the proposed five methods

	T_τ	l	h
Theorem 1	$T_\tau\geq 3\tau$	NA	$\frac{1}{2}\left(T_\tau-\tau\right)$
Theorem 2	$T_\tau>2\tau$	NA	$\frac{1}{2}(T_\tau-2\tau)$
Theorem 3	$T_\tau>3\tau$	$\left(0,\frac{1}{2}\left(T_\tau-3\tau\right)\right]$	$\frac{1}{2}\left(T_\tau-(\tau+2l)\right)$
Theorem 4	$T_\tau>2\tau$	$\left(0,\frac{1}{2}(T_\tau-2\tau)\right)$	$\frac{1}{2}(T_\tau-(2\tau+2l))$
Theorem 5	$T_\tau>3\tau$	NA	$\frac{1}{2}(T_\tau-3\tau)$

V. On a Class of Linear Delay Systems with Periodic Instantaneous Feedback

In this section, we will extend the results in Section III (without predictor) to linear systems with multiple input delays, particularly, we are interested in designing the periodic *instantaneous* feedback

$$u(t)=-K(t)x(t),\ t\geq 0, \tag{76}$$

where $K(t)$ is a time-varying feedback gain. It has been clear for us that such an extension in the general case is very involved and even impossible. Thus we will consider a special case, namely,

$$\dot{x}(t)=Ax(t)+B_0u(t)+B_1u(t-\tau), \tag{77}$$

where $(A,B_0,B_1)\in(\mathbf{R}^{n\times n},\mathbf{R}^{n\times m},\mathbf{R}^{n\times m})$, and $\tau>0$ is a constant delay. In case of either $B_0=0$ or $B_1=0$, such a system reduces to either (1) or (6). Thus we assume

$$B_0\neq 0,\ B_1\neq 0. \tag{78}$$

System (77) is a special case of (10), and is controllable if and only if $(A,\mathscr{B}_1)$ is controllable [1], where $\mathscr{B}_1=B_0+\mathrm{e}^{-A\tau}B_1$.

If $B_0=0$, it follows from Theorem 1 that there exists a $K(t)$ such that system (77) is 3τ-finite-time stabilized, while if $B_1=0$, it follows from Lemma 5 that there exists a $K(t)$ such that system (77) is stabilized at any prescribed time $T>0$. In this section, we will investigate the possibility of using (76) to stabilize system (77) at finite-time under condition (78).

Assume that $K(t)$ is 2τ-periodic and takes the form

$$K(t)=\left\{\begin{array}{ll}0, & t\in[0,\tau),\\ K_0(t), & t\in[\tau,2\tau),\end{array}\right. \tag{79}$$

where $K_0(t)\in\mathbf{C}^r_{m\times n}$ for some integer $r\geq 1$. The closed-loop system consisting of (77) and (76) is given by, for all $t\geq\tau$,

$$\dot{x}(t)=\left(A-B_0K(t)\right)x(t)-B_1K(t-\tau)x(t-\tau). \tag{80}$$

Denote the delay-free part of the above system by

$$\dot{x}_0(t)=\left(A-B_0K_0(t)\right)x_0(t), \tag{81}$$

whose state transition matrix is denoted by $\Phi_0(t,s)$. Then we have the following result whose proof is similar to the proof of Theorem 1, and will be given in Appendix A6 for completeness.

Proposition 2: Let $K(t)$ be a 2τ-periodic function satisfying (79). Denote

$$\Delta_0\left(\tau\right)=\mathrm{e}^{2A\tau}\left(I_n-\int_\tau^{2\tau}\mathrm{e}^{A(\tau-\theta)}\mathscr{B}_1K_0(\theta)\Phi_0\left(\theta,\tau\right)\mathrm{d}\theta\right). \tag{82}$$

Then the closed-loop system (80) is exponentially stable if and only if

$$\rho\left(\Delta_0\left(\tau\right)\right)<1, \tag{83}$$

and is fixed-time stable if and only if

$$\rho\left(\Delta_0\left(\tau\right)\right)=0. \tag{84}$$

Moreover, if $\nu\geq 1$ is the minimal integer such that $\Delta_0^\nu\left(\tau\right)=0$, then $x(t)=0,\forall t\geq 2\nu\tau+\tau$, namely, the system is $(2\nu\tau+\tau)$-fixed-time stable in the sense of Definition 1.

The inequality (83) and equation (84) are nonlinear in terms of $K(t)$ and thus are not easy to verify. To understand this problem well, we consider the special case of scalar coefficients.

Proposition 3: Assume that $n = m = 1$ and denote

$$\kappa_+ = \frac{\mathrm{e}^{-A\tau} B_1}{\mathscr{B}_1} + \frac{\mathrm{e}^{-2A\tau} |B_0|}{|\mathscr{B}_1|}, \kappa_- = \frac{\mathrm{e}^{-A\tau} B_1}{\mathscr{B}_1} - \frac{\mathrm{e}^{-2A\tau} |B_0|}{|\mathscr{B}_1|}.$$

1) There exists a 2τ-periodic function $K(t)$ satisfying (79) such that (83) is satisfied if and only if

$$\kappa_+ > 0. \tag{85}$$

In this case, the function $K(t)$ can be chosen according to

$$\kappa_- < \exp\left(-B_0 \int_\tau^{2\tau} K_0(s)\mathrm{d}s\right) < \kappa_+. \tag{86}$$

2) There exists a 2τ-periodic function $K(t)$ satisfying (79) such that (84) is satisfied if and only if

$$\frac{\mathscr{B}_1}{B_1} = \frac{B_0 + \mathrm{e}^{-A\tau} B_1}{B_1} > 0. \tag{87}$$

In this case, the function $K_0(t)$ can be chosen according to

$$\int_\tau^{2\tau} K_0(s)\mathrm{d}s = \frac{1}{B_0}\left(A\tau + \ln\left(\frac{\mathscr{B}_1}{B_1}\right)\right). \tag{88}$$

The proof of this proposition is given in Appendix A7. It follows that the determination of the function $K(t)$ satisfying either (83) or (84) is rather involved even in the scalar case.

Remark 11: Notice that condition (87) implies (85) but the converse is not true. We point out that both (85) and (87) can be satisfied for any $\tau > 0$ if B_0 and B_1 have the same sign. Now we assume that B_0 and B_1 have opposite signs. Without loss of generality, let $B_0 < 0$ and $B_1 > 0$. Then (87) is equivalent to

$$A\tau < \ln\left(-\frac{B_1}{B_0}\right) = \ln B_1 - \ln|B_0|. \tag{89}$$

We consider three cases:

1) **Case** $A > 0$. Then (89) can be satisfied for some $\tau > 0$ if and only if $B_1 > |B_0|$ and

$$\tau < \frac{1}{A}(\ln B_1 - \ln|B_0|),\ A > 0, B_1 > |B_0|. \tag{90}$$

2) **Case** $A < 0$. Then (89) is equivalent to

$$\tau > \frac{1}{|A|}(\ln|B_0| - \ln B_1),\ A < 0. \tag{91}$$

It is interesting to notice that, provided $|B_0| > B_1$, such a system can be 3τ-finite-time stabilized only for large τ satisfying (91).

3) **Case** $A = 0$. Then (89) is satisfied if and only if $B_1 > |B_0|$. In this case, the delay τ can be arbitrarily large.

Remark 12: Under condition (87), the function $K(t)$ can be easily computed. In fact, for any integer $r \geq 0$, we choose any function $K_*(t) \in \mathbf{S}_1^{(r)}(\tau)$ such that $\int_\tau^{2\tau} K_*(s)\mathrm{d}s = k_* \neq 0$. Then a function $K(t) \in \mathbf{S}_1^{(r)}(\tau)$ satisfying (88) can be chosen as

$$K(t) = \frac{1}{k_* B_0}\left(A\tau + \ln\left(\frac{\mathscr{B}_1}{B_1}\right)\right)K_*(t). \tag{92}$$

For the purpose of comparison, we consider for system (77) the *static instantaneous* feedback

$$u(t) = -K_0 x(t), \tag{93}$$

where K_0 is a constant. The closed-loop system then reads

$$\dot{x}(t) = (A - B_0 K_0)x(t) - B_1 K_0 x(t-\tau). \tag{94}$$

Regarding the stability of system (94), we can give the following result whose proof is given in Appendix A8.

Lemma 9: Assume that $n = m = 1, A > 0$, and $B_1 = 1$ (without loss of generality). Then the closed-loop system (94)

1) is exponentially stable for all $\tau > 0$ if and only if one of the following two conditions is satisfied

$$\text{(i)}: B_0 < -1 \text{ and } K_0 < \frac{A}{B_0 + 1}, \tag{95}$$

$$\text{(ii)}: B_0 > 1 \text{ and } K_0 \geq \frac{A}{B_0 - 1}, \tag{96}$$

2) and is exponentially stable for any $\tau < \tau_* < \infty$ if and only if one of the following two conditions is satisfied

$$\text{(iii)}: B_0 \in (-1, 1] \text{ and } K_0 > \frac{A}{B_0 + 1}, \tag{97}$$

$$\text{(iv)}: B_0 > 1 \text{ and } \frac{A}{B_0 + 1} < K_0 < \frac{A}{B_0 - 1}. \tag{98}$$

In both cases (97) and (98), τ_* is given by

$$\tau^* = \frac{\arccos\left(\frac{A - B_0 K_0}{K_0}\right)}{\sqrt{K_0^2 - (A - B_0 K_0)^2}} < \frac{1 + B_0}{A}. \tag{99}$$

We now make a comparison between the periodic instantaneous feedback (76) and the static instantaneous feedback (93). Assume $A > 0, B_0 \neq 0$, and $B_1 = 1$ (without loss of generality).

1) For $B_0 > 0$, (76) is better than (93) since the former can achieve FTS for any $\tau > 0$.
2) For $B_0 \in (-1, 0)$, it follows from (90) that (76) can achieve FTS if $\tau < -\ln(-B_0)/A$, which is better than (99) since $-\ln(-B_0) > 1 + B_0, \forall B_0 \in (-1, 0)$. Thus (76) is also better than (93).
3) For $B_0 < -1$, then (93) might be better than (76) since the latter one cannot achieve FTS (but may achieve exponential stabilization).

VI. Two Numerical Examples

In this section, we provide two numerical examples to demonstrate the effectiveness of the proposed approaches. To save spaces, we only consider state feedback.

A. A Scalar Linear System

Consider a scalar linear system in the form of (77) where $n = m = 1$ and $B_1 = 1$. We will consider three cases.

- **Case 1**: $A > 0$ and $B_0 = 0$.

In this case, the system (77) reduces to system (1) where $B = B_1$. It follows from Lemma 9 that this system can be stabilized exponentially by the static instantaneous feedback (93) if and only if $K_0 > A$ and $\tau < 1/A$. Thus, for $\tau \geq 1/A$, there is no constant instantaneous feedback in the form of (93) such that the system is stabilized exponentially.

According to Theorem 1, for any $A > 0$ and $\tau > 0$, the system can be T_τ-finite-time stabilized for any $T_\tau \geq 3\tau$ by the PDF (28). For simplicity, we choose $T_\tau = 3\tau$ and thus $h = \tau$. Then

$$u(t) = -K_{(A,h)}(t)x(t),\ t \geq 0, \tag{100}$$

where $K_{(A,h)}$ is designed according to (27). If we choose an $\mathbf{S}_1^{(1)}(h)$ function $R_h(t)$ as in (26) where $a_1 = a_2 = A$, then the $2h$-periodic gain can be obtained as $K_{(A,h)}(t) = 0, t \in [0, h]$ and, for $t \in [h, 2h]$,

$$K_{(A,h)}(t) = \frac{A\mathrm{e}^{A(5h-2t)}\left(\mathrm{e}^{A(t-h)} - 1\right)^2\left(\mathrm{e}^{A(t-2h)} - 1\right)^2}{Ah - 3\mathrm{e}^{2Ah} + 4Ah\mathrm{e}^{Ah} + Ah\mathrm{e}^{2Ah} + 3}. \tag{101}$$

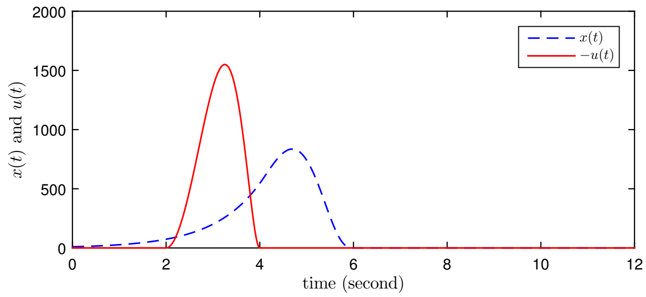

Fig. 1. The state and control signals for system (77), (100) and (101) with $A = 1, B_1 = 1, B_0 = 0$ and $\tau = 2$

In our simulation, we choose $A = 1$ and $\tau = 2$ which is larger than $1/A$. Let the initial condition be $x(0) = 10$ and $\upsilon(s) = 0, \forall s \in [-\tau, 0)$. Then the state and control signals are shown in Fig. 1, from which we clearly see that the state converges to zero at $t = 3\tau = 6$s. From this figure we can observe a quite large overshoot in $x(t)$ and $u(t)$, which, as we have mentioned in Subsection III-C (see equation (53)), is reasonable since the system operates in "open-loop" in the first $2\tau = 4$ seconds.

- **Case 2**: $A = 0$ and $B_0 = 0$.

In this case the closed-loop system consisting of (77) (or system (1)) and the constant instantaneous feedback (93) is exponentially stable if and only if $0 < K_0 < \pi/(2\tau)$ [33], and the convergence rate of the closed-loop system is maximized if and only if (see, for example, [46])

$$K_0 = K_0^* = \frac{1}{\tau \mathrm{e}}. \tag{102}$$

We also choose $T_\tau = 3\tau$. Thus, to apply the PDF (28), we should choose $h = \tau$. Let $R_h(t)$ be chosen as (26) where $a_1 = a_2 = 1$. Then the $2h$-periodic gain defined in (27) can be obtained as $K_{(A,h)}(t) = 0, t \in [0, h]$ and, for $t \in [h, 2h]$,

$$K_{(A,h)}(t) = \frac{6\left(\mathrm{e}^{t-h} - 1\right)^2 \left(\mathrm{e}^{t-2h} - 1\right)^2}{6h + \sinh(2h) - 8\sinh(h)}. \tag{103}$$

We also apply the predictor-based PDF (40) to solve the same problem. Different from the PDF (28), according to (47), we should choose $h = (T_\tau - 2\tau)/2 = \tau/2$. Let $R_h(t) \in \mathbf{S}_1^{(1)}(h)$ be the same as in (26) where $a_1 = a_2 = 1$ and $K_0 = 1$. Then the $2h$-periodic gain defined in (39) can be obtained as $K_{(A_c,h)}(t) = 0, t \in [0, h]$ and, for $t \in [h, 2h]$,

$$K_{(A_c,h)}(t) = \frac{60\mathrm{e}^{2t-h}\left(\mathrm{e}^{t-h} - 1\right)^2 \left(\mathrm{e}^{t-2h} - 1\right)^2}{\left(\mathrm{e}^h - 1\right)^5 \left(\mathrm{e}^h + 1\right)}. \tag{104}$$

For the predicted state $w(t)$, the integral (56) will be approximated by the left-hand rectangle method with the integration step $\frac{\tau}{N}$ where $N = 200$, and $\overline{\upsilon}(s) = 0, s \in [-\tau, 0)$.

In our simulation, we also let $\tau = 2$s and the initial condition be $x(0) = 10$ but $\upsilon(s) = 10, s \in [-\tau, 0)$. For purpose of comparison, the state and control signal of the closed-loop system controlled by (93) and (102), by (100) and (103), by (40) and (104), and by the distributed (predictor-based) feedback controller

$$u(t) = -K_0 w(t) = -K_0 \left(\mathrm{e}^{A\tau} + \int_0^\tau \mathrm{e}^{As} Bu(t-s)\mathrm{d}s\right), \tag{105}$$

are plotted respectively in Fig. 2, from which we can see that the predictor-based PDF (40) outperforms the PDF (28) in terms of transient performance and control effort (see the explanation in Subsection III-C). We can also observe that both of these two PDF

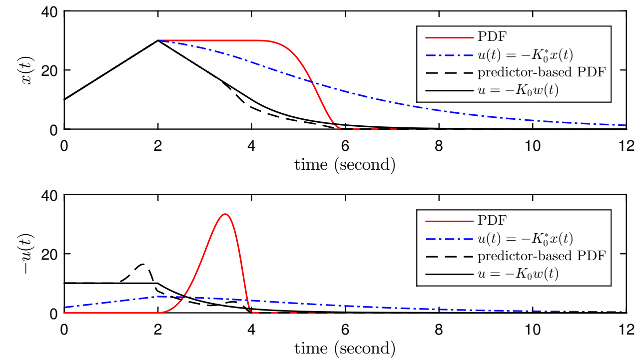

Fig. 2. The state and control signals for system (77) with $A = 0, B_1 = 1, B_0 = 0$ and $\tau = 2$ with different controllers

lead to the 6-fixed-time stability and is thus significantly better than the static instantaneous feedback (93) and the distributed feedback (105).

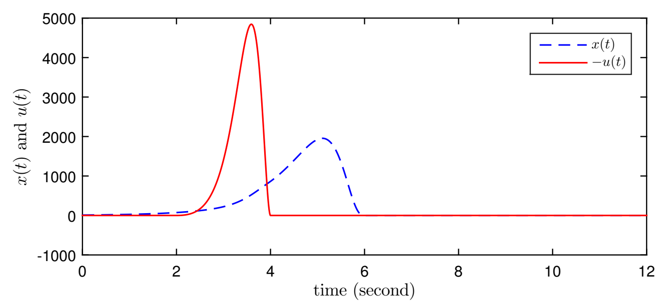

Fig. 3. The state and control signals for system (77) with $A = 1, B_1 = 1, B_0 = -\mathrm{e}^{-3}$ and $\tau = 2$

- **Case 3**: $A > 0$ and $B_0 \neq 0$.

Let $B_0 = -\mathrm{e}^{-3} \in (-1, 0)$. According to Proposition 3, it follows from (90) that this system can be 3τ-finite-time stabilized by the periodic instantaneous feedback (76) if and only if

$$\tau < \frac{1}{A}\left(\ln B_1 - \ln\left(|B_0|\right)\right) = \frac{3}{A}. \tag{106}$$

On the other hand, according to Lemma 9, it follows from (99) that this system can be exponentially stabilized by the static instantaneous feedback (93) if and only if

$$\tau < \frac{1 + B_0}{A} = \frac{1 - \mathrm{e}^{-3}}{A}. \tag{107}$$

We choose $A = 1$ and $\tau = 2$ as before. Thus (106) is fulfilled but (107) is not, namely, this system with these parameters can be 3τ-finite-time stabilized by the periodic instantaneous feedback (76) but cannot be exponentially stabilized by the static instantaneous feedback (93).

According to Remark 12, we choose $K_*(t) = R_h(t) \in \mathbf{S}_1^{(1)}(h)$ as in (26) where $a_1 = a_2 = 1$ and $h = \tau$. Then $k_* = h + \sinh(2h)/6 - 4\sinh(h)/3$ and it follows from (92) that, for $t \in [h, 2h]$,

$$K(t) = -\frac{6\mathrm{e}^3 \ln\left(1 - \mathrm{e}^{h-3}\right)\left(\mathrm{e}^{t-h} - 1\right)^2 \left(\mathrm{e}^{t-2h} - 1\right)^2}{6h + \sinh(2h) - 8\sinh(h)}.$$

For $x(0) = 10$ and $\upsilon(s) = 0, s \in [-\tau, 0)$ as before, the state and control signals of the closed-loop system are shown in Fig. 3 from which we can indeed observe the 3τ-fixed-time stability.

B. The Spacecraft Rendezvous Control System

We consider the spacecraft rendezvous control system with an input delay [8]

$$\begin{cases} \ddot{x}_1 = 2\omega_0\dot{x}_2 + \omega_0^2 J_1 - J_2\mu J_1 + u_1(t-\tau), \\ \ddot{x}_2 = -2\omega_0\dot{x}_1 + \omega_0^2 x_2 - J_2\mu x_2 + u_2(t-\tau), \\ \ddot{x}_3 = -J_2\mu x_3 + u_3(t-\tau), \end{cases} \tag{108}$$

where (x_1, x_2, x_3) is the relative distance between the target and chaser spacecraft, $J_1 = J_0 + x_1$, $J_2 = ((J_0+x_1)^2 + x_2^2 + x_3^2)^{-\frac{3}{2}}$, μ is the gravitational parameter, J_0 is the radius of the target spacecraft orbit, ω_0 is the orbit rate of the target orbit, $u = [u_1, u_2, u_3]^{\rm T}$ is the normalized acceleration vector due to thrust forces on the chaser, and τ denotes the input delay. The linearized system is in the form of (1) with $(n, m) = (6, 3)$, $x = [x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3]^{\rm T}$ and [8]

$$A = \begin{bmatrix} 0 & I_3 \\ \omega_0^2 A_{21} & 2\omega_0 A_{22} \end{bmatrix}, B = \begin{bmatrix} 0 \\ I_3 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The spacecraft rendezvous problem can be translated into the stabilization problem [8], [47]. In the absence of input delay, the finite-time/fixed-time stabilization of such a system has been considered in recent papers [47] and [48], where time-varying high-gain feedback are designed based on the PLE (48). As studied in [47], [48], to meet practical requirements we also assume that the control satisfies

$$\sup_{t\geq 0} \{|u_i(t)|\} \leq 0.1, \ i = 1, 2, 3. \tag{109}$$

We will use the proposed PDF (40) to solve the FTS problem. The instantaneous feedback gain K_0 is chosen as (50), where the parameter γ is determined according to the size of the initial condition, and is such that (109) are satisfied. Let the target spacecraft be on a geosynchronous orbit with the orbit rate $\omega = 7.2722 \times 10^{-5}$rad/s [47] and $x(0) = [-1000, 1000, 1000, 2, -2, 2]^{\rm T}$ which is the same as that in [47], [48], $\upsilon(s) = 0, s \in [-\tau, 0)$, and $\tau = 20$s. Then by simulation on the closed-loop system consisting of the original nonlinear model (108) and $u(t) = -K_0 w(t)$, in which the integral is approximated by the left-hand rectangle method with the integration step $\frac{\tau}{N}$ where $N = 200$, and $\overline{\upsilon}(s) = 0, s \in [-\tau, 0)$, it is found that the constraint (109) is satisfied if and only if $\gamma \leq 0.00806$. Thus we will chose $\gamma = 0.00806$.

To compute the periodic gain $K_{(A_c,h)}(t)$ in (39), we choose $R_h(t)$ as in (26) where $a_1 = a_2 = c\gamma$ with $c = 0.01$ and $r = 1$. Via simulation on the closed-loop system consisting of the original nonlinear model (108) and the PDF (40), it is found that the constraint (109) is satisfied if and only if $h \geq 186$. Thus we will choose $h = 186$, which, according to Theorem 2, means that the state will converge to zero at $T_\tau = 2\tau + 2h = 412$s. This is indeed the case as can be observed from Fig. 4, where the state trajectories and control signals are recorded (again the simulations are carried out on the original nonlinear model (108)).

VII. Conclusion

This paper has established a periodic delayed feedback (PDF) approach and a predictor-based PDF approach for fixed-time stabilization (FTS) of controllable linear systems with input delay. Both state feedback and observer-based output feedback have been considered. As a by-product, the FTS problem for a controllable linear system without input delay has been re-solved by the PDF. Different from the existing approaches which use either nonlinear feedback or discontinuous/non-smooth feedback, the main feature of the proposed approach is that the resulting controllers are linear

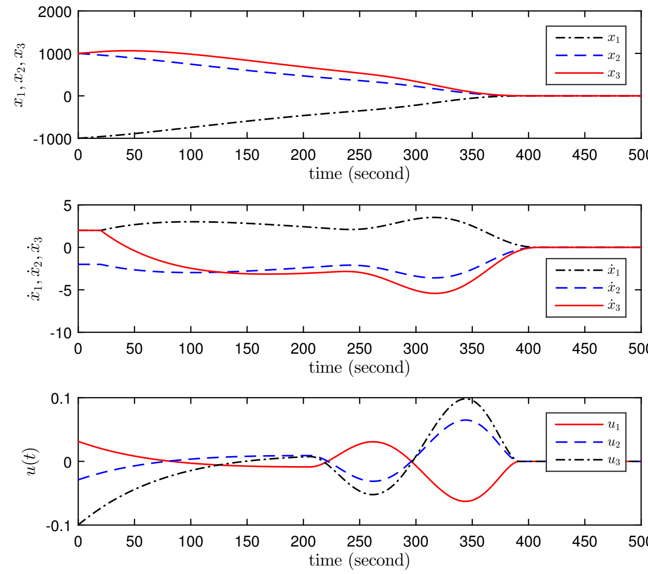

Fig. 4. The state and control signals for the spacecraft rendezvous control system controlled by the proposed PDF (40)

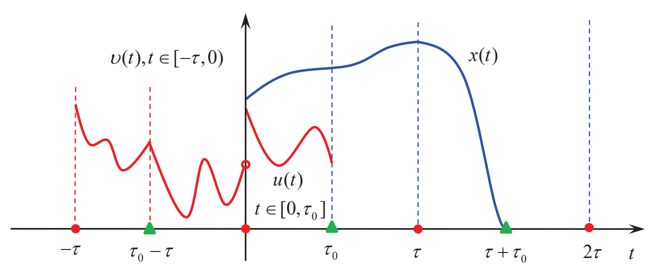

Fig. 5. An illustration of the proof of Lemma 1

and can be chosen continuous, continuously differentiable, and even smooth. The idea of periodic feedback has also been extended to solve the FTS problem of linear systems with both instantaneous and delayed controls. Two numerical examples have demonstrated the effectiveness of the proposed approaches.

Appendix

A1: The Proof of Lemma 1

We will show by contradiction that $x(t)$ cannot be driven to zero by any $u(t)$ in less than 2τ seconds. Suppose that there is a constant $\tau_0 \in (0, \tau)$ and a control $u(t) = u(t, x_{[0,t]}, u_{[0,t]}), t \in [0, \tau_0]$ such that $x(t) = 0, \forall t \geq \tau + \tau_0$ (see Fig. 5 for an illustration). Then by (1) we have

$$0 = x(\tau + \tau_0) = {\rm e}^{A\tau_0} x(\tau) + \int_0^{\tau_0} {\rm e}^{A(\tau_0 - s)} Bu(s){\rm d}s. \tag{110}$$

On the other hand, by (2) we have

$$x(\tau) = {\rm e}^{A(\tau-\tau_0)} x(\tau_0) + \int_{\tau_0-\tau}^{0} {\rm e}^{-As} B\upsilon(s){\rm d}s. \tag{111}$$

Inserting (111) into (110) gives

$$0 = \int_{\tau_0-\tau}^{0} {\rm e}^{A(\tau_0-s)} B\upsilon(s){\rm d}s + k(\tau_0), \tag{112}$$

where

$$k(\tau_0) = \mathrm{e}^{A\tau}x(\tau_0) + \int_0^{\tau_0} \mathrm{e}^{A(\tau_0-s)}Bu(s)\mathrm{d}s$$
$$= \mathrm{e}^{A\tau}x(\tau_0) + \int_0^{\tau_0} \mathrm{e}^{A(\tau_0-s)}Bu\left(s, x_{[0,s]}, u_{[0,s]}\right)\mathrm{d}s.$$

By using (2) again we have, for all $s \in [0, \tau_0]$,

$$x(s) = \mathrm{e}^{As}x(0) + \int_{-\tau}^{s-\tau} \mathrm{e}^{A(s-\tau-\theta)}B\upsilon(\theta)\mathrm{d}\theta.$$

It follows that $k(\tau_0)$ is independent of $\upsilon(s), s \in (\tau_0 - \tau, 0)$. Since (A, B) is controllable, for any given vectors $k(\tau_0)$ and $k_1 \in \mathbf{R}^n$ which is nonzero, we can find $\upsilon(s), s \in (\tau_0 - \tau, 0)$ such that

$$\int_{\tau_0-\tau}^{0} \mathrm{e}^{A(\tau_0-s)}B\upsilon(s)\mathrm{d}s = k_1 - k(\tau_0). \quad (113)$$

For example, we can choose

$$\upsilon(s) = B^{\mathrm{T}}\mathrm{e}^{-A^{\mathrm{T}}s}G_{\mathrm{c}}^{-1}(A, B, \tau_0 - \tau)\,\mathrm{e}^{-A\tau_0}(k(\tau_0) - k_1),$$

where $s \in (\tau_0 - \tau, 0)$ (we point out that it is also possible to find a continuous function $\upsilon(s), s \in [-\tau, 0)$ such that (113) is satisfied). Notice that (113) contradicts with (112). The proof is finished.

A2: The Proof of Lemma 2

If $t \in [0, \tau)$, then it follows from (2) that [1]

$$\begin{aligned}\dot{w}(t) =& \mathrm{e}^{A\tau}\dot{x}(t) + B\overline{u}(t) - \mathrm{e}^{A\tau}B\overline{u}(t-\tau) \\ &+ A\int_t^{t+\tau} \mathrm{e}^{A(t+\tau-s)}B\overline{u}(s-\tau)\,\mathrm{d}s \\ =& \mathrm{e}^{A\tau}(Ax(t) + B\upsilon(t-\tau)) - \mathrm{e}^{A\tau}B\overline{\upsilon}(t-\tau) \\ &+ Bu(t) + A\int_t^{t+\tau} \mathrm{e}^{A(t+\tau-s)}B\overline{u}(s-\tau)\,\mathrm{d}s,\end{aligned}$$

which is (5). If $t \geq \tau$, then both $\upsilon(t-\tau)$ and $\overline{\upsilon}(t-\tau)$ disappear. Thus equation (6) is proven. For $t \in [0, \tau)$, it follows from the definition of $w(t)$ that

$$\begin{aligned}w(t) =& \mathrm{e}^{A\tau}x(t) + \int_{t-\tau}^{t} \mathrm{e}^{A(t-\theta)}B\overline{u}(\theta)\mathrm{d}\theta \\ =& \mathrm{e}^{A\tau}x(t) + \int_0^t \mathrm{e}^{A(t-\theta)}Bu(\theta)\mathrm{d}\theta + \int_{t-\tau}^{0} \mathrm{e}^{A(t-\theta)}B\overline{\upsilon}(\theta)\mathrm{d}\theta.\end{aligned} \quad (114)$$

On the other hand, it follows from system (1) that

$$\begin{aligned}x(t+\tau) =& \mathrm{e}^{A\tau}x(t) + \int_{t-\tau}^{t} \mathrm{e}^{A(t-\theta)}Bu(\theta)\mathrm{d}\theta \\ =& \mathrm{e}^{A\tau}x(t) + \int_0^t \mathrm{e}^{A(t-\theta)}Bu(\theta)\mathrm{d}\theta + \int_{t-\tau}^{0} \mathrm{e}^{A(t-\theta)}B\upsilon(\theta)\mathrm{d}\theta,\end{aligned}$$

substituting of which into (114) gives the first equation of (7). Again, if $t \geq \tau$, then both $\upsilon(t-\tau)$ and $\overline{\upsilon}(t-\tau)$ disappear, and thus the second equation of (7) is also proven.

A3: Proofs for Lemmas 4 and 6

We first prove Lemma 4. Let $S(\sigma) = \mathrm{e}^{A\sigma}S_1(\sigma)\mathrm{e}^{A^{\mathrm{T}}\sigma}, \sigma \in [0, h]$, where

$$S_1(\sigma) = \int_0^{\sigma} \mathrm{e}^{-As}BR_h(h+s)B^{\mathrm{T}}\mathrm{e}^{-A^{\mathrm{T}}s}\mathrm{d}s.$$

Then $S(0) = 0$ and

$$\begin{aligned}\dot{S}(\sigma) =& A\mathrm{e}^{A\sigma}S_1(\sigma)\mathrm{e}^{A^{\mathrm{T}}\sigma} + \mathrm{e}^{A\sigma}S_1(\sigma)\mathrm{e}^{A^{\mathrm{T}}\sigma}A^{\mathrm{T}} + \mathrm{e}^{A\sigma}\dot{S}_1(\sigma)\mathrm{e}^{A^{\mathrm{T}}\sigma} \\ =& AS(\sigma) + S(\sigma)A^{\mathrm{T}} + BR_h(h+\sigma)B^{\mathrm{T}},\end{aligned}$$

which is just (37). By the definition of $W_{\mathrm{c}}(A, h)$, we obtain

$$\begin{aligned}W_{\mathrm{c}}(A, h) &= \int_0^h \mathrm{e}^{-A(s+h)}BR_h(h+s)B^{\mathrm{T}}\mathrm{e}^{-A^{\mathrm{T}}(s+h)}\mathrm{d}s \\ &= \mathrm{e}^{-Ah}S_1(h)\mathrm{e}^{-A^{\mathrm{T}}h} = \mathrm{e}^{-2Ah}S(h)\mathrm{e}^{-2A^{\mathrm{T}}h},\end{aligned}$$

substituting of which into (27) gives (38).

We then prove Lemma 6. Rewrite the PLE (48) as $A_{\mathrm{c}} = A - BB^{\mathrm{T}}P = -P^{-1}(A + \gamma I_n)^{\mathrm{T}}P$ from which we have

$$\mathrm{e}^{A_{\mathrm{c}}s} = \mathrm{e}^{-P^{-1}(A+\gamma I_n)^{\mathrm{T}}Ps} = P^{-1}\mathrm{e}^{-A^{\mathrm{T}}s}P\mathrm{e}^{-\gamma s}. \quad (115)$$

With this the Lyapunov differential equation (37) can be rewritten as

$$\begin{aligned}\dot{S}(\sigma) =& -P^{-1}(A + \gamma I_n)^{\mathrm{T}}PS(\sigma) - S(\sigma)P(A + \gamma I_n)P^{-1} \\ &+ BR_h(h+\sigma)B^{\mathrm{T}},\end{aligned}$$

where $\sigma \in [0, h]$ and $S(0) = 0$. By setting $S_\gamma(\sigma) = PS(\sigma)P$, the above equation is exactly in the form of (51). Finally, inserting (115) into (39) gives (52). The proof of Lemma 6 is finished.

A4: The Proof of Lemma 7

It is trivial to show (53). Thus we only need to prove the last two equations in (54) and equation (55). Denote $K_{\mathrm{c}}(t) = K_{(A_{\mathrm{c}},h)}(t)$.

- For $t \in [0, \tau)$, we consider two cases. **Case 1**: $\tau < h$. Then we have from (40) and (7) that

$$\begin{aligned}u(t) &= -K_0w(t) - K_{\mathrm{c}}(t-\tau)w(t-h) \\ &= -K_0w(t) = -K_0(x(t+\tau) + \delta(t)),\end{aligned} \quad (116)$$

based on which the closed-loop system can be written as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B(-K_0(x(t) + \delta(t-\tau))) \\ &= A_{\mathrm{c}}x(t) - BK_0\delta(t-\tau),\ t \in [\tau, 2\tau).\end{aligned} \quad (117)$$

Case 2: $\tau \geq h$. For $t \in [0, h)$, we have from (40) and (7) that

$$\begin{aligned}u(t) &= -K_0w(t) - K_{\mathrm{c}}(t-\tau)w(t-h) \\ &= -K_0w(t) = -K_0(x(t+\tau) + \delta(t)),\end{aligned}$$

which is the same as (116) and thus the closed-loop system is also (117) with $t \in [\tau, \tau + h)$. For $t \in [h, \tau)$, we have from (40) and (7) that

$$\begin{aligned}u(t) =& -K_0w(t) - K_{\mathrm{c}}(t-\tau)w(t-h) \\ =& -K_0(x(t+\tau) + \delta(t)) \\ &- K_{\mathrm{c}}(t-\tau)(x(t+\tau-h) + \delta(t-h)),\end{aligned}$$

based on which the closed-loop system can be written as, for all $t \in [\tau + h, 2\tau)$,

$$\begin{aligned}\dot{x}(t) =& A_{\mathrm{c}}x(t) - BK_{\mathrm{c}}(t-2\tau)x(t-h) \\ &- B(K_0\delta(t-\tau) + K_{\mathrm{c}}(t-2\tau)\delta(t-\tau-h)).\end{aligned}$$

Combining the above two cases gives (55).

- For $t \in [\tau, \tau + h)$, we have from (40) and (7) that

$$\begin{aligned}u(t) &= -K_0w(t) - K_{\mathrm{c}}(t-\tau)w(t-h) \\ &= -K_0w(t) = -K_0x(t+\tau),\end{aligned}$$

based on which the the closed-loop system can be written as $\dot{x}(t) = A_{\mathrm{c}}x(t), t \in [2\tau, 2\tau + h)$.

- For $t \in [\tau + h, \tau + 2h)$, we have from (40) and (7) that

$$\begin{aligned}u(t) &= -K_0w(t) - K_{\mathrm{c}}(t-\tau)w(t-h) \\ &= -K_0x(t+\tau) - K_{\mathrm{c}}(t-\tau)x(t+\tau-h),\end{aligned}$$

based on which the the closed-loop system can be written as

$$\dot{x} = A_{\mathrm{c}}x(t) - BK_{\mathrm{c}}(t-2\tau)x(t-h), t \in [2\tau + h, 2\tau + 2h).$$

Combining the above two cases gives (54) and completes the proof.

A5: The Proof of Lemma 8

The proof is dual to that of Theorem 1. For simplicity, we denote $L(t) = L_{(A_{\rm o},l)}(t)$ and consider two cases.

Case 1: $0 < l \leq \tau$. For $t \in [0, l)$, we have from (60) that

$$\begin{aligned}\dot{e}(t) =& A_{\rm o}e(t) - L(t-\tau)\left(\widetilde{\varphi}(t-l) - C\widetilde{\psi}(t-l)\right) \\ &+ B\left(\upsilon(t-\tau) - \widetilde{\upsilon}(t-\tau)\right),\ t \in [0, l),\end{aligned}$$

and for $t \in [l, \tau)$, we have from (60) that

$$\begin{aligned}\dot{e}(t) =& A_{\rm o}e(t) - L(t-\tau)Ce(t-l) \\ &+ B\left(\upsilon(t-\tau) - \widetilde{\upsilon}(t-\tau)\right),\ t \in [l, \tau).\end{aligned}$$

Similarly, for $t \geq \tau$, we have from (60) that

$$\dot{e}(t) = A_{\rm o}e(t) - L(t-\tau)Ce(t-l),\ t \geq \tau. \tag{118}$$

Case 2: $l > \tau$. For $t \in [0, \tau)$, we have from (60) that

$$\begin{aligned}\dot{e}(t) =& A_{\rm o}e(t) - L(t-\tau)\left(\widetilde{\varphi}(t-l) - C\widetilde{\psi}(t-l)\right) \\ &+ B\left(\upsilon(t-\tau) - \widetilde{\upsilon}(t-\tau)\right),\ t \in [0, \tau),\end{aligned}$$

and for $t \in [\tau, l)$, we have from (60) that

$$\begin{aligned}\dot{e}(t) &= A_{\rm o}e(t) - L(t-\tau)\left(\widehat{y}(t-l) - C\widehat{x}(t-l)\right) \\ &= A_{\rm o}e(t),\end{aligned} \tag{119}$$

where we have noticed that $L(t-\tau) = 0, \forall t \in [\tau, l)$. Finally, for $t \geq l$, we have from (60) that

$$\dot{e}(t) = A_{\rm o}e(t) - L(t-\tau)Ce(t-l),\ t \geq l. \tag{120}$$

Clearly, systems (119) can (120) can be written together as (118) since $L(t-\tau) = 0, t \in [\tau, l)$.

System (118) is exactly in the dual form of (33). Thus the remaining of the proof is similar to that of Theorem 1. Denote $s = t - \tau$ and $\varepsilon(s) = e(s+\tau)$. Then system (118) can be written as

$$\dot{\varepsilon}(s) = A_{\rm o}\varepsilon(s) - L(s)C\varepsilon(s-l),\ s \geq 0. \tag{121}$$

Notice that (121) takes a similar form as (34) which is exactly in the form of (11) with $F(t) = A_{\rm o}$, $G(t) = -L(s)C$ and $h = l$. Thus, by Proposition 1, the associated finite-dimensional monodromy matrix is given by (see (15))

$$\begin{aligned}\Delta_L(h) &= {\rm e}^{2A_{\rm o}l} - \int_l^{2l} {\rm e}^{A_{\rm o}(2l-s)}L(s)C{\rm e}^{A_{\rm o}(s-l)}{\rm d}s \\ &= {\rm e}^{2A_{\rm o}l}\left(I_n - M\int_l^{2l} {\rm e}^{A_{\rm o}^{\rm T}(s-l)}C^{\rm T}H_l(s)C{\rm e}^{A_{\rm o}(s-l)}{\rm d}s\right) \\ &= {\rm e}^{2A_{\rm o}l}\left(I_n - M\int_0^{l} {\rm e}^{A_{\rm o}^{\rm T}s}C^{\rm T}H_l\left(s+l\right)C{\rm e}^{A_{\rm o}s}{\rm d}s\right) \\ &= {\rm e}^{2A_{\rm o}l}\left(I_n - MW_{\rm o}(A_{\rm o}, l)\right).\end{aligned}$$

The remaining of the proof is similar to that of Theorem 1 and is thus omitted.

A6: The Proof of Proposition 2

The proof is also analogous to the proof of Theorem 1. By denoting $t - \tau = s$, system (80) can be written as

$$\dot{X}(s) = \left(A - B_0K(s+\tau)\right)X(s) - B_1K(s)X(s-\tau), \tag{122}$$

where $X(s) = x\left(s+\tau\right) = x(t), s \geq 0$. System (122) is exactly in the form of (11) where $F(t) = A - B_0K(t+\tau), G(t) = -B_1K(t)$ and $h = \tau$. Let $\Phi\left(t, s\right)$ be the state transition matrix for the linear periodic system

$$\dot{x}(t) = \left(A - B_0K(t+\tau)\right)x(t). \tag{123}$$

For $t \in [0, \tau]$, system (123) can be written as $\dot{x}(t) = (A - B_0K_0(t+\tau))x(t)$ which is in the form of (81). Thus, by assumption, we have

$$x(t) = \Phi_0\left(t+\tau, \tau\right)x\left(0\right) = \Phi\left(t, 0\right)x\left(0\right),\ \forall t \in [0, \tau].$$

For $t \in [\tau, 2\tau]$, system (123) can be written as $\dot{x}(t) = Ax(t)$, whose solution is given by $x(t) = {\rm e}^{A(t-s)}x(s) = \Phi\left(t, s\right)x(s), \forall s, t \in [\tau, 2\tau]$. Thus we get $\Phi(2\tau, 0) = {\rm e}^{A\tau}\Phi_0(2\tau, \tau), \Phi(2\tau, s) = {\rm e}^{A(2\tau-s)}, \forall s \in [\tau, 2\tau]$ and $\Phi(s-\tau, 0) = \Phi_0(s, \tau)$. According to (15), the monodromy matrix $\Delta_0\left(\tau\right)$ for system (122) is

$$\begin{aligned}\Delta_0\left(\tau\right) &= \Phi\left(2\tau, 0\right) - \int_\tau^{2\tau} \Phi\left(2\tau, s\right)B_1K_0(s)\Phi\left(s-\tau, 0\right){\rm d}s \\ &= {\rm e}^{A\tau}\left(I_n - \Delta_1\left(\tau\right)\right)\Phi_0\left(2\tau, \tau\right),\end{aligned} \tag{124}$$

where

$$\Delta_1\left(\tau\right) \triangleq \int_\tau^{2\tau} {\rm e}^{A(\tau-\theta)}B_1K_0(\theta)\Phi_0\left(\theta, 2\tau\right){\rm d}\theta.$$

By using $\mathscr{B}_1 = B_0 + {\rm e}^{-A\tau}B_1$, we have

$$\begin{aligned}\Delta_1\left(\tau\right) &= \int_\tau^{2\tau} {\rm e}^{A(2\tau-\theta)}\left(\mathscr{B}_1 - B_0\right)K_0(\theta)\Phi_0\left(\theta, 2\tau\right){\rm d}\theta \\ &= \int_\tau^{2\tau} {\rm e}^{A(2\tau-\theta)}\mathscr{B}_1K_0(\theta)\Phi_0\left(\theta, 2\tau\right){\rm d}\theta \\ &\quad - \int_\tau^{2\tau} {\rm e}^{A(2\tau-\theta)}A\Phi_0\left(\theta, 2\tau\right){\rm d}\theta \\ &\quad + \int_\tau^{2\tau} {\rm e}^{A(2\tau-\theta)}\left(A - B_0K_0(\theta)\right)\Phi_0\left(\theta, 2\tau\right){\rm d}\theta \\ &= \int_\tau^{2\tau} {\rm e}^{A(2\tau-\theta)}\mathscr{B}_1K_0(\theta)\Phi_0\left(\theta, 2\tau\right){\rm d}\theta \\ &\quad + \int_\tau^{2\tau} {\rm e}^{A(2\tau-\theta)}\frac{{\rm d}\Phi_0\left(\theta, 2\tau\right)}{{\rm d}\theta}{\rm d}\theta \\ &\quad + \int_\tau^{2\tau} \frac{{\rm d}{\rm e}^{A(2\tau-\theta)}}{{\rm d}\theta}\Phi_0\left(\theta, 2\tau\right){\rm d}\theta \\ &= \int_\tau^{2\tau} {\rm e}^{A(2\tau-\theta)}\mathscr{B}_1K_0(\theta)\Phi_0\left(\theta, 2\tau\right){\rm d}\theta \\ &\quad + \int_\tau^{2\tau} {\rm d}\left({\rm e}^{A(2\tau-\theta)}\Phi_0\left(\theta, 2\tau\right)\right) \\ &= \Delta_2\left(\tau\right) + I_n - {\rm e}^{A\tau}\Phi_0\left(\tau, 2\tau\right),\end{aligned} \tag{125}$$

where

$$\Delta_2\left(\tau\right) = \int_\tau^{2\tau} {\rm e}^{A(2\tau-\theta)}\mathscr{B}_1K_0(\theta)\Phi_0\left(\theta, 2\tau\right){\rm d}\theta.$$

Inserting equation (125) into (124) gives (82). Thus, by Proposition 1, system (122) is exponentially stable if and only if (83) is true, and is fixed-time stable if and only if (84) is satisfied. Moreover, by the definition of ν, it follows from (18) that $X(s) = 0, s \geq 2\nu\tau$, which is just $x(t) = 0, \forall t \geq 2\nu\tau + \tau$.

A7: The Proof of Proposition 3

Since $n = m = 1$, we have

$$\Phi_0\left(t, s\right) = \exp\left(\int_s^t \left(A - B_0K_0(\theta)\right){\rm d}\theta\right),$$

by which we can compute

$$\begin{aligned}\Delta_3\left(\tau\right) &\triangleq \int_\tau^{2\tau} {\rm e}^{A(\tau-\theta)}\mathscr{B}_1K_0(\theta)\Phi_0\left(\theta, \tau\right){\rm d}\theta \\ &= \mathscr{B}_1\int_\tau^{2\tau} K_0(\theta)\exp\left(-\int_\tau^\theta B_0K_0(s){\rm d}s\right){\rm d}\theta \\ &= -\frac{\mathscr{B}_1}{B_0}\int_\tau^{2\tau} {\rm d}\left(\exp\left(-\int_\tau^\theta B_0K_0(s){\rm d}s\right)\right)\end{aligned}$$

$$= \frac{\mathscr{B}_1}{B_0}\left(1 - \exp\left(-\int_{\tau}^{2\tau} B_0 K_0(s)\mathrm{d}s\right)\right).$$

Thus condition (83) can be rewritten as

$$\begin{aligned} 1 &> \left|\mathrm{e}^{2A\tau}\left(1-\Delta_3\right)\right| \\ &= \mathrm{e}^{2A\tau}\left|\frac{\mathscr{B}_1}{B_0}\right|\left|\exp\left(-\int_{\tau}^{2\tau} B_0 K_0(s)\mathrm{d}s\right) - \frac{\mathrm{e}^{-A\tau}B_1}{\mathscr{B}_1}\right|, \end{aligned}$$

which is equivalent to (86), which can be satisfied for some $K_0(t)$ if and only if (85) is satisfied.

On the other hand, (84) is satisfied if and only if $\Delta_3(\tau) = 1$, which is equivalent to

$$\exp\left(B_0\int_{\tau}^{2\tau} K_0(s)\mathrm{d}s\right) = \mathrm{e}^{A\tau}\frac{\mathscr{B}_1}{B_1}. \tag{126}$$

Clearly, there exists a $K_0(s)$ satisfying (126) if and only if (87) is satisfied, and if this is satisfied, we clearly have (88).

A8: The Proof of Lemma 9

We first recall the following result from page 123 in [33].

Lemma 10: Consider the scalar linear time-delay system

$$\dot{x}(t) = -\alpha x(t) - \beta x(t-\tau), \tag{127}$$

where α, β and $\tau > 0$ are constants. Then the system

- is exponentially stable for any $\tau \geq 0$ if and only if

$$\alpha + \beta > 0 \text{ and } \alpha \geq |\beta|, \tag{128}$$

- and is exponentially stable for any $\tau < \tau^* < \infty$ if and only if

$$\beta > |\alpha| \text{ and } \tau^* = \frac{\arccos\left(-\frac{\alpha}{\beta}\right)}{\sqrt{\beta^2 - \alpha^2}}. \tag{129}$$

Notice that system (94) is exactly in the form of (127) with $\alpha = B_0K_0 - A$ and $\beta = K_0$. We first consider condition (128) which is equivalent to $B_0K_0 - A + K_0 > 0$ and $B_0K_0 - A \geq |K_0|$, or,

$$\left\{\begin{array}{l} A < (B_0+1)\,K_0, \\ A \leq B_0K_0, \\ A \leq (B_0+1)\,K_0, \\ A \leq (B_0-1)\,K_0, \end{array}\right. \tag{130}$$

and the first condition in (129) is equivalent to $K_0 > |B_0K_0 - A|$, or equivalently,

$$\left\{\begin{array}{l} 0 < K_0, \\ A < (B_0+1)\,K_0, \\ A > (B_0-1)\,K_0. \end{array}\right. \tag{131}$$

We consider the following 7 cases.

- **Case 1**: $B_0 < -1$. Then (130) is equivalent to $K_0 < A/(B_0 + 1)$ and (131) is impossible.
- **Case 2**: $B_0 = -1$. Then both (130) and (131) are impossible as $A > 0$.
- **Case 3**: $B_0 \in (-1, 0)$. Then (130) is impossible and (131) is equivalent to $K_0 > A/(B_0 + 1)$.
- **Case 4**: $B_0 = 0$. Then (130) is impossible and (131) is equivalent to $K_0 > A = A/(B_0 + 1)$.
- **Case 5**: $B_0 \in (0, 1)$. Then (130) is impossible and (131) is equivalent to $K_0 > A/(B_0 + 1)$.
- **Case 6**: $B_0 = 1$. Then (130) is impossible and (131) is equivalent to $K_0 > A/(B_0 + 1)$.
- **Case 7**: $B_0 > 1$. Then (130) is equivalent to $K_0 \geq A/(B_0 - 1)$ and (131) is equivalent to $A/(B_0 + 1) < K_0 < A/(B_0 - 1)$.

Combining the above 7 cases proves (95)-(98). It remains to prove (99). From (129) we have

$$\tau^* = \frac{\arccos\left(\frac{A - B_0K_0}{K_0}\right)}{\sqrt{K_0^2 - (A - B_0K_0)^2}} = \frac{(\lambda + B_0)\arccos(\lambda)}{A\sqrt{1-\lambda^2}} = \tau^*(\lambda),$$

where $\lambda = (A - B_0K_0)/K_0 = A/K_0 - B_0 > -B_0$. Under condition (97), we have $\lambda < A/\left(A/(B_0+1)\right) - B_0 = 1$, while under condition (98), we have $-1 < \lambda < 1$. As $\tau^*(\lambda)$ is an increasing function of λ for $\lambda \in (-1, 1)$, we have

$$\sup_{\lambda\in(-B_0,1)} \tau^*(\lambda) = \sup_{\lambda\in(-1,1)} \tau^*(\lambda) = \lim_{\lambda\uparrow 1}\tau^*(\lambda) = \frac{1+B_0}{A},$$

which is just (99). This completes the proof.

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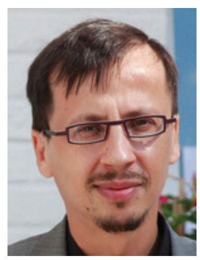

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