To add or to multiply in open problems?

Unraveling children's relational preference using a mixed-method approach

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Abstract

Previous research demonstrated that some children inappropriately solve multiplicative missing-value word problems additively, while others inappropriately solve additive missing-value word problems multiplicatively. Besides lacking skills, children's preference for additive or multiplicative relations has been shown to contribute to those errors. The present research investigated the nature of upper primary school children's relational preference by empirically examining characteristics of intuitions that had been postulated previously, using a mixed-method approach. After administering a pre-test, selected children who preferred additive or multiplicative relations further participated in one of two studies using open problems for which both types of relations were appropriate: either a reaction times study (*n*=110) in which children's acceptance behavior and reaction times were measured, or a semi-structured individual interview study (*n*=18) in which their answers, verbalizations, and conviction scores were collected. Results of both studies revealed that relational preference was perseverant and exerted a coercive effect on children's reasoning: Children mostly considered only the preferential type of relation as an appropriate answer in open problems and rejected alternative answers. Furthermore, relational preference appeared as immediate, self-evident, and certain: Children rejected the non-preferential answer more quickly than an irrelevant distractor of comparable size, experienced difficulties in justifying why they gave their preferential answer, and were very convinced of this preferential answer. While this characterization held for both relational preferences, it was especially prominent for the multiplicative one. These results have implications for research on and educational practice in multiplicative and additive reasoning, but also for the measurement of relational preference.

Keywords: additive reasoning, multiplicative reasoning, open problems, relational preference

1. Theoretical background

1.1 Multiplicative and additive reasoning in missing-value word problems

Multiplicative reasoning plays a pivotal role in everyday life as well as in school mathematics (Lamon, 1993; Lesh, Post, & Behr, 1988; Vergnaud, 1988), where it lays the foundation for learning more advanced mathematics topics such as proportionality, fractions, functions, probability, and algebra (Vergnaud, 1988). Multiplicative missing-value word problems take a central place in the instruction in multiplicative reasoning. Take the following multiplicative missing-value problem of Kaput and West (1994): “A car of the future will be able to travel 8 miles in 2 minutes. How far will it travel in 6 minutes?”, which consists of three given quantities and an unknown fourth one that has to be found by identifying the multiplicative relation between two given quantities – either of the same (2 minutes × 3 = 6 minutes, so 8 miles × 3 = 24 miles) or a different nature (i.e., 2 minutes × 4 miles/minute = 8 miles, so 6 minutes × 4 miles/minute = 24 miles) – and applying this relation to the third given quantity (Cramer & Post, 1993; Harel & Behr, 1989; Kaput & West, 1994; Karplus, Pulos, & Stage, 1983; Noelting, 1980; Tourniaire & Pulos, 1985;Vergnaud, 1983, 1988). In such problems, the multiplicative relations within and between pairs of quantities are characterized by an invariant ratio (Behr & Harel, 1990).

However, not all word problems that contain three quantities and in which the goal is to find the fourth one should be solved multiplicatively, as in the following problem: “Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?” (Cramer, Post, & Currier, 1993, p. 159). Sue is six laps ahead of Julie the first time, and since they are running equally fast, this difference stays the same. In this *additive* missing-value problem, the differences within and between pairs of quantities are invariant (Behr & Harel, 1990).

Additive and multiplicative missing-value problems have in common that they contain three quantities and the goal is to find the fourth one, by identifying the relation between two given quantities and applying this relation to the third given quantity. The quantities in the problem co-vary, while the relation between the quantities remains invariant (Behr & Harel, 1990; Lamon, 2008). In the car-of-the-future introduced above, the car has covered more distance when more time has passed by, and the car drives a fixed number of miles per minute. Likewise, in the runner problem introduced above, the more laps Sue has run, the more laps Julie has run, and the difference between the laps run by the two stays the same.

Much research attention has been devoted to both types of missing-value problems. Numerous studies have shown that children frequently inappropriately give additive solutions to multiplicative missing-value problems (e.g., Hart, 1981; Kaput & West, 1994; Karplus et al., 1983; Lesh et al, 1988; Noelting, 1980; Vergnaud, 1983, 1988). They answer 12 miles (e.g., 2+4=6, so 8+4=12) instead of 24 to the car-of-the-future problem introduced above. In this case, the addition or subtraction operation is not used as an informal strategy to find the solution to a multiplicatively modeled situation (as would be the case when using a repeated addition strategy in the aforementioned car-of-the-future problem, as in "2+2+2 minutes to drive 8+8+8 miles"). Rather, they inappropriately rely on additive relations between quantities instead of multiplicative relations. Inappropriate additive reasoning is most typical for younger children – who have limited instruction with multiplicative relations – but it still occurs after instruction (Van Dooren, De Bock, & Verschaffel, 2010).

The inverse error has been documented as well. Children making this error give multiplicative solutions in additive problems. They would answer 45 (e.g., 3×5=15, so 9×5=45) instead of 21 laps to the runner problem introduced before. This error is mainly made by children in upper primary education (Fernández, Llinares, Van Dooren, De Bock, & Verschaffel, 2012; Van Dooren, De Bock, & Verschaffel, 2010). Many children pass through an intermediate stage in between the stage in which they solve multiplicative word problems additively and additive word problems multiplicatively, wherein they simultaneously make both kinds of errors (Van Dooren, De Bock, & Verschaffel, 2010).

1.2 Explaining inappropriate missing-value word problem solving behavior in terms of skills

The finding that the first error, that is inappropriate additive reasoning in multiplicative missing-value problems, especially occurs in younger children has often been interpreted as evidence for "an 'additive phase' in children's solution to multiplicative reasoning problems" (Nunes & Bryant, 2010, p. 11), in which children are not yet able to think of relations in a multiplicative way (Clark & Kamii, 1996; Siemon, Breed, & Virgona, 2005). This is based on the Piagetian assumption that young "children first think of quantifying relations additively and can only think of relations multiplicatively at a later age" (Nunes and Bryant, 2010, p. 11). More recently too, authors explicitly pointed towards the transition from additive to multiplicative reasoning (Clark & Kamii, 1996; Jacob & Willis, 2003; Siemon et al., 2005). According to this view, young children still have to develop the skill to make multiplicative computations to derive the correct answer, as well as to discriminate between additive and multiplicative situations. Those two skills together are crucial to multiplicative reasoning (Lamon, 2008; Nunes & Bryant, 2010; Nunes, Bryant, Barros, & Sylva, 2012).

The literature documenting the inverse error of giving multiplicative answers to additive word problems often referred to children's lacking skill to discriminate between additive and multiplicative models in word problems, rather than the computation skill (see, e.g., Van Dooren, De Bock, Janssens, & Verschaffel, 2008). In this respect, Hoffer (1988) already argued that "being able to perform mechanical operations with proportions does not necessarily mean the students understand the underlying ideas of proportional thinking" (p. 293). Hence, children who give multiplicative answers to additive word problems seem to fail in discriminating between additive and multiplicative models in word problems.

More recently, however, it has been questioned whether the aforementioned explanations in terms of lacking skills are sufficient. First, there is growing evidence for children's early multiplicative reasoning skills (e.g., Boyer, Levine, & Huttenlocher, 2008; Jeong, Levine, & Huttenlocher, 2007; Nunes & Bryant, 2010; Resnick & Singer, 1993), demonstrating that many are able to execute multiplicative operations to derive the correct answer, as well as to correctly discriminate between additive and multiplicative relations, from early age on. Hence, children's inappropriate additive reasoning in multiplicative problems cannot solely be attributed to children's lacking skills. A second indication is that children perform better at classifying additive and multiplicative word problems than at solving them, suggesting that they are able to discriminate between additive and multiplicative ones, but they do not necessarily exhibit this skill when solving word problems (Van Dooren, De Bock, Vleugels, & Verschaffel, 2010). Taken together, this evidence suggests that children's inappropriate multiplicative reasoning in additive problems cannot solely be attributed to children's lacking skills – and as argued above, particularly children's lacking discrimination skill.

1.3 Preference as complementary explanation for inappropriate missing-value word problem solving behavior

As a complementary explanation for children's errors in word problems, Resnick and Singer (1993) interpreted children's additive solutions to multiplicative missing-value word problems as an indication that these children "prefer to apply additive as opposed to multiplicative relationships" (p. 124). Similarly to interpreting children's additive errors in multiplicative word problems in terms of a "strong *preference* for additive relationships among numbers" (p. 126), other children's multiplicative errors in additive missing-value word problems may be explained by a preference for multiplicative relations between numbers. Many other authors referred to similar explanations for children's errors in word problems, by referring to "tendencies" (Misailidou & Williams, 2003; Van Dooren, De Bock, & Verschaffel, 2010) or "inclinations" (Modestou & Gagatsis, 2010) towards either additive relations or multiplicative relations. In what follows, in line with Resnick and Singer (1993), the term preference for additive relations or multiplicative relations is used to stress that one type of relations "has precedence over" the other (Pellegrino & Glaser, 1982, p. 310): additive relations may have precedence over multiplicative ones, or vice versa, multiplicative relations may have precedence over additive ones.

Our research interest in children's relational preference does not stand on its own. Research on judgement and decision making in other domains, such as psychology, economics, law, politics, or medicine, has already focused on the idea of a preference, its impact on behavior, and its nature. Take, for instance, consumers' preferences for certain objects, patients' preferences for certain medical treatments, or civilians' preferences for certain policy topics and political candidates. The traditional assumption underlying the preference research in those domains was that people have stable and well-articulated preferences that are revealed in the elicitation process. That is, when confronted with situations in which people need to express their preference, they rely on a readily available preference to give an answer. However, more recently, this classical assumption has been challenged, and a new conception of preference arose, wherein preference is conceived as the result of a constructive process that is more labile and reversible than traditionally thought (Lichtenstein & Slovic, 2006; Slovic, 1995). This need for this preference construction especially arises when the decisions, judgments, and choices that need to be made are important, complex, and unfamiliar (Slovic, 1995). Particularly in these cases, preferences are not readily available, but are constructed on the spot. In a minority of cases, preferences may be stable and well-articulated, especially when the situation is familiar to people, when no conflicts between options occur and no trade-offs need to be made, and when preferences do not need to be translated in numerical answers (e.g., by stating the price one is willing to pay, by giving a score, etc.) (Lichtenstein & Slovic, 2006).

Not only in the domains of psychology, economics, law, politics, and medicine, but also in mathematics education the idea of preference is investigated in numerous studies. These studies typically distinguish between preferences for certain solution approaches on the one hand and the accompanying skills on the other. This distinction is based on the idea that preference and skill do not coincide, although they relate to and interact with each other (e.g., Bailey, Littlefield, & Geary, 2012; Pellegrino & Glaser, 1982). Take, for instance, the many studies on preferences of a certain strategy over others in single-digit addition and subtraction (e.g., Bailey et al., 2012), in multi-digit addition and subtraction (e.g., Torbeyns & Verschaffel, 2013), and in multiplication and division problems (e.g., Mulligan, 1992). In all those studies, preference is typically measured by means of tasks that allow one to use a variety of solution approaches.

In the specific domain of additive and multiplicative reasoning, recent work of Degrande, Verschaffel, and Van Dooren (2018, 2019) provided evidence for the existence and development of a preference for additive or multiplicative relations. Mainly younger children preferred additive relations between numbers and older children multiplicative ones, although inter-individual differences in relational preference were found within grades. Evidence has also been found for the unique contribution of children’s relational preference to inappropriate word problem solving behavior, for children who possessed the computation and discrimination skills (Degrande et al., 2019). In these studies, relational preference (Degrande et al., 2018, 2019) was not measured by means of classical word problems. Such word problems unmistakably contain an underlying additive or multiplicative mathematical model, and therefore, in such problems, children’s skill to detect the correct underlying mathematical model – and thus to correctly discriminate between additive and multiplicative situations – may be involved as well, besides relational preference and computation skill. Instead, problems in which additive and multiplicative relations are equally valuable and correct were used. One example of such an open problem is a schematic presentation of three given numbers and a fourth missing one, and two arrows that point out the relational structure between the numbers, as shown in Figure 1 (Degrande et al., 2019).

These schematic problems thus consist of two number pairs, whose numbers can be linked additively or multiplicatively, or still in another way, as Pellegrino and Glaser (1982, p. 302) stated it: "Consider for example the pair 2:16, which can be represented as having several relationships, e.g., +14, ×8 or 2^4", and even other relationships could be considered. As such they do not contain any indication for an additive answer (i.e., 24 in Figure 1) or multiplicative answer (i.e., 48 in Figure 1), so both answers (and any other answer) are equally correct and valuable. Therefore, these schematic problems too are open to additive and multiplicative relations between numbers.

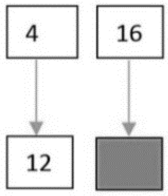

Figure 1. Example of a schematic open problem.

By administering diverse types of open problems in paper-and-pencil tests to large groups of children, previous research extensively demonstrated the existence, development, and importance of children's additive or multiplicative preference in inappropriate word problem solving behavior (e.g., Degrande et al., 2018, 2019). However, far less is known about the *nature* of this relational preference. One theory that has been used to unravel the nature of preferences across domains is the dual-process theory. Two distinct types of processes are typically distinguished. Analytic processes are conscious and effortful, so generating the answer takes place slowly and in a controlled way (e.g., Evans, 2008). In contrast, heuristic or intuitive processes happen by default, unconsciously, and immediately when confronted with a situation, leading to a rapid and automatic answer. The idea that intuitions play a role in mathematical reasoning in specific, dates back to Fischbein's work (1987, 1999). He postulated a list of characteristics of intuitions that are closely related to each other: He characterized intuitions as global and universal views of a situation, which rely on extrapolation (i.e., generalizing based on incomplete information). The processes by which intuitions arise are implicit (i.e., tacit and unconscious). Further, intuitions have been described as immediate (i.e., quick and spontaneous), self-evident (i.e., self-explanatory, obvious, directly acceptable without needing any justification) and accepted as certain (i.e., generating a subjective feeling of great confidence and conviction). This self-evidence and certainty determine the perseverance of intuitions (i.e., robust and resistant to change), as well as the coercive effect that they exert on one's reasoning too (i.e., apparently absolute, imperative, so that alternative answers are rejected as unacceptable).

2. Rationale and research aims

The present research aimed to characterize the nature of a preference for additive or multiplicative relations, hereby relying on previous research and theorizing on intuitions (e.g., Evans, 2008; Fischbein, 1987). Previous research in the domain of additive and multiplicative reasoning has already repeatedly suggested that the aforementioned systematic errors in missing-value word problems have at least partly an intuitive nature (for an overview, see Van Dooren et al., 2008), but direct and systematic empirical evidence is lacking. One exception is the study of Gillard, Van Dooren, Schaeken, and Verschaffel (2009) that demonstrated the intuitive nature of the multiplicative error in additive word problems in adults. This was done by experimental designs that were inspired by the dual-process theory and in which reaction times were limited or working memory was experimentally burdened, leading to an increase of inappropriate multiplicative answers. To the best of our knowledge, explicit evidence for the intuitive nature of this multiplicative error as well as the reversed error in children is lacking, let alone evidence for the nature of relational preference. Related to that, Degrande et al. (2018) showed that many children only gave either the additive or the multiplicative answer to an open problem, even when the problem was presented in a multiple-choice format and children were asked to indicate all possibly correct answer alternatives. However, it remained unclear how children who prefer additive or multiplicative relations would react to alternative answers to open problems that were separately and subsequently presented. Relying on the characteristics of intuitions as they were postulated and defined by Fischbein (1987, 1999), we aimed to investigate whether relational preference is *perseverant* by looking at the extent to which children stick to their preferential answer, but also whether it exerts a *coercive* effect on children's reasoning by examining the extent to which they are reluctant to accept alternative answers than their preferential one in open problems. Further, we aimed to investigate whether relational preference is *immediate* by looking at how quickly children accept or reject the preferential and alternative answers, *self-evident* by looking at children's verbalizations when giving their preferential answers and being confronted with alternative answers, and *certain* by investigating how high the conviction scores are that children allocate to their preferential answers as compared to alternative answers.

To achieve these research aims, we used a mixed-method research approach. As defined by Tashakkori and Teddlie (2003), mixed-method research relies on “qualitative and quantitative data analysis techniques in either parallel or sequential phases” (p. 11). In the present article, we report on two studies that were conducted in parallel, but relied on very different methods to answer the same research questions. This way, mixed-method research leads to multiple results, which may confirm or complement each other, by means of triangulation and complementarity (Tashakkori & Teddlie, 2003). While Study 1 relied on quantitative data, also qualitative data were collected in Study 2. More specifically, Study 1 aimed to provide evidence for the nature of children’s relational preference via the measurement of their reaction times in open problems, in addition to their actual acceptances and rejections of preferential and alternative answers. Reaction time analysis is very common in the investigation of reasoning processes in formal logic, and it is becoming more and more common in studies in the domain of mathematical reasoning too (e.g., Babai, Levyadun, Stavy, & Tirosh, 2006; Csíkos, 2016; Vamvakoussi, Van Dooren, & Verschaffel, 2012). Study 2, in contrast, used semi-structured individual interviews in which children were increasingly prompted to consider alternative answers. Besides children’s answers throughout the interview, confidence ratings of children’s answers and verbalizations of the underlying reasons for children’s answers and conviction scores were gathered. Despite these different but complementary methodologies, both studies looked at similarities and differences between the responses of children who demonstrated a preference for additive relations or a preference for multiplicative relations between numbers. The first enabled us to find out more about the nature of a relational preference in general, the second allowed us to reveal what is more typical for a preference for additive relations and a preference for multiplicative relations.

3. Study 1: Reaction times study

3.1 Research questions

This reaction times study consisted of, first, a pre-test, in which upper primary school children who preferred additive or multiplicative relations were identified and invited to participate in a second part, that was the actual reaction times experiment. This way, this reaction times study aimed to reveal how children with a preference for additive or multiplicative relations between numbers react when being confronted with several open problems in which one of the following answer alternatives was given: the additive answer, the multiplicative answer, or a distractor. Distractors are answers that cannot be reached by additive nor multiplicative computations, and for which finding another relevant computation is almost impossible for children in upper primary school.

First, we wondered whether children with a preference for additive or multiplicative relations between numbers would merely stick to their preferential answer (and hence, would reject all other answers – including the non-preferential one), or would rather consider to also accept the non-preferential answer in addition to the preferential one when this answer is shown to them. This relates to the *perseverance* and *coerciveness* of children's relational preference in open problems. To get a view on this, we investigated children's acceptance and rejection behavior (see RQ1 and RQ2 below). Second, we aimed to get a view on the *immediacy* of children's relational preference. For that purpose, we closely examined the reaction time it takes for children with an additive or multiplicative preference (as determined by a pre-test) to accept or reject several subsequently presented answer alternatives (see RQ3 to RQ6 below). More specifically, we were interested in the comparison of the reaction time to (1) accept the preferential answer as compared to the non-preferential answer or to the distractor, or to (2) reject the non-preferential answer as compared to distractors – depending on which of the latter two behaviors occurred in the reaction times experiment. We assumed hereby that longer reaction times indicate that children considered that answer alternative more extensively, as compared to other answer alternatives, even in cases when they ultimately rejected it. This led to the following research questions, whereby RQ3 and RQ5 are mutually exclusive (see RQ1):

1. Do children with a preference for additive or multiplicative relations between numbers (as determined by a pre-test) exclusively accept answers that are in line with their preference in the reaction times experiment; or do they also accept other answers, if so, which ones: the non-preferential answer, or also distractors?
2. Do both preference groups differ in terms of acceptances of the preferential additive or multiplicative answer, as well as other possibly accepted answer alternatives?

3. If children with an additive or multiplicative preference typically not only *accept* their preferential answer but also the non-preferential answer (or a distractor), do the reaction times of accepting these answers differ?
4. Does the reaction time needed to *accept* the preferential answer (or maybe other answer alternatives) differ between both preference groups?
5. If children with an additive or multiplicative preference typically *reject* all answers that are not in line with their preference (including the non-preferential one or a distractor), does the reaction time of rejecting the non-preferential answer differ from the reaction time to reject a distractor?
6. Does the reaction time needed to *reject* an answer that is not in line with a child's preference (be it the non-preferential one or a distractor) differ between both preference groups?

3.2 Method

Participants. Participants were 344 children from 20 classrooms in six elementary schools in Flanders (147 fourth and 197 fifth graders). Four schools were situated in small-sized villages, two were located in a middle-sized city. Based on their responses to the pre-test, participants were further selected to participate in the individual interviews. Here we focus on the 110 children who systematically preferred additive or multiplicative relations on the pre-test.

Instruments. The pre-test consisted of 20 items measuring relational preference that were previously designed and validated in research (Degrande et al., 2018, 2019). This number of items allowed us to distinguish different groups of children's in terms of their preference, without stimulating response tendencies across problems. The specific problems used were the ones presented above in Figure 1, in a constructed-response format, that is a format in which children were asked to fill out the missing number. This could be the additive answer, the multiplicative answer, or even another answer, and all were equally correct and valuable. Therefore, these problems were considered *open* problems. The problems contained small and integer number ratios, to avoid an impact of children's computation skill in the preference test.

The reaction times experiment consisted of 64 items that were shown on a computer-screen using the software package E-Prime. Each item contained a schematic problem similar to the ones used in the pre-test, except for the verification format in which these problems were presented here. That is, an answer was filled out in the shaded box in the presented problem, as shown in Figure 2, and children were asked to indicate by pressing a key whether that answer could be a possible answer in the problem at hand. The total of 64 items consisted of 16 sets of experimental item quartets that were presented in a quasi-randomized order. Each quartet had the same numbers, except for the answer alternative filled out in the shaded box: the additive answer, the multiplicative answer, or one of two distractors. Both distractors were answers that could not be reached by additive nor multiplicative computations, and for which finding another relevant computation was almost impossible for children in upper primary school. The intermediate distractor was a number situated somewhere in between the additive and multiplicative answer, and was a member of the table of multiplication of the upper left number in the arrow scheme, which always was the smallest one. The large distractor was a very large number in between 500 and 1000. Children were not told where the answer that was filled out in the shaded box came from.

a) b) c) d)

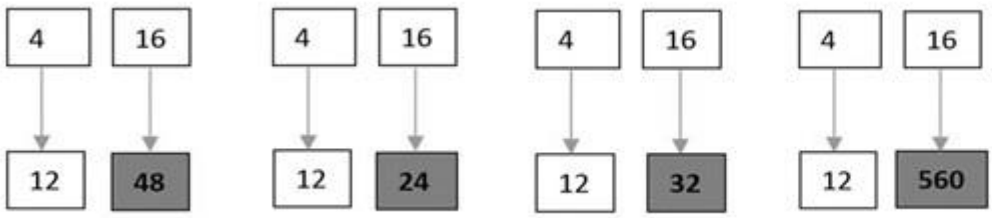

Figure 2. Example of one set of experimental items, containing (a) the multiplicative answer, (b) the additive answer, (c) the intermediate distractor and (d) the large distractor.

Procedure. First, the pre-test was administered collectively per classroom to all children in the sample. This allowed us to identify the children who showed – as compared to the rest of the sample – the most pronounced additive or multiplicative preference, and to select these children for the reaction time experiment. Children belonging to the additive group (n = 29) gave, relative to the sample, the largest number of additive and hardly any multiplicative answers (resulting in min. 10 out of 20 additive answers and max. 1 multiplicative answer). Likewise, the multiplicative group (n = 81) gave, relative to the sample, the largest number of multiplicative (min. 18 out of 20) and no additive answers. Second, the computerized reaction times experiment was administered maximum one month after the pre-test, and several computers were available to allow for parallel testing in small groups when possible. To get children acquainted with the instructions, two practice trials – that involved geometrical figures instead of numbers – were offered before administering the actual test items. The actual experiment consisted of two parts of 32 items each. Children could take a short break in between the two parts. There was no time constraint, but children were asked to fulfil the task as quickly and as correctly as possible. During the experiment, the acceptance behavior as well as the reaction times were logged by the software.

Analyses. All outliers that were further than 3 standard deviations away from a child's average reaction time were deleted (1.88% of the data). Data were further analyzed using Generalized Estimating Equations analyses, because this analysis technique allows to take into account the repeated measures per participant. More specifically, a repeated measures logistic regression analysis was used to test children's acceptance behavior, and a repeated measures linear regression analysis was run to test children's reaction times upon acceptance or rejection, after the positively skewed reaction times were logarithmically transformed. Note that for reasons of interpretation, the reported descriptive statistics of reaction times are based on untransformed values.

To answer all research questions regarding differences between answer alternatives within each preference group, several analyses were run, either with the answer alternatives as independent variable and the acceptance behavior as dependent variable, or with the typically accepted or rejected answer alternatives within a particular preference group as independent variable and the acceptance or rejection reaction times as dependent variable. Furthermore, to compare the two preference groups, several analyses were run with group as independent variable and acceptance behavior, reaction time in case of acceptance, and reaction time in case of rejection as dependent variables, for all relevant answer alternatives separately.

3.3 Results

Acceptance and rejection behavior. First, to answer RQ1, we give an overview of the acceptance and rejection behavior within the group of children who preferred additive relations and within the group of children who preferred multiplicative relations on the pre-test. As can be seen in Table 1, children who preferred additive relations accepted additive answers more often ($p<.001$) than multiplicative answers. Multiplicative answers were still more often accepted by the additive group than the intermediate distractor ($p=.004$), which was in its turn more often accepted than the large distractor ($p=.006$, main effect *Wald* $\chi^2(3)=92.073$, $p<.001$). Likewise, children who preferred multiplicative relations accepted multiplicative answers more often (p<.001) than additive answers. Additive answers were more often accepted by the multiplicative group than the intermediate distractor (p=.017), but we did not find any differences in acceptances of the intermediate and large distractor (p=.665, main effect *Wald* $\chi^2(3)$=342.654, p<.001).

Second, when comparing the acceptances between the two preference groups (RQ2), we found that acceptances of the large distractor (*Wald* $\chi^2(1)$=.691, p=.406) did not differ between groups, while the intermediate distractor was more often accepted by the additive than the multiplicative group (*Wald* $\chi^2(1)$=4.137, p=.042). The additive answer was more often (*Wald* $\chi^2(1)$=73.297, p<.001) and the multiplicative answer less often (*Wald* $\chi^2(1)$=84.479, p<.001) accepted by the additive than multiplicative group. When comparing the acceptances of the preferential answer between groups, the additive group accepted their preferential answer less often (*Wald* $\chi^2(1)$=16.286, p<.001) and their non-preferential answer more often (*Wald* $\chi^2(1)$=10.667, p=.001) than the multiplicative group.

Table 1

Acceptance and Rejection Behavior (in %) of the Additive and Multiplicative Group per Answer Type

	Additive group		**Multiplicative group**	
	Acceptance	**Rejection**	**Acceptance**	**Rejection**
Additive answer	75.1	24.9	9.0	91.0
Multiplicative answer	27.5	72.5	93.6	6.4
Intermediate distractor	10.4	89.6	4.5	95.5
Large distractor	2.6	97.4	3.9	96.1

This discrepancy may be due to a smaller percentage of children in the additive (n = 17, 58.6%) than in the multiplicative group (n = 69, 85.2%) whose acceptance behavior was consistent with their pre-test profile, that is, accepting most additive and hardly any multiplicative answers or distractors, or vice versa, accepting most multiplicative and hardly any additive answers or distractors, respectively. The remaining children of the additive pre-test group either accepted most additive and multiplicative answers and rejected most distractors (6.9%), switched to a multiplicative profile by accepting most multiplicative answers and hardly any additive answers or distractors (6.9%), or showed another behavior that could not be classified in one of the aforementioned categories, such as rejecting or accepting several answer alternatives (27.6%). Amongst the multiplicative group, only 1.2% accepted most additive and multiplicative answers and rejected most distractors, and another 1.2% switched to an additive profile by accepting most additive answers and hardly any multiplicative answers or distractors. Another 12.3% showed another behavior that could not be classified in one of the aforementioned categories. These results already indicate that the multiplicative group more often held on to their preference than the additive group, which seems to be more open towards other answers than the preferential one. For the purpose of our study, only the consistent additive and multiplicative children were of interest for the analysis of reaction times. Therefore, in what follows, we zoom in on the reaction times of the children whose acceptance and rejection behavior was in line with their profile of the pre-test (n = 17 out of 29 for the additive group and n = 69 out of 81 for the multiplicative group).

Reaction times. First, given that children with an additive or multiplicative preference only rarely accepted the non-preferential answer or a distractor (see Table 1), a comparison of the time it took for a child with a given preference to accept the preferential answer on the one hand, and the non-preferential answer or a distractor on the other, as RQ3 stated, is not meaningful. Hence, we immediately turn to a comparison of the reaction times of both preference groups to accept their preferential answer (RQ4). These data reveal that it took longer for children with an additive preference to accept the additive answer than for children with a multiplicative preference to accept the multiplicative answer (*Wald* $\chi^2(1)=9.281$, $p=.002$; see Table 2).

Second, to answer RQ5, we compare the reaction times to reject the non-preferential answer and the two distractors within each preference group. Comparing these reaction times may reveal to what extent children doubt between several answer alternatives, or instead, to what extent relational preference is immediate. As Table 2 indicates, our data reveal that children with an additive preference rejected the multiplicative answer more quickly than the intermediate distractor (p=.026), but more slowly than the large distractor (p<.001). The intermediate distractor was in its turn more slowly rejected than the large one (p<.001, main effect *Wald* $\chi^2(2)=41.672$, p<.001). For children with a multiplicative preference, we found that the additive answer was rejected more quickly than the intermediate distractor (*p*<.001), but more slowly than the large distractor (*p*<.001). The intermediate distractor was in its turn more slowly rejected than the large one (p<.001, main effect *Wald* $\chi^2(2)=89.923$, *p*<.001).

Third, when comparing the rejection times between both preference groups (RQ6), we found that children with a multiplicative preference rejected the non-preferential additive answer more quickly than children with an additive preference rejected the non-preferential multiplicative answer (*Wald* $\chi^2(1)=4.942$, *p*=.026). The intermediate distractor was rejected more quickly by the multiplicative than additive group (*Wald* $\chi^2(1)=7.060$, *p*=.008), while we did not find any group differences in rejecting the large distractor (*Wald* $\chi^2(1)=.244$, *p*=.621).

Table 2

Average Reaction Times of the Additive and Multiplicative Groups, per Answer Type

	Additive group		**Multiplicative group**	
	Mean	**SD**	**Mean**	**SD**
Additive answers	8366.0	8198.0	5505.0	3220.1
Multiplicative answers	7972.8	6847.8	5503.8	3490.8
Intermediate distractor	8418.5	6950.5	5959.8	3544.3
Large distractor	4926.2	4693.2	4603.7	3218.1

Note. Acceptance reaction times are indicated in grey, rejection reaction times in white.

3.4 Conclusion and discussion

This study revealed how children who preferred additive or multiplicative relations, as determined by a pre-test, react when separately and subsequently being confronted with additive answers, multiplicative answers, or distractors in open problems. First, based on children's acceptance and rejection behavior, we found that both preference groups generally stuck to their preferential answer, and that they rejected most other answers, including the non-preferential answer and all distractors. While the first finding demonstrates the *perseverance* of children's relational preference in open problem, the second supports its *coerciveness*. This was more pronounced in the multiplicative group, in which children more often solely held on to their preferential answer of the pre-test than the additive group. Second, relying on the accompanying reaction times, we characterized relational preference as *immediate*, as children rejected the non-preferential answer more quickly than the intermediate distractor. Further, children in the multiplicative group accepted their preferential answer and rejected the non-preferential answer more quickly than the additive group, suggesting that the multiplicative preference is more immediate than the additive one. Moreover, and importantly, this difference was not due to overall quicker reaction times of the multiplicative group, since the large distractor was rejected equally fast in both groups. The more frequent and quicker rejection of the intermediate distractor in the multiplicative than additive group may be due to the fact that this distractor is a member of the table of multiplication, and may therefore more easily be detected as irrelevant by the multiplicative than additive group.

4. Study 2: Interview study

4.1 Research questions

Study 2 aimed at replicating the results of Study 1, but using a very different methodology. Children who preferred additive or multiplicative relations (as determined by a pre-test) were invited to participate in a semi-structured interview focused on open problems. Comparing the responses of children with an additive and multiplicative preference throughout the interview, in which they were increasingly stimulated to consider the non-preferential answer too, allowed us to characterize the nature of this preference. First, we aimed to reveal how children with an additive or multiplicative preference answered the open problems initially and throughout the interview, which relates to the *perseverance* and *coerciveness* of relational preference (see RQ1 and RQ2 below). Second, we wanted to get a view on the *self-evidence* of children's relational preference by means of children's verbalizations of their initial answer and the alternative being offered, initially and throughout the interview process (see RQ3 and RQ4 below), and third, on its *certainty* by relying on the accompanying conviction scores (see RQ5 and RQ6 below). This led to the following research questions:

1. Do children with an additive or multiplicative preference initially only give answers that are in line with their preference, or do they consider the non-preferential answer as a possible answer alternative too? How easily do they consider this non-preferential answer throughout the interview process?
2. Do both preference groups differ in terms of considering the non-preferential answer as a possible answer alternative, initially and throughout the interview process?

3. How do children with an additive or multiplicative preference verbalize the underlying reasons for giving their answer? And how do those verbalizations change throughout the interview process?
4. Do both preference groups differ in those verbalizations throughout the interview process?
5. Do the conviction scores of children with an additive or multiplicative preference indicate a discrepancy in conviction of the additive and multiplicative answer? If so, does this discrepancy remain throughout the interview process?
6. Do both preference groups differ in those conviction scores throughout the interview process?

4.2 Method

Participants. Participants of the pre-test were 145 children (76 fifth and 69 sixth graders) from 7 classrooms in three Flemish primary schools. These were other schools than the ones involved in Study 1. One school was situated in a small-sized village, two were located in a middle-sized city. Based on their responses to the pre-test, children were further selected to participate in the individual interviews. As in Study 1, we focus on 18 children who systematically preferred additive or multiplicative relations on the pre-test.

Materials. The pre-test measured children's relational preference, by means of 32 open problems that contained integer number ratios. Diverse open problems were used, including the arrow schemes as in Study 1. The interview protocol was restricted to two open problems, which were presented as two subsequent trials, in order to avoid stimulating response tendencies across problems. Both open problems contained small and integer ratios, to minimize the interference of children's computation skill[1]. More specifically, an arrow scheme as in Study 1 was initially offered in a constructed-response format (Cronbach, 1984; see Figure 1) rather than in a verification format, so that children could come up with any answer they could possibly think of (and thus more than one answer if they wanted to). Throughout the interview protocol (i.e., in Phases 3 and 4, see Procedure below), however, this constructed-response format changed into a verification format, when the non-preferential answer alternative (i.e., the additive answer if the child gave a multiplicative one and vice versa) from

[1] At the end of the pre-test and the interview, some items containing non-integer ratios were also included, but these are not considered in the present paper. We focus on the items that are similar to these administered in Study 1.

a fictitious pupil was given. Children were then told that this answer was given by another pupil from another school.

Procedure. Similarly to Study 1, the pre-test was administered collectively per class group. Based on the pre-test results, we identified the children with the most distinct profiles in terms of preferences, and invited them to participate in individual semi-structured interviews. This led to an additive group consisting of 10 children who mainly gave additive answers and hardly any multiplicative answers to these integer pre-test items (min. 18 additive and max. 6 multiplicative answers), and a multiplicative group consisting of 8 children who mainly gave multiplicative answers and hardly any additive answers to these integer pre-test items (min. 26 multiplicative and max. 3 additive answers). In general, the semi-structured interviews took place about three weeks after the pre-test in a quiet room at school, in a one-to-one interview situation. The interviews were audiotaped. The interview around each of the two open problems consisted of four phases, in which children were increasingly stimulated to consider the non-preferential answer too. It was stopped in the phase where both the additive and the multiplicative answers were accepted as possible answers to these open problems.

The interview protocol was inspired by the one applied by study of De Bock, Van Dooren, Janssens, and Verschaffel (2002) and which had been used there to analyse the nature of children's erroneous multiplicative answers to non-multiplicative problems. The protocol consisted of four phases. In *Phase 1* of each trial, each child was first asked to answer the open problem, to explain how this answer was found, and to indicate and explain how convinced (s)he was of that answer by choosing a position on a four-point scale (from "not at all convinced" to "very convinced"). If the child only gave one answer in Phase 1, *Phase 2* was initiated. In this phase, the child was asked whether any *other* answer could be a possible answer in the problem at hand and if so, which one. If (s)he thought this was the case, the same questions as in Phase 1 were also asked about that alternative answer. If the child thought there was no other answer possible, (s)he proceeded to *Phase 3*, where (s)he was shown the non-preferential answer alternative (i.e., the additive answer if the child gave a multiplicative one and vice versa) from a fictitious pupil and was asked about his/her thoughts about that answer. If a child still could not explain the fictitious child's answer by the end of Phase 3, the computations leading to the non-preferential answer alternative were explained by the interviewer in *Phase 4*, and the child could again react to it. Whenever the non-preferential answer alternative entered the picture – whether in Phase 1, 2, or 3 – the child was asked to explain this answer and indicate on another four-point scale how convinced (s)he was of that answer – which was explicitly compared with the conviction score of the child's initial answer. Children were told that they could change the conviction scores for either answer at any time during the interview. The interview for the first trial ended when the non-preferential answer was given by the child to the open problem or when it was accepted as a possible answer too. Then a second open problem was presented, and for this second trial, the interview protocol – as presented above – restarted at phase one and continued until the non-preferential answer was given or accepted.

Analyses. Data were analysed by systematically looking at similarities and differences between additive and multiplicative groups as they emerged throughout the interview, in terms of the *answer* that was given or accepted, the *conviction scores* of the preferential and the non-preferential answer, and the accompanying *verbalizations*.

4.3 Results

First trial. When initially confronted with the open problem in *Phase 1*, reactions were as could be expected based on the pre-test; almost all children with an additive preference solely gave the additive answer, and likewise, almost all children with a multiplicative preference solely gave the multiplicative answer (see Table 3). Only one child from each preference group gave both the additive and multiplicative answer. Children of both preference groups were, moreover, strongly convinced of their answers, as indicated by their very high conviction scores (on the highest or second highest level, by 6 out of 7 children in the additive and 8 out of 8 in the multiplicative group). This especially held for the multiplicative group, where 6 out of 8 children initially gave conviction scores at the highest level, while 4 out of 7 children of the additive group initially gave conviction scores on the second highest level. Despite their high conviction scores, children from both preference groups had great difficulties explaining *why* they gave their multiplicative or additive answers, merely relying on superficial explanations referring to implicit task-related expectations, *"It is plus 6 here, and here you maybe have to do plus 6 too"* (A3) or *"I look at what happened from 5 to 15, that was times 3. Then I also did 10 times 3"* (M119), or the difficulty level of the task, *"It is actually pretty easy"* (M17). Especially children with a multiplicative preference sometimes referred to the correctness of their answers, "*Because I am sure that it is correct, I just know that it is correct"* (M115) or "*Because I think I am sure that this is correct*" (M114).

In *Phase 2*, children rarely gave another answer. Most children responded *"No"* or *"I don't think so"* when asked whether any other answer was possible (resp. 4 out of 6 and 6 out of 7 remaining children in the additive or multiplicative group). The few children who did come up with the non-preferential answer had a hard time justifying it. As in Phase 1, they referred to the difficulty level of the task *"because again, it was [an] easy [question]"* (M17) or the correctness of their answer *"because I am almost sure that it is correct"* (A11). As a result, 3 out of 7 children of the additive group and only 2 out of 8 of the multiplicative group had given both answers by the end of Phase 2.

Table 3

Number of Children of the Additive or Multiplicative Groups who Gave or Accepted Both Answers per Phase per Trial

	Trial 1				**Trial 2**			
Preference group	Phase 1	Phase 2	Phase 3	Phase 4	Phase 1	Phase 2	Phase 3	Phase 4
Additive (*n*=7)	1	2	1	3	3	3	1	0
Multiplicative (*n*=8)	1	1	2	4	3	5	0	0

Note: Three children of the additive group were excluded from the analyses, as they only gave the non-preferential multiplicative answer in Phase 1 of the interview.

Only very few children who were confronted with the non-preferential answer of a fictitious pupil in *Phase 3* accepted it (resp. one and two of the additive and multiplicative group). However, the underlying reason for this behavior tended to differ between both preference groups. 2 of the 3 children of the additive group who did not accept the non-preferential answer reported that they did not know how the fictitious pupil obtained this solution, *"I think this [answer] is strange. […] Because I don't know what he [the other pupil] does here"* (A3) and "*I do not know [what the other pupil did]. Yes, I just cannot explain it*" (A74), whereas all four remaining children of the multiplicative group explicitly discarded the (additive) solution as being incorrect: *"A bit wrong […]. Because you need to do the same with this [arrow]. And she [other pupil] has not done that. She did times 2, and here you should have done times 4"* (M115). This also resulted in lower scores on the conviction scale for this non-preferential answer, as compared to the preferential one, in both preference groups (i.e., 2 of the 3 of the additive and 4 out of 4 children of the multiplicative group who did not accept the non-preferential answer). A similar discrepancy in the conviction scores for the preferential and non-preferential answer was found in the two children of the multiplicative group who did accept the non-preferential answer. They were still more convinced of the preferential multiplicative than the additive answer, *"Yes, this [the additive answer] is good but I tend to work with × more quickly than with +. […] Because I rather think that × is correct than +"* (M130). However, we did not find a similar discrepancy in the conviction scores in the only child of the additive group who accepted the non-preferential answer in Phase 3.

At the end of Phase 3, 3 children of the additive group and 4 children of the multiplicative group still had not given both answers yet (see Table 3). Those children proceeded to *Phase 4*, where the computations behind the alternative answer were explained by the interviewer. This led to an increased conviction score of the non-preferential answer in almost all children in the additive group (2 out of 3) and multiplicative group (4 out of 4). The explanation of the fictitious answer functioned as an "Aha-Erlebnis" (Wertheimer, 1945), *"Aaaaah! Now I know it. […] Very smart. […] Yes, because I had not seen it myself and she [other pupil] sees that"* (M115). They called this alternative answer smart, *"Also smart. […] that she has seen well that you multiply by 3 here"* (A93), and admitted that they had not thought about it before, *"[…] because I had not thought about plus"* (M114), or *"because only now, I see that this goes like this [plus] as well"* (M115). While many children referred to the equality of both answers *"Because she has calculated it in another way […] which is also just fine*" (A93), or "*Because both are good, I think*" (M 114), some children still privileged the preferential answer "*Yes, + is possible too, but I think it is ×*" (M83).

All children were, at some point in the first trial, brought to understand the alternative answer as well, and to get to a certain degree convinced of that answer. However, when comparing the conviction scores of the preferential and non-preferential answer of both preference groups by the *end of this first trial*, it was found that the multiplicative group tended to privilege their preferential answer more strongly than the additive group. In the additive group, only one child gave a higher conviction score to the preferential than non-preferential one, while two others even gave higher conviction scores to the non-preferential than preferential answer. The inverse trend was observed in the multiplicative group, where three children gave a higher conviction score to their preferential than non-preferential answer, and only one gave a higher conviction score to the non-preferential one. The other four children of each of both preference groups gave equal conviction scores to the preferential and non-preferential answer by the end of this first trial. We were then interested in children's reaction when they were involved in a second, very similar trial. We report the results for the second trial much more briefly, focusing on the new information that was obtained compared to the first trial.

Second trial. In *Phase 1* of the second trial, 5 out of 8 children of the multiplicative group answered solely multiplicatively, while only 2 out of 7 children of the additive group solely gave the additive answer. In addition, none of the children of the multiplicative group only gave the additive answer, while 2 out of 7 children belonging to the additive group initially solely gave the multiplicative answer. These relative differences suggest that the multiplicative group stuck more strongly to their preferential answer than the additive group. Moreover, the children who answered either additively or multiplicatively in the respective groups were strongly convinced of their preferential answers (resp. 2 out of 2 and 3 out of 5 gave the highest possible conviction score), although they had difficulties in explaining why, indicating that they just "saw" what was expected based on the given numbers, such as *"Here they have done 2 times 4, and then here I have done 6 times 4"* (M119) or *"Because I do plus here, and also here I do plus. Cause there you add 10, so here you add 10 too"* (A3). As in the first trial, some children of the multiplicative group who solely gave their preferential answer again stressed the correctness of their own answer, but now implicitly considering the other answer too: *"Because I don't know which one is correct. But I think that mine is correct, but I am still doubting"* (M83) or "*Because my answer is correct too, that's why I am very convinced"* (M115), while children of the additive group who solely gave their preferential answer did not refer to the correctness of their answer, but rather to implicit task-related expectations. Most children (2 of the 3 children of the additive group, and all 3 children of the multiplicative group) who gave both answers in Phase 1 allocated the same conviction scores to both answers. Although this was often reflected in children's verbalizations, "*Because there are again, multiple possibilities*" (A24) or "*This is correct too, it is just another way of thinking, but it is simply all correct because it is not written here whether you need to multiply or add. Actually all of this is correct."* (A78) or *"Actually there is still another way…"* (M17), children sometimes still seemed to privilege their preferential answer *"Because I do think that this one is good, but I don't know for sure whether it is completely correct. I just think this is good."* (M114).

In *Phase 2*, almost all children came up with the alternative answer. While 2 of the 3 children in the additive group and 3 out of 5 in the multiplicative group did not report a discrepancy in the conviction scores of both answers, the verbalizations of others still expressed that they attached different values to both answers. This occurred in the additive group,*"Uhm, because of the first one [solution], I find it somewhat better. And I find this [the second one] a little bit stranger. There you need to think more about it. Well, in my case…"* (A3) and in the multiplicative group*, "This is the first time that I used plus. I do think that it is also correct, but I am not very convinced"* (M95), or *"Because I am a little less convinced, I am not quickly inclined to work with plus"* (M130). Those children literally expressed their preference but seemed unable to explain *why* this was the case.

Only one child of the additive group proceeded to *Phase 3*, but not to Phase 4 (see Table 3). This child had initially given the multiplicative answer in Phase 1 of the second trial, but accepted the additive answer upon presented: "*Yes, it goes like this, but then you have half here, but not here. […] But actually, it does go like this... […] Plus 10 and plus 10*" (A11).

4.4 Conclusion and discussion

In Study 2, we investigated how children with an additive or multiplicative preference – as determined by a pre-test – responded (i.e., their answer, their verbalizations, and their conviction scores) throughout a semi-structured interview in which they were increasingly stimulated to consider the non-preferential answer too in two subsequent open problems. First, related to the *perseverance* of children's relational preference in open problems, we found that most children in both preference groups initially only gave the preferential answer. The *coerciveness* of their preference was further shown, since most of them also did not come up with another answer when explicitly asked, and many of them did not accept the alternative answer when it was initially shown. Even in a second, very similar trial, many children fell back to their preferential answer at first without acknowledging both the additive and multiplicative answer as possibilities. However, this perseverance and this coerciveness were most apparent amongst the multiplicative group. The additive group tended to consider the non-preferential answer somewhat earlier in the interview of the first trial, and even in a second trial, solely giving the preferential answer was less prominent in the additive group, as compared to the multiplicative group. With respect to the verbalizations, we found that, although both preference groups gave their preferential answer, it seemed hard to justify why, which is an indication for the *self-evidence* of relational preference. Even when finally accepting the alternative answer, some verbalizations of both preference groups still suggested a preference for either additive or multiplicative relations between numbers. However, the multiplicative group, as compared to the additive group, more often called their preferential answer "correct" and the alternative answer "incorrect" and kept on privileging their preferential answer in their verbalizations. Furthermore, regarding the *certainty* of their relational preference, both preference groups initially allocated very high conviction scores to the preferential answer, resulting in a discrepancy between the conviction scores of the preferential and non-preferential answer in the third phase of the interview. The multiplicative group not only gave higher initial conviction scores to their preferential answer than the additive group, the discrepancy in conviction scores amongst the multiplicative group even remained by the end of the first trial, while the additive group was at that moment in general at least equally convinced of the non-preferential answer.

5. General conclusion and discussion

5.1 Conclusion

Previous studies already investigated the existence and development of relational preference as well as its importance in explaining inappropriate word problem solving by means of cross-sectional studies in which paper-and-pencil tests were collectively administered to large groups of children. While these studies clearly revealed the existence, development, and importance of relational preference, they also evoked questions related to its nature. As in the research on preferences in other domains, the present research aimed to characterize the nature of a relational preference, and was hereby inspired by theorizing and research on intuitions. While previous research in the domain of additive and multiplicative reasoning has already repeatedly suggested that the aforementioned systematic errors in missing-value word problems have at least partly an intuitive nature (for an overview, see Van Dooren et al., 2008), systematic empirical evidence was lacking, especially for relational preferences. Based on the characteristics of intuition as postulated and defined by Fischbein (1987), we investigated whether relational preference is *perseverant*, by looking at the extent to which children stuck to their preferential answer, and whether it exerted a *coercive* effect on their reasoning by investigating the extent to which children were reluctant to accept alternative answers than their preferential one to open problems. Further, we investigated whether relational preference is *immediate* by looking at how quickly children accepted or rejected the preferential and alternative answers, *self-evident* by looking at the justifications that children gave for giving their preferential answers and when being confronted with alternative answers, and *certain* by investigating how high the conviction scores were that children allocated to their preferential answers as compared to alternative ones. This was investigated by means of a mixed-method approach consisting of two studies using very different but complementary methodologies: a reaction times study and a study using semi-structured individual interviews.

Regarding the *perseverance and coerciveness* of children's relational preference, the results of Study 1 revealed that children who had demonstrated a preference in a pre-test most often accepted their preferential answer and rejected all alternative answers, whether it was one of the distractors or the non-preferential additive or multiplicative answer. Likewise, in Study 2, children mostly came up with the preferential answer and did not often spontaneously give an alternative answer in a first trial. Most of them did not give the alternative answer when explicitly asked to do so, nor did they easily accept it when it was presented to them. In both studies, children perseveringly gave precedence to their preferential type of relations, and were thus resistant to change their preferential answer. Further, they held on to it when being confronted with alternative answers, which demonstrates the coercive effect of relational preference on children's reasoning.

These indications of perseverance and coerciveness were even more prominent in the multiplicative than additive group. Compared to the multiplicative group, the additive group accepted their preferential answer less often and their non-preferential answer more often in Study 1. Likewise, in Study 2, the additive group tended to consider the non-preferential answer somewhat earlier in the interview of the first trial, and even in a second trial, solely giving the preferential answer was less prominent in the additive group, as compared to the multiplicative group. Overall, children who preferred additive relations stuck less strongly to their preferential answer and were more open towards the non-preferential answer than children who preferred multiplicative relations. Or, put differently, the multiplicative group was even more resistant to change their preferential answer and thus to additionally consider the non-preferential answer than the additive group.

Results regarding *immediacy* indicated that it took longer for both preference groups to reject the non-preferential answer than to reject the large distractor in Study 1, which is not that surprising given that this distractor is a very large number that children can at first sight already discard. However, the non-preferential answer, for which it is relatively easy to find the correct computation, was more quickly rejected than the intermediate distractor, for which finding a relevant computation leading to that answer is almost impossible for children in upper primary school. So, the non-preferential answer was considered less extensively than a distractor of comparable size that cannot be obtained additively or multiplicatively before it was rejected. Further, related to the *self-evidence* and *certainty* of relational preference, in Study 2 it seemed hard for children to justify why they had given this preferential answer although they were initially very convinced of this preferential answer.

When comparing both preferences in terms of immediacy, certainty, and self-evidence, this was more pronounced for the multiplicative preference than for the additive one. In Study 1 the additive group accepted the preferential answer and rejected the non-preferential answer more slowly than the multiplicative group, and in Study 2 the multiplicative group more often discarded the non-preferential additive answer as being incorrect without being able to explain why, whereas the additive group more often reported that they did not know where the non-preferential multiplicative answer came from. At the same time, especially the multiplicative group kept on privileging their preferential answer in their verbalizations, while the additive group tended to gradually acknowledge the validity of the non-preferential answer throughout the interview process. For some children in the additive group, this was accompanied by even higher conviction scores for the non-preferential than for the preferential answer by the end of the first trial, and even with giving the multiplicative answer solely or as well as the additive answer in a second trial.

5.2 Theoretical, methodological and educational implications

Theoretical implications. The term relational preference represents the phenomenon that one type of relations takes precedence over another, in a way that can be characterized as perseverant, coercive, immediate, self-evident, and certain. Hence, both the additive preference and the multiplicative one may be characterized as intuitive in nature, which was supported by empirically examining the aforementioned characteristics of intuitions postulated by Fischbein (1987, 1993). Despite this common core of the phenomenon of a preference for additive or multiplicative relations between numbers, we also revealed that the aforementioned characteristics of relational preference seem to be more pronounced for the multiplicative than additive preference in these upper primary school children. Those differences between preferences may be due to their differential origins. While it still remains an open question to what extent other child characteristics contribute to the differences between two preference groups, an analysis of the educational curriculum reveals some educational factors that may contribute to the development of preference. Given that multiplicative missing-value word problems take a very prominent place in primary school whereas additive missing-value word problems seem absent, the multiplicative preference may be more strongly fed by education than the additive one. Hence, in line with Fischbein's (1987) distinction, these preferences may be identified as primary versus secondary intuitions, respectively. Primary intuitions arise from implicit learning in all kinds of daily life experiences, while secondary intuitions "are acquired, not through natural experience, but through some educational intervention" (p. 71). Nevertheless, one needs to bear in mind the context-specificity of these students' everyday life experiences as well as their school mathematics experiences. Despite that the selected children who preferred additive or multiplicative relations belonged to many different families and classrooms, one should be cautious in generalizing these results to other children, particularly in other countries and/or educational systems. Further research could show to what extent our findings are generalizable.

In this respect, the upper grades of primary school may be a rather late time to measure children's preference for additive relations, given that mainly children in third and fourth grade of primary education have been found to prefer additive relations (Degrande et al., 2018, 2019). However, we focused on upper primary school children's relational preference, and particularly in integer items where both additive and multiplicative computations are relatively easy, to minimize the interference of lacking computation skill in preference tasks. Although a preference for additive relations between numbers is harder to investigate in children in middle primary school as compared to children in upper primary school because their computation skills are less well developed, this preference for additive relations may occur more often and may be more pronounced in this group. It thus remains an open question whether the nature of an additive preference would have a similar characterization in middle than the upper primary school children involved in our two studies.

Although research on the nature of relational preference is still in its infancy as compared to the well-established research lines of preferences in the domains of psychology, economics, law, politics, or medicine, we can already cautiously relate our results to what is known about the nature of preference in these domains. The finding that relational preferences can not only be characterized as immediate, certain, and self-evident, but also as perseverant and coercive, implies a certain degree of stability of relational preference within a child. Hence, relational preferences may be considered as somewhat more stable than other kinds of preferences in the aforementioned domains. Nevertheless, given the evidence base on the nature of preference in these other domains (Lichtenstein & Slovic, 2006; Slovic, 1995), also the utterance of children's relational preference is at least partly dependent on the concrete situation in which one is. So caution is warranted when generalizing our findings on the nature of relational preference to other open problems, as well as to other groups of children with relational preferences who bring other home environmental and educational experiences with them in solving these problems.

Methodological implications. In contrast to previous research on children's relational preference that mainly involved cross-sectional studies using paper-and-pencil tests, here we used two distinct but complementary methods: reaction times and semi-structured individual interviews. As argued by Tashakkori and Teddlie (2003), mixed-method research benefits from the complementary strengths of methods, and can offset their disadvantages. Reaction times, as used in Study 1, are recognized as an important physical measure of mental processes (Welford, 1980). However, reaction times are often criticized because of the indirect nature of the measurement of processes, and this is especially an issue in more complex mathematical tasks that involve several steps (Welford, 1980). Hence linking reaction times to other behavioral measures is necessary. Although the tasks of Study 1 are relatively simple, we still looked at reaction times in relation to the acceptance and rejection of the distinct answer alternatives. Moreover, the reaction time data of Study 1 were triangulated with verbal reports gathered via semi-structured individual interviews in Study 2.

Compared to the implementation of reaction time measures, the use of verbal reports in mathematics education research has a long tradition, but verbal reports have their drawbacks too (e.g., Ericsson & Simon, 1980; Nisbett & Wilson, 1977). First, verbal reports may be incomplete, because the cognitive process taking place may be unknown to the child. This way, the child does not have a clear or a correct idea of the process that preceded the answer. To avoid this drawback, we used conviction scales too, and we explicitly asked children to compare conviction scores for different answer alternatives, in order to make their reasoning explicit. The finding that children were, despite these measures, seemingly still not able to verbalize their reasoning was considered as a first major result. It indicates the self-evidence of children's relational preference. This self-evidence is considered by Fischbein (1987) as the fundamental characteristic of intuitions, and can be explained by the fact that the processes on which an intuition are based are tacit and thus remain implicit to the child. Given these characteristics of intuitions, it is logical that children's reasons for giving their preferential answers are to a certain extent inaccessible for introspection or reflection. Second, and relatedly, verbal reports may yield information that is not in line with the actual solution process, especially when participants are asked to verbalize ways of reasoning that are hard to verbalize. To overcome this drawback, we asked children for a global rather than a very detailed description of their way of reasoning. The self-evident nature of children's relational preference resulted in rather superficial explanations of their answering behavior, since the preferential answer was considered by the child as self-explanatory and therefore did not require any further justification. We don't have any indications that these explanations would not be in line with the actual solution process and hence with the final answers. Third, asking for a verbal report may distort the spontaneous solution process because it may elicit more reflective solution behavior, which may in its turn affect the answer that is finally given. This impact was limited by relying on retrospective verbalizations that were gathered immediately *after* task completion and by using tasks of short duration.

Educational implications. These two studies were primarily set up to understand the nature of relational preference, but some tentative lessons can be drawn regarding mathematics education nonetheless. First, given the nature of preference, it does not seem self-evident at all to de-construct preferences once they have been established. This is nevertheless important, since those preferences underlie at least partly the frequent errors children make in word problems (Degrande et al., 2019). Relying on the interview process of Study 2 and on literature on remedying intuitions (Fischbein, 1987; Nelissen, 2013), it may be feasible to de-construct preferences – albeit very slowly and gradually. This may be done by repeatedly using a variety of educational approaches in which children are asked to solve, classify, or pose word problems; are confronted with or are asked to come up with other answers than their preferential one that are valid as well in open problems; are asked to discuss diverse solutions of problems and hereby to verbalize the considerations they make when deciding on the appropriateness of a solution, etc. Such educational activities may evoke a conflict between a child's relational preference and the underlying mathematical model or between children with different preferences. Such an experience may initiate a process in which children may overcome the reliance on their preferential type of relations. Second, ideally this goes hand-in-hand with enlarging the "example space" (Watson & Mason, 2005), which is the whole collection of example problems a child has access to, by relying on open problems besides additive or multiplicative ones. More specifically, not only open problems that are suitable for research – as the ones used in these two studies – could be used. Also other open problems that represent real-life comparison and change situations that can be viewed additively as well as multiplicatively could be engaged in this kind of instruction (Lamon, 2008; Nunes & Bryant, 2010; Van Dooren et al., 2018). Take, for instance, the following problem: "Suppose there are 12 girls and 18 boys in a class and they are assigned to single-sex groups during French lessons. If there were not enough books for all of them and the Head Teacher decided to give 4 books to the girls and 6 books to the boys, would this be fair?" (Nunes & Bryant, 2010, p. 4). While children who prefer additive relations would answer that this is unfair, since only eight girls but twelve boys don't have a book, children who prefer multiplicative relations would claim that this is fair, since three boys share one book and three girls share one book.

Second, besides de-constructing relational preferences once they have been established, one may rather want to prevent the development of preferences. While we acknowledge that completely preventing the development of preferences in children is not possible – since preferences are an inherent part of children's learning and human's decision making – intervening from an early age on may have an impact on the nature of these preferences. Especially given the recent evidence that not only additive but also multiplicative reasoning skills emerge from an early age on (e.g., Boyer et al., 2008; Boyer & Levine, 2012; Jeong et al., 2007), it seems advisable to explicitly build not only on children's informal knowledge of additive relations but also on their knowledge of multiplicative ones, and to progressively develop to more abstract ways of presenting these relations, involving numerical quantification and by using word problems besides using non-quantified quantities and real world situations. Further, there still seem to be elements of the current educational environment that may trigger and even reinforce the intuitive nature of these relational preferences and that therefore may need to be avoided. A first one is the stereotyped offer of problems that may shape children's relational preference, as is the case in the current mathematics education curriculum in Flanders, where most multiplicative word problems in upper primary education do have a missing-value format while additive missing-value problems are completely missing (George, 2015). In the light of the result of the present studies, it seems valuable to include both additive and multiplicative word problems in the curriculum. These should moreover involve diverse problem characteristics such as the problem format, problem context, number characteristics, etc. Secondly, in terms of instructional approaches, not only the fluent execution of computation skills should be emphasized – as is nowadays especially the case for the operation of multiplication, but attention needs to be paid at the applicability of these operations in the problem at hand as well. These measures, related to the mathematics curriculum and the instructional approach, may seem especially important to prevent the development of a multiplicative preference, given that it may be more strongly fed by education.

References

Babai, R., Levyadun, T., Stavy, R., & Tirosh, D. (2006). Intuitive rules in science and mathematics: A reaction time study. *International Journal of Mathematical Education in Science and Technology,* 37, 913–924. doi:10.1080/00207390600794958

Bailey, D. H., Littlefield, A., & Geary, D. C. (2012). The codevelopment of skill at and preference for use of retrieval-based processes for solving addition problems: Individual and sex differences from first to sixth graders. *Journal of Experimental Child Psychology, 113,* 78-92. doi:10.1016/j.jecp.2012.04.014

Behr, M., & Harel, G. (1990). Understanding the multiplicative structure. In G. Booker, P. Cobb, & T.N. de Merldicutti (Eds.), *Proceedings of 14th conference of the International Group for the Psychology of Mathematics Education* (pp. 27-34). Mexico, Mexico: Consejo Nacional de Ciencia y Technologia, Gobierno del Estado de Morelos.

Boyer, T. W., Levine, S. C., & Huttenlocher, J. (2008). Development of proportional reasoning: Where young children go wrong. *Developmental Psychology, 44,* 1478–1490. doi:10.1037/a0013110

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. *Journal for Research in Mathematics Education, 27*, 41-51. doi:10.2307/749196

Cramer, K., & Post, T. (1993). Connecting research to teaching proportional reasoning. *Mathematics Teacher, 86*(5), 404-407.

Cramer, K., Post, T., & Currier, S. (1993). Learning and teaching ratio and proportion: Research implications. In D. T. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 159-178). New York, NY: Macmillan Publishers.

Cronbach, L. J. (1984). *Essentials of psychological testing* (4th ed.). New York, NY: Harper & Row.

Csíkos, C. (2016). Strategies and performance in elementary students' three-digit mental addition. *Educational Studies in Mathematics, 91,* 123-139. doi:10.1007/s10649-015-9658-3

De Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics, 50*, 311-334, doi:10.1023/A:1021205413749

Degrande, T., Verschaffel, L., & Van Dooren, W. (2018). Beyond additive and multiplicative reasoning abilities: How preference enters the picture. *European Journal of Psychology of Education, 33,* 559-576. doi:10.1007/s10212-017-0352-y

Degrande, T., Verschaffel, L., & Van Dooren, W. (2019). To add or to multiply? An investigation of the role of preference in children's solutions of word problems. *Learning and Instruction, 61,* 60-71. doi:10.1016/j.learninstruc.2019.01.002

Ericsson, K. A., & Simon, H. A. (1980). Verbal reports as data. *Psychological Review, 87*, 215-251. doi: 10.1037/0033-295X.87.3.215

Evans, J. S. (2008). Dual-processing accounts of reasoning, judgment and social cognition. *Annual Review of Psychology, 59,* 255-278. doi:10.1146/annurev.psych.59.103006.093629

Fernández, C., Llinares, S., Van Dooren, W., De Bock, D., & Verschaffel, L. (2012). The development of students' use of additive and proportional methods along primary and secondary school. *European Journal of Psychology of Education, 27,* 421-438. doi:10.1007/s10212-011-0087-0

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach.* Dordrecht, the Netherlands: Reidel Publishing Company.

Fischbein, E. (1999). Intuitions and schemata in mathematical reasoning. *Educational Studies in Mathematics, 38*, 11–50. doi:10.1023/A:1003488222875

George, S. (2015). *When to add and when to multiply: A textbook analysis on task characteristics in additive and proportional reasoning* [Unpublished master's thesis]. KU Leuven, Belgium.

Gillard, E., Van Dooren, W., Schaeken, W., & Verschaffel, L. (2009). Proportional reasoning as a heuristic-based process: Time pressure and dual-task considerations. *Experimental Psychology, 56*, 92-99. doi:10.1027/1618-3169.56.2.92

Harel, G., & Behr, M. (1989). Structure and hierarchy of missing value proportion problems and their representations. *Journal of Mathematical Behavior, 8*(1), 77–119.

Hart, K. (1981). Ratio and proportion. In K. Hart (Ed.), *Children's understanding of mathematics: 11-16* (pp. 88-101). London, United Kingdom: John Murray.

Hoffer, A. (1988). Ratios and proportional thinking. In T. Post (Ed.), *Teaching mathematics in grades K-8: Research based methods* (pp. 285-313). Boston, MA: Allyn & Bacon.

Jacob, L., & Willis, S. (2003). The development of multiplicative thinking in young children. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia*. Geelong, Australia: Mathematics Education Research Group of Australasia.

Jeong, Y., Levine, S., & Huttenlocher, J. (2007). The development of proportional reasoning: Effect of continuous vs. discrete quantities. *Journal of Cognition and Development, 8*, 237–256. doi:10.1080/15248370701202471

Kaput, J. J., & West, M. M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 235-287). New York, NY: State University of New York Press.

Karplus, R., Pulos, S., & Stage, E. (1983). Proportional reasoning of early adolescents. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes* (pp. 45–89). New York, NY: Academic Press.

Lamon, S. J. (1993). Ratio and proportion: Connecting content and children’s thinking. *Journal for Research in Mathematics Education, 24,* 41-61.doi:10.2307/749385

Lamon, S. J. (2008). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (2nd ed.). New York, NY: Taylor & Francis Group.

Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93-118). Reston, VA: Lawrence Erlbaum & National Council of Teachers of Mathematics.

Lichtenstein, S., & Slovic, P. (2006). The construction of preference: An overview. In S. Lichtenstein, & P. Slovic (Eds.), *The construction of preference* (pp. 1-40). New York, NY: Cambridge University Press. doi:10.1017/CBO9780511618031

Misailidou, C., & Williams, J. (2003). Children’s proportional reasoning and tendency for an additive strategy: The role of models. *Research in Mathematics Education, 5,* 215-247. doi:10.1080/14794800008520123

Modestou, M., & Gagatsis, A. (2010). Cognitive and metacognitive aspects of proportional reasoning. *Mathematical Thinking and Learning, 12*, 36-53. doi:10.1080/10986060903465822

Mulligan, J. (1992). Children's solutions to multiplication and division word problems: a longitudinal study. *Mathematics Education Research Journal, 4,* 24-41, doi:10.1007/BF03217230

Nelissen, J. M. C. (2013). Intuition and problem solving. *Curriculum and Teaching, 28*(2), 27-44.

Nisbett, R. E., & Wilson, T. D. (1977). Telling more than we can know: Verbal reports on mental processes. *Psychological Review, 84*, 231-259. doi:10.1037/0033-295X.84.3.231

Noelting, G. (1980). The development of proportional reasoning and the ratio concept: Part 1. Differentiation of stages. *Educational Studies in Mathematics, 11*, 217-253. doi:10.1007/BF00304357

Nunes, T., & Bryant, P. (2010). *Understanding relations and their graphical representation.* Retrieved from http://www.nuffieldfoundation.org/sites/defaukt/files/P4.pdf

Nunes, T., Bryant, P., Barros, R., & Sylva, K. (2012). The relative importance of two different mathematical abilities to mathematical achievement. *British Journal of Educational Psychology, 82*, 136–156. doi:10.1111/j.2044-8279.2011.02033.x

Pellegrino, J. W., & Glaser, R. (1982). Analyzing aptitudes for learning: Inductive reasoning. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 2, pp. 269-345). Hillsdale, NJ: Lawrence Erlbaum.

Resnick, L. B., & Singer, J. A. (1993). Protoquantitative origins of ratio reasoning. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 107-130). Hillsdale, NJ: Lawrence Erlbaum.

Siemon, D., Breed, M., & Virgona, J. (2005). From additive to multiplicative thinking: The big challenge of the middle years. In J. Mousley, L. Bragg, & C. Campbell (Eds.), *Proceedings of the 42nd Conference of the Mathematical Association of Victoria* (pp. 278-286). Melbourne, Australia: Mathematical Association of Victoria.

Slovic, P. (1995). The construction of preference. *American Psychologist, 50*, 364-371. doi:10.1037/0003-066X.50.5.364

Tashakkori, A., & Teddlie, C. (2003). *Handbook of mixed methods in social and behavioral research*. Thousand Oaks, CA: Sage Publications.

Torbeyns, J., & Verschaffel, L. (2013). Efficient and flexible strategy use on multi-digit sums: a choice/no-choice study. *Research in Mathematics Education, 15*, 129-140. doi:10.1080/14794802.2013.797745

Tourniaire, F., & Pulos, S. (1985). Proportional reasoning: A review of the literature. *Educational Studies in Mathematics, 16*, 181–204. doi:org/10.1007/BF02400937

Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2012). Naturally biased? In search for reaction time evidence for a natural number bias in adults. *The Journal of Mathematical Behavior, 31*, 344-355. doi:10.1016/j.jmathb.2012.02.001

Van Dooren, W., De Bock, D., Janssens, D., & Verschaffel, L. (2008). The linear imperative. An inventory and conceptual analysis of students' overuse of linearity. *Journal for Research in Mathematics Education, 39,* 311-342. doi:10.2307/30034972

Van Dooren, W., De Bock, D., & Verschaffel, L. (2010). From addition to multiplication… and back. The development of students' additive and multiplicative reasoning skills. *Cognition and Instruction, 28,* 360-381. doi:10.1080/07370008.2010.488306

Van Dooren, W., De Bock, D., Vleugels, K., & Verschaffel, L. (2010). Just answering... or thinking? Contrasting pupils' solutions and classifications of proportional and non-proportional word problems. *Mathematical Thinking and Learning, 12,* 20–35. doi:10.1080/10986060903465806

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127-174). New York, NY: Academic Press.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141-161). Reston, VA: Lawrence Erlbaum & National Council of Teachers of Mathematics.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples.* Mahwah, NJ: Lawrence Erlbaum.

Welford, A. T. (1980). *Reaction times.* London, United Kingdom: Academic Press.

Wertheimer, M. (1945). *Productive thinking.* New York, NY: Harper and Brothers.