STOP-LOSS ORDERING FOR SCALE AND POWER MIXTURES
OF DISTRIBUTIONS

by

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ABSTRACT

It is the custom to introduce mixtures of distributions for explaining non-obvious variations in observed outcomes from a collective of risks. In the present contribution we consider some special kind of mixtures, namely scale and power mixtures. It is shown that the stop-loss criterion for ordering of risks plays an important role for ordering the mixing distribution.

1. INTRODUCTION

It is well known that in the actuarial field mixtures of distributions (the Urns of Urns Paradigm)

$$(1) \quad F_X(x) = \int F_Y(y, w) dF_W(w)$$

are introduced to explicate certain non-obvious variations in observed outcomes from a portfolio, or collective of risks [1,2] $F_W(\cdot)$ denotes the so called structure function, describing the distribution function of the risk parameter W .

In what follows we will restrict ourselves to the study of two special classes of mixtures, namely the scale and power mixtures [3].

i) Scale mixtures

A restricted class of mixtures is obtained from (1) by considering the special choice

$$(2) \quad F_Y(x, w) = F_Y\left(\frac{x}{w}\right)$$

where $F_Y(\cdot)$ is a distribution function and $F_W(\cdot)$ a mixing distribution with support in $[0, \infty)$. The corresponding probability distribution function and characteristic function takes the form

$$(3) \quad f_X(x) = \int_0^{\infty} f_Y\left(\frac{x}{w}\right) \frac{1}{w} dF_W(w)$$

$$(4) \quad \phi_X(t) = \int_0^{\infty} \phi_Y(tw) dF_W(w)$$

Of course in terms of random variables one has

$$(5) \quad X =_d YW$$

where $=_d$ denotes equality in distribution.

ii) Power mixtures

If ϕ_Y is an infinitely divisible characteristic function, then ϕ_Y^w is a characteristic function for every $w \geq 0$. Randomizing with respect to w the class of power mixtures is obtained as follows

$$(6) \quad \phi_X(t) = \int_0^{\infty} \{\phi_Y(t)\}^w dF_W(w).$$

More general, one may consider characteristic functions of the form (6), where ϕ_Y^w is a characteristic function for all w in the support of F_W . This is e.g. the case when F_W is the distribution function on the positive integers, consequently the compound distributions belong to the class of power mixtures.

Let us still recall the definition of stop-loss ordering. Let F_X, F_Y be any distributions on the real line. Then we say that $X < Y$ ($F_X < F_Y$) if

$$1) \quad \int_0^{\infty} x dF_X(x) < \infty$$

$$2) \quad \int_t^{\infty} (x-t) dG(x) \leq \int_t^{\infty} (x-t) dH(x) \quad -\infty < t < \infty$$

2. UPPER BOUNDS ON STOP-LOSS PREMIUMS FOR SCALE MIXTURES

2.1. Let M_Y denote the set of distributions F_X with

$$(7) \quad F_X(x) = \int_a^b F_Y\left(\frac{x}{w}\right) dF_W(w)$$

for given F_Y .

We then consider the following problem :

$$(8) \quad p(z) = \sup_{F_X \in M_Y} \left(\int_0^{\infty} (x-t)_+ dF_X(x) \right), \quad \int_0^{\infty} dF_X(x) = 1, \quad \int_0^{\infty} x dF_X(x) = z$$

For solving this problem we apply the method of F. De Vylder [4].

Therefore we consider

$$(9) \quad f_1(w) = \int_0^{\infty} x dF_Y\left(\frac{x}{w}\right) = w \int_0^{\infty} y dF_Y(y)$$

$$(10) \quad f_2(w) = \int_0^{\infty} (x-t)_+ dF_Y\left(\frac{x}{w}\right) = w \int_0^{\infty} \left(y - \frac{t}{w}\right)_+ dF_Y(y)$$

let $\varphi(w) = \int_0^{\infty} \left(y - \frac{t}{w}\right)_+ dF_Y(y)$, then we have to consider the following problem

$$(11) \quad p(z) = \sup_{F_W \in \mathfrak{F}[a,b]} \left(\int_a^b w \varphi(w) dF_W(w), \int_a^b dF_W(w) = 1, \int_a^b w dF_W(w) = z \right)$$

where $\mathfrak{F}[a,b]$ denotes the set of all distributions on $[a,b]$. Without loss of generality it is assumed that $E(Y) = 1$.

It is readily seen that $w\varphi(w)$ represents an increasing convex function.

Hence according to the results obtained by De Vylder in ref. [4] the sur-
premium is determined by the straight line r with equation

$$(12) \quad Y = a\varphi(a) + \frac{b\varphi(b) - a\varphi(a)}{b-a} (X-a)$$

Hence

$$(13) \quad p(z) = a \int_0^{\infty} (y - \frac{t}{a})_+ dF_Y(y) \frac{b-z}{b-a} + b \int_0^{\infty} (y - \frac{t}{b})_+ dF_Y(y) \frac{z-a}{b-a}$$

Consequently the upper bound on the stop-loss premium for a scale mixture is completely determined by the stop-loss premium of the distribution F_Y .

2.2. Let M'_Y denote the set of distributions F_X with

$$(14) \quad F_X(x) = \int_a^b F_Y(\frac{x}{w}) dF_W(w), \text{ for given } F_Y$$

and where now $E(Y) = 1$, F_W is a unimodal distribution with $E(W) = z$ (hence $E(X) = z$).

We consider

$$(18) \quad p(z) = \sup_{F_W \in \mathfrak{F}'_{[a,b]}} \left(\int_a^b w \varphi(w) dF_W(w) \right), \quad \int_a^b dF_W(w) = 1, \quad \int_a^b w dF_W(w) = z$$

where $\mathfrak{F}'_{[a,b]}$ denotes the set of unimodal distributions on $[a, b]$.

We get, taking into account a result of ref. [5].

$$(19) \quad p(z) = \frac{b+a-2\mu}{b-a} \varphi\left(\frac{b+a-2\mu}{b-a}\right) + \int_a^b \frac{2(\mu-a)}{(b-a)^2} w \varphi(w) dw$$

hence :

$$(20) \quad p(z) = \int_0^{\infty} \left(\frac{b+a-2\mu}{b-a} y-t \right)_+ dF_Y(y) + \frac{2(\mu-a)}{(b-a)^2} \int_0^{\infty} \frac{1}{2y} [(by-t)_+^2 - (ay-t)_+^2] dF_Y(y)$$

Hence, in this case, the supremum can be expressed by means of an integral also appearing in the definition of second order stop-loss dominance [6].

2.3. Particular cases

1. Mixtures of gamma-distributions of order $\alpha (0 < \alpha < \infty)$

$$(21) \quad \phi_X(t) = \int_0^{\infty} (1-itw)^{\alpha} dF_W(w)$$

For the present case (13) reads

$$p(z) = \frac{a}{\Gamma(a)} \left[\Gamma(\alpha+1, \frac{t}{a}) - \frac{t}{a} \Gamma(\alpha, \frac{t}{a}) \right] \frac{b-z}{b-a} \\ + \frac{b}{\Gamma(b)} \left[\Gamma(\alpha+1, \frac{t}{b}) - \frac{t}{b} \Gamma(\alpha, \frac{t}{b}) \right] \frac{b-z}{b-a}$$

where, as usual, $\Gamma(\alpha, x)$ denotes the incomplete Γ -function.

2. Mixtures of exponential distributions

$$(22) \quad f_X(x) = \int_0^{\infty} w^{-1} e^{-\frac{x}{w}} dF_W(w)$$

This is the class of completely monotonic densities. For the present case (13) reads :

$$p(z) = a \frac{b-z}{b-a} e^{-\frac{t}{a}} + b \frac{z-a}{b-a} e^{-\frac{t}{b}}$$

3. The monotonicity and convexity classes

$$(23) \quad f_X(x) = \int_{x(n+1)}^{\infty} \left(1 - \frac{x}{(n+1)w}\right)^n \frac{1}{w} dF_W(w)$$

For $n=0$ one obtains all non-increasing densities on $(0, \infty)$. For $n=1$ all unimodal densities on $(0, \infty)$ that are convex. For n integer and different from zero and one we get the class of distributions having an n^{th} order derivative $F_X^{(n)}(x)$ in $(0, \infty)$ with $(-1)^{n+1} F_X^{(n-1)}(x)$ convex in $(0, \infty)$.

2.4. In 2.1. and 2.2. we have seen how the integrals appearing in the definitions of ordering of risks according to the net stop-loss ordering and the stop-loss dominance criteria appear for deriving upper bounds on stop-loss premiums for scale mixtures, introducing some structural constraints on the structure function. Now we will give an interpretation to the expression of the stop-loss premium of the structure function. Consider the class of mixtures κ_Y consisting of the scale-mixtures of Y -distributions, i.e. the set

$$(24) \quad \kappa_Y = \{F_X : X = YW ; Y, W \text{ independent}, W \geq 0\}$$

A distance function or metric can be associated with κ_Y in a simple way, giving rise to a metric space. Therefore the following space [3] is introduced

$$(25) \quad \kappa_Y^r = \{F_X : X = YW ; Y, W \text{ independent}; W \geq 0, E(W) = 1, E(W^r) < \infty\}$$

Let F_1, F_2 in κ_Y^r be associated with W_1, W_2 , then we define the distance function

$$(26) \quad \rho_\theta(F_1, F_2) = \int |w-1|^\theta |dF_1(w) - dF_2(w)| \quad \theta \leq r$$

One easily shows that for $0 \leq \theta \leq r$, $\rho_\theta(F_1, F_2)$ is a metric for distributions in κ_Y^r , i.e. for all $F_j \in \kappa_Y^r$ we have :

- a) $\rho_\theta(F_1, F_2) \geq 0$
- b) $\rho_\theta(F_1, F_2) = 0 \iff F_1 = F_2$
- c) $\rho_\theta(F_1, F_2) = \rho_\theta(F_2, F_1)$
- d) $\rho_\theta(F_1, F_3) \leq \rho_\theta(F_1, F_2) + \rho_\theta(F_2, F_3)$

Let us still remark that without restricting the generality we may assume $E(W) = 1$ and $E(Y) = z$. When W_1 has all support at $w = 1$ $\rho_\theta(F_1, F_2)$ represents a measure of distance to the "pure" distribution. We now have as a special case :

$$\begin{aligned}
 (27) \quad \rho_1(F_1, F_2) &= \int_0^{\infty} |w-1| \, dF_{W_2}(w) \\
 &= 2 \int_1^{\infty} (w-1)_+ \, dF_{W_2}(w)
 \end{aligned}$$

$$\text{Consequently} \quad \int_{E(W_2)}^{\infty} (w-E(W_2))_+ \, dF_{W_2}(w)$$

is a measure for the distance of $X = YW_2$ to the pure $X = Y$.

3. UPPER BOUNDS ON STOP-LOSS PREMIUMS FOR POWER MIXTURES

3.1. Let us consider next the compound distribution function

$$(28) \quad F_S(x) = \sum_{n=0}^{\infty} p(N=n) F_X^{nx}(x)$$

It is well known [6] that

$$(29) \quad \begin{cases} X < X' \implies S < S' \\ N < N' \implies S < S' \end{cases}$$

As a special case of (28) we consider the weighted compound Poisson distribution

$$(30) \quad F_S(x) = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-w} \frac{w^n}{n!} \, dF_W(w) F_X^{nx}(x)$$

We will prove the following

Theorem

$$W < W' \implies S < S'$$

Demonstration

Let $p_n = \int_0^{\infty} e^{-w} \frac{w^n}{n!} dF_W(w)$ and $p'_n = \int_0^{\infty} e^{-w} \frac{w^n}{n!} dF_{W'}(w)$ denote

the probabilities $p(N=n) = p_n$ and $p(N'=n) = p'_n$.

According to (29) the theorem holds if one shows $N < N'$.

We therefore consider the difference :

$$(31) \quad p_n - p'_n = \int_0^{\infty} e^{-w} \frac{w^n}{n!} [dF_W(w) - dF_{W'}(w)]$$

By means of two successive partial integrations the r.h.s. of (31) can be cast into the form

$$(32) \quad p_n - p'_n = \int_0^{\infty} \left(\frac{d^2}{dw^2} e^{-w} \frac{w^n}{n!} \right) \varphi(w) dw$$

$$\text{where } \varphi(w) = \int_w^{\infty} (1 - F_W(y)) dy - \int_w^{\infty} (1 - F_{W'}(y)) dy$$

$$\text{Because } \frac{d^2}{dw^2} e^{-w} \frac{w^n}{n!} = e^{-w} \left[\frac{w^n}{n!} - 2 \frac{w^{n-1}}{(n-1)!} + \frac{w^{n-2}}{(n-2)!} \right]$$

one obtains :

$$(33) \quad \sum_{k=0}^n \sum_{r=0}^k \left(\frac{d^2}{dw^2} e^{-w} \frac{w^r}{r!} \right) = \sum_{k=0}^n \left[\frac{w^k}{k!} - \frac{w^{k-1}}{(k-1)!} \right] e^{-w}$$

$$= \frac{w^n}{n!} e^{-w}$$

Consequently

$$(34) \quad \sum_{k=0}^n \sum_{r=0}^k (p_{rV} - p'_{rV}) = \int_0^{\infty} e^{-w} \frac{w^n}{n!} \varphi(w) dw < 0$$

Now the l.h.s. of (34) can successively be transformed as follows :

$$\begin{aligned}
\sum_{k=0}^n \sum_{v=0}^k (p_v - p'_v) &= \sum_{v=0}^n \sum_{k=v}^n (p_v - p'_v) \\
&= \sum_{v=0}^n (n-v) (p_v - p'_v) \\
&= - \sum_{v=0}^{\infty} (n-v) (p_v - p'_v) + \sum_{v=p}^n (n-v) (p_v - p'_v) \\
&= \sum_{v=n+1}^{\infty} (v-n) (p_v - p'_v)
\end{aligned}$$

Hence

$$(35) \quad \sum_{v=n+1}^{\infty} (v-n) p_v < \sum_{v=n+1}^{\infty} (v-n) p'_v$$

Next we consider the power mixture (6). Because $\phi_Y(.)$ is supposed to be infinitely divisible it can be cast into the form

$$(36) \quad \phi_Y(v) = e^{-w + w \phi_Z(t)}$$

Consequently we can apply the preceeding theorem in order to get

$$(37) \quad W < W' \implies X < X'$$

when X is the W power mixture of Y and X' the W' power mixture of Y .

Remark

The result of the theorem still holds in case one considers

$$p_n = \int p_n(\theta) dF_W(\theta)$$

for the class of probability laws for which

$$\sum_{k=0}^n \sum_{v=0}^k p''_v(\theta) \geq 0$$

3.2. In what follows we will solve the following problem :

$$(38) \quad p(\mu, I, r) = \min_{\{p_j\} \in I_\mu} \sum_{k=r}^{\infty} (k-r)p_k$$

where I_μ denotes the class of infinitely divisible lattice distributions with given mean μ

$$(39) \quad \mu = \sum_{j=0}^{\infty} jp_j$$

Let $P(u)$ denote the generating function of $\{p_j\}$

Because $P(u)$ denotes the generating function of an infinitely divisible distribution one has :

$$(40) \quad P(u) = \sum_{j=0}^{\infty} u^j p_j = \left(\sum_{j=0}^{\infty} u^j p_j^{(n)} \right)^n$$

and of course

$$(41) \quad \sum_{j=0}^{\infty} jp_j^{(n)} = \frac{\mu}{n}$$

Consequently, in case we choose $n > [\mu]$, where the r.h.s. denotes the largest integer contained in μ we can apply the following result

$$(42) \quad \{p_j^{(n)}\} > \left\{1 - \frac{\mu}{n}, \frac{\mu}{n}\right\}$$

where $>$ denotes as usual the stop-loss ordering [6], which implies that :

$$(43) \quad \sum_{j=r}^{\infty} (j-r)p_j^{(n)} \geq \sum_{j=r}^{\infty} (j-r) \left[\delta_{j,0} \left(1 - \frac{\mu}{n}\right) + \delta_{j,1} \frac{\mu}{n} \right]$$

where as usual $\delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

But from [6] we know that stop-loss ordering is preserved under convolution or addition.

Hence,

$$(44) \quad \{p_j\} > \left\{1 - \frac{\mu}{n}, \frac{\mu}{n}\right\}^{*n}$$

where of course the r.h.s. denotes a binomial distribution for n trials with a probability of success $\frac{\mu}{n}$.

Next we compare $\left\{1 - \frac{\mu}{n}, \frac{\mu}{n}\right\}^{*n}$ with $\left\{1 - \frac{\mu}{n+1}, \frac{\mu}{n+1}\right\}^{*(n+1)}$ where the

$$(45) \quad \begin{cases} \frac{\mu}{n+1} = \frac{1}{n+1} \frac{\lambda}{n} + \dots + \frac{1}{n+1} \cdot \frac{\lambda}{n} + \frac{1}{n+1} \cdot 0 \\ \vdots \\ \frac{\mu}{n+1} = \frac{1}{n+1} \frac{\lambda}{n} + \dots + \frac{1}{n+1} \cdot \frac{\lambda}{n} + \frac{1}{n+1} \cdot 0 \end{cases}$$

As is shown in ref. [4] we get :

$$(46) \quad \left\{1 - \frac{\mu}{n}, \frac{\mu}{n}\right\}^{*n} < \left\{1 - \frac{\mu}{n+1}, \frac{\mu}{n+1}\right\}^{*(n+1)}$$

Hence :

$$(47) \quad p(\mu, I, r) = \sum_{j=r}^{\infty} (j-r) \frac{\mu^j}{j!} e^{-\mu} \quad r = 0, 1, \dots$$

because $\lim_{n \rightarrow \infty} \left\{1 - \frac{\mu}{n}, \frac{\mu}{n}\right\}^{*n}$ gives rise to a Poisson law with parameter μ .

Remarks

1. Because of the result of section 3.1. we have

$$\sum_{j=r}^{\infty} (j-r) \frac{\mu^j}{j!} e^{-\mu} = \min_{\{p_j\} \in W_{\mu}} \sum_{j=r}^{\infty} (j-r) p_j$$

where W_μ denotes the set

$$\{p_j \mid p_j = \int_0^\infty e^{-w} \frac{w^j}{j!} dF_W(w), \int_0^\infty w dF_W(w) = \mu\}$$

with of course

$$I_\mu \subset W_\mu$$

2. From the result in 3.2. we derive the following new result in the framework of infinitely divisibility : let $\{p_j\}$ be a lattice distribution with mean value μ ; in case

$$\sum_{k=r}^{\infty} (k-r)p_j < \sum_{k=r}^{\infty} (k-r) \frac{\mu^k}{k!} e^{-\mu} \quad \text{for some } r$$

then $\{p_j\}$ is not infinitely divisible.

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