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PARETO OPTIMALITY OF NONCOOPERATIVE EQUILIBRIUM
IN A TIME DEPENDENT MULTI-PERIOD GAME

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1. INTRODUCTION

The purpose of this paper is to derive necessary and sufficient conditions for a noncooperative equilibrium to be Pareto optimal. If these conditions are not fulfilled, there is room for improvement for the players of the game. This potential betterment can effectively be realized only if there exists a possibility of binding and enforceable agreements. (See Harsanyi [8]). This problem of providing effective incentives to keep agreements, i.e. of changing noncooperative games into cooperative games, falls outside the scope of this paper.

We will use a time dependent multi-period game (or supergame) where each player wants to control a dynamical system - the evolution of which is described by a system of linear difference equations - in such a way as to minimize a quadratic cost functional. This linear-quadratic game can be looked upon as an unnecessary and unrealistic simplification. However, a theoretical analysis of the conditions for Pareto optimal noncooperative equilibria

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should preferably be based on analytical results, not on a set of numbers which are obtained by some iterative solution method. Because of its mathematical tractability and the possibility of obtaining analytical solutions, the linear-quadratic game is well suited to this purpose.

Our time dependent multi-period game is very close to the one analyzed by Friedman [5, (Chapter 9)]. Though our model is non-stochastic, it also resembles the work of Sobel [16, 17], especially if the "transition laws" of the latter are specified in a way to eliminate the stochastic elements. In both Friedman's and Sobel's work there are specific aspects which are more and others which are less general than in our model.

The type of game we use in this paper can also be considered as a discrete time version of some of the continuous time models in the differential game literature, some basic references for which are, among others, Case [2], Starr [18], Starr and Ho [19, 20] and Ho [9].

In Section 2 the time dependent multi-period linear-quadratic game is set forth. Section 3 deals with the noncooperative equilibrium and in Section 4 we derive the set of Pareto optimal solutions. The conditions for a non-cooperative equilibrium to be Pareto optimal are analyzed in Section 5. Finally, conclusions, possible applications and some directions for future work are reported on in Section 6.

2. THE GAME DESCRIPTION

The evolution of the dynamical system to be controlled by the N players $i = 1, \dots, N$, is described by the system of linear difference equations :

$$(1) \quad x_{t+1} - x_t = Ax_t + \sum_{j=1}^N B_j u_{jt} + z_t$$

$$x_0 = \varphi$$

where, for $t = 0, \dots, T$, $x_t \in R^n$ is the vector of state variables, $u_{it} \in R^{m_i}$ is the vector of control variables of player i and $z_t \in R^n$ is a known vector of uncontrollable exogenous variables. A is an $n \times n$ matrix of constants and B_j is an $n \times m_j$ matrix of constants ($j = 1, \dots, N$). x_0 is given.

In each time period player i chooses a control law or strategy $u_{it}(x_t)$ so as to minimize his cost functional

$$(2) \quad J_i = \frac{1}{2} x_T' Q_{iT} x_T + \frac{1}{2} \sum_{t=0}^{T-1} (x_t' Q_{it} x_t + u_{it}' R_{it} u_{it})$$

where $u_{it} = u_{it}(x_t)$, Q_{it} is an $n \times n$ positive semi-definite matrix and R_{it} is an $m_i \times m_i$ positive definite matrix. Q_{it} and R_{it} are known.

The class of supgame strategies considered is of the following sort :

$$s_i = (u_{i0}(x_0), \dots, u_{i,T-1}(x_{T-1})) \quad i = 1, \dots, N$$

where each function u_{it} maps the state x_t into an element of R^{m_i} , i.e. the strategy of a player depends on the prevailing state. In control theory, such strategies are known as closed-loop or feedback strategies (see Intriligator [10, (299-301)]).

3. NONCOOPERATIVE EQUILIBRIUM STRATEGIES

For noncooperative games, a basic solution concept is that of a Nash equilibrium, which is defined by the property that no player can perform better by deviating from his equilibrium control if all the other players stay at their Nash controls. Formalizing this concept, a set $\{u_1^x, \dots, u_N^x\}$ is called a noncooperative equilibrium strategy set if, for $i = 1, \dots, N$, remembering that J_i has to be minimized,

$$(3) \quad J_i(u_1^x, \dots, u_i^x, \dots, u_N^x) \leq J_i(u_1^x, \dots, u_i, \dots, u_N^x)$$

Such a strategy set is said to yield an equilibrium solution for noncooperative games in the sense that no player has an incentive to deviate unilaterally from his equilibrium control.

The existence of a unique Nash noncooperative equilibrium in time dependent multi-period linear-quadratic games where all players use feedback strategies was first established by Basar [1] using a backward induction approach. Pindyck [12] derived necessary conditions for the noncooperative equilibrium by applying the discrete-time minimum principle directly to the problem of minimizing (2) - for all i - given the other players' strategies². Since we will proceed along the same way, we quote Pindyck's results in the following theorem.

² For the discrete-time minimum principle, see Pindyck [13, (Chapter 2)].

THEOREM 1. If $(s_1^x, \dots, s_N^x)$ are the supergame strategies of the noncooperative equilibrium then we have for $i = 1, \dots, N$ and for $t = 0, \dots, T-1$ ³.

$$(4) \quad \left. \frac{\partial H_i}{\partial u_{it}} \right|_x = 0$$

$$(5) \quad p_{i,t+1}^x - p_{it}^x = - \left. \frac{\partial H_i}{\partial x_t} \right|_x - \sum_{j \neq i}^N \frac{\partial u_{jt}'}{\partial x_t} \left. \frac{\partial H_i}{\partial u_{jt}} \right|_x$$

$$(6) \quad x_{t+1}^x - x_t^x = \left. \frac{\partial H_i}{\partial p_{i,t+1}} \right|_x = Ax_t^x + \sum_{j=1}^N B_j u_{jt}^x + z_t$$

$$(7) \quad x_0^x = \varphi$$

$$(8) \quad p_{iT}^x = Q_{iT} x_T^x$$

where H_i , the Hamiltonian for player i , is defined as

$$(9) \quad H_i(x_t, u_{1t}, \dots, u_{Nt}, p_{i,t+1}) = \frac{1}{2}(x_t' Q_{it} x_t + u_{it}' R_{it} u_{it}) \\ + p_{i,t+1}' (Ax_t + \sum_{j=1}^N B_j u_{jt} + z_t)$$

and where p_{it} is an $n \times 1$ vector of costate variables, which can be considered as dynamic Lagrange multipliers and hence can be interpreted analogously as shadow prices.

Equation (8), which is valid for period T only, is the result of the transversality condition which provides the additional conditions required because

³ In the sequel, the expression $\left. \frac{\partial f}{\partial w} \right|_x$ represents the partial derivative of the function f with respect to the variable w , evaluated at the optimum.

x_T is not a priori specified. It should be remarked that in the optimal control problem, where $N = 1$, the second term in (5) is absent. This is also the case if all players use open-loop controls. Equation (5) can be interpreted in the following way. It is a well-known result from control theory that p_{it}^x measures the sensitivity of the i -th player's optimal cost to a change in the optimal state vector x_t^x . If all other players are using closed-loop strategies, an infinitesimal perturbation of x_t^x will cause them to change their controls by the amount $(\partial u_{jt}^i / \partial x_t^x)$. These reactions will in turn influence the Hamiltonian of player i , which will lead him to change his control.

We can now prove

THEOREM 2. The unique noncooperative equilibrium strategies for player $i = 1, \dots, N$ and for $t = 0, \dots, T-1$ are given by⁴

$$(10) \quad u_{it}^x = -R_{it}^{-1} B_{it}' K_{i,t+1} E_{t+1}^{-1} \left[(I+A)x_t^x - \sum_{j=1}^N Z_{jt} g_{j,t+1} + z_t \right] - R_{it}^{-1} B_{it}' g_{i,t+1}$$

where

$$(11) \quad Z_{jt} = B_j R_{jt}^{-1} B_j'$$

$$(12) \quad E_{t+1} = I + \sum_{j=1}^N Z_{jt} K_{j,t+1}$$

⁴ In Theorem 2 we apply the necessary conditions of Theorem 1. From the basic result of Basar [1], it follows that the candidate we obtain for the non-cooperative equilibrium does exist and is unique.

$$(13) K_{it} = Q_{it} + (I+A)'(E_{t+1}^{-1})'(K_{i,t+1} + K_{i,t+1}'Z_{it}K_{i,t+1})E_{t+1}^{-1}(I+A)$$

$$(14) g_{it} = (I+A)'(E_{t+1}^{-1})'(I+K_{i,t+1}Z_{it})[K_{i,t+1}E_{t+1}^{-1}(z_t - \sum_{j=1}^N Z_{jt}g_{j,t+1}) + g_{i,t+1}]$$

The boundary conditions for equations (13) and (14) are given by

$$(15) K_{iT} = Q_{iT}$$

$$(16) g_{iT} = 0$$

PROOF. Condition (4) yields

$$\left. \frac{\partial H_i}{\partial u_{it}} \right|_{\mathbf{x}} = R_{it}u_{it}^{\mathbf{x}} + B_{it}'p_{i,t+1}^{\mathbf{x}} = 0$$

from which it follows that

$$(17) u_{it}^{\mathbf{x}} = -R_{it}^{-1}B_{it}'p_{i,t+1}^{\mathbf{x}}$$

The square matrix R_{it} , assumed to be positive definite certainly has an inverse.

Equations (5) and (6), looked upon as a system of difference equations, suggest that $x_t^{\mathbf{x}}$ and $p_{it}^{\mathbf{x}}$ are related to each other. We will assume the following linear relationship to hold for all t ⁵

⁵ As stated here, relation (18) is an assumption. However, since Basar [1] proved that the unique noncooperative equilibrium strategy for each player is linear in the state variables, it follows that (18) is indeed true.

$$(18) \quad p_{it}^{\bar{x}} = K_{it} \bar{x}_t + g_{it}$$

where K_{it} is an $n \times n$ matrix and g_{it} an $n \times 1$ vector, which are given concrete form below. Substituting (18) into (17) yields

$$(19) \quad u_{it}^{\bar{x}} = -R_{it}^{-1} B_i' (K_{i,t+1} \bar{x}_{t+1} + g_{i,t+1})$$

Using (6) and (19) and rearranging, we have

$$(20) \quad \bar{x}_{t+1}^{\bar{x}} = E_{t+1}^{-1} \left[(I+A) \bar{x}_t^{\bar{x}} - \sum_{j=1}^N Z_{jt} g_{j,t+1} + z_t \right]$$

where Z_{jt} and E_{t+1} are specified in (11) and (12). E_{t+1} is a non-singular $n \times n$ matrix as is proved below.

Substitution of (20) into (19) yields the basic result (10) from which it follows that

$$(21) \quad \left. \frac{\partial u_{jt}^{\bar{x}}}{\partial \bar{x}_t} \right|_{\bar{x}} = -(I+A)' (E_{t+1}^{-1})' K_{j,t+1}' B_{j,R}^{-1}$$

Furthermore, we have that

$$(22) \quad \left. \frac{\partial H_i}{\partial \bar{x}_t} \right|_{\bar{x}} = Q_{it} \bar{x}_t^{\bar{x}} + A' p_{i,t+1}^{\bar{x}}$$

$$(23) \quad \left. \frac{\partial H_i}{\partial u_{jt}^{\bar{x}}} \right|_{\bar{x}} = B_j' p_{i,t+1}^{\bar{x}}$$

Evaluating (5), using (21) to (23), we get

$$(24) \quad p_{i,t+1}^{\mathbf{x}} - p_{it}^{\mathbf{x}} = -Q_{it}x_t^{\mathbf{x}} - A'p_{i,t+1}^{\mathbf{x}} + \sum_{j \neq i}^N (I+A)'(E_{t+1}^{-1})'K_{j,t+1}'Z_{jt}p_{i,t+1}^{\mathbf{x}}$$

Using (18) and (20) in (24), and rearranging

$$(25) \quad K_{it}x_t^{\mathbf{x}} + g_{it} = (I+A)'[K_{i,t+1} - \sum_{j \neq i}^N (E_{t+1}^{-1})'K_{j,t+1}'Z_{jt}K_{i,t+1}]E_{t+1}^{-1} \\ \times [(I+A)x_t^{\mathbf{x}} - \sum_{j=1}^N Z_{jt}g_{j,t+1} + z_t] + Q_{it}x_t^{\mathbf{x}} + (I+A)'g_{i,t+1} \\ - \sum_{j \neq i}^N (I+A)'(E_{t+1}^{-1})'K_{j,t+1}'Z_{jt}g_{i,t+1}$$

Since equation (25) is a result of the necessary conditions it must hold for any initial state φ and hence for any $x_t^{\mathbf{x}}$. Therefore it follows from (25) that

$$(26) \quad K_{it} = Q_{it} + (I+A)'K_{i,t+1}E_{t+1}^{-1}(I+A) \\ - \sum_{j \neq i}^N (I+A)'(E_{t+1}^{-1})'K_{j,t+1}'Z_{jt}K_{i,t+1}E_{t+1}^{-1}(I+A)$$

and

$$(27) \quad g_{it} = (I+A)'[K_{i,t+1} - \sum_{j \neq i}^N (E_{t+1}^{-1})'K_{j,t+1}'Z_{jt}K_{j,t+1}]E_{t+1}^{-1} \\ \times (z_t - \sum_{t=1}^N Z_{jt}g_{j,t+1}) + (I+A)'[g_{i,t+1} - \sum_{j \neq i}^N (E_{t+1}^{-1})'K_{j,t+1}'Z_{jt}g_{i,t+1}]$$

Equations (26) and (27) are known as the matrix Ricatti equation and the tracking equation, respectively. The boundary conditions (15) and (16) for these equations follow immediately from (18) and the transversality condition (8).

We still have to prove that E_t is a non-singular matrix. Equation (26) can be written as follows:

$$(28) \quad K_{it} = Q_{it} + (I+A)' [I - (I+H'_{t+1})^{-1} H'_{t+1}] K_{i,t+1} E_{t+1}^{-1} (I+A) \\ + (I+A)' (E_{t+1}^{-1})' K'_{i,t+1} Z_{it} K_{i,t+1} E_{t+1}^{-1} (I+A)$$

$$\text{where } H_{t+1} = \sum_{j=1}^N Z_{jt} K_{j,t+1} \quad \text{and} \quad I + H_{t+1} = E_{t+1}$$

From matrix theory we know that the following identity holds

$$(29) \quad (I_n + XY')^{-1} = I_n - X(I_r + Y'X)^{-1} Y'$$

where X and Y are $n \times r$ matrices, with $r \leq n$, and I_n and I_r are identity matrices of order n and r respectively. Putting $X = I$ and $Y = H_{t+1}$ we have that

$$(I+H'_{t+1})^{-1} = I - (I+H'_{t+1})^{-1} H'_{t+1}$$

Substitution of this identity in (28) results in (13), the final equation for K_{it} . Solving the matrix Ricatti equation backwards in time, taking into

account boundary condition (15) and the assumptions made about Q_{it} and R_{it} , it can be easily verified that K_{it} is a symmetric positive semi-definite matrix for all i and t . It follows then from (12) that E_t is non-singular for all t .

Finally, applying to equation (27) the same line of reasoning which enabled us to transform (26) into (13), we can obtain result (14) for the tracking equation.

Q.E.D.

Before concluding this section, let us recapitulate in a schematical way the different steps in the calculation of the noncooperative equilibrium strategies. First, solve the matrix Ricatti equation (13) and the tracking equation (14) backwards in time to get values for K_{it} and g_{it} for all i and t . Second, compute x_{i0}^x from equation (10), using $x_0^x = \varphi$, and then use (6) to solve for x_1^x . This second step is then repeated $T-1$ times to cover all time periods.

4. PARETO OPTIMAL STRATEGIES

In noncooperative games, the Nash solution yields an equilibrium point in the sense that it is secure against any attempt by one player to unilaterally alter his control, since he cannot improve his performance by doing so. However, if all players cooperate, they might end up in a situation where each of them is able to achieve a lower cost as compared with the noncooperative equilibrium. When we consider the possibility of cooperation, the assumption of group rationality leads to the solution concept of the set of

Pareto optimal strategies. This set P is characterized by the property that any solution not belonging to P is dominated by some element of P in the sense that at least one player can do better while none of the players does worse.

A strategy N -tuple $\hat{u} = \{\hat{u}_1, \dots, \hat{u}_N\}$ belongs to the Pareto optimal set P if there exists no control N -tuple $\tilde{u}$ such that

$$\begin{aligned} J_i(\tilde{u}) &\leq J_i(\hat{u}) && \text{for all } i = 1, \dots, N \\ (30) \quad J_i(\tilde{u}) &< J_i(\hat{u}) && \text{for some } i \end{aligned}$$

It is clear from (30) that $\hat{u}$ is a solution of the problem of minimization of the vector-valued functional $[J_1(u), \dots, J_N(u)]$. Necessary and sufficient conditions for reducing the minimization problem of a vector-valued function to the minimization of a class of scalar-valued functions can be found in Da Cunha and Polak [3] and in Takayama [21, (112-117)].

For the problem at hand, the elements of the Pareto optimal set can be found by solving the following family of scalar-valued criterion problems :

$$(31) \quad \min J = \sum_{i=1}^N \mu_i J_i \quad \text{sub (1)}$$

$$u_1, \dots, u_N$$

for some vector $\mu = \{\mu_1, \dots, \mu_N\}$ satisfying

$$\sum \mu_i = 1 \quad \text{and} \quad \mu_i > 0 \quad \text{for all } i = 1, \dots, N$$

Problem (31) is a usual optimal control problem with one controller. Direct application of the discrete-time minimum principle leads to the following solution for the Pareto optimal strategies :

$$(32) \quad \hat{u}_{it} = - \frac{1}{\mu_i} R_{it}^{-1} B_i' S_{t+1} F_{t+1}^{-1} [(I+A) \hat{x}_t - \sum_{j=1}^N \frac{1}{\mu_j} Z_{jt} f_{t+1} + z_t] - \frac{1}{\mu_i} R_{it}^{-1} B_i' f_{t+1}$$

where

$$(33) \quad F_{t+1} = I + \sum_{j=1}^N \frac{1}{\mu_j} Z_{jt} S_{t+1}$$

$$(34) \quad S_t = \sum_{j=1}^N \mu_j Q_{jt} + (I+A)' S_{t+1} F_{t+1}^{-1} (I+A)$$

$$(35) \quad f_t = (I+A)' S_{t+1} F_{t+1}^{-1} (z_t - \sum_{j=1}^N \frac{1}{\mu_j} Z_{jt} f_{t+1}) + (I+A)' f_{t+1}$$

and Z_{jt} is defined in (11).

The boundary conditions for the matrix Ricatti equation (34) and the tracking equation (35) are :

$$(36) \quad S_T = \sum_{j=1}^N \mu_j Q_{jT}$$

$$(37) \quad f_T = 0$$

It can be shown that S_t is a symmetric positive semi-definite matrix for all t . Hence, it follows from (33) that F_t is non-singular for all t .

5. CONDITIONS FOR A PARETO OPTIMAL NONCOOPERATIVE EQUILIBRIUM

It is recognized in the literature that these conditions will probably be very stringent. However, as far as we know, they have not been stated explicitly, not even for the special kind of linear-quadratic games analyzed in this paper. In this section, we will assume that $I+A$ is non-singular. If this restriction is put on the model, we can characterize the conditions very precisely.

From (10) and (32) it appears that the noncooperative equilibrium belongs to the Pareto optimal set if and only if there exists a strictly positive vector $\mu = \{\mu_1, \dots, \mu_N\}$ such that

$$\begin{aligned}
 (38) \quad & B_i' K_{i,t+1} E_{t+1}^{-1} \left[(I+A) x_t^* - \sum_{j=1}^N Z_{jt} g_{j,t+1} + z_t \right] + B_i' g_{i,t+1} \\
 &= \frac{1}{\mu_i} B_i' S_{t+1} F_{t+1}^{-1} \left[(I+A) \hat{x}_t - \sum_{j=1}^N \frac{1}{\mu_j} Z_{jt} f_{j,t+1} + z_t \right] + \frac{1}{\mu_i} B_i' f_{i,t+1} \\
 &\qquad\qquad\qquad \text{for } i = 1, \dots, N \\
 &\qquad\qquad\qquad \text{and } t = 0, \dots, T-1
 \end{aligned}$$

We first prove the following Theorem :

THEOREM 3. The noncooperative equilibrium is Pareto optimal if and only if the two following conditions hold for $t = 1, \dots, T$ and $i = 1, \dots, N$

$$(39) \quad B_i' K_{it} = \frac{1}{\mu_i} B_i' S_t$$

$$(40) \quad B_i' g_{it} = \frac{1}{\mu_i} B_i' f_t$$

PROOF. If (39) holds, it follows from (12) and (33) that $E_t = F_t$. Starting at $x_0^x = \hat{x}_0 = \varphi$ and using conditions (39) and (40), it can be verified that (38) holds, i.e. $u_{i0}^x = \hat{u}_{i0}$ for all i . From (1) we then have that $x_1^x = \hat{x}_1$ and the argument can be repeated for this and the other time periods.

To prove necessity, first remark that if $u_{it}^x = \hat{u}_{it}$ for all i and t - i.e. if (38) holds - and given the initial condition $x_0^x = \hat{x}_0 = \varphi$, it follows from (1) that $x_t^x = \hat{x}_t$ for all t . Furthermore, since condition (38) must hold for any value of the initial state and since none of the parameters in this condition depends upon the initial state, one can separate (38) into the two following conditions :

$$(41) \quad B_i' K_{it} E_t^{-1} = \frac{1}{\mu_i} B_i' S_t F_t^{-1}$$

$$(42) \quad B_i' K_{it} E_t^{-1} (z_{t-1} - \sum_{j=1}^N Z_{j,t-1} g_{jt}) + B_i' g_{it} = \frac{1}{\mu_i} B_i' S_t F_t^{-1} (z_{t-1} - \sum_{j=1}^N \frac{1}{\mu_j} Z_{j,t-1} f_t) + \frac{1}{\mu_i} B_i' f_t$$

Premultiplying both sides of (41) and (42) by $B_i R_{i,t-1}^{-1}$, summing over all i , making use of (12) and (33) and applying the matrix identity

$Y(Y+X)^{-1} = I - X(Y+X)^{-1}$ we get

$$(43) \quad E_t^{-1} = F_t^{-1}$$

$$(44) \quad E_t^{-1} (z_{t-1} - \sum_{j=1}^N Z_{j,t-1} g_{jt}) = F_t^{-1} (z_{t-1} - \sum_{j=1}^N \frac{1}{\mu_j} Z_{j,t-1} f_t)$$

Condition (39) follows from (41) and (43). Using (39) and (44) in (42), we obtain condition (40).

Q.E.D.

In determining the necessary and sufficient conditions for a Pareto optimal noncooperative equilibrium or, put equivalently, in evaluating (39) and (40), we will start at period T and proceed backwards in time since this is the only way to solve the matrix Ricatti equation and the tracking equation. Furthermore, we will concentrate only upon (39) since, as is shown in Theorem 6, (39) implies (40).

We can prove :

LEMMA 1. Suppose that equation (39) holds for all i and for the periods $t = T, \dots, T-\rho$ where $0 \leq \rho \leq T-2$. Then we have that :

$$(45) \quad S_t = \sum_{i=1}^N \mu_i K_{it} \quad \text{for } t = T, \dots, T-\rho-1$$

PROOF. Equation (45) is true for $t = T$ because of the boundary conditions (15) and (36). Using equation (13) we have that :

$$\begin{aligned} \sum_{i=1}^N \mu_i K_{i,T-1} &= \sum_{i=1}^N \mu_i Q_{i,T-1} + (I+A)' (E_T^{-1})' \left(\sum_{i=1}^N \mu_i K_{iT} + \sum_{i=1}^N \mu_i K_{iT} Z_{i,T-1} K_{iT} \right) \\ &\times E_T^{-1} (I+A) = \sum_{i=1}^N \mu_i Q_{i,T-1} + (I+A)' (E_T^{-1})' (S_T + H_T S_T) E_T^{-1} (I+A) \end{aligned}$$

where $H_T = \sum_{i=1}^N Z_{i,T-1} K_{iT}$ and $I + H_T = E_T$

Since $E_T = F_T$ under the assumption of the Lemma, we obtain

$$\sum_{i=1}^N \mu_i K_{i,T-1} = \sum_{i=1}^N \mu_i Q_{i,T-1} + (I+A)' S_T F_T^{-1} (I+A) = S_{T-1} \quad \text{from (34)}$$

This completes the proof, since the same steps can be repeated for the other relevant time periods.

Q.E.D.

We just quote the following Lemma since its proof is analogous to that of Lemma 1.

LEMMA 2. Suppose that equations (39) and (40) hold for all i and for the periods $t = T, \dots, T-\rho$ where $0 \leq \rho \leq T-2$. Then we have that

$$f_t = \sum_{i=1}^N \mu_i g_{it} \quad \text{for } t = T, \dots, T-\rho-1$$

We can now prove :

THEOREM 4. Condition (39) is satisfied if and only if

$$(46) \quad B_i' K_{jt} = 0 \quad \text{for all } i, j \text{ and } t ; i \neq j$$

PROOF. Since $S_T = \sum_{i=1}^N \mu_i K_{iT}$, condition (39) for $t = T$ can be written as

$$\mu_i B_i' K_{iT} = B_i' \sum_{j=1}^N \mu_j K_{jT}$$

or

$$(47) \quad B_i' \left(\sum_{j \neq i}^N \mu_j K_{jT} \right) = 0$$

The sufficiency part of the theorem is obvious. To prove necessity, post-multiply (47) by B_i

$$B_i' \left(\sum_{j \neq i}^N \mu_j K_{jT} \right) B_i = 0$$

Take an arbitrary column b_{ik} of B_i , where $k = 1, \dots, m_i$. Since $\mu_j K_{jT}$ is positive semi-definite for all j , it follows then that

$$b_{ik}' K_{jT} b_{ik} = 0 \quad \text{for all } i \text{ and } j, i \neq j$$

Since matrix K_{jT} is symmetric positive semi-definite, it can be written as $D_{jT}' D_{jT}$ where the matrix D_{jT} has the same rank as K_{jT} . We have then

$$b_{ik}' D_{jT}' D_{jT} b_{ik} = 0$$

from which we obtain $b'_{ik} D'_{jT} = 0$ and hence $b'_{ik} K_{jT} = 0$. Since the choice of b_{ik} was arbitrary, it follows that $B'_i K_{jT} = 0$. As we are proceeding backwards in time, we take (39) as given for $t = T$. It follows then from Lemma 1 that $S_{T-1} = \sum_{i=1}^N \mu_i K_{i,T-1}$ and the proof can be repeated for $t = T - 1$.

In this way, Theorem 4 can be proved for all t .

Q.E.D.

Now we will evaluate E_t^{-1} and $K_{i,t-1}$ on the assumption that (46) is satisfied for period t . We can write

$$E_t = I + \sum_{j=1}^N B_j R_{j,t-1}^{-1} B'_j K_{jt} = I + XY'$$

where $X = [B_1, B_2, \dots, B_N]$

and $Y = [K_{1t} B_1 R_{1,t-1}^{-1}, K_{2t} B_2 R_{2,t-1}^{-1}, \dots, K_{Nt} B_N R_{N,t-1}^{-1}]$

Applying identity (29) we obtain, after some manipulation,

$$(48) \quad E_t^{-1} = I - \sum_{j=1}^N B_j (I + R_{j,t-1}^{-1} B'_j K_{jt} B_j)^{-1} R_{j,t-1}^{-1} B'_j K_{jt}$$

From this and the assumed validity of (46) for period t , we obtain the following expression for $K_{i,t-1}$, where use has been made of (13),

$$(49) \quad K_{i,t-1} = Q_{i,t-1} + (I+A)' K_{it} (I+Z_{i,t-1} K_{it})^{-1} (I+A)$$

We can now prove :

THEOREM 5. Condition (39) or the equivalent condition (46) is satisfied if and only if, for $t = 1, \dots, T$ and for $i, j = 1, \dots, N$ and $i \neq j$,

$$(50) \quad B_j'(I+A')^s Q_{it} = 0 \quad s = 0, \dots, t-1$$

PROOF. For period T we have that $K_{iT} = Q_{iT}$ and so $B_j' K_{iT} = 0$ if and only if $B_j' Q_{iT} = 0$.

For $t = T-1$, we obtain from (49)

$$K_{i,T-1} = Q_{i,T-1} + (I+A)' Q_{iT} (I + Z_{i,T-1} K_{iT})^{-1} (I+A)$$

Since both terms on the right-hand side are symmetric positive semi-definite we will have that $B_j' K_{i,T-1} = 0$ if and only if $B_j' Q_{i,T-1} = 0$ and $B_j' (I+A)' Q_{iT} = 0$.

In general, solving (49) backwards in time by repeated substitution we get, for $\rho \in [1, T-1]$,

$$K_{i,T-\rho} = Q_{i,T-\rho} + \sum_{k=1}^{\rho} [(I+A')^k Q_{i,T-\rho+k} \prod_{m=\rho-k}^{\rho-1} L_{i,T-m}]$$

where $L_{it} = (I + Z_{i,t-1} K_{it})^{-1} (I+A)$

A necessary and sufficient condition for $B_j' K_{i,T-\rho} = 0$ is then that

$$B_j' (I+A')^k Q_{i,T-\rho+k} = 0 \quad \text{for } k = 0, 1, \dots, \rho$$

If we let ρ take values ranging from 1 to $T-1$ we arrive at the general condition (50).

Q.E.D.

Finally, we can prove :

THEOREM 6. If one of the mutually equivalent conditions (39), (46) or (50) holds, then condition (40) is satisfied.

PROOF. This is obvious for period T because of the boundary conditions (16) and (37). Under the assumption of the theorem and making use of (16) and (48), we can obtain the following result for period $T-1$ from (14) or (27)

$$B_j' g_{i,T-1} = B_j' (I+A') g_{iT} + B_j' (I+A') U_{jT} g_{iT} = 0 \quad j \neq i$$

where
$$U_{jT} = K_{jT} B_{jT} R_{j,T-1}^{-1} (I + B_j' K_{jT} B_{jT} R_{j,T-1}^{-1})^{-1} B_j'$$

Since (39) and (40) are satisfied for $t = T$, we have from Lemma 2 that

$$f_{T-1} = \sum_{i=1}^N \mu_i g_{i,T-1}$$

and hence $B_j' g_{i,T-1} = 0$ implies that (40) holds for period $T-1$.

Proceeding to period $T-2$, we obtain

$$B_j' g_{i,T-2} = B_j' (I+A')^2 g_{iT} + B_j' (I+A')^2 U_{jT} g_{iT} + B_j' (I+A') U_{j,T-1} g_{i,T-1}$$

Because of (16) and $B_j' g_{i,T-1} = 0$, it follows that $B_j' g_{i,T-2} = 0$. Again, since (39) and (40) are satisfied for T and $T-1$, we have from Lemma 2 that

$$f_{T-2} = \sum_{i=1}^N \mu_i g_{i,T-2}$$

and hence we can conclude that (40) holds for period $T-2$. Backward substitution leads to

$$B_j' g_{i,T-\rho} = B_j' (I+A')^\rho g_{iT} + B_j' \sum_{k=1}^{\rho} (I+A')^k u_{j,T-\rho+k} g_{i,T-\rho+k}$$

and the same line of reasoning can be applied to cover the entire planning period.

Q.E.D.

Hence we have established that the noncooperative equilibrium will be Pareto optimal if and only if (50) is satisfied. Since this condition has to hold for all possible values of the initial state x_0 , (50) can also be stated as follows :

$$(51) \quad x_t' Q_{it} (I+A)^s B_j = 0 \quad s = 0, \dots, t-1$$

It can be easily verified that $(I+A)^s B_j$ is the matrix of interim multipliers of order $s + 1$ of the control vector of player j , i.e.

$$(I+A)^s B_j = \frac{\partial x_t}{\partial u_{j,t-s-1}'}$$

Furthermore, since $x_t' Q_{it} = \frac{\partial J_i}{\partial x_t'}$, (51) reduces to

$$(52) \quad \frac{\partial J_i}{\partial u_{jt}} = 0 \quad i \neq j, \quad t = 0, \dots, T-1$$

6. CONCLUSIONS

We can now state the main result of this paper. Given that

- (a) all players have a quadratic cost functional (2),
- (b) the dynamic behavior of the system to be controlled can be described by the system of linear difference equations (1), with $I+A$ non-singular and
- (c) all players use closed-loop strategies,

then the noncooperative equilibrium will be Pareto optimal if and only if the cost functional of any one player is not influenced, directly or indirectly through the system of difference equations, by the controls pursued by the other players. This is a very stringent condition since it basically boils down to the absence of the fundamental characteristic of game theory, i.e. the conflict of interest.

The model used in this paper is well suited to analyze the nature and the efficiency of the decision-making process in institutionally decentralized systems of economic policy. These systems are characterized by two basic properties. First, the various policy-makers have conflicting objectives concerning the desired trajectories of at least some economic variables. Second, the various systems, the welfare of which the respective policy-makers want to promote, are not independent. This implies that no authority is able to completely control the time path of the target variables in which he is interested. This second characteristic has become more important because of the strong interdependencies of the various decentralized policy-systems, which can lead to significant spillovers of locally designed policies.

The result of this paper subscribes closely to the finding in international and regional economics that, in an interdependent economic setting, the

pursuance of a stabilization policy - or some other economic policy - by each country or each region individually often leads to suboptimal outcomes (see Hamada [6,7] and Engerman [4]). In the specific game we have analyzed, it has been shown that the equilibrium of the noncooperative game has its price since the equilibrium strategies are not efficient in the sense of Pareto, except under extreme and unreasonable conditions. This conclusion stresses the importance of changing noncooperative games into cooperative games. This means that one should provide the players with effective incentives such that agreements can be made binding.

Although our analysis provides some basic insight into the problem, the model we used is restrictive in some ways. It should be interesting to know how the conclusions of this paper are affected if $I+A$ is not restricted to be non-singular, if the problem is generalized to an infinite horizon and if the linear and quadratic functional forms are generalized. Another question concerns the Pareto optimality of the noncooperative Stackelberg equilibrium⁶. All these problems are interesting topics for future research.

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⁶ See Simaan and Cruz [14,15] and Kydland [11] .

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