ON THE REPRESENTATION OF ADDITIVE PRINCIPLES
OF PREMIUM CALCULATION

BY

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Abstract

Two characterizations of mixtures of Escher principles in terms of additivity combined with other conditions are given.

1. Introduction

In the following a random variable of bounded range will be called a risk. A principle of premium calculation, say H , is a rule that assigns a real number to any risk X . Symbolically :

$$H : X \rightarrow H(X)$$

In connection with an insurance contract, $H(X)$ should be interpreted as the premium that is quoted for a payment of X , the claims.

Some examples are :

The net premium principle

$$H(X) = E(X) \quad (1)$$

The variance principle

$$H(X) = E(X) + \alpha \text{Var}(X) \quad (2)$$

The maximal loss principle

$$H(X) = \text{Max}(X) \quad (3)$$

The exponential principle

$$H(X) = \frac{1}{|a|} \log E(e^{aX}) \quad (4)$$

The Esscher principle

$$H(X) = \frac{E(Xe^{hX})}{E(e^{hX})} \quad (5)$$

In the last example $H(X)$ is the mean of the Esscher transform (with parameter h) of the distribution of X .

Typically, the parameters α , a and h are positive, so that the premium provides for a loading.

A premium H is said to be additive, if

$$H(X_1 + X_2) = H(X_1) + H(X_2) \quad (6)$$

whenever X_1 and X_2 are independent.

An obvious requirement is that

$$H(c) = c \text{ for any constant } c \quad (7)$$

All the examples above have these two properties.

Instead of describing a risk directly by its distribution, we prefer its specification in terms of the derivative of the cumulant generating function :

$$\phi_X(t) = \frac{d}{dt} \log E(e^{tX}) , \quad -\infty < t < +\infty \quad (8)$$

One can interpret $\phi_X(t)$ as the mean, and $\phi_X'(t)$ as the variance, of the Esscher transform (with parameter t) of the distribution of X . It follows that for any risk X , say with $a = \text{Min}(X)$ and $b = \text{Max}(X)$, $\phi_X(t)$ is a function that increases from a to b as t goes from $-\infty$ to $+\infty$, and that

$$\phi_X'(t) \leq \frac{1}{4} (b-a)^2 ,$$

Let $\tilde{\Phi} = \{\phi(t) \mid \phi(t) = \phi_X(t) \text{ for some risk } X\}$

From now on we consider $\tilde{\Phi}$ as the domain of the functional H .

Symbolically :

$$H : \phi \rightarrow H(\phi)$$

In this new notation, conditions (6) and (7) can be restated as follows :

$$H(\phi_1 + \phi_2) = H(\phi_1) + H(\phi_2) , \quad \phi_1, \phi_2 \in \tilde{\Phi} \quad (6')$$

$$H(c) = c \text{ for any constant } c. \quad (7')$$

The formulae for the examples cited above take on the following form :

$$H(\phi) = \phi(0) \quad (1')$$

$$H(\phi) = \phi(0) + c \phi'(0) \quad (2')$$

$$H(\phi) = \phi(\infty) \quad (3')$$

$$H(\phi) = \frac{1}{a} \int_0^a \phi(t) dt \quad (4')$$

$$H(\phi) = \phi(b) \quad (5')$$

We shall consider a special family of distributions.

Let $0 < p < 1$, and $q = 1 - p$.

For $z > 0$ and $-\infty < t < +\infty$

let X have the following two point distribution :

$$P(X=z) = 1 - P(X=0) = \frac{qe^{-zh}}{p + qe^{-zh}} \quad (8)$$

let $\phi(z, h)(t) = \phi_X(t)$, i.e.

$$\phi(z, h)(t) = \frac{zqe^{z(t-h)}}{p + qe^{-zh}} \quad (9)$$

Note that $\phi(z, h)(h) = zq$, and (more importantly) that

$$\lim_{z \rightarrow \infty} \frac{\phi(z, h)(t)}{z} = \begin{cases} 0 & \text{if } t < h \\ 1 & \text{if } t > h \end{cases} \quad (10)$$

2. Mixtures of Esscher principles

A principle H is said to be a mixture of Esscher principles if it is of the form :

$$H(\phi) = F(-\infty)\phi(-\infty) + \int_{-\infty}^{+\infty} \phi(h) dF(h) + [1-F(\infty)]\phi(\infty) \quad (11)$$

where $F(h)$ is a nondecreasing function with $0 \leq F \leq 1$.

With the exception of the variance principle, all the examples above are of this type. After an integration by parts (11) can be written as :

$$H(\phi) = \phi(-\infty) + \int_{-\infty}^{+\infty} \phi'(h)[1-F(h)]dh. \quad (11')$$

Clearly, a mixture of Esscher principles satisfies properties (6') and (7'). In the following two sections we shall discuss conditions that lead to mixtures of Esscher principles. But first we discuss a comparison theorem :

Theorem 1:

Let H_1, H_2 be mixtures of Esscher principles generated by F_1, F_2 , respectively. Then $H_1(\phi) \leq H_2(\phi)$ for all $\phi \in \mathcal{F}$ iff $F_1(h^-) \geq F_2(h^-)$ for all h .

Proof:

The "if" part follows immediately from (11').

For the "only if" part we start with the inequality

$$H_1(\phi^{(z,h)}) \leq H_2(\phi^{(z,h)}).$$

Thus

$$\int_{-\infty}^{+\infty} \phi^{(z,h)}(t) dF_1(t) + [1-F_1(\infty)]z \leq \int_{-\infty}^{+\infty} \phi^{(z,h)}(t) dF_2(t) + [1-F_2(\infty)]z$$

Now divide by z and let $z \rightarrow \infty$, see (10), to obtain

$$1 - F_1(h^-) \leq 1 - F_2(h^-), \quad \text{q.e.d.}$$

Corollary:

A mixture of Esscher principles generated by F provides for a nonnegative loading, i.e. $H(\phi) \geq \phi(0)$ for all $\phi \in \mathcal{F}$, iff $F(0-) = 0$.

3. First Characterization

Theorem 2 : A principle H is a mixture of Escher principles, iff properties (6') and (7') are satisfied, and for any $\phi_1, \phi_2 \in \Phi$

$$\phi_1(t) \leq \phi_2(t), -\infty < t < +\infty, \text{ implies } H(\phi_1) \leq H(\phi_2). \quad (13)$$

Proof:

The "if" part is trivial. For the "only if" part first observe that (7') and (13) imply that

$$\phi(-\infty) \leq H(\phi) \leq \phi(+\infty), \phi \in \Phi \quad (14)$$

It follows that

$$0 \leq \frac{H(\phi(z, h))}{z} \leq 1.$$

Hence there is a sequence $\{z_k\}$, with $z_k \rightarrow \infty$ for $k \rightarrow \infty$, and a function $F(h)$, $0 \leq F(h) \leq 1$ such that

$$\lim_{k \rightarrow \infty} \frac{H(\phi(z_k, h))}{z_k} = 1 - F(h) \quad (15)$$

for all rational h . Since $\phi^{(z, h)}(t)$ is a decreasing function of h , it follows from (13) that $H(\phi^{(z, h)})$ is a nonincreasing function of h .

Hence we may assume that F is a nondecreasing function and that (15) holds at all points of continuity of F . By modifying F at the points of discontinuity (and possibly switching to a subsequence of $\{z_k\}$) we can assume that (15) holds for all h .

Let $\phi_1, \phi_2, \dots, \phi_n \in \Phi$ and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be positive rational numbers. We claim that

$$\left| \phi(t) - \sum_{i=1}^n \alpha_i \phi_i(t) \right| \leq c \quad (16)$$

implies that

$$\left| H(\phi) - \sum_{i=1}^n \alpha_i H(\phi_i) \right| \leq c. \quad (17)$$

To show this, let $a_i = a/m$, where $m, a_1, \dots, a_n$ are positive integers.

Then (16) can be written as :

$$m \phi(t) - m c < \sum_{i=1}^n a_i \phi_i(t) \leq m \phi(t) + m c$$

From (13) it follows that :

$$H(m\phi - mc) \leq H\left(\sum_{i=1}^n a_i \phi_i\right) \leq H(m\phi + mc)$$

or, because of (6') and (7'), that

$$mH(\phi) - mc \leq \sum_{i=1}^n a_i H(\phi_i) \leq mH(\phi) + mc.$$

Now (17)'s obtained after a division by m .

By reasons of continuity, the conclusion from (16) to (17) is also valid if the a_i 's are reals.

Now we are ready to show that (11) holds for any $\phi \in \Phi$. If $\phi(t)$ is constant, (11) is trivially satisfied. Otherwise, we define for $n = 1, 2, \dots$ the numbers h_{in} ($i=1, 2, \dots, 2^{n-1}$) by the condition that

$$\phi(h_{in}) = a + i 2^{-n} (b-a)$$

where $a = \phi(-\infty)$, $b = \phi(\infty)$. The idea is to approximate $\phi(t)$ by

$$\psi(z, n)(t) = a + 2^{-n} (b-a) \sum_{i=1}^{2^{n-1}} \frac{\phi(z, h_{in})(t)}{z}$$

Because of (10),

$$|\psi(t) - \psi(z, n)(t)| \leq 2^{-n} (b-a) + \epsilon(z, n)$$

where $\epsilon(z, n) \rightarrow 0$ for $z \rightarrow \infty$. It follows that

$$\left| H(\phi) - a - 2^{-n} (b-a) \sum_{j=1}^{2^{n-1}} \frac{H(\phi(z, h_{jn}))}{z} \right| \leq 2^{-n} (b-a) + \epsilon(z, n)$$

Setting $z = z_k$ and letting $k \rightarrow \infty$, we obtain from this and (15) that

$$\left| H(\phi) - a - 2^{-n} (b-a) \sum_{i=1}^{2^{n-1}} [1 - F(h_{in})] \right| \leq 2^{-n} (b-a)$$

Now let $n \rightarrow \infty$. Then (11') follows from this and the observation that

$$\sum_{i=1}^{2^n} 2^{-n} [1 - F(h_{in})] \rightarrow \int_{-\infty}^{\infty} \phi'(u) [1 - F(u)] du, \quad \text{q.e.d.}$$

Remarks

1. The proof could be simplified somewhat by using the representation theorem (of F. Riesz) for linear functionals on $C[-\infty, +\infty]$ (Since (16) implies (17), H can be extended to a linear functional) However, the proof given here is independent and seems to be of interest by its own.
2. One might argue that the Esscher principles are contained in condition (13) in a very transparent way, and that the result is therefore not at all that surprising. In this sense the characterization of the next section, which is also easier to prove, is more satisfactory.

4. Second characterization

This characterization will be in terms of two conditions :

Condition 1 : $H(\phi^{(z,h)})$ is a nonincreasing function of h with
 $0 \leq H(\phi^{(z,h)}) \leq z$.

Condition 2 : For every $\phi \in \mathcal{F}$

$$\lim_{z \rightarrow \infty} \int_{-\infty}^{+\infty} \phi'(h) \frac{H(\phi^{(z,h)})}{z} dh + \phi(-\infty) = H(\phi)$$

Then the following result holds :

Theorem 3 :

A principle H is a mixture of Esscher principles iff conditions 1 and 2 are satisfied.

Proof :

The "if" part is straightforward. For the "only if" part we conclude from condition 1 that there is a sequence $\{z_k\}$ with $z_k \rightarrow \infty$ for $k \rightarrow \infty$ and a nondecreasing function $F(h)$ such that (15) holds a.e. Now we use condition 2 (with z replaced by z_k) and the dominated convergence theorem to obtain (11'). q.e.d.

Remarks :

1. Condition 1 is a very natural requirement : increasing h amounts to decreasing $P(X=z)$, see (8).
2. Condition 2 implies (6') and (7') and provides for a certain continuity of H . Its motivation stems from the identity

$$\phi(t) = \int_{-\infty}^{+\infty} \phi'(h) \lim_{z \rightarrow \infty} \frac{\phi^{(z,h)}(t)}{z} dh + \phi(-\infty)$$

See (10), and the requirement that the mapping H should be linear and (in a certain sense) continuous.

3. Condition 2 is satisfied for a wide class of principles. For example, the variance principle satisfies condition 2. Let $V(\phi) = \phi'(0)$:

Then

$$\int_{-\infty}^{+\infty} \phi'(h) \frac{V(\phi(z, h))}{z} dh = \int_{-\infty}^{+\infty} \phi'(h) \frac{z p q e^{-zh}}{(p+q e^{-zh})^2} dh .$$

Outside any open interval containing 0, the integrand tends to 0 uniformly for $z \rightarrow \infty$. Hence, the limit for $z \rightarrow \infty$ is

$$\phi'(0) \int_{-\infty}^{+\infty} \frac{z p q e^{-zh}}{(p+q e^{-zh})^2} dh = \phi'(0) = V(\phi) .$$

With V also the variance principle satisfies condition 2.

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References

1. Borch, K "The Safety Loading of Reinsurance Premiums",
Skandinavisk Aktuarietidskrift (1960), pp. 163 - 184.
2. ——— "Equilibrium in a Reinsurance Market",
Econometrica (1962), pp. 424 - 444.
3. ——— "A Contribution to the Theory of Reinsurance Markets",
Skandinavisk Aktuarietidskrift (1962), pp. 176-189.
4. Coovaerts M. and De Vylder Fl. "A note on additive premium calculation principles", forthcoming in
Bulletin de l'Association Royal des Actuaire Belges
5. Gerber, H.U. "On additive premium calculation principles",
Astin Bulletin 7 (1974), pp. 215 - 222.

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