Pseudo Predictor Feedback Stabilization of Linear Systems with Both State and Input Delays

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ABSTRACT

This paper is concerned with the stabilization of linear systems with both input and state delay by utilizing the predictor based delay compensation method. A pseudo predictor feedback control scheme is proposed, where the targeted closed-loop system is used for the prediction. This avoids the explicit dependence of the control input on past input data, and related instability problems in the implementation of the control law. It is proved that the stability of the closed-loop system is determined by the stability of the target closed-loop system, along with the stability of an associated integral delay system, which describes the additional dynamics induced by the pseudo predictor. In the considered case where the input delay is larger than the state delay the control law explicitly depends on the fundamental matrix of the system. It is shown that in case of commensurate delays, the control law can be directly expressed as a function of the original system matrices, while the stability of the integral delay system can be connected with the stability of an associated system with discrete delay, widening the applicability of analysis and design tools. For the synthesis of the feedback gains, a numerical procedure is presented to solve the simultaneous stabilization problem of the target system and the associated integral delay system. Numerical experiments are carried out to illustrate the effectiveness of the methodology.

KEYWORDS

Pseudo predictor feedback, retarded-type linear delay system, stabilization, integral delay system, time-varying systems.

1. Introduction

In many engineering applications such as networked control systems (Heemels et al., 2010; Lu et al., 2012), video transmission (Khalek, Caramanis & Heath, 2014) and wireless communications (Ai et al., 2014), delay phenomena are encountered which may adversely affect the stability and performance of the system. For time-delay systems, the stability analysis and stabilization problem have attracted massive attention during the past decades and enormous investigations have been conducted in the existing literature. The Lyapunov-Krasovskii functional based method is an important method for addressing the asymptotic stability analysis and stabilization problem of time-delay systems, and it has been widely investigated in the literature (for instance,

one can refer to Egorov & Kharitonov (2018); Krstic (2008); Zhang et al. (2019) and references therein). This method often gives rise to tractable criteria for analysis and controller synthesis in terms of linear matrix inequalities (LMIs). However, in most cases this approach only provides sufficient conditions for stability/stabilization of time-delay systems, while imposing structure on the controller is difficult. At the same time advances in matrix computations have led to a complementary, eigenvalue based approach for the analysis and control of linear time-invariant time-delay systems (Michiels & Niculescu, 2014). Using this methodology, the location of the spectrum, stability, performance and robustness measures are directly optimized as a function of the available controller parameters. For example, stability is assessed by computing the rightmost eigenvalues, while stabilization is achieved by minimizing the spectral abscissa (the real part of the rightmost eigenvalue) as a function of the controller parameters. Such an approach is non-conservative, as it is grounded in a necessary and sufficient condition for stability, and structure or sparsity can be easily imposed on the controller parameterization (Dileep et al., 2018), at the price that the optimization problems for the synthesis may be non-convex.

Delays in control loops may introduce significant challenges and limitations. An established and intuitive approach for handling delays in the control input consists of predictor feedback control, where one aims at compensating the effect of the input delay by using a prediction of the system’s state for feedback. Numerous studies focusing on different design problems were conducted and various results were published in the literature. For example, the exponential stability of predictor feedback in multi-input systems with distributed delay was investigated in Bekiaris-Liberis & Krstic (2010a) by constructing a corresponding Lyapunov functional; an observer-predictor control is proposed in Sorribes & Gil (2019) to stabilize uncertain systems subject to unknown but bounded arbitrarily-fast time-varying input and output delays; the pseudo-predictor feedback (PPF) approach was proposed in Zhou (2014b) for stabilization of time-varying linear systems with only a single time-varying input delay. Differently from the traditional predictors, the PPF approach predicts future states by using the artificial closed-loop system dynamics corresponding to the absence of input delay. As a result, the controller is finite dimensional and easy to be implemented. Finite-dimensional controllers involving a closed-loop system matrix for a linear system with a single input delay also appeared in Cacace, Germani, & Manes (2014) for achieving partial spectrum assignment, and in Carravetta, Palumbo, & Pepe (2013) for solving the infinite horizon optimal control problem. Different from Cacace, Germani, & Manes (2014) and Carravetta, Palumbo, & Pepe (2013), the idea of PPF in Zhou (2014b) also allows to establish memory/infinite-dimensional controllers for linear systems with multiple pointwise and distributed input delays (Liu & Zhou, 2019; Zhou & Cong, 2016), and for neutral time-delay systems with both input and state delays (Zhou, Liu, & Michiels, 2019).

Linear systems with both state and input delay are more complicated to deal with. In Rodríguez-Guerrero, Kharitonov, & Mondié (2016), the robust stability properties of a dynamic predictor scheme for state and input delay systems were analyzed in the frequency domain, and robust stability bounds for matrix parameters, input or state delay uncertainties were also given. For the case where the delays are of unknown length, a Lyapunov-based adaptive controller was designed in Bekiaris-Liberis & Krstic (2010b) for linear feedforward systems with simultaneous state and input delays. In Zhou (2014a), nested prediction approach was proposed to compensate large input delays. By predicting the future states in a nested fashion, arbitrarily large yet bounded input delay can be completely compensated. At the same time, a totally different

prediction scheme for the same problem was presented in Kharitonov (2014), building on the concept of fundamental matrix. Very recently, an observer-predictor based approach without using input filters was established in Zhou, Liu, & Mazenc (2017).

Inspired by the fundamental matrix based approach proposed in Kharitonov (2015), in this paper we extend the PPF approach presented in Liu & Zhou (2019); Zhou (2014b); Zhou, Liu, & Michiels (2019) for linear systems with only input delays to the case of linear systems with both state and input delays. By the PPF controller developed in this paper, arbitrarily large yet bounded input delays can be compensated properly. It is shown that the stability of the closed-loop system controlled by the PPF control can be recast into the stability of the target closed-loop system (without input delay) and the stability of a certain integral delay system. Differently from traditional predictor feedback, the distributed terms in the proposed PPF control scheme can be safely implemented without using of input filters. We also show how the aforementioned eigenvalue based approach can be used to synthesize stabilizing controller gains. The main difference of the problem considered in this paper and the one considered in Zhou, Liu, & Michiels (2019) is that the input delay in the present paper is larger than the state delay, while in Zhou, Liu, & Michiels (2019) the input delay equals the state delay. As a result, the fundamental matrix has to be employed to predict the future states, instead of the closed-form state transition matrix used in Zhou, Liu, & Michiels (2019). However, we will show that in case of commensurate delay the control law can be reformulated in terms of the controller gains and system matrices of the uncontrolled system, thereby avoiding the explicit use of the fundamental matrix.

The remainder of the paper is organized as follows: the problem formulation and some preliminaries are given in Section 2. The PPF approach is then developed in Section 3. The computation of the feedback gains is addressed in Section 4. Numerical examples are given in Section 5 to verify the efficiency of the proposed method and we present the conclusions in Section 6. Some preliminary developments regarding the results in 3 have been reported in Zhang, Zhou & Michiels (2019).

2. Problem formulation and preliminaries

Consider the retarded-type linear system in the form of

$$\dot{x}(t) = A_1 x(t) + A_2 x(t - r) + Bu(t - h), t \geq 0, \quad (1)$$

where $A_i \in \mathbb{R}^{n \times n}$, $i = 1, 2$ and $B \in \mathbb{R}^{n \times m}$ are known coefficient matrices, $r > 0$ and $h > 0$ are exactly known state delay and input delay, respectively. We assume that $h > r$ since the case $h \leq r$ can be treated by the method in Zhou, Liu, & Michiels (2019). Denoting by x_t the function segment $x(t + \theta)$, $\theta \in [-r, 0]$, we address the following problem.

Problem 1. Design a control law of the form $u(t) = \mathcal{F}(x_t)$ for the time-delay system (1) such that the corresponding closed-loop system is asymptotically stable, where $\mathcal{F}$ is a bounded linear operator.

The adopted approach is based on compensation of the input delay, in order to ease the stabilization task. More precisely, if we consider a controller of the form

$$u(t) = F_1 x(t + h) + F_2 x(t + h - r), t \geq 0, \quad (2)$$

then we arrived at the *target closed-loop system*

$$\dot{x}(t) = (A_1 + BF_1)x(t) + (A_2 + BF_2)x(t - r), t \geq 0. \quad (3)$$

However, designing gains F_1 and F_2 such that (3) is asymptotically stable is not sufficient. Because the controller (2) is acausal, what we should do is making it implementable, by predicting the future state $x(t + h)$ and $x(t + h - r)$ from the present state $x(t)$.

Function $K_o : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ is called the fundamental matrix (Bellman & Cooke, 1963) of system (1) in the absence of input u if it satisfies

$$\begin{cases} \dot{K}_o(t) = A_1 K_o(t) + A_2 K_o(t - r), t \geq 0, \\ K_o(t) = 0_{n \times n}, t < 0, \\ K_o(0) = I_{n \times n}. \end{cases} \quad (4)$$

Then the traditional methods predict the future states by using the dynamics of the open loop system (1) as in Kharitonov (2014), which leads to the expressions.

$$x(t + h) = K_o(h)x(t) + \int_{-r}^0 K_o(h - \theta - r)A_2x(t + \theta)d\theta + \int_{-h}^0 K_o(-\theta)Bu(t + \theta)d\theta, \quad (5)$$

and

$$\begin{aligned} x(t + h - r) = & K_o(h - r)x(t) + \int_{-r}^0 K_o(h - \theta - 2r)A_2x(t + \theta)d\theta \\ & + \int_{-h}^{-r} K_o(-r - \theta)Bu(t + \theta)d\theta. \end{aligned} \quad (6)$$

Substituting the predicted states (5) and (6) into the controller (2), one can obtain

$$u(t) = Q_o(h)x(t) + \int_{-r}^0 Q_o(h - r - \theta)A_2x(t + \theta)d\theta + \int_{-h}^0 Q_o(-\theta)Bu(t + \theta)d\theta, t \geq 0, \quad (7)$$

where $Q_o(t) = F_1 K_o(t) + F_2 K_o(t - r)$. It can be seen that the third term in the right hand side of (7) involves the integration of $u(t)$ in the interval $[-h, 0]$, which needs to be done on-line by numerical integration.

Remark 1. According to Michiels, Mondié & Roose (2003); Van et al. (1999), when implementing the traditional controller (7) by discretizing the distributed term with sufficient precision, a necessary condition guaranteeing the asymptotic stability of the closed-loop system is that the integral-type system

$$u(t) = \int_{-h}^0 Q_o(-\theta)Bu(t + \theta)d\theta \quad (8)$$

is asymptotically stable.

Recently, pseudo-predictor feedback (PPF) was established to solve the stabilization problem for time-varying linear systems with a single input delay (Zhou, 2014b). Differently from the traditional predictor feedback, the PPF predicts the future state

by using the dynamic of the virtual closed-loop system (3), which results in a controller without distributed terms, and is thus easy to implement. In this paper, the PPF approach will be adopted to solve the stabilization problem for retarded-type linear systems with both state and input delays.

3. Pseudo Predictor Feedback

3.1. Derivation of the control law

Let K be the fundamental matrix of the closed-loop system (3) and denote $A_{1c} \triangleq A_1 + BF_1$, $A_{2c} \triangleq A_2 + BF_2$. By using (5) and (6) with $K_0 \rightarrow K$, $A_1 \rightarrow A_{1c}$, $A_2 \rightarrow A_{2c}$, and $u = 0$, we can obtain

$$x(t+h) = K(h)x(t) + \int_{-r}^0 K(h-\theta-r)A_{2c}x(t+\theta)d\theta \quad (9)$$

and

$$x(t+h-r) = K(h-r)x(t) + \int_{-r}^0 K(h-\theta-2r)A_{2c}x(t+\theta)d\theta, t \geq 0. \quad (10)$$

Substituting (9) and (10) into the controller (2), one can obtain

$$\begin{aligned} u(t) &= F_1x(t+h) + F_2x(t+h-r) \\ &= (F_1K(h) + F_2K(h-r))x(t) + F_1 \int_{-r}^0 K(h-\theta-r)A_{2c}x(t+\theta)d\theta \\ &\quad + F_2 \int_{-r}^0 K(h-\theta-2r)A_{2c}x(t+\theta)d\theta, t \geq 0. \end{aligned} \quad (11)$$

The obtained controller (11) is referred to as PPF controller because the future states are predicted by using the artificial closed-loop system (3), rather than the original open-loop system (1). It can be observed that the PPF controller eliminates the terms involving integration of $u(t)$ in the right side of (11).

Remark 2. If the state delay r in system (1) is zero, then the controller (11) reduces to

$$u(t) = F\Phi_{A+BF}(h)x(t),$$

where $F = F_1 + F_2$, $A = A_1 + A_2$ and $\Phi_{A+BF}(t) = e^{(A+BF)t}$ is the state transfer matrix of the closed-loop system (1) in the absence of input $u(t)$. Such a controller is exactly the time-invariant version of the controller designed in Zhou (2014b). On the other hand, if the state delay r in system (1) equals the input delay h , namely, $r = h$, we can use the state transfer matrix of the closed-loop system (3) instead of the fundamental matrix, the PPF controller (11) reduces to

$$u(t) = F_2x(t) + F_1 \int_{t-r}^t e^{A_{1c}(t-s)} A_{2c}x(s)ds,$$

which takes the same form as the PPF controller proposed in Zhou, Liu, & Michiels (2019).

3.2. Stability analysis of the closed-loop system

The closed-loop system consisting of the linear retarded time-delay system (1) and the PPF controller (11) can be formulated as

$$\begin{aligned} \dot{x}(t) = & A_1 x(t) + A_2 x(t-r) + B(F_1 K(h) + F_2 K(h-r)) x(t-h) \\ & + \int_{-r}^0 B(F_1 K(h-\theta-r) + F_2 K(h-\theta-2r)) A_{2c} x(t-h+\theta) d\theta, t \geq 0. \end{aligned} \quad (12)$$

Regarding the stability of the closed-loop system (12), we can provide the following theorem,

Theorem 3.1. *The closed-loop system (12) is asymptotically stable if and only if both target system (3) and the following integral delay system (IDS),*

$$\rho(t) = - \int_{t-h}^t (F_1 K(t-\theta) + F_2 K(t-r-\theta)) B \rho(\theta) d\theta \quad (13)$$

are asymptotically stable, where $K(t)$ is the fundamental matrix of system (3).

Proof. *Sufficiency.* Consider the closed-loop system (12), which can be rewritten as

$$\dot{x}(t) = A_{1c} x(t) + A_{2c} x(t-r) + B \bar{\rho}(t), \quad (14)$$

where

$$\begin{aligned} \bar{\rho}(t) = & -F_1 x(t) - F_2 x(t-r) + (F_1 K(h) + F_2 K(h-r)) x(t-h) \\ & + \int_{-r}^0 (F_1 K(h-\theta-r) + F_2 K(h-\theta-2r)) A_{2c} x(t-h+\theta) d\theta, t \geq 0. \end{aligned} \quad (15)$$

From (14), the Cauchy formula gives, for all $t \geq 0$,

$$x(t) = K(h) x(t-h) + \int_{-r}^0 K(h-\theta-r) A_{2c} x(t-h+\theta) d\theta + \int_{t-h}^t K(t-\theta) B \bar{\rho}(\theta) d\theta, \quad (16)$$

and for all $t \geq r$,

$$\begin{aligned} x(t-r) = & K(h-r) x(t-h) + \int_{-r}^0 K(h-\theta-2r) A_{2c} x(t-h+\theta) d\theta \\ & + \int_{t-h}^{t-r} K(t-r-\theta) B \bar{\rho}(\theta) d\theta. \end{aligned} \quad (17)$$

Notice that $K(t) = 0$ for all $t < 0$. Thus the limits of the third term on the right hand of (17) can be expanded as $\int_{t-h}^t K(t-r-\theta) B \bar{\rho}(\theta) d\theta$. Substituting (16) and (17) into

(15) gives

$$\bar{\rho}(t) = - \int_{t-h}^t (F_1 K(t-\theta) + F_2 K(t-r-\theta)) B \bar{\rho}(\theta) d\theta, t \geq 0,$$

which takes the form of (13). The closed-loop system (14) (or, equivalently, (12)) is thus asymptotically stable.

Necessity. Consider the following auxiliary system, for all $t \geq 0$,

$$\begin{cases} \dot{y}(t) = A_{1c}y(t) + A_{2c}y(t-r) + B\varrho(t), \\ \varrho(t) = - \int_{t-h}^t (F_1 K(t-\theta) + F_2 K(t-r-\theta)) B \varrho(\theta) d\theta. \end{cases} \quad (18)$$

The solutions to the first equation of (18) for is given by $t \geq 0$

$$y(t) = K(h)y(t-h) + \int_{-r}^0 K(h-\theta-r) A_{2c} y(t-h+\theta) d\theta + \int_{t-h}^t K(t-\theta) B \varrho(\theta) d\theta, \quad (19)$$

and for all $t \geq r$,

$$\begin{aligned} y(t-r) = & K(h-r)y(t-h) + \int_{-r}^0 K(h-\theta-2r) A_{2c} y(t-h+\theta) d\theta \\ & + \int_{t-h}^t K(t-r-\theta) B \varrho(\theta) d\theta. \end{aligned} \quad (20)$$

Substituting (19) and (20) into the second equation of (18) gives

$$\begin{aligned} \varrho(t) = & F_1 (K(h)y(t-h) - y(t)) + F_2 (K(h-r)y(t-h) - y(t-r)) \\ & + \int_{-r}^0 (F_1 K(h-\theta-r) + F_2 K(h-\theta-2r)) A_{2c} y(t-h+\theta) d\theta, t \geq 0. \end{aligned} \quad (21)$$

According to (21) we can obtain, for all $t \geq 0$,

$$\begin{aligned} \dot{y}(t) = & A_{1c}y(t) + A_{1c}y(t-r) + B F_1 (K(h)y(t-h) - y(t)) \\ & + B F_2 (K(h-r)y(t-h) - y(t-r)) \\ & + \int_{-r}^0 B (F_1 K(h-\theta-r) + F_2 K(h-\theta-2r)) A_{2c} y(t-h+\theta) d\theta \\ = & A_1 y(t) + A_2 y(t-r) + B (F_1 K(h) + F_2 K(h-r)) y(t-h) \\ & + \int_{-r}^0 B (F_1 K(h-\theta-r) + F_2 K(h-\theta-2r)) A_{2c} y(t-h+\theta) d\theta. \end{aligned}$$

By assumption, y converges to zero exponentially. Then, from (21) we conclude that ϱ (and thus also ρ) are converging exponentially to zero. Thus the proof is completed. $\square$

Remark 3. The IDS (13) takes the form of the IDS (27) in Zhou (2014b), where $G(\theta) = (F_1 K(-\theta) + F_2 K(-r-\theta))$. Thus the sufficient condition guaranteeing stability of the IDS obtained in Zhou (2014b) is also applicable to (13), and omitted for brevity.

Remark 4. Under assumption of stability of the target system (3) the stability of (13) is a sufficient and necessary condition for stability of the closed-loop system (12), while the stability of (8) is only a necessary condition for the stability of closed-loop system consisting of (1) and (8). Furthermore, unlike the integral kernel in (8), the kernel of (13) depends on the controller parameters, which allows additional tuning.

Remark 5. The PPF control scheme presented in this paper can also be extended to retarded-type time-varying linear systems with both time-varying state and input delays, as worked in Zhang, Zhou & Michiels (2019).

We rewrite the PPF controller (11) as

$$u(t) = (F_1 K(h) + F_2 K(h - r)) x(t) + \int_0^r (F_1 K(h - r + \theta) + F_2 K(h - 2r + \theta)) A_{2c} x(t - \theta) d\theta, t \geq 0, \quad (22)$$

which still involves an integral term that should be approximately computed by a finite sum in the implementation. However, differently from the traditional predictor feedback, such an approximation will not lead to any instability problems. More precisely, let N be an appropriate positive integer. Then the integral term in (22) can be approximated as

$$u(t) = (F_1 K(h) + F_2 K(h - r)) x(t) + \sum_{i=1}^N \eta(i) (F_1 K(h - r + \theta_i) + F_2 K(h - 2r + \theta_i)) A_{2c} x(t - \theta_i), \quad (23)$$

where $\{\eta(i), \theta_i\}_{i=1}^N$ are the nodes and weights of the quadrature rule. The following proposition states that controller (22) can be implemented safely.

Proposition 3.2. *Assume the closed-loop system consisting of (1) and PPF controller (22) is asymptotically stable, and the selected quadrature rule satisfies convergence property (27) in Mondié & Michiels (2003). Then there exists an appropriate integer N^* such that the closed-loop system consisting of (1) and the approximated PPF (23) is asymptotically stable for all $N > N^*$.*

The underlying reason for the robustness against small implementation errors is that, unlike in traditional prediction based controllers, there is no direct feed-through term in the control law ($u(t)$ linearly depending on past values of $u(t)$). A formal proof can be constructed following the lines of the proof of Proposition 1 in Zhou, Liu, & Michiels (2019).

3.3. Refined stability conditions for the commensurate delays case

We now consider the case where the delays (r, h) are commensurate, by taking the following assumption.

Assumption 1. There exist integers $n_1 > 0$ and $n_2 > 0$ and a constant $\delta > 0$ such that

$$r = n_1 \delta, \quad h = n_2 \delta.$$

This assumption will induce refinements of Theorem 3.1 in two directions:

- It allows to express Equation (13) directly in terms of the original system matrices (instead of indirectly via the fundamental matrix).
- It allows to connect the stability of (13) with the stability of an associated delay systems with discrete delays.

Furthermore, as we shall motivate in Remark 7 given later on, such an assumption is not restrictive from an application point of view.

Let us partition the fundamental matrix K , for $t \in [0, h]$, in the following way,

$$\begin{cases} K_1(s) = K(s), \\ K_2(s) = K(s + \delta), \\ K_3(s) = K(s + 2\delta), \\ \vdots \\ K_{n_2}(s) = K(s + (n_2 - 1)\delta), s \in [0, \delta], \end{cases}$$

and define $\tilde{K}(s) = [K_1^T(s) \cdots K_{n_2}^T(s)]^T$. Then we can introduce the following $\lceil n_2/n_1 \rceil \times \lceil n_2/n_1 \rceil$ block matrix,

$$\tilde{A} = \begin{bmatrix} A_{1c} & & & \\ & \ddots & & \\ & & A_{1c} & \\ A_{2c} & & & \\ & \ddots & & \\ & & A_{2c} & \\ & & & A_{1c} \\ & & & & \ddots & \\ & & & & & \ddots & \\ & & & & & & A_{2c} & 0 & A_{1c} & \\ & & & & & & & \ddots & & \\ & & & & & & & & A_{2c} & 0 & & & A_{1c} \end{bmatrix}$$

$\in \mathbb{R}^{(n_2 n) \times (n_2 n)},$

where all blocks except for the last block row and block column have dimensions $(n_1 n) \times (n_1 n)$. One observes that function $\tilde{K}$ satisfies

$$\dot{\tilde{K}}(s) = \tilde{A} \tilde{K}(s), \quad s \in [0, \delta]. \quad (24)$$

At the same time the condition $K(0) = I$ and the continuity of K at multiples of time-instant δ lead us to the boundary condition

$$\tilde{K}(0) + N \tilde{K}(\delta) = W, \quad (25)$$

where

$$N = \begin{bmatrix} 0 & \cdots & 0 \\ -I & & \\ & \ddots & \\ & & -I & 0 \end{bmatrix}, \quad W = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

From (24) and (25) we obtain the following explicit expression for $\tilde{K}$ (inducing an explicit expression for K),

$$\tilde{K}(s) = e^{\tilde{A}s} \tilde{B}, s \in [0, \delta], \quad (26)$$

where

$$\tilde{B} = \left(I + N e^{\tilde{A}\delta} \right)^{-1} W. \quad (27)$$

Using a change of variables, Equation (13) can be written as

$$\rho(t) = - \int_0^h (F_1 K(u) + F_2 K(u-r)) B \rho(t-u) du, \quad (28)$$

with characteristic equation

$$\det \left(I + \int_0^h (F_1 K(u) + F_2 K(u-r)) B e^{-us} du \right) = 0. \quad (29)$$

Using the notation

$$E_i = \begin{cases} (e_i \otimes I_n), & i = 1, \dots, n_2, \\ 0_{n_2, n n_2}, & i < 1, \end{cases}$$

with e_i the i -the unit vector in $\mathbb{R}^{n_2}$, we can express

$$\begin{aligned} \int_0^h K(u) e^{-us} du &= \sum_{i=1}^{n_2} \int_{(i-1)\delta}^{i\delta} K(u) e^{-us} du \\ &= \sum_{i=1}^{n_2} \int_{(i-1)\delta}^{i\delta} K_i(u - (i-1)\delta) e^{-us} du \\ &= \sum_{i=1}^{n_2} \int_0^\delta E_i^T \tilde{K}_i(u) e^{-(i-1)\delta s} du \\ &= \left(\sum_{i=0}^{n_2-1} E_i^T e^{-i\delta s} \right) \int_0^\delta e^{(\tilde{A}-sI)u} \tilde{B} du. \end{aligned} \quad (30)$$

Similarly we obtain

$$\begin{aligned}
\int_0^h K(u-r)e^{-us}du &= \sum_{i=n_1+1}^{n_2} \int_{(i-1)\delta}^{i\delta} K(u-r)e^{-us}du \\
&= \sum_{i=n_1+1}^{n_2} \int_{(i-1)\delta}^{i\delta} K_{i-n_1}(u-(i-1)\delta)e^{-us}du \\
&= \left(\sum_{i=n_1}^{n_2-1} E_{i+1-n_1}^T e^{-i\delta s} \right) \int_0^\delta e^{(\tilde{A}-sI)u} \tilde{B} du. \tag{31}
\end{aligned}$$

Substituting (30) and (31) into (29) and using the identity $\det(I+XY) = \det(I+YX)$ for matrices X and Y of compatible dimensions, we obtain an alternative expression for the characteristic equation for (13),

$$\det \left(I + \int_0^\delta e^{(\tilde{A}-sI)u} du F(s) \right) = 0, \tag{32}$$

where

$$\begin{aligned}
F(s) &= \tilde{B}B \left(F_1 \sum_{i=0}^{n_2-1} E_{i+1}^T e^{-i\delta s} + F_2 \sum_{i=n_1}^{n_2-1} E_{i+1-n_1}^T e^{-i\delta s} \right) \\
&= \sum_{i=0}^{n_2-1} \tilde{B}B (F_1 E_{i+1}^T + F_2 E_{i+1-n_1}^T) e^{-i\delta s}. \tag{33}
\end{aligned}$$

This brings us to the following proposition.

Proposition 3.3. *The zero solution of (13) is asymptotically stable if and only if the zeros of (32) are confined to the open left half plane.*

It is important to note that, as a consequence of the explicit expression

$$\int_0^\delta e^{(\tilde{A}-sI)u} du = (sI - \tilde{A})^{-1} \left(I - e^{\tilde{A}\delta} e^{-\delta s} \right),$$

characteristic equation (32) is closely related to the spectrum of the following system with discrete delay,

$$\begin{aligned}
\dot{\xi}(t) &= \left(\tilde{A} - \tilde{B}B F_1 E_1^T \right) \xi(t) \\
&+ \sum_{i=0}^{n_2-1} \left(-\tilde{B}B [F_1 E_{i+1}^T + F_2 E_{i+1-n_1}^T] + e^{\tilde{A}\delta} \tilde{B}B [F_1 E_i^T + F_2 E_{i-n_1}^T] \right) \xi(t-i\delta) \\
&+ e^{\tilde{A}\delta} \tilde{B}B [F_1 E_{n_2}^T + F_2 E_{n_2-n_1}^T] \xi(t-n_2\delta). \tag{34}
\end{aligned}$$

More precisely, denoting by $\sigma(\cdot)$ the spectrum of a matrix and by $\sqcup$ the *disjunct union* of two sets, the following result holds.

Theorem 3.4. *Let Σ_{ID} denote the spectrum of (13), and let Σ_{DD} denote the spectrum of discrete delay system (34). Then we have*

$$\Sigma_{\text{DD}} = \Sigma_{\text{ID}} \sqcup \sigma(\tilde{A}).$$

Proof. The characteristic function of (34) satisfies

$$\begin{aligned} & \det \left(sI - \tilde{A} + F(s) - e^{\tilde{A}\delta} F(s) e^{-s\delta} \right) \\ &= \det \left(sI - \tilde{A} + \left(I - e^{\tilde{A}\delta} e^{-s\delta} \right) F(s) \right) \\ &= \det(sI - \tilde{A}) \det \left(I + (sI - \tilde{A})^{-1} \left(I - e^{(\tilde{A}-sI)\delta} \right) F(s) \right) \\ &= \det(sI - \tilde{A}) \det \left(I + \int_0^\delta e^{(\tilde{A}-sI)u} du F(s) \right), \end{aligned}$$

from which the assertion directly follows. $\square$

Since $\tilde{A}$ is Hurwitz if and only if A_{1c} is Hurwitz we can conclude the following.

Corollary 3.5. *Assume that $A_1 + BF_1$ is Hurwitz. Then integral delay system (13) is asymptotically stable if and only if the discrete delay system (34) is asymptotically stable.*

From a computational point of view, Theorem 3.4 enables algorithms for computing eigenvalues of delay equations with discrete delay, such as Michiels (2011); Wu & Michiels (2012), in order to assess the spectrum location and stability of IDS (13).

Remark 6. In the special case $r = h$, we can take $\delta = h$, which leads to $\tilde{A} = A_{1c} = A_1 + BF_1$ and $\tilde{B} = I$. In this way equation (34) simplifies to

$$\dot{\xi}(t) = A\xi(t) + \left(e^{(A_1+BF_1)h} BF_1 \right) \xi(t-h),$$

which is consistent with the results in Zhou, Liu, & Michiels (2019).

Remark 7. In practice the assumption of commensurate delays is not restrictive, as it can always be satisfied by introducing an additional input delay. More precisely, if delays (r, h) in (1) are non-commensurate, we choose $\Delta h > 0$ such that delays $(r, h + \Delta h)$ are commensurate. Letting $u(t) = v(t - \Delta h)$, system (1) takes the form

$$\dot{x}(t) = A_1 x(t) + A_2 x(t-r) + Bv(t - (h + \Delta h)), \quad (35)$$

and we can design a feedback law for auxiliary input v in the first place.

The choice of the additional delay Δh in (35) induces an interesting trade-off between the dimension of discrete delay system (34), and the amount of the additional delay (the later usually making the stabilization problem of (35) more challenging). This trade-off is also relevant if the original delays are already commensurate, as we illustrate with an example. Letting $(r, h) = (1, 1.11)$, we conclude the following:

- no additional delay ($\Delta h = 0$): $\delta = 0.01$, $(n_1, n_2) = (100, 111)$, dimension of (34) equals $111n$;
- additional delay $\Delta h = 0.09$: $\delta = 0.2$, $(n_1, n_2) = (5, 6)$, dimension of (34) equals $6n$;

- additional delay $\Delta h = 0.89 : \delta = 1$, $(n_1, n_2) = (1, 2)$, dimension of (34) equals $2n$.

4. Computation of the feedback gains

By Theorem 3.1 the design of a stabilizing controller requires the determination of gains F_1 and F_2 such that both the zero solution to system (3) and the zero solution to system (13) are asymptotically stable. To solve this simultaneous stabilization problem, we adopt the direct optimization approach, as developed in Michiels (2011) and the references therein. More precisely we determine the gains by solving the optimization problem

$$\min_{F_1, F_2} \alpha(F_1, F_2), \quad (36)$$

where

$$\alpha(F_1, F_2) = \max(\alpha_1(F_1, F_2), \alpha_2(F_1, F_2)), \quad (37)$$

with α_1 , respectively α_2 being spectral abscissa of (4) and (16),

$$\begin{aligned} \alpha_1(F_1, F_2) &= \sup_{s \in \mathbb{C}} \{ \Re(s) : \det(sI - (A_1 + BF_1) - (A_2 + BF_2)e^{-sr}) = 0 \}, \\ \alpha_2(F_1, F_2) &= \sup_{s \in \mathbb{C}} \left\{ \Re(s) : \det \left(I + \int_0^h (F_1 K(u) + F_2 K(u-r)) B e^{-su} du \right) = 0 \right\}. \end{aligned} \quad (38)$$

The closed-loop system is stabilizable if the global minimum is strictly negative, and for exponential stability it is sufficient to find gain values resulting in a strictly negative objective function.

Optimization problem (36) is in general non-convex. The objective function may be not everywhere differentiable, due to the maximum operators in (37). At points where a rightmost eigenvalue is multiple and defective, it may even be not locally Lipschitz continuous. These properties preclude the use of standard optimization methods, developed for smooth problems. Instead we use a combination of the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm with weak Wolfe line search and gradient sampling, as implemented in the MATLAB code Hybrid Algorithm for Non-smooth Optimization (HANSO Overton (2009)). This code only requires a routine for the evaluation of the objective function, as well as its derivatives with respect to the controller parameters, *whenever* it is differentiable. The latter is satisfied if in the closed upper complex plane there is only one rightmost eigenvalue of multiplicity one. The value of the objective function α is obtained by computing the rightmost eigenvalue using a procedure similar as in Wu & Michiels (2012), and the gradient with respect to the controller gains is obtained from the corresponding left and right eigenvectors, see Michiels (2011) for the details. Instead of explaining the algorithm in full detail, we highlight two points where tailoring the approach of Michiels (2011) to the problem under consideration deserves special attention and necessitates adaptations to the software.

First, as the software corresponding to Michiels (2011) has been designed for systems with discrete delays, we need to rely on (34) to evaluate the spectral abscissa of (13). Therefore, we assume that Assumption 1 is satisfied (if not we can always proceed as spelled out in Remark 7). The rightmost eigenvalues of (13) are then computed as

the rightmost eigenvalues of (34), followed by an automated removal of eigenvalues of matrix $\tilde{A}$, which is in accordance to Proposition 3.4.

Second, we need to provide a routine for computing the derivative of an isolated eigenvalue $\hat{s}$ of system (34) with respect to the elements of F_1 and F_2 , which we group in a vector $f = [f_1 \ \cdots \ f_{2mn}]$. Hereto we can use expression (55) in Michiels (2011),

$$\frac{d\hat{s}(f)}{df_j} = -\frac{\hat{u}^* \frac{\partial R(\hat{s}, f)}{\partial f_j} \hat{v}}{\hat{u}^* \frac{\partial R(\hat{s}, f)}{\partial \hat{s}} \hat{v}}, \quad j = 1, \dots, 2mn, \quad (39)$$

where $R(s, f)$ is the characteristic matrix of (34),

$$R(s, f) = sI - \left(\tilde{A} - \tilde{B}BF_1E_1^T \right) \quad (40)$$

$$\begin{aligned} & - \sum_{i=0}^{n_2-1} \left(-\tilde{B}B [F_1E_{i+1}^T + F_2E_{i+1-n_1}^T] + e^{\tilde{A}\delta} \tilde{B}B [F_1E_i^T + F_2E_{i-n_1}^T] \right) e^{-i\delta s} \\ & - e^{\tilde{A}\delta} \tilde{B}B [F_1E_{n_2}^T - F_2E_{n_2-n_1}^T] e^{-n_2\delta}, \end{aligned} \quad (41)$$

and $\hat{u}$ and $\hat{v}$ are its left and right null vectors corresponding to eigenvalue $\hat{s}$. As part of computing the numerator in (39) we need to determine the partial derivatives of matrices

$$M_1(f) = \tilde{B}B, \quad M_2(f) = e^{\tilde{A}\delta} \tilde{B}B, \quad (42)$$

where $\tilde{A}$ explicitly depends on f via its definition, while $\tilde{B}$ depends on $\tilde{A}$ via (27). As starting point we express

$$M_1(f) = \tilde{K}(0; f)B; \quad M_2(f) = \tilde{K}(\delta; f)B, \quad (43)$$

where function $s \mapsto \tilde{K}(s; f)$ satisfies boundary value problem (24)-(25). Note here that we stress the dependence on parameters f by a second argument separated by a dot-comma, whenever needed for clarity. Letting $\tilde{K}_{f_i}(s; f) = \frac{\partial \tilde{K}}{\partial f_i}(s; f)$, we can express

$$\frac{\partial M_1}{\partial f_i}(f) = \tilde{K}_{f_i}(0; f)B, \quad \frac{\partial M_2}{\partial f_i}(f) = \tilde{K}_{f_i}(\delta; f)B, \quad i = 1, \dots, 2mn. \quad (44)$$

Taking the derivative of (24)-(25) with respect to parameter f_i leads to a new boundary value problem:

$$\begin{cases} \dot{\tilde{K}}_{f_i}(s; f) = \tilde{A}\tilde{K}_{f_i}(s; f) + \frac{d\tilde{A}}{df_i}\tilde{K}(s; f), & s \in [0, \delta], \\ \tilde{K}_{f_i}(0; f) + N\tilde{K}_{f_i}(\delta; f) = 0. \end{cases} \quad (45)$$

Now, letting

$$\Xi(s) = \begin{bmatrix} \tilde{K}(s) \\ \tilde{K}_{f_1}(s) \\ \vdots \\ \tilde{K}_{f_{2mn}}(s) \end{bmatrix},$$

we can compactly rewrite (24)-(25), combined with (45) for $i = 1, \dots, 2mn$, in the following form,

$$\dot{\Xi}(s) = \begin{bmatrix} \tilde{A} & 0 & \cdots & 0 \\ \frac{d\tilde{A}}{df_1} & \tilde{A} & & \\ \vdots & & \ddots & \\ \frac{d\tilde{A}}{df_{2mn}} & 0 & \cdots & \tilde{A} \end{bmatrix} \Xi(s), \quad s \in [0, 1], \quad (46)$$

supplemented with

$$\Xi(0) + \begin{bmatrix} N & & & \\ & N & & \\ & & \ddots & \\ & & & N \end{bmatrix} \Xi(\delta) = \begin{bmatrix} W \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (47)$$

From (43)-(44) we are interested in computing $\Xi(0)B$ and $\Xi(\delta)B$. For this, it is important to note that (46)-(47) has exactly the same structure as (24)-(25). Hence, it can be solved in the same way (analytically in terms of matrix exponentials or numerically), allowing to evaluate functions $M_1(f)$, $M_2(f)$ and all their derivatives *simultaneously*.

5. Numerical experiments

In this section numerical examples are given to illustrate the effectiveness and the advantages of the proposed PPF control scheme. The example is borrowed from Zhou (2014a), in which the model for a liquid propellant rocket motor is linearized into a system of form (1), with

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Note that the pair (A_1, B) is not controllable, which further motivates the terms in the control law related to feedback gains F_2 .

In the first experiment we take delay values $(r, h) = (1, 1.5)$. Accordingly, we can choose $\delta = 0.5$ in Assumption 1, resulting in dimension 12 of discrete-delay system (34). For $F_1 = 0$ and $F_2 = 0$ the zero solution of the closed-loop system is unstable with spectral abscissa equal to 0.1126. Initiating the optimization algorithm discussed in Section 4 with these gain values, we reach a local minimum of (36) after 345 iterations, characterized by $\alpha = -1.2473$ and

$$F_1 = [1.9593 \ 0.4193 \ 4.8893 \ -3.5585], \quad F_2 = [-0.0404 \ 0.2910 \ -1.2324 \ 0.9252].$$

In Figure 1 we depict the evolution of the spectral abscissae of (3) and (13) as a function of the iterations of the optimization algorithm. In Figure 2 we show the rightmost eigenvalues in three situations: without control, after 1 iterations step and after convergence. Note that already after 1 iteration step the spectrum of (13) is clearly visible

Figure 1. Spectral abscissa α_1 of (3) and spectral abscissa α_2 of (13), as a function of the iterations in solving (36). The delay values satisfy $(r, \tau) = (1, 1.5)$.

in the depicted region of the complex plane. The optimum is characterized by right-most eigenvalues of the closed-loop system, related to both target system (3) and (13), illustrating the trade-off between the conflicting requirements of minimizing α_1 and α_2 .

For the second experiment we add an additional input delay of 0.5 seconds, resulting in

$$(r, h) = (1, 2).$$

According to Remark 7, the additional delay could also be intentional, since we can now take $\delta = 1$ in Assumption 1, resulting in a reduction of dimension of System (34) from 12 to 8, and thus facilitating the design of the gains. Starting again from zero gains, we reach, after 385 iterations, an optimum characterized by $\alpha = -0.7339$,

$$F_1 = [7.9630 \ 2.5589 \ 15.2332 \ -8.8436], \quad F_2 = [4.3438 \ -0.0043 \ -8.8519 \ -1.2075].$$

The closed-loop spectrum corresponding to the optimum is shown in Figure 3.

Finally, we consider delays $(r, \tau) = (1, 1.2)$ and compute stabilizing gains for (3) in an alternative way, by applying Lemma 1 in Zhou, Liu, & Mazenc (2017). This results in

$$F_1 = [0.2598 \ -3.1485 \ 2.4424 \ 1.0341], \quad F_2 = [-2.6372 \ -0.0231 \ 2.5234 \ -0.0012].$$

The verification of stability of (13) is then checked a posteriori (the spectral abscissae satisfy $\alpha_1 = -0.1215$ and $\alpha_2 = -0.0450$). We approximate the integrals in the control law (11) by applying the left-hand rectangle methods with $N = 100$. Then the state trajectory of the controlled system is recorded in Figure 4, for initial condition $x(0) = [0.1 \ 0.1 \ 0.1 \ 0.1]^T$ and $u(s) = 0, s \leq 0$. For the purpose of comparison, the corresponding state trajectory under the control scheme proposed in Kharitonov (2015) is also recorded in the figure. By simulation we find that the closed-loop system

Figure 2. The rightmost eigenvalues of (3) are indicated by blue pluses, the rightmost eigenvalues of (13) by red circles. Their union forms the spectrum of the closed-loop system, and needs to be located in the open left half plane for asymptotic stability. The green circles denote the eigenvalues of $A_1 + BF_1$. Together with the spectrum of (13), they form the spectrum of (34). The sub-figure at top left location corresponds to the uncontrolled system, the sub-figure at top right location to the spectrum location after one iteration, while the two sub-figures below correspond to the optimized spectrum (the right figure is a zoom of the left figure). The delay values satisfy $(r, \tau) = (1, 1.5)$.

Figure 3. Optimized spectrum for delay values $(r, \tau) = (1, 2)$. The right figure is a zoom of the left figure). The meaning of the marking symbols is explained in the caption of Figure 2.

Figure 4. Euclidean norm of state variable of the linearized under different controllers.

under the traditional controller

$$\begin{aligned}
 u(t) = & (F_1 K(h) + F_2 K(h - r)) x(t) \\
 & + \int_0^h (F_1 K(\theta) + F_2 K(-r + \theta)) B u(t - \theta) d\theta \\
 & + \int_0^r (F_1 K(h - r + \theta) + F_2 K(h - 2r + \theta)) A_2 x(t - \theta) d\theta, \quad (48)
 \end{aligned}$$

where the integrals are also discretized with the same rule, is unstable.

From Figure 4 we can see that the convergence rate of the system under the PPF controller is comparable to that of the system under the control scheme proposed in Kharitonov (2015) and the computational time required by the PPF is shorter than that in Kharitonov (2015). On the other hand, it can be noticed that the linearized system can not be stabilized by traditional controller (48), which indicates that this control scheme faces the problem of implementation that has to be handled by some additional methods, such as adding an input filter Mondié & Michiels (2003).

6. Conclusion

The stabilization problem of retarded-type linear systems with both state and input delays was considered. By using the fundamental matrix, a pseudo predictor feedback (PPF) control scheme was developed to compensate arbitrarily large yet bounded input delay, which can be safely implemented by replacing the integral terms with finite sums. The control law was explicitly expressed in terms of the original system matrices for the commensurate delays case, and an optimization based procedure for computing the gains was presented. Numerical examples demonstrated the efficiency of the proposed method. The application of the proposed method to nonlinear systems, the extension to the case where the system are subject to multiple delays and the consideration of robustness to system uncertainty are some directions of future research.

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