Expertise in developing students' expertise in mathematics: Bridging teachers' professional knowledge and instructional quality

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Abstract

This paper provides a survey of the research on the relationship between teachers' professional knowledge and their instructional quality in the domain of mathematics education. First, an overview is provided of the way in which mathematics professional knowledge has been conceptualized. Second, the relation between mathematics teachers' professional knowledge and instructional practice is outlined, with a special focus on the mediating role of teachers' professional beliefs and teachers' situation-specific knowledge and skills. Third, we describe the model of mathematics teachers' competence as an integrative model including knowledge, beliefs, situation-specific skills and observable classroom behavior. Fourth, we describe the current research on bridging teachers' professional knowledge and instructional quality, with special attention to recent issues with respect to the validation, development, and impact of professional competence. We end with some challenges for future research.

Key words

Professional knowledge, instructional practice, professional competence, mathematics education

1. Introduction

This narrative review provides an overview of the current state of the art and the latest developments in the domain of mathematics teachers' professional knowledge. We start with an overview of the ways in which teachers' professional knowledge has been conceptualized since its introduction by Shulman in 1986, and how this conceptualization has impacted the way in which we think about mathematics teachers' expertise. Secondly, we delve into two developments regarding the conceptualization of teachers' mathematics professional knowledge that have helped further to acknowledge the complexity of the relationship between knowledge and instructional practice, namely, the acknowledgement of the mediating role of teachers' professional beliefs and their more situation-specific knowledge and skills in bridging teachers' knowledge and practice. Next, we introduce a model of mathematics teachers' competence that incorporates decontextualized professional knowledge as well as professional beliefs, situation-specific knowledge and skills, and observable classroom behavior. Fourthly, recent issues in research on the assessment, development and impact of different facets of teacher professional competence are addressed. More specifically, we elaborate on the validation of more situated approaches towards mathematical professional knowledge that account for the specificities of the classroom context in which this knowledge is used, the development of different facets of teachers' professional competence, and the impact of teachers' professional knowledge on their instructional behavior. We end with some challenges for future research on the association between mathematics teachers' professional knowledge and instructional quality.

2. Teachers' expertise in developing students' learning in mathematics

Good education is largely determined by the quality of the teachers (see, e.g., Burroughs et al. 2019; Hattie 2009). By selecting and implementing curricular material and providing adequate learning and socio-emotional support to the students, the teacher contributes to a diverse range of cognitive, meta-cognitive, and motivational-affective learning outcomes. Educational researchers generally agree that teachers' professional knowledge is at the heart of their expertise (e.g., Blömeke et al. 2008; Kunter et al. 2013). This professional knowledge entails knowledge of pedagogical principles and strategies that transcend the subject matter as well as knowledge that is specific to the subject matter taught. Shulman (1986, 1987) conducted pioneering work in conceptualizing teachers' professional knowledge, resulting in a model that distinguishes among seven categories of knowledge. Empirical research regarding teachers' professional knowledge and its impact on students' learning processes and outcomes has mainly focused on the following three categories: content knowledge, pedagogical content knowledge, and—to a lesser extent—general pedagogical knowledge. Content knowledge (CK) can be described as a deep and thorough understanding of the subject matter to be taught. Pedagogical content knowledge (PCK) refers to knowledge of the subject matter for the purpose of teaching. Shulman (1986) identified two central components in PCK, namely, knowledge of instructional strategies and representations, and knowledge of students' misconceptions. General pedagogical knowledge (PK) comprises knowledge about teaching and learning that transcends subject matter, such as general strategies for classroom management. Shulman's work has been very influential in research on teaching and teacher education, especially in the domain of mathematics education (Depaepe et al. 2013).

Several scholars, both within Shulman's research group at Stanford University (Grossman 1990; Marks 1990) and at other research centers (e.g., Ball et al. 2008; Cochran et al. 1993), have refined Shulman's conceptualization of PCK. These refinements, firstly, relate to an expansion and reorganization of the components that are covered by PCK, such as the addition of knowledge of curriculum and knowledge of purposes for teaching (e.g., Grossman 1990). Further refinements included alternative conceptions of PCK. For instance, Cochran et al. (1993) used the term pedagogical content knowing (PCKg) instead of PCK to stress its dynamic nature. Within mathematics education, the most influential reconceptualization of teachers' PCK was through the construct of mathematical knowledge for teaching (MKT) (Ball et al. 2008; Hill et al. 2005; Hill et al. 2008). MKT refers to the mathematics knowledge teachers need to teach mathematics. As represented in Figure 1, MKT differs from Shulman's conceptualization of PCK in at least two ways. First, Shulman's concept of PCK was initially purely theoretical; it served "as a heuristic, as a tool for helping the field to identify distinction in teacher knowledge that could matter for effective teaching" (Ball et al. 2008, p. 392). By contrast, the concept MKT resulted from an attempt to validate PCK empirically. Second, whereas PCK and CK are distinct categories in Shulman's (1986, 1987) conceptualization, both categories are integrated within one overarching category, namely MKT, in the conceptualisation of Ball et al. (2008).

Fig. 1. The relationship between Shulman's (1987) conceptualization of teachers' knowledge base (left representation) and Ball et al.'s (2008) conceptualization of mathematical knowledge for teaching (right representation) (Depaepe et al., 2013, p. 14).

In the domain of mathematics education, three large-scale studies have dominated the empirical research base on teachers' professional knowledge and its relation to instructional quality and students' learning outcomes: the MLT project (Mathematics Teaching and Learning to Teach), the COACTIV study (professional competence of teachers, Cognitively ACTIVating instruction, and development of students' mathematical literacy), and the TEDS-M study (Teacher Education and Development Study in Mathematics). First, within the MLT project Ball, Hill and colleagues (e.g., Ball et al. 2008; Hill et al. 2005; Hill et al. 2008) developed a paper-and-pencil test to measure primary school teachers' MKT and found a significant association between primary school teachers' MKT and their instructional quality (Hill et al. 2008). More particularly, they observed a strong positive correlation between US teachers' MKT and the appropriateness of their interpretations of their students' mathematical responses and of the ways in which they addressed their misunderstandings. In addition, strong negative correlations were found between MKT and making mathematical errors and between MKT and using incorrect mathematical language in instruction. A qualitative observational case study with four primary school teachers and one middle-school teacher revealed that, in general, a high MKT score was associated with a more conceptual approach towards teaching mathematics, whereas low MKT impeded such a teaching approach. In terms of the relation between MKT and student learning outcomes, Hill et al. (2005) noticed that students' learning gains for mathematics in first and third grade US classrooms were significantly related to teachers' MKT, even after controlling for other student-level (e.g., SES, gender, ethnicity) and

teacher-level variables (e.g., number of mathematics content and mathematics methods courses taken, teaching experience).

Second, studies within the COACTIV project (e.g., Baumert et al. 2010; Kunter et al. 2013) used paper-and-pencil tests to assess separately teachers' CK and PCK in the domain of mathematics. In addition to Shulman's components of PCK (i.e., knowledge of instructional strategies and representations, and knowledge of students' misconceptions) a third component was distinguished, namely, knowledge of multiple solution paths in a particular mathematical task. Instructional quality was assessed by means of questionnaires about teachers' instructional approaches relating to their cognitive activation, learning support, and classroom management. Student cognitive outcome variables were measured by means of PISA achievement tests for mathematics at the end of Grades 9 and 10. Students' non-cognitive outcomes, such as their enjoyment of mathematics, were measured by means of a student enjoyment questionnaire. Based on results in 194 German Grade 10 classrooms, Baumert and colleagues (2010) found that both CK and PCK make a unique contribution to instructional quality and students' learning gains, with the impact of PCK being larger. Multilevel structural equation models revealed a substantial impact of PCK on students' learning gains, which was mediated by the provision of cognitive activation and individual learner support. Drawing on the same dataset but conducting additional analyses, Kunter et al. (2013) concluded that teachers' PCK had a significant positive effect not only on students' cognitive learning outcomes, but also on their mathematics enjoyment.

Third, in the TEDS-M study (e.g., Blömeke et al. 2012; Blömeke & Kaiser 2012; Kaiser & Blömeke 2013; Senk et al. 2012), which included pre-service teachers in their final year of teacher education across different Western and Eastern countries, scholars developed a CK and PCK paper-and-pencil test to assess teachers' professional knowledge. PCK was again operationalized in terms of three components: mathematical curricular knowledge (e.g., knowing the school mathematics curriculum, establishing appropriate learning goals), knowledge of planning for mathematics teaching and learning (e.g., predicting typical students' responses, planning appropriate methods for representing mathematical ideas), and knowledge of mathematics for teaching and learning (e.g., explaining or representing mathematical concepts or procedures, diagnosing students' responses). As the study involved pre-service teachers, the focus was on opportunities to learn for the development of professional knowledge and how these opportunities differed across teacher education programs and countries, rather than on the relationship between teachers' professional knowledge and their instructional quality and student learning outcomes. Based on the TEDS-M data, different competence levels were observed for CK (i.e., three levels: strong structural mathematical knowledge, sound mathematical knowledge, and no structural insight in mathematics) as well as for PCK (i.e., two levels: high PCK versus low PCK). Furthermore, Eastern pre-service teachers clearly outperformed Western pre-service teachers on both CK and PCK, a finding which is in line with a general trend in international comparisons in which Eastern students generally outperform their Western peers (Kaiser & Blömeke 2013).

To conclude, the three large-scale studies mentioned above (MLT, COACTIV and TEDS-M) contributed considerably to our understanding of in-service and pre-service teachers' professional knowledge and differences in this knowledge base within and across countries. Furthermore, these differences also account for differences in instructional quality and student learning outcomes.

3. Bridging professional knowledge and instructional quality

Despite the convincing empirical evidence regarding the importance of teachers' professional knowledge in view of teachers' instructional quality, scholars have also argued that the relationship between teachers' professional knowledge and instructional quality is far from simple and straightforward (e.g., Borko et al. 1992; Hill et al. 2008; Kaiser et al. 2015; Kersting et al. 2012; Thompson 1992; Thompson & Thompson 1994). Having professional knowledge and being able and/or willing to apply that professional knowledge in the actual classroom context are two different things. There have been at least two, interrelated, developments in the research on teachers' expertise that problematize a direct relation between teachers' professional knowledge and instructional quality, and that suggest the need for a broader conceptualization of teachers' professional expertise beyond a mere cognitive perspective.

Firstly, teachers' expertise has to be seen as a complex construct that underlies their instructional behavior. It has been argued that this expertise entails not only a cognitive side, i.e., professional knowledge, but also includes affective-motivational facets. Relying on Weinert (2001), Blömeke and colleagues (2008) described this affective-motivational side of teachers' expertise as "the motivational, volitional and social willingness and skills to apply solutions successfully and responsibly in variable situations—in terms of teachers' professional beliefs" (p. 720). Different from the ways that the field thinks about professional knowledge, a common framework for understanding teachers' professional beliefs is lacking (Depaepe & König 2018). However, based on different conceptualizations of teachers' professional beliefs in the domain of mathematics education (e.g., Baumert & Kunter 2013; Blömeke et al. 2008; Thompson 1992), one can distinguish at least the following two categories: epistemological beliefs about the nature of mathematics and mathematical knowledge, and beliefs about teaching and learning mathematics. Concerning epistemological beliefs, Ernest (2014) introduced the well-known distinction between an "absolutist" and a "fallibilist" perspective on the nature of mathematics and mathematical knowledge. From an absolutist perspective, mathematics is regarded as a fixed body of not changing—and often unrelated—facts, concepts and procedures. Mathematical conventions are considered as absolutely true, value-free and culturally independent. Fallibilists, on the other hand, assume that mathematical objects have been created by people. Consequently, fallibilists believe that mathematics is evolving and context-dependent. The distinction between an absolutist and fallibilist perspective on mathematics is similar to Blömeke and Kaiser's (2014) distinction between static and dynamic views on mathematics. With respect to teachers' beliefs about teaching and learning mathematics, one can distinguish between transmission and a constructivist views on teaching and learning mathematics (Blömeke & Kaiser 2014). Whereas a transmission view is mainly teacher-centered with students following teachers' instructions, a constructivist view conceptualizes mathematics teaching and learning as an inquiry-based process in which students are ascribed an active role. Research by Blömeke and Kaiser (2014) and Voss et al. (2013) has revealed the existence of "belief systems", with epistemological beliefs being typically associated with beliefs about mathematics teaching and learning. More specifically, teachers whose views of the nature of mathematics are more static and absolutist tend to hold beliefs about teaching and learning that are transmission-oriented, while teachers whose views of the nature of mathematics are more dynamic and fallibilistic tend to hold beliefs about teaching and learning that are more constructivist. Furthermore, teachers' professional knowledge and beliefs are associated. Blömeke and colleagues observed that strong mathematical content and pedagogical content knowledge is associated with dynamic constructivist beliefs and weak knowledge with more static transmission beliefs. In addition, teachers' beliefs have been shown to impact their instructional quality. There is some research evidence to suggest that teachers' beliefs have a mediating or moderating impact on the relation between teachers' professional knowledge and instructional behavior, as they seem to

determine how teachers interpret mathematical classroom situations and how they act in these (Hill et al. 2008; Leder et al. 2002; Schoenfeld 2011a).

Secondly, the way in which professional knowledge has been conceptualized and operationalized in studies such as MLT, COACTIV, and TEDS-M has increasingly been perceived as only partly grasping the knowledge that underlies classroom behavior. In previous research on teachers' professional knowledge, scholars typically took a 'cognitive perspective', conceiving teachers' knowledge as mainly formal, decontextualized and fragmented knowledge that can be measured through classical short-answer or multiple-choice content knowledge and pedagogical content knowledge tests. Merits of this cognitive perspective include the development of reliable instruments to assess teachers' professional knowledge independently of their instructional behavior and their predictive validity in terms of instructional behavior and student learning outcomes (Depaepe et al. 2013). However, some scholars also argued that this type of formal, decontextualized and fragmented professional knowledge is not a sufficient condition for suitable classroom instruction. For instance, Kaiser et al. (2017) stated that "it can reasonably be assumed that this knowledge is not directly transformed into performance, but mediated by cognitive skills more closely related to activities of teachers" (p. 171). Indeed, teachers need to perceive and interpret critical classroom situations and, accordingly, make a sound decision about how to act appropriately in these classroom situations (Blömeke et al. 2015). Consequently, teachers need informal, contextualized knowledge that bridges the formal, decontextualized knowledge with classroom behavior. Consequently, scholars taking such a 'situated perspective' have made a plea for assessing teachers' professional knowledge in close relation to the context (i.e., the mathematics classroom) in which it is applied, for instance through interviews with teachers regarding their instructional behavior (e.g., video-stimulated recall) or assessments based on teachers' interpretation of video segments of other teachers' real instructional behavior. This contextualized teacher professional knowledge is considered to be more informative for understanding what teachers actually do in the classroom than their decontextualized professional knowledge (Knievel et al. 2015). However, this latter type of professional knowledge is more difficult to assess, especially for large samples of teachers (Depaepe et al. 2013). Interestingly, the above distinction between cognitive and situated perspectives on teachers' professional knowledge is closely related to Fenstermacher's (1994) older distinction between "knowledge for teachers" and "knowledge of teachers". This first type of professional knowledge is considered to be standardized and measurable, with teachers having strong or weak knowledge, whereas the second type is knowledge that teachers develop based on deliberate classroom experience and is inherently connected to the context in which it is developed and used (Berry et al. 2016). Scholars have used still other terms to refer to this distinction between cognitive knowledge (knowledge for) and situated knowledge (knowledge of), such as "explicit knowledge" and "implicit knowledge" (Gasteiger et al. 2020), or "small k knowledge" and "big K knowledge" (Schoenfeld 2020).

4. From professional knowledge to professional competence

As documented in the previous section, current perspectives on teachers' expertise and its relation to classroom practice acknowledge that a conceptualization of that expertise purely in terms of formal, decontextualized knowledge, such as CK, PCK and general PK, is too limited. During the last years more and more research groups have broadened their conceptualization of teachers' expertise by also incorporating affective-motivational aspects (such as professional beliefs) and more situated-specific,

contextualized knowledge (Kaiser et al. 2015). Such a broader conceptualization of teachers' expertise might be more helpful for understanding the complex relationship between teachers' professional knowledge and instructional behavior. More specifically, this approach might combine the merits of the cognitive and situated perspectives on teachers' professional knowledge, and prevent the pitfalls of each individual perspective (Depaepe et al. 2013; Kaiser et al. 2015). On the one hand, the cognitive perspective enhances an analytical understanding of the different facets of teachers' expertise (i.e., distinct dimensions within professional knowledge and professional beliefs), how they are related, and how they contribute to teachers' instructional behavior and student learning outcomes (e.g., Ball et al. 2008; Baumert et al. 2010; Hill et al. 2005; Kunter et al. 2013). On the other hand, the situated perspective takes into account the contextualized nature of classroom life (Hodgen 2011; Mason 2008) and the complex interaction between the different facets' of teachers' expertise (e.g., CK, PCK, general PK, and professional beliefs) in the act of teaching (Petrou and Goulding 2011). The competence model developed by Blömeke and colleagues (2015) in the frame of the state-funded programme line KoKoHs on competence measurement at tertiary level, has been particularly inspiring for researchers on teachers' professional knowledge and its relation to instructional quality and students' learning outcomes. A clear merit of this competence model is that it helps to unravel the relationships between formal, decontextualized aspects and situation-specific, contextualized aspects of competence. Furthermore, it integrates cognitive as well as affective-motivational facets. Both the TEDS-M follow-up studies and the COACTIV study started from the work of Blömeke et al. (2015) to develop a research model of mathematics teachers' professional competence. Below, we first elaborate on the model developed by Blömeke et al. (2015), and next we provide an overview of the current conceptualizations of mathematics teachers' expertise underlying the TEDS-M and COACTIV studies.

4.1 A model of competence as a continuum

Framing the discussion between different forms of knowledge within the broader research base on competence, Blömeke et al. (2015) advocated moving beyond either an analytic trait definition of competence (as in the cognitive perspective mentioned above) or a holistic real-world performance view on competence (similar to the situated perspective mentioned above). As represented in Figure 2, these scholars proposed a structural model of the nature of competence consisting of three layers, i.e., dispositions (including cognitive and affective-motivational facets), situation-specific skills (i.e., perception, interpretation, and decision-making), and observable behavior.

Fig. 2. Model of teacher professional competence (Blömeke et al. 2015, p. 7)

Consequently, instead of focusing on either knowledge or behavior, the scholars claim that competence should be conceptualized as a continuum from cognitive and affective-motivational dispositions to performance. Moreover, they pay attention to the processes that connect these dispositions with performance, such as perception, interpretation, and decision-making.

Although the Blömeke et al. (2015) model is generic and not restricted to professional competence, it has been very influential in recent conceptualizations on the relationship between professional knowledge and instructional practices, also in the domain of mathematics education.

4.2 The impact model of the TEDS follow-up study (TEDS-FU)

The continuum model on competence was influential in the TEDS-M follow-up study, namely TEDS-FU, which aims to assess and understand the professional competence development of teachers at the end of teacher education and during the first years of teaching practice (Kaiser et al. 2015). Compared to TEDS-M, TEDS-FU has broadened the conceptualization of teachers' expertise by enriching the cognitively oriented framework of TEDS-M with a situated approach to mathematics teachers' professional competence (Kaiser et al. 2017). Their impact model of mathematics teachers' professional competence explicitly builds on the three layers of Blömeke et al.'s (2015) model. The first layer, i.e., dispositions, corresponds to the aspects of teachers' professional competence that were assessed in the first generation of the TEDS-M study (see also Section 2). More particularly, the cognitive facet of teachers' disposition is defined in terms of teachers' professional knowledge, including their mathematical CK, mathematical PCK, and general PK. The affective-motivational facet includes their professional beliefs, with a distinction between beliefs about the nature of mathematics and mathematical knowledge (i.e., a static versus a dynamic view) and about mathematics teaching and learning (i.e., a knowledge transmission versus a constructivist view). Test instruments and questionnaires were developed to assess, respectively, these cognitive and affective-motivational facets.

Within the TEDS-FU study, attention was also paid to the other two layers of Blömeke et al.'s (2015) competence model. The second layer, namely situation-specific skills, was conceptualized as cognitive

processes before, during, and after actual classroom behavior. A distinction was made between perception, interpretation, and decision-making. Perception includes perceiving and attending to events that happen in a mathematics classroom situation, interpretation refers to making sense of the perceived events in that classroom situation, and decision-making is more action-related and entails anticipating a response to students' mathematical activities or proposing alternative instructional strategies (Kaiser et al. 2015). Within TEDS-FU both mathematics-related situation-specific skills (M_PID) and pedagogy-related classroom demands (P_PID) were assessed by asking teachers to respond to short video vignettes dealing, respectively, with mathematics-specific and domain-general classroom situations (Kaiser et al. 2017).

The third layer, teachers' observable behavior, refers to their instructional practices as implemented in the classroom (Depaepe and König 2018). Yang et al. (2020) distinguished between two dimensions of teachers' instructional practice in the mathematics classroom, i.e., cognitive activation of mathematics and providing student learner support in mathematics. Cognitive activation in mathematics relates to two components, namely, the use of cognitively demanding tasks and stimulating students' independence. Providing student learner support in mathematics has components relating to encouraging students and providing adaptive learner support. These scholars used a questionnaire to assess teachers' self-reported practices.

In sum, this three-step model of teacher professional competence underlying TEDS-FU combines cognitive and affective-motivational dispositions and relates them to observable teacher performance through situation-specific skills.

4.3 The cascade model: an integrative model of teacher professional competence

Similarly to the recent developments within the TEDS-M study, the model of Blömeke et al. (2015) also inspired the cascade model underlying the recent COACTIV studies (Krauss et al. 2020). This model—presented in Figure 3—represents a causal chain including five links. Links 2 and 4 are new, since earlier studies within the COACTIV project typically investigated only links 1, 3 and 5.

Fig. 3. The cascade model (Krauss et al. 2020, this issue)

The first link of the chain includes teachers' dispositions, with both cognitive and non-cognitive facets. The cognitive facets refer to teachers' professional knowledge, including mathematical CK, mathematical PCK, and general PK. The non-cognitive facets pertain to teachers' professional beliefs, goals and orientations (e.g., constructivist beliefs), their motivational orientations (e.g., enthusiasm for teaching), and their self-regulation skills (e.g., regulatory style) (Kunter et al. 2013). The second link relates to teachers' situation-specific competence, including perception, interpretation and decision-making. Similarly to Blömeke et al.'s (2015) model, these situation-specific skills mediate the relationship between teachers' dispositions and instructional behavior. This instructional behavior constitutes the third link and is in line with the TEDS-M project, conceptualized in terms of the dimensions of cognitive activation and individual learner support. The cascade model modifies the TEDS-FU impact model by explicitly relating teacher competence to student outcomes. Similarly to earlier research on learning and instruction (e.g., Vermunt and Verloop 1999), the model postulates that students' learning activities (link 4 in the model) mediate teacher behavior and student learning outcomes (link 5 in the model). Student learning activities relate to students' participation in class, the extent to which they make use of learning opportunities provided by the teacher, and their homework activities. Finally, student learning outcomes are assessed in light of the teaching goals and include cognitive outcomes (i.e., achievement in mathematics) as well as non-cognitive outcomes (i.e., increase in enjoyment in mathematics and decrease in mathematics anxiety). This interest in the relationship between teachers' disposition and student learning outcomes, mediated by teachers' situation-specific skills, classroom behavior and students' learning process, is also apparent in the TEDS-Validate study "which intends to validate this complex relational model under different geographical and socio-cultural conditions" (Kaiser et al. 2017, p. 179).

5. Key issues in recent research on teachers' professional competence

Current empirical research on teachers' expertise in the domain of mathematics education has been founded on the previous conceptualizations of teachers' professional competence (see Section 4), further building on, but also broadening the earlier more cognitively-oriented perspectives on teachers' professional knowledge. In what follows, we provide a narrative review of recent research on teachers' professional competence. We elaborate on different key issues that derived from that review and these issues are structured around three research themes, namely, the assessment of teachers' competence, the development of teachers' competence, and the impact of teachers' competence on mathematical learning processes and outcomes.

5.1 Assessment of teachers' professional competence

The trend towards a more comprehensive perspective on teachers' professional knowledge also results in the methodological challenges of developing reliable and valid situated and performance-oriented instruments (Kaiser et al. 2017; see also Section 2).

In this regard, Gasteiger and colleagues (2020) developed and validated a scenario-based paper-and-pencil test that measures early childhood teachers' implicit knowledge. Although the influence of the cognitive perspective is still apparent in their approach through the use of a paper-and-pencil test with multiple choice items that enabled a large-scale administration of the test, their test also follows a more situated measurement approach as it focuses on concrete classroom situations. More particularly, the test is based on four scenarios of typical kindergarten situations accompanied by different items measuring professional knowledge regarding the mathematical abilities of children and adaptive mathematical learning activities. The test was validated with a sample of 149 German pre- and in-service early childhood teachers. The results indicated that the test allows a valid assessment of early childhood teachers' professional knowledge in mathematics.

Other approaches moved beyond paper-and-pencil multiple choice tests and developed video-based assessments to examine teachers' professional competence with high face validity (Kaiser et al. 2015). In Germany, this approach was used for both the follow-up studies of TEDS-M and COACTIV project. For instance, in the TEDS-FU project, short video-vignettes of mathematical classroom situations were used to assess teachers' perceptions, interpretations, and decision-making skills. After watching the video-vignettes, teachers were asked to answer questions based on what they had observed. Teachers' open-ended responses were coded as fully correct, partially correct, or wrong (Kaiser et al. 2015, 2017). Similarly, within the COACTIV group a video-based assessment instrument was developed to assess teachers' mathematics-specific and more general pedagogical skills from a more situated perspective, i.e., the COACTIV-video instrument (Krauss et al. 2020). This instrument was composed of short video clips of mathematics lessons and required teachers' proposed continuations for the presented lesson clips. The mathematics-specific competence of the assessment related to insight orientation (whether the teachers' answers were more indicative of conceptual understanding than on formal techniques), content-specific accuracy (whether teachers' answers were precise and correct from a mathematical perspective), and instructional opportunity (whether the educational opportunity as reflected by the video clip was clearly captured). Confirmatory factor analysis based on data from 163 German secondary mathematics teachers revealed that performance on the cognitive COACTIV paper-and-pencil tests resulted from factors that differed from the situated video-assessments. Furthermore, Krauss et al. (2020) observed a significant moderate correlation between the COACTIV PCK test and the subject-

specific competence video-test. The association between the subject-specific competence assessment and the PCK test indicated that these were more highly correlated, compared to the association between the subject-specific competence assessment and the CK test.

In the US context similar attempts have been made to develop measurement instruments that situate teacher professional knowledge as closely related to the classroom context in which it is used. In the contributions of Kersting et al. (2020) and Lee and Santagata (2020) the Classroom Video Analysis (CVA) assessment (Kersting et al. 2012) was used. This assessment captures teachers' situated-specific, contextualized knowledge for teaching mathematics by means of short video clips of authentic classroom instruction in mathematics. After watching the videos teachers are prompted to respond to short video clips. The prompts are designed to simulate common teaching tasks. Kersting et al. (2020) distinguished the CVA measure, which reflects teachers' situation-specific mathematical and general pedagogical knowledge, from the CVA-M measure that is focused only on mathematics. The video clips simulate common teaching tasks such as generating mathematical questions, identifying students' mathematical thinking, and providing mathematics-specific suggestions for improving a particular lesson episode. Given the open-ended answers of teachers to these video clips, great effort was invested in the development of reliable coding rubrics (Kersting et al. 2020; Lee and Santagata 2020).

In a study with 20 US pre-service teachers, Charalambous (2020) administered a newly developed simulation instrument to assess teachers' situation-specific skills, i.e., in the terminology of the author, their action-related competence, which is in line with a situated perspective on teachers' professional knowledge. This action-related competence goes beyond knowledge measured from a cognitive perspective in that it relates to teachers' abilities and skills as enacted in different teaching tasks. In a digital simulation, pre-service teachers have to indicate how a virtual teacher (Ms. Rebecca) should interact with virtual students during a mathematics lesson with respect to selecting and using tasks, providing explanations, using representations, analyzing student work and contributions, and responding to students' requests for help. Charalambous compared pre-service teachers' performances on the teaching simulation with their performances on the paper-and-pencil MKT test as developed by the LMT group using a cognitive teaching measurement approach. Significant positive associations were observed between pre-service teachers' performance in the simulated assessment and on the MKT test.

Similarly, Wildgans-Lang et al (2020) developed a simulated online environment in which virtual students solve mathematical problems. In their study 91 German pre-service primary school teachers were asked to diagnose the virtual students' competence levels based on a theoretical model that was empirically validated. Participants determine the competence level based on mathematical tasks that they can assign to each of the virtual students. The simulated environment seemed to have a high face-validity as participants rated it as authentic. In general, participants' competence in judging students' competence levels was fairly good, although there were large differences between pre-service teachers.

To conclude, a first important recent development within the research on teachers' professional knowledge is the development of more situated and holistic assessment instruments, such as scenario-based instruments, video-based assessments and digital simulations, each taking authentic mathematics classroom situations as their starting point. These instruments reveal individual differences in the expertise of pre- and in-service teachers. Moreover, teachers' performances on these newly developed measures are significantly positively associated with the traditional paper-and-pencil tests developed within the cognitive perspective.

5.2 Development of professional competence

Several studies have been conducted to explore how different facets of teachers' professional competence develop as a result of opportunities to learn in initial teacher education as well as throughout teachers' professional careers. As professional competence is a multidimensional construct; being more or less competent is most probably not characterized by a linear increase or decrease on one facet, but by structural changes in the diverse components of this professional competence (Blömeke et al. 2015). Research during initial teacher training has investigated the impact of different opportunities to learn in teacher education and the extent to which these opportunities are used by pre-service teachers (Kaiser and König 2019). Typically, distinction is made between four types of opportunities to learn in initial teacher education, namely, courses on the subject to teach (subjects), courses on the methods of how to teach that subject (subject-related pedagogy), courses on general educational sciences (general pedagogy), and practical experiences in schools (teaching practice) (Kaiser and König 2019; Schmidt et al. 2011). Throughout teacher education, the different facets of professional competence can vary as a result of these opportunities to learn. Furthermore, also after initial teacher education, studies have revealed differences between novice and expert teachers on all facets of teachers' professional competence (Kaiser et al. 2015). An interesting research question is whether competence changes are best characterized by linear increase or decrease, by differentiation processes from more general and basic expressions to more specialized one, or by qualitative changes—as assumed in the novice-expert paradigm. In the two latter cases, developmental trajectories would imply structural changes in the nature of competence (Blömeke et al. 2015).

With respect to the cognitive component of teachers' professional competence, Phelps et al. (2020) compared differences between pre-service and in-service US primary school teachers in teachers' professional knowledge, based on data from over 10000 participants. Following the MKT tradition (of the MLT project) the scholars compared teachers' specialized mathematical content knowledge, i.e., knowledge that is needed to be learned for teaching mathematics. Their study provided evidence that this specialized mathematical content knowledge is acquired through professional learning opportunities, as in-service teachers significantly outperformed pre-service teachers. Furthermore, this better performance for in-service teachers was specifically related to particular items and not to others, with differences both in terms of the mathematical subdomain and the dimension of this specialized content knowledge.

Besides developmental differences in teachers' professional knowledge as measured from a more cognitive perspective, scholars have also started to look at the development of professional knowledge that is more related to the classroom context. For instance, Torbeyns et al. (2020) examined the impact of opportunities to learn in teacher education by comparing differences between 162 first-to-final year Flemish pre-service preschool teachers on a scenario-based assessment instrument (see Gasteiger et al. 2020). They distinguished between the four previously described types of opportunities to learn: subjects, subject-related pedagogy, general pedagogy, and teaching practice (Kaiser and König 2019; Schmidt et al. 2011). Their findings revealed that second- and third-year pre-service teachers significantly outperformed first-year pre-service teachers, while no differences were observed between second- and third-year pre-service teachers. A content analysis of the curriculum in the participating teacher programs suggested that theoretical courses on subject-related pedagogy (which are mainly situated in the first two years of teacher training) might provide opportunities to develop this professional knowledge, whereas the impact of practical experiences (which are mainly situated in the third year of teacher training) might be more subtle.

Studies aimed at unraveling the way in which different facets of professional competence develop typically make use of cross-sectional designs comparing teachers in different phases of their professional career. More longitudinal designs might be fruitful, although more time-consuming, to track the development of professional competence in teachers, and the sources of that development, in a more fine-grained way. As an example, Lee and Santagata (2020) followed nine Kindergarten and primary school teachers before and after a teacher preparation program until the end of the first three years of teaching experience. The authors observed growth in teachers' knowledge as measured through the CVA video-based assessment instrument, both during initial teacher training and during the first three years of being in service. In addition, expert ratings of videotaped mathematics lessons revealed growth in instructional quality from the first to the third year of teaching experience. However, in order to further understand competence development, it might be useful simultaneously to track the development of different facets of teachers' professional competence. Doing so would be informative in determining whether different opportunities to learn, differently contribute to different facets of teachers' professional competence. As suggested by Kaiser et al. (2017), theoretical courses in teacher education might be more important for the development of cognitive aspects of teachers' professional competence, whereas classroom experiences might contribute more to the development of more situated aspects of that competence. In addition, there is a need for intervention research systematically investigating the impact of particular opportunities to learn for competence development, such as the content of the courses and how the courses are organized, the impact of practical experience and the role of reflection on these experiences, and the influence of professional development initiatives.

5.3 Impact of professional competence on instructional quality

Whereas in the previous section professional competence was considered as the dependent variable, being affected by different opportunities to learn in initial teacher education and beyond, professional competence can also be seen as an independent variable affecting process characteristics of teaching (Kaiser and König 2019). Theoretical models of teachers' professional competence (e.g., Blömeke et al. 2015; see Section 4) assume a causal relationship between teachers' knowledge and instructional quality, with teachers' situation-specific skills as a mediating or moderating factor. Several recent studies provide empirical evidence for these assumed relations.

In the study of Yang et al. (2020), the associations among professional knowledge as measured from a cognitive perspective, professional beliefs, and self-reported instructional practice relating to the dimensions of cognitive activation and student support are examined. These scholars administered the TEDS-M paper-and-pencil CK and PCK tests, a self-reported instructional practice questionnaire, and beliefs questionnaires about the nature of mathematics and mathematics teaching and learning, to 495 Chinese pre-service mathematics teachers. They did not observe a significant association between CK and instructional practice, and for PCK only a weak association was observed. By contrast, they did observe a significant positive association between CK and PCK, on the one hand, and teachers' dynamic beliefs about the nature of mathematics and their constructivist beliefs about teaching and learning, on the other hand. Path analyses showed no direct effect of CK and PCK on reported instructional practice, but there was evidence for a small indirect effect of PCK on reported instructional practice through the mediating role of constructivist beliefs about teaching and learning mathematics.

Lee and Santagata (2020) also investigated the relationship between teachers' professional knowledge and the quality of their instructional practice, but they did so from a situated perspective. More

particularly, they investigated knowledge-in-use through the video-based CVA assessment, and instructional quality by means of expert ratings of videotaped lessons using the MQI coding scheme. The association between professional knowledge and instructional quality was examined for nine US teachers with one to three years of teaching experience in Kindergarten to grade 4. The scholars did not observe a significant association for the first year of teaching experience, while a moderate positive association was found in year 2 and year 3. Interviews with the teachers suggested that the association between teachers' knowledge and instructional quality depended on different contextual factors, such as the curriculum being used, opportunities to teach at the same grade level over time, and the available support.

In an attempt to test their integrative, theoretical cascade model bridging teachers' professional knowledge and instructional quality (see Section 3.2), Krauss and colleagues (2020) investigated the associations between teachers' performances on the traditional COACTIV CK and PCK tests and on the COACTIV video-instrument measuring situation-specific skills, their instructional quality, and student learning outcomes. Data were collected from 163 German secondary mathematics teachers and 3806 students from 169 grade 10 classes. Instructional quality was determined based on several data sources such as teacher questionnaires, student questionnaires and expert ratings with respect to the dimensions of instructional quality, classroom management, and individual learning support. Student learning outcomes were assessed based on the learning gains on the PISA assessment at the end of grades 9 and 10. Results of path analyses revealed that teachers' performances on the PCK test and the video-based assessment significantly predicted their instructional quality, whereas no impact was observed for teachers' performances on the CK test on their instructional quality. Furthermore, when the impact of all measures of teachers' knowledge was simultaneously modeled, only the PCK test significantly predicted teachers' instructional quality. Based on path analyses of the data for the four phases of the CASCADE model, the scholars observed a significantly positive effect of CK and PCK on subject-specific situational skills and on the classroom management dimension of instructional quality on student achievement. The data seem to suggest that there is a differential impact of different facets of teacher knowledge on different dimensions of classroom instruction, and, in turn, of different dimensions of classroom instruction on student learning outcomes.

Whereas previous studies mainly relied on the investigation of linear, average associations between professional knowledge and instructional quality, some studies also account for heterogeneity in teacher competence profiles with subgroups of teachers varying in the distinct aspects of teacher competence. For instance, Blömeke and colleagues (2020) used latent profile analyses to determine different competence profiles of teachers based on their performances on different measures of professional knowledge (CK, general PK, and situation-specific skills) and professional beliefs (about the dynamic or static nature of mathematics and mathematics as a fixed ability). Based on data from 77 German secondary school mathematics teachers they found four profiles, which differed based on the levels of instructional quality (in terms of classroom management), student support, mathematics-related quality and cognitive activation.

Focusing solely on the domain-general aspects of teachers' professional competence, Nehls et al. (2020) identified two different profiles of teachers based on 462 German secondary school mathematics and non-mathematics teachers' performances on the TEDS-M general PK test. These two profiles of teachers differed significantly both quantitatively and qualitatively. Furthermore, the scholars related these different profiles of teachers to differences in professional beliefs and instructional quality. Their results

revealed that the profile with the strongest general PK showed significantly less static and teacher-oriented beliefs compared to the lowest profile, whereas no significant differences were observed for dynamic beliefs and student-oriented beliefs. Furthermore, the strongest profile showed a higher quality of cognitive activation compared to the lowest profile, whereas no significant differences were observed for classroom management and student support.

In sum, these studies provide evidence for an association between professional knowledge and instructional practice, although mediated by several variables, such as teachers' beliefs (Yang et al. 2020) and teaching experience (Lee and Santagata 2020). Finally, there is evidence for heterogeneity in teachers' competence profiles, particularly as related to differences in teachers' professional beliefs and instructional behavior (Blömeke et al. 2020; Nehls et al. 2020).

6. Major themes of our review and issues for future research

Our survey of research on bridging mathematics teachers' professional knowledge and instruction represents a plea for researchers to embrace a less purely cognitive stance, and broader and more integrated framing of teachers' expertise, in view of its relationship to instructional practice. Based on a description of the Teaching for Robust Understanding (TRU) framework, Schoenfeld (2020) identifies five dimensions of powerful mathematics classroom that help students to become knowledgeable, flexible, and resourceful thinkers and problem solvers. These dimensions relate to supporting rich mathematics, supporting productive struggle, equitable access to rich content, agency, ownership and identity, and formative assessment. Designing mathematics lessons in line with these five dimensions requires, according to Schoenfeld, teacher proficiency that includes productive teacher knowledge, beliefs, inclinations, perceptions, and goals. Within recent research, several of these facets of teacher proficiency that go beyond pure declarative, formal, and decontextualized knowledge are examined, such as teachers' contextualized knowledge (also labeled as situation-specific skills) and their beliefs about the nature of mathematics and about its teaching and learning. The contributions to this special issue are structured in three groups that correspond to the three key issues in recent research on teachers' professional competence (see Section 5).

The first theme, addressed in various contributions, involves the construction and validation of instruments aimed at assessing teachers' situated professional knowledge, ranging from scenario-based questionnaires (Gasteiger et al. 2020) to video-based assessments (Blömeke et al. 2020; Kersting et al. 2020; Krauss et al. 2020) and online simulation environments (Charalambous 2020; Wildgans-Lang et al. 2020). Moderate correlations have been observed between teachers' performances on these measures of situated knowledge and paper-and-pencil tests that are developed following a cognitive perspective on teacher knowledge (Charalambous 2020; Gasteiger et al. 2020; Krauss et al. 2020).

The second recurrent theme relates to the question of how different facets of teachers' expertise develop. Cross-sectional and longitudinal research has revealed that different facets, such as declarative knowledge, situated knowledge and skills, professional beliefs, and instructional behavior develop during the different stages of teacher training. Significant differences were observed between pre-service and in-service teachers (Phelps et al. 2020), across the different phases of pre-service teacher education (Torbeyns et al. 2020) and across the first three years of in-service practice (Lee and Santagata 2020). These three studies provide clear evidence for growth in teachers' expertise, and suggestive evidence

that this growth is due to opportunities to learn in initial and in-service teacher training, including theoretical courses and practical experiences. However, hard evidence on how particular kinds of opportunities to learn in initial and in-service teacher education affect the development of the different facets of this expertise is still missing (Schoenfeld 2020).

Third, several studies examine the impact of teachers' professional knowledge on their instructional quality, in an attempt to bridge the knowledge-practice gap. These studies have revealed that the relationship between professional knowledge and instructional practice is not straightforward but mediated or moderated by several factors. In pre-service teacher education, no significant association was observed between CK and instructional practice and only a weak association was observed between PCK and one dimension of instructional practice (Yang et al. 2020). Furthermore, the relationship between PCK and instructional practice was mediated through teachers' constructivist beliefs about teaching and learning mathematics (Yang et al. 2020). In addition, Krauss et al. (2020) concluded that a cognitive measure of PCK was more predictive for instructional quality, compared to CK and situation-specific skills. Moreover, Lee and Santagata (2020) observed that the relationship between teachers' professional knowledge and instructional behavior was dependent on teaching experience. More particularly, no association was observed among teachers with only one year of teaching experience, whereas a significant association was observed among teachers with two and three years of teaching experience. Finally, there is some evidence that different facets of teachers' professional knowledge differently impact distinct dimensions of classroom instruction (Krauss et al. 2020). Whether different dimensions of classroom instruction differently impact student cognitive and non-cognitive learning outcomes in mathematics is still an open question for future research.

Despite the merits of research on bridging teachers' professional knowledge and instructional practice, several challenges have not been addressed. We elaborate on two of these challenges, one at the conceptual level and one at the practical implication level. Regarding the conceptual level, we noticed several promising new frameworks for understanding teachers' expertise. These new frameworks typically stress the importance of extending the earlier, more cognitive perspectives on declarative and decontextualized knowledge to more situated knowledge and situational-specific skills. This reframing of teachers' professional knowledge is promising, as a broader and more comprehensive conceptualization, in which several approaches to professional knowledge are combined, contributes to a better understanding of teachers' expertise, its development, and its impact on instructional quality. However, these different conceptualizations of teachers' professional knowledge also induce a new challenge, namely, how to align these different new conceptualizations and terminologies. For instance, whereas professional competence is sometimes used to refer only to teachers' professional knowledge (from a cognitive and situated perspective; e.g., Krauss et al. 2020), other categorizations also include instructional behavior in that overarching construct (e.g., Blömeke et al. 2015), whereas still others refer to other notions that are similar to professional competence, such as teacher expertise (e.g., Schoenfeld 2011b) or teacher proficiency (e.g., Schoenfeld 2020). Similarly, in identifying the situated teachers' professional knowledge, some authors refer to situation-specific skills (e.g., Blömeke et al. 2015), others to action-related competence (e.g., Charalambous 2020; Knievel et al. 2015), and still others refer to teachers' professional vision (e.g., Seidel and Stürmer 2014) in the context of the knowledge that teachers use in noticing and interpreting critical classroom situations and making appropriate decisions. This conceptual confusion also impacted the present survey of recent empirical research on the relation between professional knowledge and instructional practice. Consequently, this review also suffers from

this challenge in its use of different terms, such as teachers' professional knowledge, teachers' professional competence, and teachers' expertise, without clearly indicating their similarities and differences. Whereas the framework of Blömeke et al. (2015) already helps to reveal the relations between the different notions in which professional knowledge is part of the broader notion of professional competence, it remains a challenge for future researchers to reflect on exactly what is missing in earlier conceptualizations, and how this new framework contributes to filling this gap.

Another challenge relates to the practical implication of the development of instruments to assess teachers' professional knowledge and instructional behavior. Whereas these instruments can be used for research purposes to assess professional knowledge and instructional practices in a valid way, and to relate these constructs to other aspects of teacher expertise, these instruments could also serve different, more practical purposes. Firstly, these measures might be used formatively to assess pre- and in-service teachers' knowledge and practice, and to raise awareness of or ameliorate gaps in this knowledge and practice. Similarly, video-based assessments and simulation environments developed to assess more contextualized knowledge in use, might provide an ideal opportunity to develop such knowledge among pre- and in-service teachers. As examples of assessments that have been used in this more practical manner, two US-based projects—Pianta and colleagues' "MyTeachingPartner" (Pianta et al. 2008), and Hill and colleagues' MQI Coaching institute—have re-purposed research instruments in the service of teacher professional development, with promising results. Secondly, these instruments—especially if used summatively—could also be used as a policy lever for both pre-service and in-service teacher education, in both productive and perhaps troubling ways. For example, assessments of pre-service teachers' competencies could provide insights that would be useful in the design or evaluation of teacher preparation programs, such as by identifying courses that are particularly useful (or that are not effective) for the development of teacher knowledge and behavior. For in-service teachers, assessments of teachers' knowledge and practice could be used to identify teachers who are exemplary (e.g., for rewarding or promoting those who excel) or those who are deficient (e.g., to terminate the employment of those who are not meeting established standards for knowledge or teaching practice). To what extent instruments developed for research purposes can or should be used to address practical or policy matters is an important question. What concerns—logistical or even ethical—may surface from such uses?

Reflecting on and addressing these conceptual and practical challenges might be a promising avenue in future research on the relationship between professional knowledge and instructional practice.

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