A simple extended compact nonlinear scheme with adaptive dissipation control

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Abstract

Weighted compact nonlinear schemes (WCNSs) [1] are a popular family of high resolution shock-capturing methods, of which the nonlinear interpolation procedure is essential for non-oscillatory shock-capturing computation. In this article, numerical dissipation of an improved compact nonlinear scheme, which is fifth-order accurate, is significantly reduced by further introducing a six-point optimized stencil in the nonlinear interpolation, without using more complex local or global smoothness indicators. Nonlinear dissipation and dispersion are then further reduced by using an adaptive strategy. In particular, the computational procedure including the proposed six-point nonlinear interpolation is almost the same as that of the improved five-point scheme, and thus their computational efforts are similar. However, significant gains in numerical performance are obtained while including the six-point optimized stencil. Benchmark cases in one and two space dimensions are used to assess performance of the present scheme, viz., shock-capturing capability and resolution in smooth regions, showing promising advantages.

Keywords: compact nonlinear schemes; nonlinear interpolation; high-order schemes; extended stencil

1. Introduction

Accurate simulations of compressible flows involving discontinuities such as shocks and interfaces, as well as small-scale flow structures, have received considerable attention in the computational fluid dynamics (CFD) research community over the past decades. Advanced high-order CFD schemes are capable of capturing discontinuities with high resolution, and simultaneously resolving complicated flow structures, as introduced by numerous publications [2, 3, 4, 5]. High-order numerical schemes have undergone extensive development, and this is still an active topic with much yet to be done as noted by Wang et al. [6].

Among high-order shock-capturing methods, a popular family of schemes is the weighted compact nonlinear schemes (WCNSs) developed by Deng and Zhang [1]. WCNSs are extension of the compact nonlinear

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scheme (CNS) of Deng and Maekawa [7], which provides excellent resolution for high-frequency waves due to the spectral-like resolution of compact schemes proposed by Lele [8], among others. As shown by Nonomura et al. [9, 10], Deng et al. [11], and Wong and Lele [12], WCNSs have several advantages over the classical finite-difference WENO scheme of Jiang and Shu [13], including: (1) resolution is slightly higher; (2) choice of numerical flux schemes is flexible [14]; and (3) WCNSs perform well with respect to freestream and vortex preservation properties on wavy grids. Besides these advantages of solving general gas dynamic problems, WCNSs are also more effective to achieve pressure equilibrium while simulating multi-component flows [15].

The classical WCNS procedure consists of three steps [1]: (1) node-to-midpoint weighted nonlinear interpolation of flow variables, (2) evaluation of fluxes at midpoints, and (3) midpoint-to-node centered flux differencing. In the first step the idea of nonlinear weighting, originating from WENO schemes [16], is used. The classical WCNS using the JS weight of Jiang and Shu [13] is therefore referred to as WCNS-JS. It has been demonstrated that the nonlinear interpolation procedure is essential for shock capturing ability and resolution of WCNSs [17, 18, 19]. Hence, improving this procedure has been attracting much attention.

Recently, focusing on improving the performance of the nonlinear interpolation, the ENO-like stencil selection used by TENO schemes [20] was incorporated in WCNS, resulting in a new compact nonlinear scheme [21], exhibiting reduced numerical dissipation and dispersion. In [21], flux differencing in the third step is performed using an explicit scheme, as used in the WCNS-E-5 [22]. In fact, as suggested by Deng et al. [23], for a WCNS, the nonlinear interpolation in step (1) dominates the resolution property, and explicit centered differencing, which does not need tridiagonal inversion for derivative calculations, is favored due to simplicity of implementation and superior computational efficiency. Further work of Nonomura et al. [24] demonstrated that the type of flux differencing employed does not significantly change resolution, even for higher-order WCNSs. Therefore, in the present work, only the nonlinear interpolation is investigated. It should be noted that, first order interpolation was recommended if the interpolated density or pressure becomes locally negative [12], thus improving the robustness of traditional WCNSs. In this work, this treatment is avoided, and robustness will be further investigated separately.

The effort of this work, again, explores development of low dissipation and dispersion compact nonlinear schemes. Inspired by the work of Zhang et al. [25, 26], we propose to implement a six-point optimized stencil without using more complex smoothness measurement, or storing extra smoothness information, prior to nonlinear interpolation. Moreover, realizing that nonlinear dissipation is controllable via the ENO-like stencil selection procedure, which acts as a discontinuity-detector [27], we apply an adaptive strategy for controlling numerical dissipation and achieving the optimal spectral property.

For introducing the ideas of this work, the remainder of this article is organized as follows. In the next section, the fundamentals and basic equations of compact nonlinear schemes are introduced, and the proposed schemes are derived in Section 3, which also includes discussion of the approximate dispersion relation (ADR) analysis [28]. Section 4 presents solutions of scalar equations and the Euler equations, and concluding remarks are given in the last section.

2. Fundamentals

The governing equations of compressible flow are hyperbolic systems. Without loss of generality, the one-dimensional scalar hyperbolic conservation law is used for discussing the numerical schemes involved in this work, and takes the form

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad (1)$$

subject to the initial condition

$$u(x, 0) = u_0(x), \quad (2)$$

where u is the dependent variable, and $f(u)$ is the flux term. The governing equation (1), together with its initial condition (2), composes an initial-value problem (IVP), which is assumed to be well-posed in the sense that its exact solution u continuously depends on the initial field and is piecewise smooth with a finite number of discontinuities.

Spatial discretization of Eq. (1) is performed on an equally-spaced one-dimensional grid, and the distance between two adjacent grid nodes is denoted as h . At each node i , $x_i = ih$, and $u_i = u(x_i, t)$. The characteristic velocity in Eq. (1) is $\partial f(u)/\partial u$, which is assumed to be positive, without loss of generality. Therefore, at each node, a semi-discrete solution can be constructed:

$$\left(\frac{\partial u}{\partial t} \right)_i = -\hat{f}'_i, \quad i = 1, \dots, n, \quad (3)$$

where $\hat{f}'_i$ is the approximation to the first spatial derivative of the numerical flux function f_i .

In WCNSs, $\hat{f}'_i$ can be approximated using a centered compact scheme [22], which is written in a general form as

$$\kappa \hat{f}'_{i-1} + \hat{f}'_i + \kappa \hat{f}'_{i+1} = \frac{a_1}{h} \left(\tilde{f}_{i+\frac{1}{2}} - \tilde{f}_{i-\frac{1}{2}} \right) + \frac{a_2}{h} \left(\tilde{f}_{i+\frac{3}{2}} - \tilde{f}_{i-\frac{3}{2}} \right) + \frac{a_3}{h} \left(\tilde{f}_{i+\frac{5}{2}} - \tilde{f}_{i-\frac{5}{2}} \right), \quad (4)$$

the solution of which involves tridiagonal inversions if $\kappa \neq 0$. As mentioned in the Introduction, an explicit formula is preferred due to its lower computational cost, and thus $\kappa = 0$ is set for the following discussions. The sixth-order explicit formula given in [10], termed midpoint-and-node-to-node difference (MND), is specifically applied in this work. The formula is

$$\hat{f}'_i = \frac{b_1}{h} \left(\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}} \right) + \frac{b_2}{h} \left(f_{i+1} - f_{i-1} \right) + \frac{b_3}{h} \left(\hat{f}_{i+\frac{3}{2}} - \hat{f}_{i-\frac{3}{2}} \right), \quad (5)$$

where the coefficients have values $b_1 = 3/2$, $b_2 = -3/10$, and $b_3 = 1/30$.

Apparently, this explicit differencing scheme is not strictly a compact scheme but is considered a special case of general compact nonlinear schemes. Moreover, the present work focuses on nonlinear interpolation of flow variables, without concern for the centered-differencing procedure employed. The present method can also be implemented with using the implicit formula in Eq. (4).

Midpoint flux terms in Eq. (5) are unknown and can be evaluated using numerical upwind flux functions. The scalar upwind flux function is given in generic form as

$$\hat{f}_{i\pm\frac{1}{2}} = \frac{1}{2} \left[\left(f(u_{R,i\pm\frac{1}{2}}) + f(u_{L,i\pm\frac{1}{2}}) \right) - |\hat{a}| \left(u_{R,i\pm\frac{1}{2}} - u_{L,i\pm\frac{1}{2}} \right) \right], \quad (6)$$

where the subscripts, viz., L and R , indicate, respectively, the variables on the left- and right-hand sides of the midpoint $x_{i\pm\frac{1}{2}}$, and $\hat{a}$ is the approximate eigenvalue of the flux Jacobian matrix. The method for calculating the midpoint variables $u_{L/R,i\pm\frac{1}{2}}$ is described in the next section.

Once the numerical approximation $\hat{f}_i'$ is given, temporal integration of Eq.(3) is performed using the third-order strongly stable Runge–Kutta method [29].

3. The present method

As mentioned above, in order to calculate $u_{L/R,i\pm\frac{1}{2}}$, we had proposed an improved compact nonlinear scheme [21] using the ENO-like stencil selection procedure [20], showing low dissipation and dispersion, in the nonlinear interpolation procedure. A further improvement focusing on nonlinear interpolation is investigated in the current work so that the overall performance of this compact nonlinear scheme can be enhanced accordingly. For simplicity, we only consider the evaluation of variables on the left- and right-hand sides of the midpoint $x_{i+\frac{1}{2}}$, here denoted as $u_{L/R,i+\frac{1}{2}}$.

3.1. The smoothness measurement

Before performing nonlinear interpolations, locations of discontinuities within the range of the candidate stencils must be determined. In order to measure the smoothness of flow field variables, local and global smoothness indicators are constructed.

Figure 1: Local smoothness indicators.

As shown in Fig. 1, evaluation of smoothness measurement is applied on a five-point full stencil $SM_{L,i+\frac{1}{2}} = \{x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}\}$, where SM denotes “the stencil used for smoothness measurement”. Just as in WENO schemes [16, 13], three three-point substencils with index k , namely,

$$SM_{L,i+\frac{1}{2},k} = \{x_{i+k-2}, x_{i+k-1}, x_{i+k}\}, \quad k = 0, 1, 2, \quad (7)$$

are used to compute three local smoothness indicators, the β_k s. Considering that each of the third-order interpolations evaluated on its corresponding substencil can be expressed in a generic form using (approximated) n^{th} derivatives ($n = 1, 2$), a local Taylor polynomial can be constructed

$$u_{L,i+\frac{1}{2},k} = u(x_i + \Delta x) = u_i + u_{i,k}^{(1)} \Delta x + u_{i,k}^{(2)} \frac{\Delta x^2}{2}, \quad (8)$$

where $\Delta x = x_{i+\frac{1}{2}} - x_i = h/2$, with first- and second-order differences, given by

$$\begin{aligned} u_{i,0}^{(1)} &= \frac{1}{2h}(u_{i-2} - 4u_{i-1} + 3u_i), \\ u_{i,1}^{(1)} &= \frac{1}{2h}(u_{i+1} - u_{i-1}), \\ u_{i,2}^{(1)} &= \frac{1}{2h}(-3u_i + 4u_{i+1} - u_{i+2}), \end{aligned} \tag{9}$$

and

$$\begin{aligned} u_{i,0}^{(2)} &= \frac{1}{h^2}(u_{i-2} - 2u_{i-1} + u_i), \\ u_{i,1}^{(2)} &= \frac{1}{h^2}(u_{i-1} - 2u_i + u_{i+1}), \\ u_{i,2}^{(2)} &= \frac{1}{h^2}(u_i - 2u_{i+1} + u_{i+2}), \end{aligned} \tag{10}$$

respectively. Then β_k is defined as [1]

$$\beta_k = \left(hu_{i,k}^{(1)}\right)^2 + \left(h^2u_{i,k}^{(2)}\right)^2. \tag{11}$$

Following the procedure in [21], the smoothness measurement for the k^{th} candidate substencil is calculated as

$$\gamma_k = \left(C + \frac{\tau_5}{\beta_k + \epsilon}\right)^q, \quad k = 0, 1, 2, \tag{12}$$

where the global smoothness indicator is defined as $\tau_5 = |\beta_0 - \beta_2|$. The small threshold value employed for the reported studies was $\epsilon = 10^{-12}$, which is consistent with the double precision arithmetic used here. The two constant parameters resulting in strong scale separation have values $C = 1$ and $q = 6$ [20]. Instead of using the original nonlinear smoothness measurement in Eq. (12), γ_k is further normalized as

$$\chi_k = \frac{\gamma_k}{\sum_{k=0}^2 \gamma_k}. \tag{13}$$

Then a sharp cut-off function can be defined as

$$\delta_k = \begin{cases} 0, & \text{if } \chi_k < C_T, \\ 1, & \text{otherwise,} \end{cases} \tag{14}$$

where C_T serves as the global reference for smoothness indicators, and a constant value $C_T = 10^{-5}$ was used for computations of several five point schemes [20, 21]. Here, if δ_k equals zero, the polynomial evaluated on the corresponding substencil is deemed to be “oscillatory”; otherwise the substencil is regarded as “smooth”. Therefore, by using this procedure, locations of discontinuities within the range of the five-point full stencil can be determined. The smoothness measurement on the right-hand side of $x_{i+\frac{1}{2}}$ can be defined symmetrically, and thus the corresponding discussion is omitted.

3.2. Adaptive nonlinear dissipation

It has also been found that appropriately distinguishing discontinuities may fully recover spectral properties of the underlying linear schemes as shown in Deng et al. [30]. In [21], as noted above, we employed a

constant value of C_T , but this may not always be optimal. and thus an alternative is the adaptive nonlinear dissipation control [31] which uses a detector m , originally from the work of Ren et al. [32], given by

$$m = 1 - \min\left(1, \frac{\eta_{i+\frac{1}{2}}}{C_r}\right), \quad (15)$$

to indicate the potential discontinuity location. Here

$$\eta_{i+\frac{1}{2}} = \min(\eta_{i-1}, \eta_i, \eta_{i+1}), \quad \eta_i = \frac{|2\Delta f_{i+\frac{1}{2}}\Delta f_{i-\frac{1}{2}}| + \epsilon}{(\Delta f_{i+\frac{1}{2}})^2 + (\Delta f_{i-\frac{1}{2}})^2 + \epsilon}, \quad \text{and} \quad \epsilon = \frac{0.9C_r}{1 - 0.9C_r}\xi^2. \quad (16)$$

Specifically, the two parameters involved in Eq. (16) take values $C_r = 0.24$ and $\xi = 10^{-3}$. The parameter C_T in Eq. (14) can now be adaptively determined using

$$g(m) = (1 - m^4)(1 + 4m), \quad (17a)$$

$$\bar{\beta} = \alpha_1 - \alpha_2(1 - g(m)), \quad (17b)$$

$$C_T = 10^{-[\bar{\beta}]}, \quad (17c)$$

where $\alpha_1 = 10$, $\alpha_2 = 5$, and $[\bar{\beta}]$ denotes the floor function, which yields the largest integer less than or equal to the variable $\bar{\beta}$. By using this discontinuity detector, C_T will be set to a small value if the flow field is likely to be smooth, further reducing nonlinear numerical dissipation and dispersion; and a large C_T which ensures shock-capturing stability will be obtained if there is potentially a discontinuity.

3.3. The new stencil-selection procedure

Once locations of discontinuities are determined using the aforementioned procedure of smoothness measurement, high-order interpolations can be constructed on only smooth stencils shown in Fig. 2, labeled SP_l (stencils used for constructing polynomials), without crossing any discontinuities. Note that SMs in the last subsection are only utilized for smoothness measurement, and the traditional concept of “convex combination” [16] is not applied in this work. Instead, the concept of “stencil-selection procedure” is applied for final high-order interpolations, yielding more flexibility in constructing new schemes [27]. The idea of the optimal candidate stencil selection is straightforward: choose the largest smooth stencil. The algorithm for constructing a five-point scheme is shown in Algorithm 1. With using this algorithm and the adaptive nonlinear dissipation for nonlinear interpolations, the corresponding compact nonlinear scheme is denoted as TCNS5-II.

3.4. Extended nonlinear interpolation: a six-point interpolation

A six-point stencil SP_6 was presented in Fig. 2. This stencil exceeds the range of $SM_{L,i+\frac{1}{2}}$, and thus cannot be applied based only on the smoothness information of the $\{\delta_k | k = 0, 1, 2\}$ on $SM_{L,i+\frac{1}{2}}$. However, a simple solution for this can be used, as shown in Fig. 3. Two five-point full stencils, which are usually used for the five-point interpolations of $u_{L,i+\frac{1}{2}}$ and $u_{R,i+\frac{1}{2}}$, respectively, cover the whole range of the centered six-point stencil SP_6 . Therefore, if both five-point full stencils are smooth, a six-point interpolation can

Figure 2: Candidate stencils for high-order interpolations.

be used for calculating $u_{L/R,i+\frac{1}{2}}$. In fact, the code structure can be elegantly designed in such a way that the smoothness information of both five-point full stencils is easily accessible for this purpose. To demonstrate this, we introduce Algorithm 2 below. With using this algorithm for nonlinear interpolations, the corresponding six-point compact nonlinear scheme is denoted as TCNS5-E.

Figure 3: Left and right SMs in the relevant range of $x_{i+\frac{1}{2}}$ for measuring smoothness.

105 3.5. High-degree polynomials

Coefficients of the high-degree polynomials $P_{l,L} = \sum_{m=i-2}^{i+2} a_{l,m} u_m$ corresponding to the candidate stencils SP_l are shown in Table 1 for interpolating $u_{L,i+\frac{1}{2}}$. The polynomials for interpolating $u_{R,i+\frac{1}{2}}$ can be given symmetrically, and thus we drop the subscript L for simplicity. For P_0 through P_5 , the polynomials are constructed for achieving the highest-order of accuracy on given stencils, whereas, a different strategy is
110 used to compute the six-point interpolation, i.e., P_6 .

A sixth-order centered scheme can be defined on SP_6 , without numerical dissipation. However, numerical

Table 1: Coefficients of polynomials for high-degree nonlinear interpolation.

P_l	$a_{l,i-2}$	$a_{l,i-1}$	$a_{l,i}$	$a_{l,i+1}$	$a_{l,i+2}$	$a_{l,i+3}$
P_0	3/128	-5/32	45/64	15/32	-5/128	0
P_1	0	-1/16	9/16	9/16	-1/16	0
P_2	1/16	-5/16	15/16	5/16	0	0
P_3	0	0	3/8	3/4	-1/8	0
P_4	0	-1/8	3/4	3/8	0	0
P_5	3/8	-5/4	15/8	0	0	0
P_6	9/640	-7/64	39/64	9/16	-11/128	3/320

dissipation in high-wavenumber regions is necessary and beneficial for numerical stability. Following the optimization method given in [20], a six-point interpolation of fifth-order accuracy is constructed as

$$P_6 = \eta p_{i+\frac{1}{2},1} + (1 - \eta) p_{i+\frac{1}{2},2}, \quad (18)$$

using two five-point polynomials

$$\begin{aligned} p_{i+\frac{1}{2},1} &= \frac{1}{128} (3u_{i-2} - 20u_{i-1} + 90u_i + 60u_{i+1} - 5u_{i+2}), \\ p_{i+\frac{1}{2},2} &= \frac{1}{128} (-5u_{i-1} + 60u_i + 90u_{i+1} - 20u_{i+2} + 3u_{i+3}), \end{aligned} \quad (19)$$

in which, $p_{i+\frac{1}{2},1}$ is a fifth-order upwind scheme, and $p_{i+\frac{1}{2},2}$ is a fifth-order downwind scheme. If the weight $\eta = 0.5$, the sixth-order centered scheme is obtained from Eq. (18), and if $\eta = 1.0$, the fifth-order upwind scheme, i.e., P_0 , is recovered. Therefore, increasing η produces a more dissipative upwind scheme, but dissipation is still reduced compared with the fifth-order upwind scheme as long as $\eta < 1$, due to the anti-
115 dissipation property of $p_{i+\frac{1}{2},2}$. In this work, $\eta = 0.6$ is applied in most numerical simulations of the following section, while other choices are also applicable. More specific discussions are given in Section 3.6, to explain the effect of the simple strategy designed for the six-point interpolation.

We can now compare the five-point and six-point methods. As mentioned above, Algorithm 1 is provided to explain the procedure of constructing a five-point scheme, where P_l is the high-order polynomial evaluated
120 on candidate stencil SP_l . **For comparison, another six-point interpolation scheme with using the optimization procedure in Ref.[20] and the adaptive nonlinear dissipation [31] is also introduced in the Appendix.**

Remark 1. In a typical WCNS, if only the central substencil, i.e. SM_1 , is deemed to be oscillatory, the other two stencils are applied in convex combination [1]. However, this combination produces an interpolation which, in fact, still crosses the discontinuity located within the range of SM_1 . This specific dilemma, however,
125 is avoided in the current method as shown below in Algorithm 1. In particular, if SM_1 is oscillatory while SM_0 and SM_2 are both smooth, SP_3 is applied for the interpolation since the midpoint $x_{i+\frac{1}{2}}$ is located within the range of SP_3 . $\square$

Data: Grid information x_i and spatial distribution u_i at time step n .

Result: Left variables $u_{L,i+\frac{1}{2}}$ at $x_{i+\frac{1}{2}}$.

```

1 begin
2   Get spatial distribution at time step  $n$  in relevant range of  $x_{i+\frac{1}{2}}$ , viz.,  $u_{i-2}$  through  $u_{i+2}$ ;
3   Calculate  $\{\delta_k | k = 0, 1, 2\}$  on  $SM_{L,i+\frac{1}{2}}$  using Eqs.(9)–(14);
4   switch  $\{\delta_k | k = 0, 1, 2\}$  do
5     case  $\{1, 1, 1\}$   $u_{L,i+\frac{1}{2}} = P_0$ 
6     case  $\{0, 1, 1\}$   $u_{L,i+\frac{1}{2}} = P_1$ 
7     case  $\{1, 1, 0\}$   $u_{L,i+\frac{1}{2}} = P_2$ 
8     case  $\{*, 0, 1\}$   $u_{L,i+\frac{1}{2}} = P_3$ 
9     // * can be either 1 or 0
10    case  $\{0, 1, 0\}$   $u_{L,i+\frac{1}{2}} = P_4$ 
11    case  $\{1, 0, 0\}$   $u_{L,i+\frac{1}{2}} = P_5$ 
12  endsw
13 end

```

Algorithm 1: Constructing the five-point scheme.

Remark 2. The underlying philosophy of designing the present algorithm is partly inspired by the MOOD paradigm [33, 34], which provides optimal accuracy while satisfying the discrete maximum principle (DMP) [33] and preserving density/pressure positivity. Although the present algorithm is not an *a posteriori* procedure as is MOOD, it provides high-order approximation with guaranteeing the ENO property (instead of satisfying DMP), and reduces the order of accuracy if the ENO property might be violated. An *a posteriori* paradigm achieving oscillation-free interpolations can also be implemented using these SPs, and details will be discussed in future work. $\square$

Remark 3. Discarding the concept of “convex combination” [16] allows use of optimal linear schemes on all SPs. Particularly, optimization can be performed for a specific interpolation, e.g., P_0 , to improve the dissipation and/or dispersion properties, without changing other interpolations. This flexibility results in minor improvement described in this section, while the idea is exploited further in the following sections. $\square$

As is shown, the selection procedure in this algorithm (lines 8 through 15) is exactly the same as that in Algorithm 1, except that both left and right interpolations are now considered at the same time. The difference lies in lines 3, 4 and 5, where a six-point interpolation can be applied for interpolating the variables on both sides of the midpoint $x_{i+\frac{1}{2}}$ in smooth flow fields.

We are able to use the six-point centered stencil in the nonlinear interpolation by making the most of smoothness measurements already available, and the present interpolation method is thus called an *extended* nonlinear interpolation procedure. This idea, as mentioned above, is motivated by [25, 26], in which proce-

Data: Grid information x_i and spatial distribution u_i at time step n .

Result: Left and right variables $u_{L/R, i+\frac{1}{2}}$ at $x_{i+\frac{1}{2}}$.

```

1 begin
2   Get spatial distribution at time step  $n$  in relevant range of  $x_{i+\frac{1}{2}}$ , viz.,  $u_{i-2}$  through  $u_{i+3}$ ;
3   Calculate  $\{\delta_k|k=0,1,2\}_L$  on  $SM_{L, i+\frac{1}{2}}$  using Eqs. (9)–(14), and calculate  $\{\delta_k|k=0,1,2\}_R$  on
    $SM_{R, i+\frac{1}{2}}$  using the symmetrical form of Eqs.(9)–(14);
4   if  $\{\delta_k|k=0,1,2\}_L = \{\delta_k|k=0,1,2\}_R = \{1,1,1\}$  then
5      $u_{L, i+\frac{1}{2}} = P_{6,L}$ ,  $u_{R, i+\frac{1}{2}} = P_{6,R}$ ;
6   else
7     foreach  $\{s|L, R\}$  do
8       switch  $\{\delta_k|k=0,1,2\}_s$  do
9         case  $\{1,1,1\}$   $u_{s, i+\frac{1}{2}} = P_{0,s}$ 
10        case  $\{0,1,1\}$   $u_{s, i+\frac{1}{2}} = P_{1,s}$ 
11        case  $\{1,1,0\}$   $u_{s, i+\frac{1}{2}} = P_{2,s}$ 
12        case  $\{*,0,1\}$   $u_{s, i+\frac{1}{2}} = P_{3,s}$ 
13        case  $\{0,1,0\}$   $u_{s, i+\frac{1}{2}} = P_{4,s}$ 
14        case  $\{1,0,0\}$   $u_{s, i+\frac{1}{2}} = P_{5,s}$ 
15      endsw
16    end
17  end
18 end

```

Algorithm 2: Constructing the six-point scheme.

dures of smoothness measurement and high-order interpolation are separated, and smoothness information must be stored before the high-order interpolation is performed. In the present work, however, extra storage of smoothness information is not required, and the algorithm is simpler to implement. In the following discussions, this new six-point scheme is simply denoted as TCNS5-E to emphasize the idea of extending the basic strategy being used.

3.6. Approximate dispersion relation

Here, the approximate dispersion relation (ADR) analysis [28] is performed to evaluate spectral properties of the numerical schemes considered in the present work, viz., WCNS-JS, TCNS5, TCNS5-II, TCNS5-E and the TCNS6 introduced in the appendix. Particularly, the ADR analysis shows the dissipation and dispersion errors of nonlinear numerical schemes in wavenumber space by solving a linear advection equation with a smooth initial condition including different Fourier modes supported on a given grid. A modified wavenumber ξ (normalized to lie in the interval $[0, \pi]$) is defined for all supported modes, and its real (ξ_R)

and imaginary (ξ_I) parts represent dispersion and dissipation properties, respectively, for the numerical scheme being evaluated. Results are shown in Fig. 4. Significant improvements of both the dispersion and dissipation properties are obtained by the TCNSs compared with WCNS-JS in intermediate and high-wavenumber regimes.

Figure 4: Approximate dispersion and dissipation properties of fifth-order schemes; (a) dispersion, and (b) dissipation.

In particular, the TCNS5-II and TCNS5-E almost fully recover the dispersion property of the underlying linear scheme, mostly due to using the adaptive strategy introduced in Section 3.2. In addition, since dispersion properties of the five-point scheme TCNS5-II and the six-point scheme TCNS5-E are essentially the same, the present TCNS5-E does not show any significant advantage in this property compared with the five-point scheme TCNS5-II. Whereas, the present TCNS5-E shows satisfying numerical dissipation property, even compared with the TCNS6. The dispersion error of the TCNS6 is a little larger compared with the TCNS5-E. The difference is produced probably due to using different parameters in the adaptive nonlinear dissipation function, which is first introduced in Ref.[31] and shown in the Appendix.

The present six-point scheme TCNS5-E showing the low-dissipation property, uses a centered stencil in the stencil-selection procedure, reducing the overall numerical dissipation. The effect of using different weights η in Eq. (18) is also investigated. As discussed in the previous subsection 3.5, the fifth-order upwind scheme is recovered when $\eta = 1.0$, and the six-order centered scheme is obtained if $\eta = 0.5$. Gradually decreasing the value of weight η from 1 to 0.5 reduces dissipation of the six-point interpolation by including the anti-dissipation effect from the downwind polynomial. A weight $\eta = 0.6$ is used in all of the following numerical cases as a compromise between controlling spurious oscillations and reducing dissipation errors. In the following numerical test cases, this adaptive strategy is applied by default for both TCNS5-II and

TCNS5-E.

4. Numerical results

A variety of canonical problems are simulated to assess performance of the present method. It is noted that TCNS5-II is used mainly for comparison purposes to highlight the advantage of the extending strategy used in TCNS5-E, which requires only a minor algorithmic modification compared with TCNS5-II. The one-dimensional linear advection equation, the one-dimensional inviscid Burgers' equation and the Euler equations of gas dynamics are used as model equations in this section. Node-to-midpoint interpolation is performed on characteristic fields to alleviate spurious oscillations [1], and the Rusanov scheme [35] is employed for computing numerical fluxes. Although previous results, e.g., Wang et al. [14], and Zhang et al. [36, 37], have shown that low-dissipation upwind flux schemes significantly improve numerical capability in resolving complex flow structures, using the more dissipative Rusanov scheme can still display the advantage of the present method. We note that a CFL number equal to 0.4 has been used in computing the test cases reported herein, except that in the linear advection problem in subsection 4.1 the CFL numbers are sufficiently small to achieve temporally stable and converged results.

4.1. Linear advection problem

The one-dimensional Gaussian pulse advection problem [38] is used to assess the order of accuracy of the proposed schemes when solutions are smooth. This problem is modeled by the linear advection equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \quad x \in [0, 1], \quad (20)$$

with periodic boundary conditions and initial condition

$$u(x, 0) = e^{-300(x-x_c)^2}, \quad x_c = 0.5. \quad (21)$$

Temporal integration is performed up to $t = 1$, corresponding to one period of the single wave propagation in time. Uniformly-spaced grids are progressively refined by a factor of two from the coarsest grid of $N = 51$ points. As noted above, the numerical simulation on each grid is conducted using sufficiently small time steps so as to achieve temporally converged results.

Table 2 and Fig. 5 illustrate numerical errors and convergence rates of the various numerical schemes used. Accuracy of TCNS5 and TCNS5-II coincides with that of the fifth-order underlying linear scheme, indicating that the optimal linear scheme is recovered in this case. WCNS-JS also shows approximate fifth-order accuracy, but its resolution is significantly lower than that of the other three nonlinear schemes. The accuracy of TCNS5-E is of fifth-order approximately, but its resolution is even higher than the fifth-order linear scheme because of the low dissipation property of the six-point interpolation.

Table 2: L^∞ -error and convergence rate of fifth-order schemes in solving the linear advection equation at $t = 1$.

N	Linear		WCNS-JS		TCNS5		TCNS5-II		TCNS5-E	
	Error	Order	Error	Order	Error	Order	Error	Order	Error	Order
51	6.74E-02	*	0.146572	*	6.56E-02	*	6.69E-02	*	4.40E-02	*
101	5.01E-03	3.75	1.56E-02	3.23	5.01E-03	3.71	5.01E-03	3.74	1.72E-03	4.68
201	1.84E-04	4.77	6.94E-04	4.49	1.84E-04	4.77	1.84E-04	4.77	4.45E-05	5.27
401	5.89E-06	4.96	2.82E-05	4.62	5.89E-06	4.96	5.89E-06	4.96	1.24E-06	5.16
801	1.85E-07	4.99	9.91E-07	4.83	1.85E-07	4.99	1.85E-07	4.99	3.76E-08	5.05
1601	5.80E-09	5.00	3.21E-08	4.95	5.80E-09	5.00	5.80E-09	5.00	1.17E-09	5.01

Figure 5: Spatial convergence rate of fifth-order schemes at $t = 1$ for L^2 - and L^∞ -errors in solutions to the linear advection equation.

4.2. Inviscid Burgers' equation

The one-dimensional inviscid Burgers' equation [39] is used to assess actual order of accuracy of the proposed schemes when applied to a nonlinear scalar equation. The governing equation is

$$\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x} = 0, \quad x \in [0, 2], \quad (22)$$

with periodic boundary conditions and initial condition

$$u(x, 0) = \frac{1}{2} + \sin(\pi x). \quad (23)$$

The exact solution is computed by solving the derived general characteristic relation given in [39]. The solution is smooth for $0 \leq t < 1/\pi$, and a discontinuity develops and starts to interact with the expansion

Figure 6: One-dimensional inviscid Burgers' equation solution on a grid of 161 points; numerical and exact solution at (a) $t = 0.2$, and (b) $t = 0.7$.

wave when $t = 1/\pi$. Results at $t = 0.2$ and $t = 0.7$ are presented to show the continuous and discontinuous parts, respectively, of the solution. Solutions computed with the above fifth-order schemes are compared with the exact solution in Fig. 6, and show good agreement for both smooth and discontinuous parts. As shown in Fig.6(b), all solutions obtained from these nonlinear schemes are free of overshoot at the discontinuity, and sharp resolution is achieved.

Moreover, L^∞ -error and convergence rate of each scheme at $t = 0.2$ are presented in Table 3. Spatial accuracy of fifth order is achieved by all methods for the refined meshes except that the TCNS5-E yields results with apparent order of accuracy slightly higher than five. Perfect agreement can be found for TCNS5, TCNS5-II and the underlying fifth-order linear scheme, indicating the optimal linear schemes are recovered by both TCNS and TCNS-II. TCNS-E is more accurate than the other fifth-order schemes. This is attributed to use of the six-point stencil for polynomial construction, which is the only difference between the TCNS5-II and TCNS5-E, and thus it is expected that asymptotically (as $h \rightarrow 0$) fifth-order accuracy can be observed.

4.3. Sod and Lax shock tube problems

Riemann initial-value problems of Sod [40] and Lax [41] are used to further evaluate shock-capturing capability of the proposed schemes employing discretization of the 1-D Euler equations

$$\begin{aligned}
 \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} &= 0, \\
 \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} &= -\frac{\partial p}{\partial x}, \\
 \frac{\partial E}{\partial t} + \frac{\partial(uE)}{\partial x} &= -\frac{\partial(up)}{\partial x},
 \end{aligned}$$

Table 3: L^∞ -error and convergence rate of fifth-order schemes in solving the 1-D inviscid Burgers equation at $t = 0.2$.

N	Linear		WCNS-JS		TCNS5		TCNS5-II		TCNS5-E	
	Error	Order	Error	Order	Error	Order	Error	Order	Error	Order
21	4.85E-03	*	3.59E-03	*	4.85E-03	*	4.85E-03	*	8.86E-03	*
41	1.17E-03	2.05	1.75E-03	1.04	1.17E-03	2.05	1.17E-03	2.05	8.82E-04	3.33
81	1.15E-04	3.35	1.78E-04	3.30	1.15E-04	3.35	1.15E-04	3.35	4.24E-05	4.38
161	4.69E-06	4.62	7.54E-06	4.56	4.69E-06	4.62	4.69E-06	4.62	1.57E-06	4.76
321	1.56E-07	4.91	2.55E-07	4.89	1.56E-07	4.91	1.56E-07	4.91	3.51E-08	5.48
641	4.67E-09	5.06	7.92E-09	5.01	4.67E-09	5.06	4.67E-09	5.06	7.38E-10	5.57

where $E = e + \frac{1}{2}u^2$ is the total energy per unit mass, and e is internal energy. The dependent variables are related through the perfect gas equation of state given by $p = (\gamma - 1)\rho e$ with $\gamma = 1.4$, thus closing the Euler equations.

The Sod shock tube problem involves a right-moving shock of Mach number 1.7, while for the Lax shock tube problem, the right-moving shock has Mach number 2.0. Initial conditions of the Sod problem are

$$(\rho, u, p) = \begin{cases} (1, 0, 1) & x \in [0, 0.5], \\ (0.125, 0, 0.1) & x \in (0.5, 1], \end{cases} \quad (24)$$

and by solving the problem on an evenly-distributed grid of $N = 101$ points the results at $t = 0.2$ are given in Fig. 7. Initial conditions for the Lax shock-tube problem are

$$(\rho, u, p) = \begin{cases} (0.445, 0.698, 3.528) & x \in [0, 0.5], \\ (0.5, 0, 0.571) & x \in (0.5, 1]. \end{cases} \quad (25)$$

This case is also simulated on an evenly distributed grid of $N = 101$ points, and the results at $t = 0.14$ are shown in Fig. 8.

Density and velocity distributions are better resolved by using TCNSs, compared with using WCNS-JS, as shown particularly in Fig.7(b). The results are free of spurious oscillations near discontinuities, showing the shock-capturing capability of these compact nonlinear schemes. WCNS-JS smears the contact discontinuity in both the Sod and Lax problems, indicating too much numerical dissipation. Minor overshoot is noticeable in the density profiles of TCNS5-II and TCNS5-E near the contact discontinuity in the Lax problem. However, such overshoots are considered to be acceptable since similar behaviors are also seen in various other high-order schemes and simply reflect “essentially” rather than “completely” non-oscillatory behavior.

4.4. Shock/density-wave interaction

The shock/density-wave interaction problem of Titarev and Toro [42] is characterized by a right-moving shock wave of Mach number 1.3 interacting with high-frequency sine waves in the density field. A multi-

Figure 7: Numerical and exact solutions of the Sod problem at $t = 0.2$; (a) density, (b) zoom in of density, (c) velocity.

scale wave structure evolves after the shock wave interacts with the oscillating density wave, and both shock-capturing and wave-resolution capabilities are evaluated for the methods considered herein via this problem.

The problem again employs the 1-D Euler equations and is initialized by

$$(\rho, u, p) = \begin{cases} (1.515695, 0.523346, 1.805), & x \in [0, 0.5], \\ (1 + 0.1\sin(20\pi(x - 5)), 0, 1), & x \in (0.5, 10]. \end{cases} \quad (26)$$

240 This case is run on a grid of $N = 1001$ points which are uniformly distributed, and the results at $t = 5$ are

Figure 8: Numerical and exact solutions of the Lax problem at $t = 0.14$; (a) density, and (b) velocity.

given. Since there is not a theoretical solution for this problem, the numerical result calculated by using the WCNS-JS on a grid of $N = 5001$ points is used as a reference solution.

As shown in Fig. 9, TCNSs produce considerably better resolved density waves behind the shock wave compared with the WCNS-JS. The result of the WCNS-JS indicates strong numerical dissipation, since small-scale wave structures are relatively smeared. More specifically, two six-point schemes, viz. TCNS5-E and TCNS6, show better resolution compared with two five-point TCNSs. Therefore, one may conclude that using the six-point low-dissipation stencil indeed dominates the performance of the present TCNS5-E, although the smoothness measurement is simple. Furthermore, in Table 4 we can see that the TCNS5-E is as efficient as the TCNS5-II, and less time consuming compared with the TCNS6.

Figure 9: Shock/density-wave interaction problem; numerical solutions and the exact solution at $t = 5$; (a) full spatial domain, and (b) zoom in on high-amplitude region.

It is also worth noting that numerical dissipation can be further reduced by using a more accurate Riemann solver in place of the dissipative Rusanov scheme. However, developing or evaluating different upwind flux schemes is beyond the scope of the present research, and will not be discussed further in the following sections.

Table 4: CPU timing (in seconds) per time step in the shock/density-wave interaction problem.

Grid size	WCNS-JS	TCNS5	TCNS5-II	TCNS6	TCNS5-E
1001	0.02	0.03	0.04	0.08	0.04

4.5. Two-dimensional Riemann problems

In this section, the configurations 3 and 6 out of 19 2-D Riemann problems used by Lax and Liu [43] are employed to evaluate performance of the currently investigated numerical schemes. In particular, these two configurations are characterized by rich small scales in the flow fields, and thus are specifically chosen to evaluate ability of the proposed numerical schemes to resolve fine flow structures. These 2-D Riemann problems are widely-solved canonical problems in the present context, analogous to the 1-D Sod shock tube problem in the previous section. The equations of motion are the 2-D Euler equations which are a

straightforward extension of the 1-D case given above:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} &= 0, \\
\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} &= -\frac{\partial p}{\partial x}, \\
\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} &= -\frac{\partial p}{\partial y}, \\
\frac{\partial E}{\partial t} + \frac{\partial(uE)}{\partial x} + \frac{\partial(vE)}{\partial y} &= -\frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y}.
\end{aligned}$$

4.5.1. Configuration 3

Initial conditions for configuration 3 are given by

$$(\rho, u, v, p) = \begin{cases} (1.5, 0, 0, 1.5) & (x, y) \in [\frac{1}{2}, 1] \times [\frac{1}{2}, 1], \\ (0.5323, 1.206, 0, 0.3) & (x, y) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1], \\ (0.138, 1.206, 1.206, 0.029) & (x, y) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}], \\ (0.5323, 0, 1.206, 0.3) & (x, y) \in [\frac{1}{2}, 1] \times [0, \frac{1}{2}]. \end{cases} \quad (27)$$

Boundary conditions are

$$\begin{aligned}
\frac{\partial \mathbf{u}(x, y, t)}{\partial x} &= 0, & x = 0, 1, & \forall y, t, \\
\frac{\partial \mathbf{u}(x, y, t)}{\partial y} &= 0, & y = 0, 1, & \forall x, t,
\end{aligned} \quad (28)$$

where the vector of primitive variables is given by $\mathbf{u} = (\rho, u, v, p)^T$. It should be noted that computations are terminated before the shock structures interacted with boundaries, and thus the discussion will focus on the central part of the flow field.

Figure 10 presents results obtained with 1024^2 grid points with uniform spacing using WCNS-JS and TCNSs, at $t = 0.3$. All TCNSs capture more vortices along contact lines than does WCNS-JS. The solution computed with TCNS5-II is slightly different from that of TCNS5, and TCNS5-E resolves a few unique small features not seen in the other figures, probably due to its low numerical dissipation. Especially, the present TCNS5-E shows similar resolution compared with the TCNS6 which uses the traditional “convex combination” concept for the nonlinear interpolation.

Moreover, a significant advantage of the present TCNS5-E is shown in Table 5. It can be found that the CPU time of using the TCNS5-E is only slightly higher than using the TCNS5-II, which is expectable since the algorithms of these two schemes are almost the same. However, the present TCNS5-E is more efficient compared with the TCNS6, since the smoothness measurement is simpler while using the extended strategy introduced in section 3.4. Therefore, we may conclude that the present TCNS5-E achieves similar performance but requires less computational resource, compared with the TCNS6.

Without physical results for comparisons, one must question whether this numerically-produced “turbulence” actually represents physics. In fact, Shi et al. [44] suggest that during a mesh refinement slip lines

Table 5: CPU timing (in seconds) per time step in the 2D Riemann problem of configuration 3

Grid size	WCNS-JS	TCNS5	TCNS5-II	TCNS6	TCNS5-E
1024 × 1024	3.72	4.95	7.35	15.06	7.54

usually are non-physical and depend on numerical scheme being used. However, these simulations are important for identifying the magnitude of numerical viscosity, and the small numerical viscosity indicated by the small structures in the flow field is a necessity to guarantee that the physical viscosity in Navier-Stokes solutions is not overwhelmed. Moreover, in this numerical case, expected symmetries are lost in all TCNS calculations while they have been maintained with WCNS-JS. Similar behaviors were found in various references [44, 27, 45], but a conclusive opinion about the correctness of numerical results has not been provided, to the best of the authors' current knowledge. We remark that testing grid-function convergence for weak solutions is not easy, and very little on this topic appears in the extant literature. We have demonstrated for 1-D problems that divergence occurs in the L^2 strong-operator norm of the residual, and that the weak scalar product form does not diverge. Further studies are ongoing and will be reported later.

4.5.2. Configuration 6

The 2-D Euler equations are again used as the governing equations, as in the last case of configuration 3, but now the initial conditions are given by

$$(\rho, u, v, p) = \begin{cases} (1, 0.75, -0.5, 1) & (x, y) \in [\frac{1}{2}, 1] \times [\frac{1}{2}, 1], \\ (2, 0.75, 0.5, 1) & (x, y) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1], \\ (1, -0.75, 0.5, 1) & (x, y) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}], \\ (3, -0.75, -0.5, 1) & (x, y) \in [\frac{1}{2}, 1] \times [0, \frac{1}{2}]. \end{cases} \quad (29)$$

Boundary conditions are the same as in the previous test case.

Results at $t = 0.3$ are shown in Fig. 11. All TCNSs resolve abundant small-scale flow structures along contact lines. Minor differences can be seen between the TCNS5-II result and that of TCNS5, and two six-point schemes yield solutions characterized with much richer small-scale structures. Apparently, the TCNS5-II which uses the adaptive nonlinear dissipation, is low-dissipative compared with the TCNS5, and the six-point stencil used in the TCNS5-E further reduces the overall numerical dissipation, thus resulting this low-dissipation scheme. The results of the TCNS5-E and TCNS6 are similar. However, as shown in Table 6, the TCNS5-E with using the present extended stencil is more efficient than the TCNS6. Of course, in the absence physical results or demonstrably accurate simulations, it is somewhat difficult to rank these procedures in modelling real physics.

4.6. Rayleigh–Taylor instability

The Rayleigh–Taylor instability problem, which contains both discontinuities and complex flow structures, has been widely studied, both experimentally and computationally, and it is used here as a final example

Figure 10: Configuration 3 of 2-D Riemann problems in [43]; 30 density contour lines ranging from 0.1 to 1.8 at $t = 0.3$.

Figure 11: Configuration 6 of 2-D Riemann problems in [43] computed on a grid of 1024×1024 points; 40 density contour lines ranging from 0.1 to 2.9 at $t = 0.3$.

Table 6: CPU timing (in seconds) per time step in the 2D Riemann problem of configuration 6

Grid size	WCNS-JS	TCNS5	TCNS5-II	TCNS6	TCNS5-E
1024× 1024	3.75	4.96	7.43	15.03	7.58

for examining performance of the presently-studied methods. Again, the 2-D Euler equations comprise the basic mathematical model. In the present case, initial conditions are given by

$$(\rho, u, v, p) = \begin{cases} (2, 0, -0.025 a \cos(8\pi x), 1 + 2y) & (x, y) \in [0, 0.25] \times [0, 0.5], \\ (1, 0, -0.025 a \cos(8\pi x), y + 3/2) & (x, y) \in [0, 0.25] \times [0.5, 1], \end{cases} \quad (30)$$

where a is the speed of sound, given by $a = \sqrt{\gamma p / \rho}$ and a different $\gamma = \frac{5}{3}$ is used for this specific case. Reflecting boundary conditions are imposed at the left- and right-hand sides of the domain. Constant boundary conditions are given for the top and bottom sides, expressed as

$$(\rho, u, v, p) = \begin{cases} (1, 0, 0, 2.5) & y = 1, \quad \forall t, x, \\ (2, 0, 0, 1) & y = 0, \quad \forall t, x. \end{cases} \quad (31)$$

Two source terms, ρ and ρv , are added to the right-hand side of the third and the fourth equations, respectively, of the 2-D Euler system given at the beginning of this subsection, as done by Shi et al. [44], among others. (Note: this does not correctly adhere to dimensional consistency of terms in the equations of motion, so any correspondence with actual physics would be accidental. Moreover, the Rayleigh–Taylor problem is 3D, so again, not much actual physics can be expected from these simulations, but only numerical properties are investigated.) Two grids were used, namely, 128×512 and 256×1024 points. Density profiles at the final time $t = 1.95$, computed using TCNS5, TCNS5-II and TCNS5-E, are shown in Fig. 12. It is rather clear that grid-function convergence, in any usual sense, is not occurring for any of the methods.

(a) Grid resolution 128×512

(b) Grid resolution 256×1024

Figure 12: Rayleigh–Taylor instability problem; 30 density contour lines ranging from 0.9 to 2.2.

310

Overall, TCNS5-E has resolved much more vortical-structure details than has either TCNS5 or TCNS5-

II, demonstrating the advantage of the low numerical dissipation of the TCNS5-E. The performance of TCNS5-II lies between that of TCNS5 and TCNS5-E. Moreover, symmetric density profiles are produced with all three different schemes, as should be expected. It is worth noting again, that for this problem, a more accurate Riemann solver would be quite helpful for improving resolution of the present simulations, as
315 investigated by San and Kara [46] for a somewhat different problem.

5. Conclusions

Following the fundamental concept of compact nonlinear schemes, a new extending strategy is introduced to implement a six-point centered stencil by making full use of currently available five-point smoothness measurements, and numerical performance can be significantly improved without applying more complex
320 smoothness indicators. Essentially, the smoothness measurement for the extending strategy simply acts as a discontinuity-location detector. With the smoothness/discontinuity-location information provided, an extended high-order, carefully optimized interpolation can be directly applied by using the improved stencil-selection procedure described in this work. It is shown that using this specific stencil-selection procedure, instead of the traditional concept of “convex combination” yields more flexibility in development of nonlinear
325 interpolations required to construct higher-order methods.

In order to further improve the spectral property of the proposed schemes, a mature adaptive dissipation control can be applied to control the magnitude of the cut-off parameter C_T , although the value of this parameter was first taken to be constant. Various numerical analyses have shown that the presented schemes are capable of simulating complex flow problems, including shock waves and hydrodynamic instabilities, for
330 instance, the Rayleigh–Taylor instability problem. Results are significantly improved compared with the traditional WCNS approach. Most importantly, the present TCNS5-E is significantly more efficient than the TCNS6 which uses the traditional “convex combination” concept for the nonlinear interpolation.

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Appendix A. The six-point nonlinear interpolation procedure for compact nonlinear schemes

Following the idea of Ref.[21], a six-point nonlinear interpolation procedure for TCNS can be designed based on the six-point scheme in Ref.[31], and the comparison can be made specifically for the new features introduced in the TCNS5-E. In the scheme introduced in this appendix, which is denoted as TCNS6, the
340 traditional “convex combination” concept is used for its nonlinear interpolation, and the numerical flux scheme or the midpoint-to-node centered flux differencing mentioned in previous sections is directly applied.

In order to measure the smoothness of sub-stencils and calculate the nonlinear weights of the convex combination procedure, the smoothness measurement needs to be given at first. The local smoothness indicators are given in a unified form

$$\beta_k = \left(hu_{i,k}^{(1)}\right)^2 + \left(h^2u_{i,k}^{(2)}\right)^2 + \left(h^3u_{i,k}^{(3)}\right)^2. \quad (\text{A.1})$$

Therefore, besides the Eqs.(9) and (10), we need to provide extra information, including

$$\begin{aligned} u_{i,3}^{(1)} &= \frac{1}{6h}(-11u_i + 18u_{i+1} - 9u_{i+2} + 2u_{i+3}), \\ u_{i,3}^{(2)} &= \frac{1}{h^2}(2u_i - 5u_{i+1} + 4u_{i+2} - u_{i+3}), \\ u_{i,3}^{(3)} &= \frac{1}{h^3}(-u_i + 3u_{i+1} - 3u_{i+2} + u_{i+3}). \end{aligned} \quad (\text{A.2})$$

It should be noted that $u_{i,0}^{(3)}$, $u_{i,1}^{(3)}$ and $u_{i,2}^{(3)}$ are all zero.

Then the high-order global smoothness indicator can be defined based on the local smoothness indicators, resulting

$$\tau_6 = |\beta_6 - \frac{1}{6}(\beta_0 + 4\beta_1 + \beta_2)|, \quad (\text{A.3})$$

where

$$\beta_6 = \left(hu_{i,6}^{(1)}\right)^2 + \left(h^2u_{i,6}^{(2)}\right)^2 + \left(h^3u_{i,6}^{(3)}\right)^2 + \left(h^4u_{i,6}^{(4)}\right)^2 + \left(h^5u_{i,6}^{(5)}\right)^2. \quad (\text{A.4})$$

Corresponding definitions used for this high-order smoothness indicator are given as

$$\begin{aligned} u_{i,6}^{(1)} &= \frac{1}{60h}(3u_{i-2} - 30u_{i-1} - 20u_i + 60u_{i+1} - 15u_{i+2} + 2u_{i+3}), \\ u_{i,6}^{(2)} &= \frac{1}{12h^2}(-u_{i-2} + 16u_{i-1} - 30u_i + 16u_{i+1} - u_{i+2}), \\ u_{i,6}^{(3)} &= \frac{1}{4h^3}(-u_{i-2} - u_{i-1} + 10u_i - 14u_{i+1} + 7u_{i+2} - u_{i+3}), \\ u_{i,6}^{(4)} &= \frac{1}{h^4}(u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}), \\ u_{i,6}^{(5)} &= \frac{1}{h^5}(-u_{i-2} + 5u_{i-1} - 10u_i + 10u_{i+1} - 5u_{i+2} + u_{i+3}). \end{aligned} \quad (\text{A.5})$$

With using the present smoothness indicators, the formulas shown in section 3, including Eqs.(12), (13) and (14), can be calculated. It should be noted that the original τ_5 shall be replaced by τ_6 . The convex combination can thus be given as

$$u_{L,i+\frac{1}{2}} = \sum_{k=0}^3 \omega_k u_{L,i+\frac{1}{2},k}, \quad (\text{A.6})$$

and

$$\omega_k = \frac{d_k \delta_k}{\sum_{k=0}^3 d_k \delta_k}, \quad (\text{A.7})$$

where

$$\begin{aligned} u_{L,i+\frac{1}{2},0} &= P_5, \\ u_{L,i+\frac{1}{2},1} &= P_4, \\ u_{L,i+\frac{1}{2},2} &= P_3, \\ u_{L,i+\frac{1}{2},3} &= \frac{1}{16}(5u_i + 15u_{i+1} - 5u_{i+2} + u_{i+3}). \end{aligned} \quad (\text{A.8})$$

and

$$d_0 = \frac{1}{32}, d_1 = \frac{15}{32}, \quad d_2 = \frac{5}{16}, \quad d_3 = \frac{3}{16}. \quad (\text{A.9})$$

In Eq.(18), the six-point interpolation is optimized to balance the numerical dissipation and dispersion. Here, this optimized interpolation is directly incorporated into the nonlinear interpolation procedure,

345 resulting

$$u_{L,i+\frac{1}{2}} = \begin{cases} P_{6,L}, & \text{if } \delta_k = 1, \quad k = 0, \dots, 3, \\ \sum_{k=0}^3 \omega_k u_{L,i+\frac{1}{2},k}, & \text{otherwise.} \end{cases} \quad (\text{A.10})$$

Finally, according to Ref.[31], part of the parameters of the adaptive procedure introduced in section 3.2 need to be redefined for the present six-point scheme. Thereinto, $\eta_{i+\frac{1}{2}}^{(6)} = \min(\eta_{i-1}, \eta_i, \eta_{i+1}, \eta_{i+2})$, $C_r^{(6)} = 0.17$, $\alpha_1^{(6)} = 10.5$ and $\alpha_2^{(6)} = 4.5$. The other parameters are directly reused without further modifications and discussion.

350 References

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