Equilibrium Asset Returns in Financial Markets

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Abstract

Return distributions in the class of pure jump limit laws are observed to reflect numerous asymmetries between the upward and downward motions of asset prices. The return distributions are modeled by self decomposable parametric laws with all parameters continuously responding to each other. Fixed points of the response functions define equilibrium distributions. The equilibrium distributions that can arise in practice are constrained by the level of return acceptability they may attain. As a consequence expected returns are the cost of purchasing the the negative of the centered return. They are then equal to the ask price in two price economies, for the centered and negated return. The asymmetries studied include differences in scale, speed, power variation, excitation and cross excitation.

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JEL Classification: G11, G12, G13.

1 Introduction

The classical perspective on asset returns recognizes that the returns are risky and seeks to explain the expected return or the mean of the return distribution as compensation for the inherent risk. Centered return distributions are purely composed of risk, with the latter being typically measured by the variance. Portfolio considerations are seen as assisting variance reduction via diversification with mean returns reflecting the compensation for systematic or undiversifiable risk as represented by the return covariance with systematic factors. The return to the systematic factors constitutes their risk premium. These are the central

themes of the capital asset pricing models of Sharpe (1964), Lintner (1965), Mossin (1966) and the subsequent development of the Ross (1976) arbitrage pricing theory. They are effectively summarized in numerous text books on the subject including Cochrane (2001), Duffie (2001), and Skiadas (2009).

But why are returns risky? Clearly, they need not be if the return expected was just the interest rate or the so called risk free rate. Expected returns that differ significantly from this risk free rate on either side will attract investors and or speculators seeking to exploit any momentum or to benefit from an anticipated mean reversion. The result of these activities will create the risk and the resulting variance. Hence not only are expected returns responding to the risk, however measured, but the risk is also responding to the expected return. Studies in this direction include Schwert (1989) and Engle, Ghysels and Sohn (2008). These interactions must result in some stable, acceptable, temporally equilibrium return distributions. In this regard, we concur with the view expressed in Weitzman (2007) of an essentially nonergodic situation with no set of true parameters for the return distributions to converge to. Whatever equilibria one effectively solves for are at best transitory.

In principle such equilibrium return distributions could be studied at the level of multivariate distributions or return generating processes. Similar assets or stocks of companies in comparable lines of business may not have independent return distributions and the joint distribution would be expected to reflect some correlation driven by market participants looking at them in tandem. Unusual movements in one may trigger corrective actions in another. The longer the return horizon the greater are the possibilities for developing the correlations. This phenomenon known as the Epps effect has been known since Epps (1979) and is further documented in Bonanno, Lillo and Mantegna (2001). Over short horizons correlations and portfolio considerations are less relevant. Most traders holding positions for a short span of time are not evaluating or exploiting cross asset return dependencies or diversification benefits, but rather seeking out short lived mispricings one asset and one moment at a time. The variance is being monitored by the short span of time for which positions are being held.

Attention is therefore focused here, not on equilibrium mean returns given risk, but on equilibrium levels for the entire return distribution taken one asset at a time for the shorter horizons. For the physical return distribution the object of study is the daily return distribution. Risk neutrally we consider mainly the maturities below a month. In both cases a parametric class of distributional possibilities is first adopted as the default. Within the chosen parametric class the equilibrium return distribution is identified on determining equilibrium levels for the parameters. For this purpose a one to one mapping between certain moments and the parameters is employed to infer equilibrium parameters from corresponding equilibrium levels for the selected moments. The moments are seen as responding to each other and the equilibrium is given by the fixed point for this response mapping. Such an equilibrium construction was made in Madan and Schoutens (2018) and it is here generalized and extended to a wider class of processes underlying the daily returns. In this way one may construct equilibrium daily return distributions for each asset each day under the physical

process for returns. Risk neutrally one seeks to construct equilibrium volatility surfaces each day for each underlying asset on which options are traded.

Ultimately one develops a time series of estimated parameters for the return distribution in the default parametric class and a time series of the estimated equilibrium parameters. Lead lag relationships between these may be investigated to identify regions of parametric momentum when the equilibrium levels lag the estimated levels and the regions of parametric mean reversion when the estimated levels lag the equilibrium ones. It may be possible for both phenomena to occur simultaneously for different parameters. Hence one may have mean reversion in drifts with momentum for volatility or the other way around. A study of these issues is a topic for future research. The objective here is to first identify the equilibrium levels.

Critical to such an exercise are the default parametric class of distributions. Clearly the class must be capable of fitting the data but yet not be so wide as to permit over fitting. Structural restrictions help reduce parameters and avoid over fitting while some parametric richness helps in differentiating possibilities away from imposing too many symmetries that may not be present. In this regard we rely on our experience in fitting return distributions both risk neutrally and in the time series data permitting the parametric richness being sought and on structural theoretical restrictions limiting overfitting.

The theoretical restriction is to employ a limit law for the distribution, on the grounds that the number of price moves involved in unit time, though finite, is quite large. The limit laws arise at infinitely many moves and have been characterized as the self decomposable laws by Lévy (1937) and Khintchine (1938). They are a subclass of the infinitely divisible distributions with a further restriction on the associated Lévy density. Arrival rates of jumps when scaled by the absolute jump size must be decreasing functions of the absolute jump size. The self decomposable laws also have provably unimodal distributions (Sato (1999)) and thereby refrain from overfitting a multiple of modes.

There are asymmetries between the way market prices rise and how they fall. It is often said that markets take the escalator up and the elevator down. We cite two references from the web. Such references may be found on the web and we cite two examples <http://allstarcharts.com/escalator-up-and-elevator-down/>, and <http://seekingalpha.com/article/3746396-market-stars-ride-escalator-elevator>. Recently these differences were explored and documented in Madan and Wang (2017), and Madan, Schoutens and Wang (2017). Here they are further expanded upon. We refrain from employing a continuous component in the price process or its logarithm. These processes impose a symmetry between the upward and downward motion and do not permit a split between the two. They are also processes of infinite variation and do not allow for the two processes of upward and downward motion to have an independent existence. Already the use of limit laws stretches reality by allowing for infinitely many moves. Infinite variation is yet another stretch away from reality. Processes in continuous time are employed for their analytical convenience recognizing that as a description of reality they are not really admissible. Physical time, eventually marked by precise time stamps, is at best discrete. Hence we work primarily with pure jump

processes of finite variation and shall document the differences to be found physically and risk neutrally in the separate processes for the upward and downward motion.

A particularly simple pure jump process with a self decomposable law at unit time is given by the variance gamma model of Madan and Seneta (1990) and Madan, Carr and Chang (1998). The jump arrival rate function when scaled by the absolute jump size is just a negative exponential of the absolute jump size, a clearly decreasing positive function. It permits a separation of the upward and downward motion, with both processes being gamma processes with separate scale parameters and the same shape or speed parameters. Madan and Wang (2017), Madan, Schoutens and Wang (2017) document that identical speeds on the two sides may be overly restrictive and they employ the bilateral gamma process of Küchler and Tappe (2008). This is a difference of two gamma processes with their own speed and scale parameters. It is observed that the upward motion has a higher speed and lower scale parameter when compared to the same for the downward motion.

The first parametric class employed in the construction of equilibrium return distributions is the bilateral gamma. However, we go on to address other asymmetries beyond scale and speed. Beyond speed and scale we go on to consider asymmetries in power variation. This is the smallest power α such that the α power variation is finite. Given that we work with semimartingales with finite quadratic variation the smallest power with a finite power variation is less than 2. For a process of independent and identically distributed increments three asymmetries are documented, scale, speed and the power variation.

One may go further in the context of a Markov process to document how the parameters of local motion respond to what has occurred to date. For diffusions this takes us to local volatility models when the volatility is made to depend on the level of the process and or calendar time. In the interests of parsimony and with a focus on local dynamics we suppress any dependence on calendar time and work with time homogeneous Markov processes. This would suggest making all of speed, scale and power variation functions of level of the process. In the interests of pursuing and documenting asymmetries we consider functions of not the price level or process level, but the aggregated positive and negative moves so far. The dependence could be in all parameters and nonlinear at that. The resulting structure is not only analytically intractable but also numerically intractable. However we may incorporate as numerically tractable the linear dependence of speed on the aggregated positive and negative moves. The infinitesimal generator is then linear in state variables permitting an exponential affine solution for the characteristic function (Duffie, Filipovic and Schachermayer (2003)) that leads to ordinary differential equations for the coefficients. The solution for the characteristic function then requires the numerical solution of thousands of ordinary differential equations to get the characteristic function. Fourier inversion then delivers the density and or the option prices. Calibration is possible in a matter of a few hours on current computing capacities. This linear dependence of speed separately on the aggregated positive and negative moves we term excitation and cross excitation depending on whether we are

referring to the speed of positive moves responding to the aggregated positive and negative moves respectively. Similarly for the speed of the negative moves. The five asymmetries studied and reported on in the paper will then be scale, speed, power variation, excitation and cross excitation. For excitation and cross excitation we report the results of estimations conducted but equilibrium constructions are not possible given the computational costs of single estimations.

Estimations are conducted both for the physical daily return distribution and risk neutrally at the shorter maturities. In both cases we identify the structure of jump arrival rates. The arrival rates may risk neutrally be lifted by running time faster and they may also be altered by changing the scale. In continuous time a measure change may change scale but not speed, but as the continuous time limit is not the reality but just a model, we permit changes in speed as well. Beyond speed and scale we consider the power variation. Again this parameter may not be changed by an equivalent change of probability in continuous time as it is embedded in the paths and may be recovered from them. However, we entertain such changes as well.

Given the classically special attention focused on explaining mean returns we turn to explaining the equilibrium mean returns of our equilibrium distributions. Given the absence of portfolio considerations at the level of daily returns the mean returns are explained by the implications for avoiding excessively good deals. The level of a deal is measured by the acceptability index of the equilibrium return distribution as constructed in Cherny and Madan (2009). Mean returns may then be related to the risk of a return distribution as measured by the ask price for the centered and negated return. Asset pricing implications along these lines are formulated and tested.

The outline of the rest of the paper is as follows. Section 2 introduces models for the five asymmetries being reported on and the details for the characteristic function constructions that are employed in estimating daily return distributions and calibrating volatility surfaces. Section 3 reports on the first three asymmetries of scale, speed and power variation in the physical process. The risk neutral counterparts are taken up in Section 4. For both constructions equilibrium return distribution and volatility surfaces are reported on in Section 5 and 6 respectively. Section 7 takes up asset pricing implications and their tests. Sections 8 reports on the physical and risk neutral estimation of excitation and cross excitation respectively. Section 9 concludes.

2 Five asymmetries of motion

The first two asymmetries of motion are those of speed and scale and these are captured in the bilateral gamma process. Let $\gamma(t)$, $t > 0$ be a path of the standard gamma process with unit scale and speed. For fixed t , the random variable $G_t = \gamma(t)$ has a gamma density $f(g)$ given by

$$f(g) = \frac{g^{t-1} e^{-g}}{\Gamma(t)}.$$

The gamma process is an infinitely divisible pure jump, increasing process with arrival rates of jumps of size x given by the arrival rate function $k_\gamma(x)$ where

$$k_\gamma(x) = \frac{e^{-x}}{x}, \quad x > 0.$$

The aggregate arrival rate is infinite and the process is one of infinite activity with infinity many arrivals in any time interval. The characteristic function $\phi_G(u)$ is given by

$$\begin{aligned} \phi_{Gt}(u) &= E[e^{iuG_t}] \\ &= \left(\frac{1}{1 - iu} \right)^t. \end{aligned}$$

The bilateral gamma process $X(t)$, $t > 0$ has four parameters b_p , c_p , b_n , and c_n that are the scale and speed parameters for the upward and downward motion respectively. For two independent standard gamma processes $\gamma_p(t)$, $\gamma_n(t)$, $t > 0$ on has

$$X(t) = b_p \gamma_p(c_p t) - b_n \gamma_n(c_n t).$$

The characteristic function for the bilateral gamma is given by

$$\begin{aligned} \phi_X(u, t) &= E[\exp(iuX(t))] \\ &= \left(\frac{1}{1 - iu b_p} \right)^{c_p t} \left(\frac{1}{1 + iu b_n} \right)^{c_n t}. \end{aligned}$$

The arrival rate of jumps $x \neq 0$ is given by $k_X(x)$ where

$$k_X(x) = c_p \frac{\exp\left(-\frac{x}{b_p}\right)}{x} \mathbf{1}_{x>0} + c_n \frac{\exp\left(-\frac{|x|}{b_n}\right)}{|x|} \mathbf{1}_{x<0}.$$

In the special case of identical speeds up and down with $c_p = c_n = 1/\nu$ one gets the variance gamma process of Madan and Seneta (1990), Madan, Carr and Chang (1998) that may be represented as Brownian motion with drift θ and volatility σ time changed by a gamma process with unit mean rate and variance rate ν . In this case

$$\begin{aligned} \theta &= \frac{b_p - b_n}{\nu} \\ \sigma^2 &= \frac{2b_p b_n}{\nu}. \end{aligned}$$

The variance gamma (VG) has an arrival rate function described in Carr, Geman, Madan and Yor (2002) as

$$k_{VG}(x) = C \frac{\exp(-Mx)}{x} \mathbf{1}_{x>0} + c_n \frac{\exp(-G|x|)}{|x|} \mathbf{1}_{x<0}$$

with $C = 1/\nu$, $G = 1/b_n$, and $M = 1/b_p$. The variance gamma has a density $f_{VG}(x)$ in closed form given by (See Carr and Madan (2014))

$$f_{VG}(x) = \frac{(GM)^C}{2^{C-1}\Gamma(C)\sqrt{2\pi}\left(\frac{G+M}{2}\right)^{C-1/2}} \exp\left(\frac{G-M}{2}x\right) |x|^{C-1/2} K_{C-1/2}\left(\frac{G+M}{2}|x|\right),$$

where $K_\nu(x)$ is the modified Bessel functions. The variance gamma may be first estimated to obtain starting values for estimating the bilateral gamma by Fourier inversion of the characteristic function. Madan and Wang (2017) and Madan, Schoutens and Wang (2017) have reported that the upward process has higher speed and lower scale when compared to the downward process.

2.1 The asymmetry of power variation

The CGMY model of Carr, Geman, Madan and Yor (2002) permits a differentiation of power variation for the process of upward moves from downward moves if the parameters C, Y are split to allow for separate values on the two sides. We denote this model PV for power variation. The arrival rate function $k_{PV}(x)$ is given by

$$k_{PV}(x) = c_p \frac{\exp(-x/b_p)}{x^{1+y_p}} \mathbf{1}_{x>0} + c_n \frac{\exp(-|x|/b_n)}{|x|^{1+y_n}} \mathbf{1}_{x<0}.$$

The additional parameters beyond the bilateral gamma are y_p, y_n and the α power variation on the positive side given by

$$\int_0^\infty x^\alpha k_{PV}(x) dx < \infty$$

for all $\alpha > y_p$. Similarly for the negative side for $\alpha > y_n$ the α power variation is finite. The characteristic function is given by

$$\phi_{PV}(u) = \frac{c_p \Gamma(-y_p)}{b_p^{y_p}} ((1 - i u b_p)^{y_p} - 1) + \frac{c_n \Gamma(-y_n)}{b_n^{y_n}} ((1 + i u b_n)^{y_n} - 1).$$

Again we first estimate the variance gamma and use the variance gamma estimates as starting values for the power variation model that is estimated by Fourier inversion of the characteristic function.

For the physical process the drift if any is constructed from the asymmetry in the jump arrival rates and hence the process is one finite variation on both sides with $y_p, y_n < 1$. Risk neutrally one adds drifts determined by the forward price to compensated jump martingales and so one may allow for infinite variation and constructed the two compensated jump martingales for the upward compensated jumps and the downward compensated jumps. We have already commented in the introduction on the possibilities for changing all parameters between the physical and risk neutral estimates.

2.2 The asymmetries of excitation and cross excitation

Let $X_p(t)$, $X_n(t)$, $t > 0$ be the aggregated upward and downward moves to time t . More general asymmetries may be modeled by making all the parameters of motion functions of these aggregated entities. However, one may lose tractability for the characteristic function either analytically or numerically in such constructions. Madan (2016) employs continuous time finite state Markov chain approximations to access probability elements in such one dimensional formulations. However, here the Markov dimension is two and such approximations would be more involved. If the infinitesimal generator is kept linear in state variables then the characteristic function is known to be in exponential affine class and some tractability may be possible.

With these considerations in mind consider the two dimensional process $X(t) = (X_p(t), X_n(t)) \in \mathbb{R}_+^2$ with jumps x_p, x_n and compensator

$$\begin{aligned} k(X_p, X_n, x_p, x_n) &= (c_p + d_p X_p + e_p X_n) \frac{\exp\left(-\frac{x_p}{b_p}\right)}{x_p} \mathbf{1}_{x_p > 0, x_n = 0} \\ &\quad + (c_n + d_n X_n + e_n X_p) \frac{\exp\left(-\frac{x_n}{b_n}\right)}{x_n} \mathbf{1}_{x_n > 0, x_p = 0} \end{aligned}$$

The speed of upward and downward motion is now responsive differentially to the accumulated upward and downward motion to date. Such a model incorporates both excitation and cross excitation for both processes. The coefficients d_p, d_n calibrate excitation while e_p, e_n calibrate the cross excitation.

For the characteristic function we wish to evaluate the exponential affine form

$$\begin{aligned} &E_t [\exp(iuX_p(1) - iuX_n(1))] \\ &= \phi(t, u) \\ &= \exp(a(1-t, u) + b(1-t, u)X_p(t) - c(1-t, u)X_n(t)) \end{aligned}$$

The infinitesimal generator is

$$\mathcal{A}f(X_p, X_n) = \int_0^\infty \int_0^\infty [f(X_p + x_p, X_n + x_n) - f(X_p, X_n)] k(X_p, X_n, x_p, x_n) dx_p dx_n.$$

The martingale property for $\phi(t, u)$ implies that

$$\phi_t + \mathcal{A}\phi = 0$$

Upon evaluation we get

$$\begin{aligned} &-a_t - b_t X_p + c_t X_n + \\ &-(c_p + d_p X_p + e_p X_n) \ln(1 - bb_p) \\ &-(c_n + d_n X_n + e_n X_p) \ln(1 + cb_n) \\ &= 0. \end{aligned}$$

The equations for the coefficients of the chacteristic function are

$$\begin{aligned} b_t &= -d_p \ln(1 - bb_p) - e_n \ln(1 + cb_n) \\ c_t &= d_n \ln(1 + cb_n) + e_p \ln(1 - bb_p) \\ a_t &= -c_p \ln(1 - bb_p) - c_n \ln(1 + cb_n) \end{aligned}$$

Define

$$\begin{aligned} y &= bb_p \\ w &= cb_n \end{aligned}$$

to get

$$\begin{aligned} y' &= -b_p d_p \ln(1 - y) - b_p e_n \ln(1 + w) \\ w' &= b_n e_p \ln(1 - y) + b_n d_n \ln(1 + w) \end{aligned}$$

This system of nonlinear ordinary differential equations may be solved for y, w subject to the initial conditions

We want to solve for initial conditions

$$\begin{aligned} y(0, u) &= iu b_p \\ w(0, u) &= iu b_n. \end{aligned}$$

The function $a(1 - t, u)$ follows on numerical integration. The characteristic function may be employed to obtain the density by Fourier inversion.

2.3 Risk Neutral Considerations

For the risk neutral calibration of models to data on option prices one needs to formulate the characteristic function for the risk neutral martingale model that is shifted to match the forward price. The martingale process associated with the jump process $X(t)$, with a finite variation arrival rate function $k_X(x, t)$, is the compensated jump process $M_X(t)$ where

$$M_X(t) = \exp \left(X(t) - \int_0^t \int_{-\infty}^{\infty} (e^x - 1) k_X(x, s) dx ds \right).$$

For processes with independent increments $X(t)$ the arrival rate function is a deterministic function of the jump size and time and the characteristic function of $M_X(t)$

$$\begin{aligned} \phi_{M_X}(u, t) &= E[\exp(iu M_X(t))] \\ &= \phi_X(u, t) \exp(iu \omega(t)) \\ \omega(t) &= -\ln(\phi_X(-i, t)). \end{aligned} \tag{1}$$

The construction in equation (1) also suffices for processes of infinite variation but the martingale construction takes the form

$$M_X(t) = \exp \left(\int_0^t b(u) du + X(t) - \int_0^t \int_{-\infty}^{\infty} (e^x - 1 - x \mathbf{1}_{|x| \leq 1}) k_X(x, s) dx ds \right)$$

for a drift function $b(t)$.

The bilateral gamma and power variation processes we work with have self decomposable laws at unit time. They are therefore infinitely divisible and have associated Lévy processes. However, recognizing that Lévy processes do not fit option surfaces across maturity we employ the associated Sato processes as introduced in Carr, Geman, Madan and Yor (2007). The law at time t is obtained by scaling the law at unit time and defining

$$X(t) \stackrel{(d)}{=} t^\gamma X(1)$$

for a scaling coefficient γ . The number of parameters for the risk neutral process is thereby increased by one to allow for the scaling coefficient. Sato (1991) shows the existence of an additive process with independent but time inhomogeneous increments that have the marginal distributions at each t consistent with the supposed scaling. The risk neutral processes are such scaled self decomposable processes. The risk neutral stock price process then takes the form

$$S(t) = F(t) \exp(X(t) + \omega(t))$$

where $F(t)$ is the forward price for maturity t .

These characteristic function constructions suffice for the bilateral gamma and power variation additive risk neutral processes. When we extend to the asymmetries of excitation and cross excitation the drift corrections must be introduced locally in the infinitesimal generator and they lead to changes in the ordinary differential equations that must be solved for the coefficients of the exponential affine form of the characteristic function.

Consider the two dimensional process $X(t) = (X_p(t), X_n(t)) \in \mathbb{R}_+^2$ with jumps x_p, x_n and compensator

$$\begin{aligned} k(X_p, X_n, x_p, x_n) &= (c_p + d_p X_p + e_p X_n) \frac{\exp\left(-\frac{x_p}{b_p}\right)}{x_p} \mathbf{1}_{x_p > 0, x_n = 0} \\ &\quad + (c_n + d_n X_n + e_n X_p) \frac{\exp\left(-\frac{x_n}{b_n}\right)}{x_n} \mathbf{1}_{x_n > 0, x_p = 0} \end{aligned}$$

For the risk neutral process we want the exponential of

$$X_p(t) - X_n(t)$$

to be a martingale and hence we need to incorporate an exponential compensation. This leads us to a drift compensation of the form

$$-g_p(X_p(t), X_n(t)) - g_n(X_p, X_n) dt$$

where

$$\begin{aligned}
g_p(X_p, X_n) &= (c_p + d_p X_p + e_p X_n) \int_0^\infty (e^{x_p} - 1) \frac{\exp\left(-\frac{x_p}{b_p}\right)}{x_p} dx_p \\
g_n(X_p, X_n) &= (c_n + d_n X_n + e_n X_p) \int_0^\infty (e^{-x_n} - 1) \frac{\exp\left(-\frac{x_n}{b_n}\right)}{x_n} dx_n \\
g_p(X_p, X_n) &= -(c_p + d_p X_p + e_p X_n) \ln(1 - b_p) \\
g_n(X_p, X_n) &= -(c_n + d_n X_n + e_n X_p) \ln(1 + b_n)
\end{aligned}$$

The infinitesimal generator of the 2 dimensional Markov process under consideration is then

$$\begin{aligned}
\mathcal{A}f(X_p, X_n) &= -g_p(X_p, X_n)f_{X_p} - g_n(X_p, X_n)f_{X_n} \\
&\quad + \int_0^\infty \int_0^\infty [f(X_p + x_p, X_n + x_n) - f(X_p, X_n)] k(X_p, X_n, x_p, x_n) dx_p dx_n. \\
g_p(X_p, X_n) &= -(c_p + d_p X_p + e_p X_n) \ln(1 - b_p) \\
g_n(X_p, X_n) &= -(c_n + d_n X_n + e_n X_p) \ln(1 + b_n)
\end{aligned}$$

We take the characteristic function in exponential affine form to be

$$\begin{aligned}
\phi^T(t, u) &= E_t [\exp(iuX_p(T) - iuX_n(T))] \\
&= \exp(a(T - t, u) + b(T - t, u)X_p(t) - c(T - t, u)X_n(t)).
\end{aligned}$$

The martingale condition for the characteristic function then yields the equation

$$\phi_t^T + \mathcal{A}\phi^T = 0.$$

Specifically we have

$$\begin{aligned}
&-a_t - b_t X_p + c_t X_n + \\
&-(c_p + d_p X_p + e_p X_n) \ln(1 - b_p) \\
&-(c_n + d_n X_n + e_n X_p) \ln(1 + b_n) \\
&-bg_p(X_p, X_n) - cg_n(X_p, X_n) \\
&= 0
\end{aligned}$$

Equivalently we write

$$\begin{aligned}
&-a_t - b_t X_p + c_t X_n + \\
&-(c_p + d_p X_p + e_p X_n) \ln(1 - b_p) \\
&-(c_n + d_n X_n + e_n X_p) \ln(1 + b_n) \\
&+b(c_p + d_p X_p + e_p X_n) \ln(1 - b_p) \\
&+c(c_n + d_n X_n + e_n X_p) \ln(1 + b_n) \\
&= 0
\end{aligned}$$

Hence the required differential equations are

$$\begin{aligned} b_t &= -d_p \ln(1 - bb_p) - e_n \ln(1 + cb_n) + bd_p \ln(1 - b_p) + ce_n \ln(1 + b_n) \\ c_t &= d_n \ln(1 + cb_n) + e_p \ln(1 - bb_p) - be_p \ln(1 - b_p) - cd_n \ln(1 + b_n) \\ a_t &= -c_p \ln(1 - bb_p) - c_n \ln(1 + cb_n) + bc_p \ln(1 - b_p) + cc_n \ln(1 + b_n) \end{aligned}$$

Let

$$\begin{aligned} y &= bb_p \\ w &= cb_n \end{aligned}$$

and write

$$\begin{aligned} y' &= -b_p d_p \ln(1 - y) - b_p e_n \ln(1 + w) + y d_p \ln(1 - b_p) + w \frac{b_p e_n}{b_n} \ln(1 + b_n) \\ w' &= b_n e_p \ln(1 - y) + b_n d_n \ln(1 + w) - y \frac{b_n e_p}{b_p} \ln(1 - b_p) - w d_n \ln(1 + b_n) \end{aligned}$$

The ordinary differential equations in y, w are solved numerically for initial conditions

$$\begin{aligned} y(0, u) &= iu b_p \\ w(0, u) &= iu b_n. \end{aligned}$$

The solution for a follows on numerical integration and hence we have the risk neutral characteristic function with the asymmetries of excitation and cross excitation.

3 Scale, Speed, and Power Variation, Physically

We report on estimates for the models for data on the top 50 underliers by option liquidity for 51 days taken 10 days apart. The ticker symbols for 50 underliers are presented in the Appendix. Two subsections present the results for the bilateral gamma model and the power variation model.

3.1 Bilateral Gamma Results

Figure 1 presents the quantiles for the scale parameter estimates for the upward and downward motion for the 2550 observations across names and days.

We observe that quite consistently the scale up is below the scale down. The mean difference is -480.3 basis points and the t -statistic is -3.58 .

Figure 2 presents the quantiles for the speed parameter up and down in the case of the bilateral gamma model.

The speed up is consistently higher than the speed down. The mean difference is 1.5280 with a t -statistic of 5.40 . These bilateral gamma results are consistent with the observations in Madan and Wang (2017) and Madan, Schoutens and Wang (2017).

Bilateral Gamma Scale		
Parameter Quantiles		
Percentile	Scale up	Scale Down
1	8.0695	15.2352
5	23.1531	29.9321
10	30.6770	38.6222
25	46.6507	56.9805
50	69.2700	82.8834
75	98.8349	108.5733
90	144.8889	145.8174
95	183.6948	181.5046
99	281.1554	248.5961

Figure 1: Percentiles for the scale parameters up and down in the bilateral gamma model estimated across 50 names on 51 days.

Bilateral Gamma Speed		
Parameter Quantiles		
Percentile	Speed up	Speed Down
1	0.8164	0.5705
5	1.0020	0.7533
10	1.3981	1.0563
25	1.7994	1.3983
50	2.5716	2.0891
75	5.6592	4.0452
90	10.2364	7.6088
95	106.1118	70.5060
99	281.1554	248.5961

Figure 2: Percentiles of speed parameters for the bilateral gamma model.

Figure 3: Median scale parameters up and down over days and across names. Blue circles are for the upward motion while red circles are for the downward motion.

Figure 4: Median scale parameters up and down over names and across days. Blue circles are for the upward motion while red circles are for the downward motion.

Figure 3 presents the median scale parameters over days and across the 50 names. Generally the scale for the upward process is smaller than for the downward process.

Figure 4 presents the median scale parameters over names and across the 51 days. The median upward scale is uniformly lower than the median downward scale.

The corresponding Figures for the speed parameters are in Figures 5 and 6. The upward speed is higher than its downward counterpart.

3.2 Power Variation Results

The results for the scale and speed in the power variation model are similar to those for the bilateral gamma. The aggregate percentiles are presented in Figures 7 and 8.

Figure 5: Median speed parameters up and down over days and across names. Blue circles are for the upward motion while red circles are for the downward motion.

Figure 6: Median speed parameters up and down over days and across names. Blue circles are for the upward motion while red circles are for the downward motion.

Power Variation Scale		
Parameter Quantiles		
Percentile	Scale up	Scale Down
1	6.0255	10.4475
5	24.5602	29.4373
10	30.8722	38.1046
25	45.9535	59.1480
50	66.5345	86.3381
75	96.6792	115.9010
90	140.5073	182.6168
95	185.0352	254.1800
99	282.0109	421.9139

Figure 7: Percentiles for the scale parameters in the estimated power variation model.

Power Variation Speed		
Parameter Quantiles		
Percentile	Speed up	Speed down
1	0.0042	0.0025
5	0.0685	0.0078
10	0.8953	0.0306
25	1.2052	0.8329
50	1.5977	1.1744
75	2.2732	1.7725
90	3.6992	2.8109
95	5.1864	4.3327
99	12.7171	11.6011

Figure 8: Percentiles for the speed parameters in the estimated power variation model.

Power Variation Parameter		
Parameter Quantiles		
Percentile	PV up	PV down
1	0.0000	0.0000
5	0.0022	0.0009
10	0.0061	0.0023
25	0.0153	0.0080
50	0.0273	0.0215
75	0.0366	0.0365
90	0.1112	0.6150
95	0.4613	0.7876
99	0.9405	0.9013

Figure 9: Percentiles for the power variation parameters as estimated in the power variation model.

Additionally we have the levels for the power variation parameters. The percentiles for these parameters are presented in Figure 9.

At the lower percentiles the power variation parameter up is higher but this reverses at the higher percentiles. The mean is negative at -0.0419 with a t -statistic of -8.55 . On this basis the power variation for the downward moves is higher and more powers have an infinite variation.

4 Scale, Speed, and Power Variation, Risk Neutrally

Risk neutrally we first present the quartiles over the days for each of the 50 underliers for the Sato time scaling parameter, the bilateral gamma scale and speed parameters and the power variations as estimated in the power variation model. The estimations were conducted daily for a period of two years using the shorter maturities below a month. There are four Figures for time scaling 10, scale 11, speed 12 and power variation 13. The time scaling covers the bilateral gamma estimation and the power variation estimation. We observe similar results in the two estimations. The scale, speed and power variations quartiles are presented in six columns, three for the upward motion and three for the downward motion. We observe that the upward power variation is generally higher than the power variation for the downward process.

	Time Scaling Parameter Quartiles					
	Bilateral Gamma			Power Variation		
	Q1	Q2	Q3	Q1	Q2	Q3
spy	0.4909	0.5439	0.6097	0.4900	0.5445	0.6082
iwm	0.4616	0.4996	0.5524	0.4574	0.4994	0.5544
qqq	0.4797	0.5480	0.5897	0.4810	0.5489	0.5903
aapl	0.4381	0.5087	0.6189	0.4380	0.5081	0.6204
fb	0.4607	0.5436	0.6204	0.4604	0.5433	0.6220
nvda	0.4268	0.4882	0.5859	0.4271	0.4940	0.5937
baba	0.4270	0.4860	0.6043	0.4279	0.4869	0.6031
mu	0.4582	0.5126	0.5839	0.4605	0.5180	0.5857
gld	0.4242	0.4688	0.4934	0.4353	0.4957	1.1599
bac	0.4275	0.4801	0.5375	0.4226	0.4773	0.5345
msft	0.4321	0.4977	0.5817	0.4334	0.4948	0.5856
nflx	0.4636	0.4984	0.5463	0.4635	0.4987	0.5464
c	0.4230	0.4822	0.5332	0.4249	0.4802	0.5281
amzn	0.4635	0.5510	0.6099	0.4635	0.5515	0.6105
eem	0.4113	0.5016	0.5790	0.3832	0.4536	0.5245
tsla	0.4488	0.5097	0.5969	0.4487	0.5096	0.5974
intc	0.4159	0.4949	0.5631	0.4184	0.4945	0.5559
xop	0.4560	0.4922	0.5506	0.4541	0.4913	0.5506
ewz	0.4549	0.5049	0.5601	0.4508	0.5051	0.5579
jpm	0.4529	0.4945	0.5599	0.4521	0.4965	0.5586
wfc	0.4366	0.4862	0.5256	0.4349	0.4836	0.5246
dis	0.4374	0.4946	0.5726	0.4375	0.4904	0.5845
gs	0.4466	0.5020	0.5424	0.4466	0.5025	0.5426
xlfi	0.4130	0.4789	0.5389	0.4062	0.4823	0.5527
t	0.4347	0.4979	0.5701	0.4336	0.4945	0.5600
xle	0.4618	0.5027	0.5810	0.4605	0.5022	0.5833
rut	0.5614	0.8102	1.4880	0.4860	0.6265	0.8334
wmt	0.4343	0.4977	0.5459	0.4300	0.4969	0.5505
jd	0.3923	0.4556	0.5341	0.3847	0.4499	0.5247
ba	0.4567	0.4882	0.5706	0.4571	0.4891	0.5725
gm	0.4345	0.4884	0.5702	0.4342	0.4912	0.5665
v	0.4440	0.4992	0.5854	0.4318	0.5013	0.5870
bidu	0.4361	0.4842	0.5367	0.4351	0.4845	0.5370
xom	0.4698	0.5133	0.5683	0.4672	0.5155	0.5621
cmg	0.4369	0.4940	0.5639	0.4368	0.4954	0.5695
ndx	0.5168	0.5911	0.6323	0.5175	0.5871	0.6338
gdx	0.4365	0.4780	0.5460	0.4348	0.4736	0.5423
amd	0.1979	0.3694	0.5601	0.1680	0.3588	0.5455
dia	0.4921	0.5439	0.5986	0.4919	0.5464	0.5970
googl	0.4546	0.5290	0.6030	0.4547	0.5287	0.6018
gild	0.4071	0.4986	0.6202	0.4070	0.4990	0.6227
twtr	0.4206	0.4831	0.5335	0.4206	0.4810	0.5371
wynn	0.4560	0.5136	0.5485	0.4559	0.5152	0.5504
pypl	0.3243	0.4028	0.5313	0.3148	0.4100	0.5299
ge	0.4223	0.4920	0.5706	0.4189	0.4880	0.5668
csc	0.4742	0.5231	0.6053	0.4670	0.5160	0.6052
goog	0.4549	0.5137	0.6077	0.4544	0.5136	0.6057
xlu	0.4197	0.4723	0.5289	0.4165	0.4637	0.5233
hd	0.4546	0.5262	0.6142	0.4548	0.5264	0.6123
cmcsa	0.4536	0.4854	0.5828	0.4568	0.4847	0.5925

Figure 10: Time scaling parameter quartiles as estimated for the bilateral gamma and power variation models.

	Bilateral Gamma Scale Parameters					
	Scale up			Scale down		
	Q1	Q2	Q3	Q1	Q2	Q3
spy	0.0017	0.0065	0.0288	0.1488	0.2011	0.2813
iwm	0.0052	0.0146	0.0498	0.1489	0.1963	0.2526
qqq	0.0028	0.0158	0.0493	0.1408	0.1975	0.2720
aapl	0.0329	0.0507	0.1065	0.1627	0.2885	0.3561
fb	0.0238	0.0554	0.0973	0.1633	0.2671	0.3597
nvda	0.0488	0.1202	0.1826	0.1175	0.2077	0.5585
baba	0.0691	0.0948	0.1489	0.1210	0.1925	0.2729
mu	0.1668	0.2133	0.3047	0.2210	0.2851	0.4521
gld	0.0001	0.0545	0.0800	0.0000	0.0322	0.0704
bac	0.0740	0.1059	0.1247	0.1141	0.1429	0.1929
msft	0.0303	0.0647	0.0929	0.1046	0.1594	0.2390
nflx	0.0779	0.1112	0.1527	0.1249	0.1841	0.3481
c	0.0638	0.0802	0.1077	0.1202	0.1453	0.2387
amzn	0.0099	0.0468	0.0838	0.1399	0.2446	0.3564
eem	0.0636	0.0823	0.1119	0.1494	0.1852	0.2408
tsla	0.0791	0.0978	0.1517	0.2734	0.3572	0.4759
intc	0.0587	0.0800	0.1079	0.0975	0.1515	0.1995
xop	0.0963	0.1171	0.1346	0.1525	0.1808	0.2200
ewz	0.0831	0.1203	0.1417	0.1802	0.2274	0.3191
jpm	0.0190	0.0629	0.0753	0.1284	0.1592	0.2820
wfc	0.0502	0.0691	0.0811	0.1047	0.1322	0.1840
dis	0.0285	0.0689	0.0926	0.1115	0.1690	0.2863
gs	0.0177	0.0322	0.0591	0.1846	0.2550	0.3509
xlfi	0.0547	0.0774	0.1039	0.1084	0.1371	0.1969
t	0.0256	0.0604	0.0760	0.1029	0.1494	0.3760
xle	0.0535	0.0716	0.0841	0.1110	0.1380	0.1783
rut	0.0090	0.0348	0.0955	0.2871	0.5400	5.2516
wmt	0.0209	0.0596	0.0772	0.0960	0.1388	0.3078
jd	0.1085	0.1670	0.2055	0.1591	0.2128	0.3194
ba	0.0072	0.0174	0.0473	0.2261	0.3548	0.4986
gm	0.0692	0.0936	0.1289	0.1265	0.1672	0.2591
v	0.0097	0.0211	0.0718	0.1775	0.3184	0.4903
bidu	0.0385	0.1052	0.1337	0.1381	0.2058	0.3141
xom	0.0180	0.0412	0.0691	0.1048	0.1673	0.2236
cmg	0.0764	0.1086	0.1862	0.1314	0.2244	0.4886
ndx	0.0007	0.0010	0.0019	0.2017	0.2373	0.3174
gdx	0.0879	0.1271	0.1792	0.1041	0.1500	0.2085
amd	0.1714	0.2501	0.5041	0.1979	0.3536	0.6845
dia	0.0011	0.0029	0.0216	0.1588	0.2480	0.3163
googl	0.0139	0.0342	0.0632	0.1205	0.2140	0.3424
gild	0.0716	0.1074	0.1632	0.1336	0.1905	0.3369
twtr	0.2079	0.3186	0.6103	0.1406	0.2054	0.4232
wynn	0.0881	0.1219	0.1685	0.1588	0.2131	0.3226
pypl	0.0592	0.1024	0.1556	0.1370	0.2072	0.3654
ge	0.0274	0.0613	0.0876	0.1049	0.1551	0.2880
csc	0.0576	0.0768	0.1055	0.1016	0.1394	0.2463
goog	0.0170	0.0377	0.0712	0.1170	0.2034	0.3491
xlu	0.0253	0.0449	0.0537	0.0716	0.0880	0.1330
hd	0.0050	0.0161	0.0571	0.1623	0.2496	0.3360
cmcsa	0.0189	0.0562	0.0878	0.1498	0.3018	0.7078

Figure 11: Bilateral Gamma Scale Parameter Quartiles for the upward and downward process.

	Bilateral Gamma Speed					
	Speed Up			Speed Down		
	Q1	Q2	Q3	Q1	Q2	Q3
spy	1.1660	2.2727	7.1745	0.4268	1.0173	1.2310
iwm	1.4599	1.8770	3.9507	1.1184	1.4611	1.8974
qqq	1.1860	1.3058	1.5436	1.0687	1.2390	1.5079
aapl	2.2648	7.5997	16.2361	0.2176	0.5145	2.1450
fb	1.8907	2.9077	7.3533	0.8714	1.8255	2.4159
nvda	2.0732	2.8918	9.9283	1.0022	2.3425	3.3220
baba	2.4672	2.8869	5.9240	1.5934	2.6323	2.9270
mu	1.3664	1.7400	2.1573	1.2263	1.6985	2.1082
gld	1.3588	1.8983	2.1974	1.6293	1.9352	2.3642
bac	1.6298	2.0364	2.3493	1.4785	2.0030	2.3277
msft	1.6131	1.9174	2.5789	1.4326	1.8077	2.0271
nflx	2.2676	2.9638	9.1310	0.8608	2.2676	2.9663
c	1.6183	1.9113	2.2428	1.6012	1.8761	2.1479
amzn	3.3951	7.0184	16.4788	0.3797	2.3287	4.1081
eem	0.7638	1.4086	1.5996	0.6065	1.1842	1.5377
tsla	3.2779	6.0747	12.2221	0.5022	0.8655	2.2228
intc	1.3363	1.6692	2.3964	1.2656	1.6050	2.3131
xop	2.0181	2.3942	3.0960	2.0181	2.3942	3.0960
ewz	1.8418	2.2983	2.8260	1.7648	2.0732	2.6004
jpm	1.4878	1.8145	2.2953	0.9914	1.6269	1.9803
wfc	1.6093	2.0671	2.5764	1.4635	1.8702	2.2930
dis	1.5347	1.8210	2.5774	0.3916	1.5133	1.7813
gs	1.9904	12.4919	24.9679	0.3844	0.6878	1.7733
xlfi	0.7495	1.1514	1.4458	0.6660	1.1317	1.3737
t	1.2687	1.9073	10.4550	0.3377	1.1865	1.6894
xle	1.7543	2.0486	2.7159	1.6998	1.9934	2.6629
rut	1.2139	2.0814	2.9077	0.7672	1.4032	2.3554
wmt	1.7819	2.0735	2.5287	1.6036	1.8595	2.2759
jd	1.1635	1.8478	2.5828	1.0753	1.8063	2.4040
ba	1.8458	2.2346	20.9592	0.4044	1.6215	1.9286
gm	1.4046	1.8246	2.2373	1.3356	1.6553	2.0688
v	1.3530	1.6548	2.1458	0.9334	1.3783	1.6730
bidu	2.0034	2.5559	4.2235	1.4217	2.2390	2.7948
xom	1.5901	2.0127	4.4866	1.2429	1.6518	2.0782
cmg	1.8774	2.7128	5.1931	0.8100	2.0914	2.6684
ndx	0.5885	1.5433	15.0897	0.7174	1.2880	1.9268
gdx	2.7627	3.1249	3.6969	2.7626	3.1249	3.6969
amd	1.0208	1.5190	1.8392	0.7965	1.5189	1.8392
dia	1.1058	1.3158	2.6515	0.9729	1.1279	1.4110
googl	2.0420	3.2984	6.9212	0.9422	2.1821	3.4433
gild	1.4755	1.7794	2.3373	0.8481	1.5186	1.8138
twtr	0.4521	1.4651	2.1824	1.4460	2.1793	4.4879
wynn	2.3121	2.9177	4.6339	1.8009	2.4848	3.8277
pypl	1.0229	1.5737	2.8127	0.5957	1.1233	1.8150
ge	1.3315	1.6996	2.3770	0.7596	1.3603	1.7693
csc	1.2855	1.5975	2.3798	1.1466	1.5123	2.2079
goog	2.1679	3.7828	15.2800	0.5975	1.6820	2.8302
xlu	1.5067	1.8145	2.5142	1.4027	1.7102	2.3535
hd	1.8468	2.9383	7.1605	0.5714	1.7242	2.2165
cmcsa	1.2818	1.5909	7.1423	0.1924	1.1917	1.4963

Figure 12: Bilateral gamma quartiles for the speed parameter of the upward and downward motion.

			Power Variation			
		PV up		PV down		
	Q1	Q2	Q3	Q1	Q2	Q3
spy	0.3906	0.9652	1.6922	0.1384	0.2701	1.0182
iwm	0.2530	0.3246	1.6985	0.1090	0.1805	0.2221
qqq	0.2693	0.4107	1.6501	0.1443	0.2002	0.2379
aapl	0.2437	1.0729	1.2716	0.0674	0.1266	0.1871
fb	0.2422	0.3246	1.4393	0.1248	0.1533	0.2052
nvda	0.2110	0.2976	0.9281	0.1317	0.1781	0.2446
baba	0.1742	0.2382	0.3495	0.1149	0.1502	0.1995
mu	0.2022	0.2483	0.3551	0.1641	0.2001	0.2473
gld	0.0901	0.1576	0.2222	0.1302	0.1868	0.4061
bac	0.1276	0.1684	0.2015	0.1406	0.1794	0.2084
msft	0.1463	0.1943	0.3980	0.1152	0.1741	0.2073
nflx	0.1740	0.3148	0.7289	0.1366	0.1910	0.3065
c	0.1645	0.1976	0.2655	0.1155	0.1497	0.1942
amzn	0.3252	0.8468	1.7576	0.1491	0.2559	0.6570
eem	0.1726	0.2511	0.4582	0.1695	0.1948	0.4094
tsla	0.4088	0.6704	0.9637	0.1595	0.2189	0.2905
intc	0.1488	0.1907	0.2465	0.1278	0.1800	0.2065
xop	0.1183	0.1610	0.2029	0.1387	0.1615	0.1828
ewz	0.1451	0.2005	0.2595	0.1507	0.1818	0.2046
jpm	0.1740	0.2605	1.3566	0.0863	0.1449	0.1765
wfc	0.1353	0.1674	0.2569	0.1096	0.1552	0.1900
dis	0.2133	0.2775	0.9464	0.1018	0.1548	0.2303
gs	0.3895	1.3240	1.5566	0.1058	0.1577	0.2432
xlf	0.1868	0.2493	0.3897	0.1337	0.1943	0.2620
t	0.1588	0.2334	0.3798	0.0791	0.1356	0.2249
xle	0.1290	0.1693	0.2084	0.1282	0.1482	0.1791
rut	0.2901	0.8035	1.2458	0.2223	0.4443	1.5857
wmt	0.1371	0.1848	0.2905	0.0955	0.1674	0.2037
jd	0.1572	0.2445	0.3467	0.1308	0.1830	0.3225
ba	0.3838	1.6012	1.9198	0.0543	0.0936	0.1994
gm	0.1608	0.2202	0.3037	0.1203	0.1609	0.2141
v	0.2673	0.4021	1.7686	0.0701	0.1430	0.2169
bidu	0.2373	0.3172	1.0920	0.1317	0.1744	0.2532
xom	0.1331	0.2469	0.3464	0.0853	0.1448	0.1820
cmg	0.2797	0.4261	0.8315	0.1719	0.2877	0.4915
ndx	0.3616	0.9888	1.8956	0.5307	1.2168	1.4136
gdx	0.0931	0.1358	0.1650	0.0956	0.1282	0.1699
amd	0.2258	0.3337	0.6271	0.1757	0.3324	0.5078
dia	0.3195	1.1030	1.9890	0.1338	0.1953	0.2865
googl	0.2780	0.4416	1.3564	0.1666	0.2254	0.3148
gild	0.1927	0.2512	0.3446	0.1370	0.1789	0.2287
twtr	0.1413	0.2249	0.3287	0.2011	0.2787	0.5796
wynn	0.2066	0.2755	0.6147	0.1042	0.1521	0.2001
pypl	0.1913	0.2859	0.3817	0.1231	0.2120	0.2941
ge	0.1660	0.2330	0.3445	0.1255	0.1816	0.2482
csc	0.1421	0.2138	0.3094	0.1169	0.1872	0.2660
goog	0.3022	0.5580	1.5669	0.1384	0.2142	0.3009
xlu	0.0974	0.1534	0.1984	0.0701	0.1233	0.1807
hd	0.2701	1.1720	1.9824	0.0622	0.1414	0.2090
cmcsa	0.2460	0.3744	1.5107	0.0885	0.1490	0.2449

Figure 13: Quartiles for the power variation parameters for the upward and downward motion.

5 Equilibrium Return Distributions

Return distributions in any parametric class may be estimated daily on the basis of immediately past return data. The result is a time series of parameters that characterize the distribution of price motion. In principle each parameter has limits set by where the other parameters are. In the simplest terms with just two parameters of mean and variance we understand that the mean is risk compensation for the risk represented by the variance. But equally given the mean the variance can get too small relative to this mean for otherwise the return would become too much of a good deal. Hence one may employ the time series of parameters to estimate how each parameter is potentially responding to the other. The relationships between parameters may however be quite nonlinear as they are entities of different scales. With a view to matching the scales one may consider the n^{th} root of the n^{th} moment. However, such a procedure requires access to densities and their moments that can get quite complicated. Furthermore, one is aggregating across all sizes and not separating the upward and downward motions. We work instead with the n^{th} root of the n^{th} power variation taken separately for the upward and downward motions. For the bilateral gamma model the first two variations on each side suffice. For power variation with finite variation on each side the first second and third variations on each side suffice. In the case of infinite variation we use the second, third and fourth variations on each side. In each case we first determine a one to one mapping from the parameters to the variations being employed and back.

We denote the variations being employed as

$$\begin{aligned} m_n^p &= \left(\int_0^\infty x^n k(x) dx \right)^{\frac{1}{n}} \\ m_n^n &= \left(\int_0^\infty x^n k(-x) dx \right)^{\frac{1}{n}} \end{aligned}$$

For the bilateral gamma we have

$$\begin{aligned} m_1^p &= b_p c_p \\ m_2^p &= b_p \sqrt{c_p} \\ m_1^n &= b_n c_n \\ m_2^n &= b_n \sqrt{c_n} \end{aligned}$$

and the reverse mapping is

$$\begin{aligned} c_p &= \left(\frac{m_1^p}{m_2^p} \right)^2 \\ b_p &= \frac{(m_2^p)^2}{m_1^p} \\ c_n &= \left(\frac{m_1^n}{m_2^n} \right)^2 \\ b_n &= \frac{(m_2^n)^2}{m_1^n}. \end{aligned}$$

For power variation in the finite variation case we have on the positive side, with similar expressions for the negative side, and suppressing the subscript p on the b, c, y parameters and the superscript p on the variations, that

$$\begin{aligned} m_1 &= b^{1-y} (c\Gamma(1-y)) \\ m_2 &= b^{1-y/2} (c\Gamma(2-y))^{\frac{1}{2}} \\ m_3 &= b^{1-y/3} (c\Gamma(3-y))^{\frac{1}{3}} \end{aligned}$$

It follows that

$$\begin{aligned} \log \left(\frac{m_3^3 m_1}{m_2^4} \right) &= \ln \Gamma(3-y) + \ln \Gamma(1-y) - 2 \ln \Gamma(2-y) \\ b &= \frac{m_2^2 \Gamma(1-y)}{m_1 \Gamma(2-y)} \\ c &= \frac{m_1}{b^{1-y} \Gamma(1-y)}. \end{aligned}$$

For power variation and infinite variation we have

$$\begin{aligned} m_2 &= b^{1-y/2} (c\Gamma(2-y))^{\frac{1}{2}} \\ m_3 &= b^{1-y/3} (c\Gamma(3-y))^{\frac{1}{3}} \\ m_4 &= b^{1-y/4} (c\Gamma(4-y))^{\frac{1}{4}} \end{aligned}$$

and for the reverse

So

$$\begin{aligned} \log \left(\frac{m_4^4 m_2^2}{m_3^6} \right) &= \log(\Gamma(4-y)) + \log(\Gamma(2-y)) - 2 \log(\Gamma(3-y)) \\ b &= \frac{\Gamma(2-y) m_3^3}{\Gamma(3-y) m_2^2} \\ c &= \frac{m_2^2}{b^{2-y} \Gamma(2-y)}. \end{aligned}$$

Given a time series of parameters we may employ linear regression to estimate each parameter as a linear function of the others with a constant term estimated by the vector κ for all the variations used. One may then seek a fixed point of these response functions for the variations v . The result is an equation system of the form

$$(I - B)v = \kappa$$

with fixed point solution for the variations of

$$v = (1 - B)^{-1} \kappa.$$

The equilibrium variations may then be transformed into equilibrium parameters using the inverse mappings outlined. This generally works well but on occasion the matrix being inverted may be ill conditioned with solutions producing outliers that may be deleted or interpolated from the other good points.

With a view towards bypassing these inversion issues we replaced the linear regression by a nonlinear support vector machine regression based on a Gaussian kernel. These nonlinear predictions are generally superior to linear regression and more importantly map into a compact range that is appropriate for variations derived from parameters of motion. The result is a nonlinear response mapping of the form

$$v' = F(v)$$

and a fixed point is easily as

$$v^* = \lim_{n \rightarrow \infty} F^n(v_0)$$

where F^n is an n -fold composition of F with itself. For a starting value we used the median point in the data set employed in estimating F . We employed the past 126 days or six months of data on parameter values and variations to estimate the support vector machine regressions delivering the response function F . Two subsections present the results on equilibrium return distributions using bilateral gamma parameters, power variation parameters with finite variation imposed. Infinite variation was not observed in the physical process and this case is taken up risk neutrally.

5.1 Bilateral Gamma Equilibrium Distributions

Figure 14 presents a graph of the four bilateral gamma parameters as they were estimated from the time series data on the ticker *SPY* along with the equilibrium levels based on three months of support vector machine regressions and six months of linear regressions.

We also present in four Figures 15, 16, 17 and 18 with four parameter values and the two associated equilibrium values at intervals of a month or 21 days.

Figure 14: Bilateral Gamma parameter Values in Black and the support vector machine regression equilibrium values in red. Also presented are equilibrium values based on linear regression in blue.

5.2 Power Variation under Finite Variation

For power variation equilibrium distributions of finite variation results presented are for the ticker *MSFT*. Figure 19 presents graphs of the six parameters and the two equilibrium constructions.

Additionally we present Figures 20 and 21 providing the estimated and equilibrium values for the finite variation time scaling coefficients.

6 Equilibrium Volatility Surfaces

Risk neutrally one works with compensated jump martingales adjusted by an external drift to fit the forward price. Here we allow for infinite variation and present in this section the equilibrium volatility surfaces for power variation in the infinite variation context. The option pricing model is extended away from a Lévy process to that of a time scaled self decomposable law with an additional scaling parameter. In developing the equilibrium levels we added this scaling parameter into the linear and support vector machine parametric response regressions before solving for the required fixed points.

The results presented are for the ticker *AMZN*. Figure 22 presents the graphs for the data and the equilibrium on the six power variation parameters. Figure 23 present the data and equilibrium for the time scaling parameter.

We observe that the equilibrium construction smooths out many extremal observations. The equilibrium estimation of the power variation parameters is somewhat noisy but here we used an interpolation procedure for the estimation

Scale Up in Basis Points			
dates	data	rsvm	linear
20160705	56.6688	55.2337	54.8597
20160803	56.0323	55.2900	55.4632
20160901	52.8721	56.6588	55.4128
20161003	47.0783	55.9848	54.0870
20161101	43.7929	49.3531	50.7198
20161201	44.6093	47.9262	48.9162
20170103	51.0365	51.2783	50.9231
20170202	34.9028	47.0671	49.0542
20170306	29.1535	46.0736	44.4320
20170404	28.3679	28.9968	40.4409
20170504	27.4871	28.7808	37.7133
20170605	23.4722	28.1506	32.7626
20170705	17.9729	25.3980	28.4060
20170803	17.3575	20.5326	25.3334
20170901	19.4758	20.6744	22.2434
20171003	3.5550	17.2317	20.9757
20171101	3.3070	17.6325	18.0325
20171201	4.0786	5.0293	12.7998

Figure 15: Scale up in basis points at monthly intervals along with support vector machine regression and linear regression equilibrium values based on three and six months of past parameter estimates.

	Speed Up		
dates	data	rsvm	linear
20160705	1.7900	1.7721	1.7638
20160803	1.7335	1.7826	1.7665
20160901	1.6545	1.6669	1.7483
20161003	1.8656	1.6572	1.7303
20161101	1.8466	1.7915	1.7562
20161201	1.7846	1.7630	1.7677
20170103	1.3199	1.4234	1.5584
20170202	1.9628	1.4933	1.5608
20170306	2.2682	1.4893	1.5598
20170404	2.2725	2.2388	1.6641
20170504	2.3368	2.2565	1.7870
20170605	2.8311	2.2802	2.0210
20170705	4.2562	2.5502	2.3247
20170803	4.1801	3.3457	2.6174
20170901	3.3915	3.4093	3.0160
20171003	65.3316	4.3146	3.2503
20171101	19.4531	4.2712	4.0449
20171201	7.9364	16.8505	5.9183

Figure 16: Speed up at monthly intervals along with support vector machine regression and linear regression equilibrium values based on three and six months of past parameter estimates.

Scale Down in Basis Points			
dates	data	rsvm	linear
20160705	97.5166	92.4537	92.1008
20160803	104.5373	94.5953	93.9652
20160901	93.7104	111.9504	96.0882
20161003	84.4108	99.7066	95.3995
20161101	87.5904	95.6318	92.6894
20161201	87.7581	87.7544	91.4929
20170103	4.8382	24.8247	59.4491
20170202	42.4881	15.2961	50.0202
20170306	31.9689	12.0869	23.1589
20170404	26.6133	28.5216	19.4829
20170504	12.0212	26.9024	23.9987
20170605	6.4355	19.8677	18.6010
20170705	2.1819	13.6362	17.2025
20170803	4.9405	7.2429	14.4124
20170901	8.1844	5.6022	9.1565
20171003	3.0685	3.1646	6.9629
20171101	5.2836	3.1877	4.9735
20171201	1705.8958	5.7175	4.5978

Figure 17: Scale down up in basis points at monthly intervals along with support vector machine regression and linear regression equilibrium values based on three and six months of past parameter estimates.

Speed Down			
dates	data	rsvm	linear
20160705	0.7000	0.7090	0.7051
20160803	0.5964	0.6931	0.6975
20160901	0.5018	0.5156	0.6570
20161003	0.5876	0.5296	0.6034
20161101	0.5203	0.5041	0.5584
20161201	0.4676	0.5217	0.5336
20170103	0.9014	0.3256	0.5244
20170202	0.5395	0.5617	0.5296
20170306	0.6689	0.7688	0.5990
20170404	0.7522	0.7223	0.6772
20170504	1.4506	0.7633	0.6848
20170605	3.1123	0.9646	0.9192
20170705	14.8612	1.4015	1.1367
20170803	6.1002	3.3534	1.4765
20170901	2.7761	4.6725	2.4932
20171003	63.6707	9.6594	3.4505
20171101	6.0913	9.7170	5.9497
20171201	0.0000	8.4147	7.6773

Figure 18: Speed down up points at monthly intervals along with support vector machine regression and linear regression equilibrium values based on three and six months of past parameter estimates.

Figure 19: Power Variation parameter Values in Black and the support vector machine regression finite variation equilibrium values in red. Also presented are equilibrium values based on linear regression in blue.

of this parameter as opposed to a proper equation solve. We anticipate better results with a proper equation solve.

7 Asset Pricing Implications

Classically asset pricing models seek to explain differences in expected returns in terms of return covariations with systematic factors representing undiversifiable risks. Alternatively, the absence of arbitrage links to martingale conditions under a measure change and it is the covariation with the change of measure density that is critical in explaining expected returns. The two may be linked by expressing the return on the change of measure density as being driven by candidate systematic factor selections in constructing multifactor arbitrage pricing models. Both the equilibrium and arbitrage pricing approaches to asset pricing take an economy wide perspective seeking portfolios that maximize expected utilities or seek arbitrage opportunities. The asset pricing theory is an implication of economy wide perspectives defining efficient portfolios or excluding arbitrages. Explaining the cross section of expected returns has been a classical objective in finance theory.

The focus of attention in this section is somewhat different. We seek to explain, not expected returns, but the full equilibrium return distribution. The equilibrium return distribution construction recognizes that each aspect of the return distribution is continuously responding to and reacting to the other aspects. The question then is to ask what equilibrium return distributions can prevail for asset returns taken one asset at a time for a fixed and small return horizon. Generally, covariations and correlations occur over the longer time

Power Variation Up			
dates	data	rsvm	linear
20160705	0.0105	0.0310	0.0440
20160803	0.0153	0.0230	0.0420
20160901	0.0118	0.0240	0.0290
20161003	0.0139	0.0610	0.0010
20161101	0.0181	0.0100	0.0010
20161201	0.0160	0.0300	0.0200
20170103	0.0198	0.0140	0.0160
20170202	0.0179	0.0250	0.0220
20170306	0.0235	0.0210	0.0240
20170404	0.0342	0.0020	0.0010
20170504	0.0426	0.0350	0.0010
20170605	0.5563	0.0620	0.0480
20170705	0.0266	0.5270	0.0900
20170803	0.0370	0.3860	0.1490
20170901	0.0342	0.2400	0.0630
20171003	0.0254	0.0410	0.0640
20171101	0.0191	0.0300	0.0680
20171201	0.0161	0.0010	0.0360

Figure 20: Power Variation up at monthly intervals along with support vector machine regression and linear regression equilibrium values based on three and six months of past parameter estimates.

Power Variation Down			
dates	data	rsvm	linear
20160705	0.0020	0.0160	0.0240
20160803	0.0049	0.0210	0.0200
20160901	0.0029	0.0030	0.0180
20161003	0.0058	0.0110	0.0010
20161101	0.0095	0.0010	0.0010
20161201	0.0086	0.0010	0.0110
20170103	0.0124	0.0130	0.0070
20170202	0.0124	0.0130	0.0110
20170306	0.0134	0.0450	0.0150
20170404	0.0202	0.0150	0.0010
20170504	0.0266	0.0600	0.0010
20170605	0.4593	0.1310	0.0350
20170705	0.0156	0.2940	0.0590
20170803	0.0171	0.2680	0.1030
20170901	0.0154	0.2470	0.0380
20171003	0.0085	0.0190	0.0410
20171101	0.0055	0.0210	0.0420
20171201	0.0040	0.0320	0.0170

Figure 21: Power Variation down at monthly intervals along with support vector machine regression and linear regression equilibrium values based on three and six months of past parameter estimates.

Figure 22: Power Variation risk neutral parameter Values in Black and the support vector machine regression finite variation equilibrium values in red.

steps that permit markets to react to each other. Portfolio and diversification considerations are then important as devices for controlling or managing the risk. Over short horizons risk exposure is being limited one asset at a time by reducing the holding period and portfolio considerations are less important. The correlations may also be reduced at the lower horizons. Many trading strategies with short holding periods seek to bias the gain probability in ones direction slightly to then benefit from doing a very large number of such trades. The assessment of the gain probability is done one asset at a time with little attention to portfolio considerations. Sometimes, as in pairs trading, one may consider a groups of assets taken together to seek a gain probability for a particular portfolio position and the attention shifts to the portfolio return distribution and we could add equilibrium portfolio returns distributions to our collection of equilibrium return distributions being studied. To some extent this is already present in the equilibrium distribution for an index or an exchange traded fund.

What collections of equilibrium return distributions do we expect to encounter in financial markets is our question. The question may be asked of both the physical return distributions and the risk neutral ones. For tractability we condition on fixed parametric classes of distributions that are here taken to be the bilateral gamma and power variation models already reported on. For each such distribution one may evaluate an arbitrage consistent performance measure. Here we employ the acceptability index introduced in Cherny and Madan (2009). The acceptability index is infinite for an arbitrage. It is defined by an increasing family $\mathcal{D}_\gamma$ of test probability measures $Q \in \mathcal{D}_\gamma$ with a random variable X being acceptable at level γ just if

$$E^Q[X] \geq 0, \text{ for all } Q \in \mathcal{D}_\gamma.$$

Figure 23: The data and equilibrium time scaling parameter values for amzn.

The acceptability index is the highest level $\gamma^*(X)$ such that X is acceptable for all $\gamma \leq \gamma^*$. The family of test measures increases towards the set of all measures and only an arbitrage is acceptable at all levels $\gamma > 0$.

For an increasing sequence of concave distribution functions $\Psi^\gamma(u)$, $0 \leq u \leq 1$, Cherny and Madan (2009) and Madan, Pistorius and Stadjje (2017) introduced for a base probability measure P , the sets

$$\mathcal{D}_\gamma = \{Q \mid Q(A) \leq \Psi^\gamma(P(A)), \text{ for all } A\}$$

for which acceptability at level γ for a distribution function $F_X(x)$ reduces to a positive distorted expectation or

$$\mathcal{E}^\gamma(X) = \int_{-\infty}^{\infty} x d\Psi^\gamma(F_X(x)) \geq 0. \quad (2)$$

The particular concave distortion we employ is the distortion termed *minmaxvar* in Cherny and Madan (2009) where we have

$$\Psi^\gamma(u) = 1 - \left(1 - u^{\frac{1}{1+\gamma}}\right)^{1+\gamma}.$$

The acceptability index based on the distortion minmaxvar is the highest stress level γ such that the distorted expectation $\mathcal{E}^\gamma(X)$ (2) is nonnegative. We anticipate that expected returns for equilibrium return distributions observed in the market are positive. As a consequence the acceptability indices for these distributions will be positive. However, we do not expect the acceptability indices of equilibrium return distributions observed in markets to get to high or be anywhere near an arbitrage.

For each of our 50 underliers we evaluated for each of 438 days the acceptability index for the equilibrium return distribution in the bilateral gamma class. We thus obtained a 438 by 50 matrix of acceptability indices. For each day we evaluated the quartiles across the 50 tickers of the equilibrium acceptability indices. Figure 24 presents these quartiles over time.

We observe that over this period the acceptability indices have risen from a median value 0.1 towards a median near 0.25. The interquartile range near the end of the data period is between 0.15 and 0.3.

The acceptability index of a return distribution has implications for the level of expected returns and how they are related to the risk. Furthermore there are implications for how this risk is to be measured. Consider the distorted expectation $\mathcal{E}^\gamma(X)$. The distorted expectation by construction is

$$\mathcal{E}^\gamma(X) = \inf_{Q \in \mathcal{D}_\gamma} E^Q[X].$$

It follows that

$$\mathcal{E}^\gamma(X) = E^P[X] + \mathcal{E}^\gamma(X - E^P[X]).$$

We may then write the distorted expectation in the form of reward less risk by noting that

$$\begin{aligned} \mathcal{E}^\gamma(X) &= E^P[X] - (-\mathcal{E}^\gamma(-(E^P[X] - X))) \\ &= E^P[X] - \sup_{Q \in \mathcal{D}_\gamma} (E^Q[X] - X). \end{aligned}$$

Figure 24: Quartiles for the equilibrium acceptability indices.

The risk is now

$$\mathcal{R}_\gamma(X) = \sup_{Q \in \mathcal{D}_\gamma} (E^P[X] - X).$$

From the perspective of two price economies this risk is the ask price for the negated and centered random variable. Basically the centered random variable has no reward and is just risk that can be eliminated by buying its negative at the cost of the ask price for it. Hence the ask price for the negated centered random variable is the financial risk.

At the level of the acceptability index γ^* it is the case that $\mathcal{E}^{\gamma^*}(X) = 0$ and hence we have that

$$E^P[X] = \mathcal{R}_{\gamma^*}(X). \quad (3)$$

This is a distributionally based asset pricing equation where required returns are equal to the risk as measured by the ask price for the negated and centered random variate. One may hypothesize that each day there exists a market target γ_t^* that best explains the cross section of expected returns as this measure of risk for the said return. The level of γ_t^* may be estimated from the cross section of equilibrium return distributions for the day.

For the equilibrium return distributions in the bilateral gamma class constructed using support vector machine regressions applied to three months of past data we estimate the cross sectional relation

$$\mu_{it} = \mathcal{R}_{\gamma_t^*}(R_{it}) + \varepsilon_{it}$$

where μ_{it} is the mean return on the i^{th} equilibrium return distribution on day t and R_{it} is the i^{th} bilateral gamma distributed on day t . The estimation was conducted by least squares. Figure 25 presents the time series for γ_t^* .

We observe that the stress level consistent with the cross sectional asset pricing equation has been increasing over the period but was generally below

Figure 25: Stress level implied by cross sectional asset pricing model based on ask price for negated centered risk.

the median level. We regressed the mean returns cross sectionally on the risk charges $\mathcal{R}_{\gamma_t^*}(R_{it})$ to test for a unit slope and a zero intercept. Both these hypotheses were rejected. Figures 26 and 27 present the associated t-statistics.

Figure 28 presents the t-statistics for a zero slope. The zero slope is rejected in the latter period.

8 Excitation and Cross Excitation

The model with excitation and cross excitation requires a large number of ordinary differential equations to be solved with the output that then has to be sent for Fourier inversion. The estimation takes a few hours so it was done for 8 days on the first four underliers. Figure 29 presents the results for the time series data. There is evidence of some excitation but little cross excitation. The results for the risk neutral estimation are presented in Figures 8 and 8. Figure 8 presents the dates and underliers for which the calibration converged to an absolute percentage error below 10%, along with the fit statistics and the number of options employed with maturities below a month. Figure 8 presents the parameter estimates for the power variation model enhanced to allow for excitation and cross excitation. We observe that risk neutrally speed and scale are higher for the down moves as is the level of power variation. Also the down side has greater excitation and cross excitation.

Figure 26: t-statistics for a zero intercept cross sectionally each day.

Figure 27: t-statistics for a unit slope coefficient.

Figure 28: t-statistic for a zero slope.

9 Conclusion

Asset return distributions are taken to belong to the class of pure jump limit laws. Asymmetries between upward and downward motions are identified and documented. The return distributions are then modeled by self decomposable parametric laws where all the parameters are seen to be continuously responding to each other. Fixed points of the response functions define equilibrium distributions. The distributions that can arise in practice are seen to be constrained by the level of return acceptability they may attain. As a consequence expected returns reflect risk charges embedded in the ask price for the centered and negated return. The asymmetries studied include differences in scale, speed, power variation, excitation and cross excitation.

Power Variation with Excitation and Cross Excitation											
	dates	bp	cp	bn	cn	yp	yn	dp	dn	ep	en
spy	20160104	37.1824	2.1246	64.7113	0.7162	0.0106	0.0007	0.1257	0.2702	0.0631	0.1289
	20160405	38.3210	1.9392	68.6987	0.7202	0.0255	0.0031	0.1089	0.1958	0.0817	0.1168
	20160705	37.4567	1.9662	69.2027	0.6848	0.0242	0.0025	0.1106	0.2013	0.0778	0.1191
	20161003	37.0723	2.0638	74.3142	0.5970	0.0120	0.0006	0.1387	0.2827	0.0650	0.1279
	20170103	36.6645	1.9789	71.8798	0.5848	0.0173	0.0009	0.1300	0.2579	0.0692	0.1223
	20170404	37.8021	1.8319	71.0658	0.5267	0.0178	0.0013	0.1298	0.2625	0.0648	0.1307
	20170705	49.7614	1.1334	2.7343	1.4442	0.0289	0.0360	0.1050	0.1017	0.0987	0.1031
	20171003	37.4607	1.5181	65.7003	0.4269	0.0359	0.0056	0.1026	0.1970	0.0726	0.1259
iwm	20160104	7.3047	16.4973	21.8410	14.4071	0.1383	0.0054	0.0524	0.1531	0.0783	0.1151
	20160405	27.1545	4.8622	55.3686	2.1844	0.0233	0.0001	0.6245	0.2217	1.0847	0.2910
	20160705	25.9480	4.9156	54.6635	2.2065	0.0293	0.0002	0.2291	0.8613	0.0283	0.0753
	20161003	27.0693	4.5928	59.6226	1.8808	0.0254	0.0001	0.3722	1.5110	0.0084	0.0510
	20170103	25.6747	5.1147	57.0495	2.0686	0.0238	0.0000	1.4511	1.1719	1.5395	3.0996
	20170404	25.5070	5.1100	50.3895	2.5278	0.0331	0.0004	0.2415	0.8201	0.0209	0.0730
	20170705	29.7803	3.9268	53.5041	2.0816	0.0348	0.0021	0.0993	0.2461	0.0722	0.1086
	20171003	30.8578	3.6658	56.4464	1.8377	0.0318	0.0020	0.1029	0.2334	0.0751	0.1091
qqq	20160104	41.2941	2.2150	77.4032	0.7677	0.0111	0.0005	0.1616	0.2845	0.0740	0.1189
	20160405	45.7831	1.9302	85.3603	0.6889	0.0159	0.0028	0.1507	0.2259	0.0853	0.1140
	20160705	46.1286	1.8821	86.7764	0.6357	0.0145	0.0019	0.1486	0.2374	0.0830	0.1179
	20161003	44.7814	1.8891	89.6698	0.5889	0.0148	0.0010	0.1499	0.2670	0.0794	0.1199
	20170103	43.2314	1.9415	88.6391	0.5870	0.0145	0.0007	0.1456	0.2884	0.0748	0.1234
	20170404	42.6551	1.9306	92.5252	0.5023	0.0099	0.0002	0.1307	0.3498	0.0495	0.1443
	20170705	44.2784	1.7560	95.6202	0.4429	0.0136	0.0003	0.1784	0.3504	0.0601	0.1324
	20171003	44.1175	1.7877	95.0997	0.4352	0.0099	0.0002	0.1819	0.3580	0.0542	0.1376
aapl	20160104	78.4674	1.9515	92.1684	1.3397	0.0176	0.0121	0.1257	0.1596	0.0999	0.0979
	20160405	80.7389	1.7919	99.4135	1.2155	0.0232	0.0147	0.1070	0.1414	0.1009	0.0983
	20160705	73.1112	2.0850	94.5364	1.3371	0.0156	0.0097	0.1281	0.1814	0.0979	0.0996
	20161003	77.8730	1.8801	97.5428	1.2171	0.0151	0.0099	0.1318	0.1863	0.0952	0.1035
	20170103	72.8947	1.8933	93.6449	1.1582	0.0183	0.0126	0.1252	0.1631	0.0988	0.1002
	20170404	74.7961	1.7009	100.0083	0.9307	0.0145	0.0092	0.1422	0.1858	0.0976	0.1025
	20170705	76.4239	1.5958	105.0884	0.8228	0.0117	0.0067	0.1569	0.2058	0.0973	0.1037
	20171003	73.5527	1.6610	102.3910	0.8339	0.0070	0.0037	0.1811	0.2397	0.0985	0.1029

Figure 29: Parameter estimates for power variation extended to include excitation and cross excitation.

Risk Neutral Results for PVECE					
dates	underlier	rmse	aae	ape	nop
20160104	spy	0.2999	0.2126	0.0788	1008
20160405	spy	0.2144	0.1766	0.0631	1015
20161003	spy	0.1721	0.1375	0.0579	1164
20170404	spy	0.1866	0.1467	0.0672	1165
20170404	iwm	0.1278	0.1055	0.0580	1114
20160104	iwm	0.1647	0.1357	0.0670	781
20171003	iwm	0.2106	0.1826	0.0935	989

PVECE Risk Neutral Parameter estimates									
cp	cn	bp	bn	yp	yn	dp	dn	ep	en
0.1415	0.2969	0.0029	0.2076	0.0591	0.8900	0.0100	0.5338	0.0796	0.0252
0.0165	0.0656	0.4614	0.4498	0.0582	1.1598	0.0422	0.1010	0.7245	0.0002
0.0195	0.0954	0.5634	0.2290	0.0317	1.0564	0.0044	0.5630	0.9828	0.7648
0.0022	0.0218	1.4900	0.6309	0.0258	1.3664	2.2335	0.1364	0.2215	1.2225
0.9303	0.0770	0.0303	0.3140	0.0731	1.2614	0.0539	0.1093	3.1071	0.6385
0.0010	0.0226	0.0076	7.8068	0.0315	1.6293	0.0459	0.0302	0.3733	0.7681
0.0004	0.0114	0.0121	15.8016	0.0267	1.6592	0.0399	0.0135	0.6610	1.1446

APPENDIX

The ticker symbols for the underliers reported on are presented in Figure 30.

Underlying Asset Ticker Symbols				
spy	msft	wfc	gm	gild
iwm	nflx	dis	v	twtr
qqq	c	gs	bidu	wynn
aapl	amzn	xlf	xom	pypl
fb	eem	t	cmg	ge
nvda	tsla	xle	ndx	csc
baba	intc	rut	gdx	goog
mu	xop	wmt	amd	xl
gld	ewz	jd	dia	hd
bac	jpm	ba	googl	cmcsa

Figure 30: Ticker Symbols for the Assets reported on for both the physical returns distributions and their risk neutral counterparts.

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